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LAVAURS ALGORITHM FOR CUBIC SYMMETRIC POLYNOMIALS

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ABSTRACT. As discovered by W. Thurston, the action of a complex one-variable polynomial on its Julia set can be modeled by a geodesic lamination in the disk, provided that the Julia set is connected. It also turned out that the parameter space of such dynamical laminations of degree two gives a model for the bifurcation locus in the space of quadratic polynomials. This model is itself a geodesic lamination, the so called *quadratic minor lamination* of Thurston. In the same spirit, we consider the space of all *cubic symmetric polynomials* $f_\lambda(z) = z^3 + \lambda^2 z$ in three articles. In the first one we construct the *cubic symmetric comajor lamination* together with the corresponding quotient space of the unit circle. As is verified in the third paper, this yields a monotone model of the *cubic symmetric connectedness locus*, i.e., the space of all cubic symmetric polynomials with connected Julia sets. In the present paper, the second in the series, we develop an algorithm for generating the cubic symmetric comajor lamination analogous to the Lavaurs algorithm for constructing the quadratic minor lamination.

1. INTRODUCTION

We use standard notation ($\mathbb{R}, \mathbb{C}$ for the real/complex numbers, $\mathbb{D}$ for the unit disk centered at the origin, etc). The Riemann sphere is denoted by $\hat{\mathbb{C}}$. The boundary (in $\mathbb{C}$) of a set $X \subset \mathbb{C}$ is denoted by $\mathrm{Bd}(X)$. We consider *only complex* polynomials P; for such a P, let J_P be its Julia set and K_P be its filled Julia set. We normalize the circle so that its length is 1, and identify numbers of $[0, 1)$ with points on the circle and with the corresponding angles (so, we talk about the angle $\frac{1}{2}$ rather than angle π, etc). A *chord* is a closed straight line segment

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with endpoints on the unit circle $\mathbb{S} = \mathrm{Bd}(\mathbb{D})$. The reader is referred to [Mil06, Thu85] for basic notions of complex polynomial dynamics on $\mathbb{C}$, including Fatou and Julia sets, external rays, landing etc.

The connectedness locus $\mathcal{M}_d$ is the space of polynomials of degree d, up to affine conjugacy, with connected Julia sets. A fundamental problem is to understand the structure of $\mathcal{M}_d$. Major progress has been made for $d = 2$ but much less is known for $d > 2$. Thurston [Thu85] introduced *invariant laminations* to provide a combinatorial model for $\mathcal{M}_2$. A *lamination* $\mathcal{L}$ is a compact set of chords, called *leaves*, that are pairwise disjoint in $\mathbb{D}$ (equivalently, do not *cross*). Given a lamination $\mathcal{L}$, one can consider an equivalence relation $\sim_{\mathcal{L}}$ on $\mathbb{S}$ where $x, y \in \mathbb{S}$ are equivalent if there is a finite chain of leaves of $\mathcal{L}$ connecting x and y. If all $\sim_{\mathcal{L}}$-classes of equivalence are finite and all leaves of $\mathcal{L}$ are edges of their convex hulls, then we say that $\mathcal{L}$ is a *q-lamination*.

Thurston constructs the *quadratic minor lamination* QML whose leaves tag all invariant quadratic laminations (for $d \geq 2$, a lamination is *invariant* if it is invariant under the map $\sigma_d(z) = z^d$ restricted to $\mathbb{S}$, see Definition 2.6). He shows that QML is a q-lamination; moreover, the quotient space $\mathcal{M}_2^{Comb} = \mathbb{S}/QML$ of the unit circle $\mathbb{S}$ by the equivalence relation defined by QML is a monotone image of $\mathrm{Bd}(\mathcal{M}_2)$ (conjecturally, this map is a homeomorphism), cf. [Thu85]. No such models exist for $d > 2$.

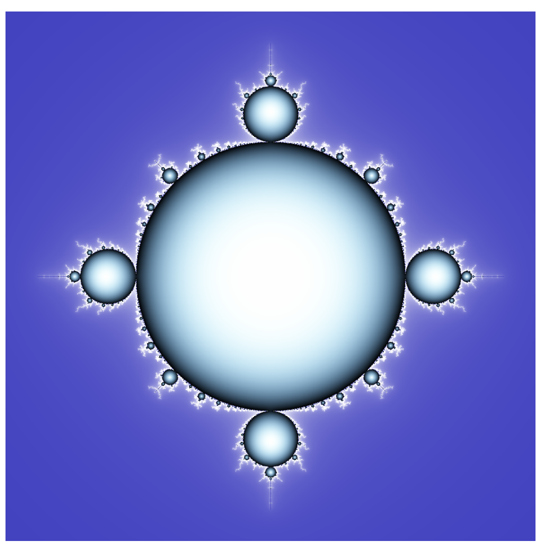

FIGURE 1. The parameter space of symmetric cubic polynomials $\mathcal{M}_{3,s}$.

