Theorem 4.16. For any $A \in \mathbb{C}^{m \times n}$ we have

$$\|A\|_2^2 = \|A^*\|_2^2 = \|A^*A\|_2 = \|AA^*\|_2 = \lambda_{\text{max}}$$

where $\lambda_{\text{max}}$ is the largest eigenvalue of both $A^*A$ and $AA^*$.

In the proof, $\|\cdot\|$ will always denote the 2-norm.

Lemma. For every vector $z \in \mathbb{C}^n$ we have $\|z\| = \sup_{\|y\|=1} |\langle y, z \rangle|$.

Proof. Indeed, by the Cauchy-Schwarz inequality

$$|\langle y, z \rangle| \leq \langle y, y \rangle^{1/2} \langle z, z \rangle^{1/2} = \|z\|$$

and the equality is attained whenever $y$ is parallel to $z$, we can set $y = \pm \frac{z}{\|z\|}$. \hfill \Box

Step 1. To prove that $\|A\| = \|A^*\|$ we write

$$\|A\| = \sup_{\|x\|=1} \|Ax\| = \sup_{\|x\|=1} \sup_{\|y\|=1} |\langle y, Ax \rangle| = \sup_{\|x\|=1} \sup_{\|y\|=1} |\langle A^*y, x \rangle|$$

$$= \sup_{\|y\|=1} \sup_{\|x\|=1} |\langle x, A^*y \rangle| = \sup_{\|y\|=1} \|A^*x\| = \|A^*\|$$

Step 2. To prove that $\|A\|^2 = \|A^*A\|$ we write

$$\|A^*A\| = \sup_{\|x\|=1} \|A^*Ax\| = \sup_{\|x\|=1} \sup_{\|y\|=1} |\langle y, A^*Ax \rangle| = \sup_{\|x\|=1} \sup_{\|y\|=1} |\langle Ay, Ax \rangle|$$

Then again by the Cauchy-Schwarz inequality

$$|\langle Ay, Ax \rangle| \leq \|Ay\| \|Ax\| \leq \|A\| \|A\| = \|A\|^2$$

hence $\|A^*A\| \leq \|A\|^2$. On the other hand, setting $x = y$ gives

$$\sup_{\|x\|=1} \sup_{\|y\|=1} |\langle Ay, Ax \rangle| \leq \sup_{\|x\|=1} |\langle Ax, Ax \rangle| = \|A\|^2,$$

hence $\|A^*A\| \geq \|A\|^2$. Therefore, $\|A^*A\| = \|A\|^2$.

Step 3. Using an obvious symmetry we conclude that $\|A^*\|^2 = \|AA^*\|$

Lemma. Let $B$ be a Hermitian matrix. Then

$$\|B\| = \max_{1 \leq i \leq n} |\lambda_i|,$$

where $\lambda_1, \ldots, \lambda_n$ denote the eigenvalues of $B$. 

Proof. By the Spectral Theorem, \( B = Q^* \Lambda Q \), where \( Q \) is a unitary matrix and \( \Lambda \) a diagonal matrix whose diagonal entries are \( \lambda_i \)'s. We know (from earlier homework assignments) that

\[
\|B\| = \|Q^* \Lambda Q\| = \|\Lambda\|.
\]

Now for any vector \( x = (x_1, \ldots, x_n) \) we have \( \Lambda x = (\lambda_1 x_1, \ldots, \lambda_n x_n) \), hence

\[
\|\Lambda x\|^2 = |\lambda_1|^2 |x_1|^2 + \cdots + |\lambda_n|^2 |x_n|^2
\]

Now if \( \|x\| = 1 \), then

\[
\|\Lambda x\|^2 \leq \max_{1 \leq i \leq n} |\lambda_i|^2
\]

On the other hand, if \( |\lambda_j| = \max_{1 \leq i \leq n} |\lambda_i| \) then we choose \( x = e_j \) and obtain \( \|\Lambda x\|^2 = |\lambda_j|^2 \). Lemma is proven. \( \square \)

This completes the proof of 4.16. Note that \( A^*A \) and \( AA^* \) are positive-semidefinite matrices, so their eigenvalues are \( \geq 0 \), so \( \max_{1 \leq i \leq n} |\lambda_i| \) is simply the largest eigenvalue, we denote it by \( \lambda_{\text{max}} \).

A little modification of the previous Lemma:

**Lemma.** Let \( B \) be a Hermitian positive-semidefinite matrix with eigenvalues \( \lambda_1, \ldots, \lambda_n \). Then

\[
\sup_{\|x\| = 1} \langle Bx, x \rangle = \lambda_{\text{max}} = \max_{1 \leq i \leq n} \lambda_i,
\]

and if \( x \) is a vector such that

\[
\|x\| = 1 \quad \text{and} \quad \langle Bx, x \rangle = \lambda_{\text{max}},
\]

then \( x \) is a corresponding eigenvector: \( Bx = \lambda_{\text{max}} x \).

**Proof.** Again, we use the Spectral Theorem to reduce the problem to a diagonal matrix \( \Lambda \), then the proof is just a direct inspection. \( \square \)

**Corollary.** If \( \lambda_{\text{max}} \) again denotes the largest eigenvalue of \( A^*A \), then

\[
\|Ax\|_2 = \|A\|_2 \|x\|_2 \quad \iff \quad A^*Ax = \lambda_{\text{max}} x.
\]

**Proof.** On the one hand

\[
\|Ax\|^2 = \langle Ax, Ax \rangle = \langle A^*Ax, x \rangle
\]

and on the other hand

\[
\|A\|^2 = \lambda_{\text{max}},
\]

so for any vector \( x \) with \( \|x\| = 1 \) we have

\[
\|Ax\|_2 = \|A\|_2 \quad \iff \quad \langle A^*Ax, x \rangle = \lambda_{\text{max}}.
\]

Then we use the previous lemma. \( \square \)