A natural next object of study is $\mathcal{M}_3$, i.e., the space of all cubic polynomials with connected Julia sets, or its subspaces. Notice that polynomials from $\mathcal{M}_3$ are associated, in a natural fashion, with invariant cubic laminations. Similarly to the quadratic case, one can expect that to provide a model for a subspace of $\mathcal{M}_3$ one may need to describe the appropriate subspace of cubic laminations. We adopt this approach in a series of papers in which we consider *symmetric cubic polynomials* $P(z) = z^3 + \lambda^2 z$ with connected Julia sets; these form the *cubic symmetric connected locus* denoted by $\mathcal{M}_{3,s}$ (see Figure 1)

It is easy to see that the natural association between polynomials from $\mathcal{M}_{3,s}$ and their laminations leads to the space of all cubic invariant *symmetric* laminations defined as cubic invariant laminations that are also invariant under the map that sends each leaf ℓ to the leaf $-\ell$ (i.e., under the rotation of the unit circle by the angle π). In [BOSTV1] we define the "parametric" q-lamination C_sCL (this stands for *cubic symmetric comajor lamination*) together with the induced factor space $\mathbb{S}/C_sCL$ of the unit circle $\mathbb{S}$. This lamination parameterizes all cubic invariant symmetric laminations similar to how QML parameterizes all quadratic invariant laminations. Then, in [BOSTV3], we verify that $\mathbb{S}/C_sCL$ is a monotone model of $\mathcal{M}_{3,s}$.

The present paper is devoted to the construction of C_sCL and aims at understanding its structure and at obtaining suitable pictures of it. To this end, we obtain two main results. We state them here in the Introduction to make reading more focused and purposeful (we thank the referee for this suggestion).

Let us normalize the circle length to 1. For each chord $\ell = \overline{ab}$ let $|\ell|$ be the length of the shorter of the two circle arcs with endpoints a and b. Let c be a non-degenerate chord of length at most $\frac{1}{6}$. It is easy to see that there are two chords M_c and M_c' that are disjoint, have the same σ_3-image as c, and have lengths at least $\frac{1}{6}$. Denote by $\mathcal{S}(M_c)$ the component of $\mathbb{D} \setminus (M_c \cup M_c')$ that contains both M_c and M_c' in its boundary. Non-degenerate chords $\{c, -c\}$ of length at most $\frac{1}{6}$ such that the chords from the σ_3-orbits of c and $-c$ do not cross and never enter the set $\mathcal{S}(M_c) \cup -\mathcal{S}(M_c)$ form a *legal pair* (see Definition 3.18). A chord c such that $\{c, -c\}$ is a legal pair is said to be a *comajor*. .

The lamination C_sCL is formed by all legal pairs and is in fact a q-lamination [BOSTV1]. Consider a special subset of C_sCL that consists of *co-periodic* comajors (a leaf is *co-periodic* if it is not periodic but its image is). In the first main result of the paper, Theorem 4.13, we show that co-periodic comajors are dense in the entire C_sCL. To state our second main result we need some definitions.

A $2n$-periodic point x of σ_3 with $\sigma_3^n(x) = -x$ is said to be *of type B*. All other periodic points of σ_3 are said to be *of type D*. For example, $\frac{1}{4}$ is a periodic point of type B while $\frac{1}{26}$ is a periodic point of type D. Our choice of symbols B and D for the types of periodic orbits follows Milnor's notation for polynomial hyperbolic components and stands for Bi-transitive and Disjoint, respectively. A periodic leaf of a symmetric lamination is *of type B* if its endpoints are of type B, and *of type D* if its endpoints are of type D. By Lemma 5.9 all periodic leaves of symmetric laminations are of type B or of type D. A co-periodic leaf of a symmetric lamination is *of type B* if its image is a periodic leaf of type B, and *of type D* otherwise. A periodic point (leaf) of type B and period $2n$ has *block period* n; so, $\frac{1}{4}$ is a periodic point of period 2 but of block period 1. A periodic point (leaf) of type D and period n has *block period* n; so, $\frac{1}{26}$ is a periodic point of period 3 and block period 3. A co-periodic leaf is said to be *of block period* n if its image is of block period n.

Given a chord $\ell = \overline{ab}$ with $|\ell| < \frac{1}{2}$, set $H(\ell)$ to be a circle arc of length $|\ell|$ with endpoints a and b. If ℓ and ℓ' are chords disjoint inside $\mathbb{D}$ with $H(\ell') \subset H(\ell)$ then we write $\ell' \prec \ell$. Suppose that co-periodic comajors c and c' are such that $c' \prec c$, both c and c' are either of type B or type D, and c and c' have the same block period n. In Theorem 5.13 we prove that then there exists a co-periodic comajor d with $c' \prec d \prec c$ such that d is of block period $j < n$. This yields an algorithm allowing one to inductively construct the family of all co-periodic comajors (dense in C_sCL as we know). We call it the *L-algorithm*.

The L-algorithm is similar to the famous Lavaurs algorithm [Lav86, Lav89] that defines a dense (in QML) set of pairwise disjoint σ_2-periodic chords. The co-periodic comajors play for C_sCL the same role as the periodic minors for QML. In a nutshell, the L-algorithm is as follows. Start with marking the co-periodic comajors of block period 1, namely, the chords,

$$\overline{\frac{1}{6}\frac{1}{3}}, \quad \overline{\frac{2}{3}\frac{5}{6}}, \quad \overline{\frac{5}{12}\frac{7}{12}}, \quad \overline{\frac{11}{12}\frac{1}{12}}.$$

Of these four leaves, the first two are of type D, and the last two are of type B, cf. Figure 2. Once all co-periodic comajors of block periods from 1 to $k-1$ are constructed, generate those of block period k as follows. Mark all type D points of preperiod 1 and block period k. Next, connect these points consecutively while bypassing the already generated leaves. Similarly, mark type B points of block period k and connect them. Every time there is a choice between longer connections and shorter ones, the latter must be preferred. Details are given in by Theorem 5.13.

Figure 2 shows three initial steps of the construction.

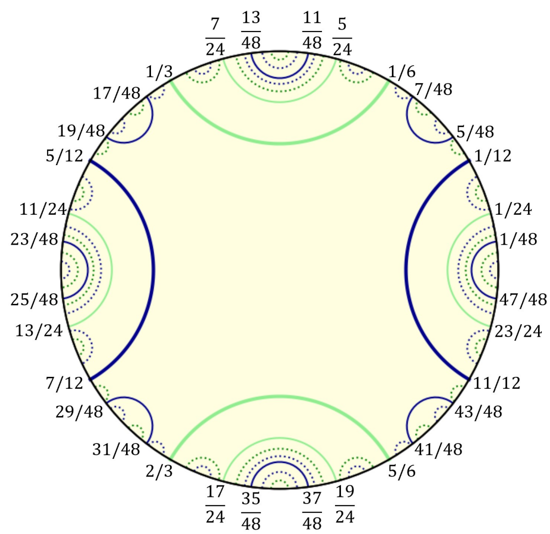

FIGURE 2. Three initial steps in the construction of the Symmetric Cubic Comajor Lamination C_sCL. Boldface curves indicate leaves of block period 1 constructed on the first step, normal thickness leaves are leaves of block period 2 constructed on the second step, and the dotted leaves are of block period 3 constructed on the third step. Darker leaves are co-periodic comajors of type B, lighter leaves are co-periodic comajors of type D.

The L-algorithm defines an involution on the family of all co-periodic comajors reminiscent of the quadratic case and the title of Lavaurs' paper [Lav86] (we thank the referee for bringing this to our attention).

Now we would like to address the issue of the tools used in the paper. We follow Thurston's approach, implemented so successfully in his seminal preprint that started circulating in 1985 and was, in our view, a key step in the development of polynomial dynamics. We were influenced by it, and decided to apply similar ideas to cubic symmetric polynomials. Thus, our choice here was partially a matter of taste. Another reason for not using analytic machinery when constructing a model space for $\mathcal{M}_{3,s}$ was that while Thurston's approach is involved, it is also elementary and for this reason can be potentially accessible to a wider audience. Finally, combining analytic and combinatorial methods in one construction seems to us less attractive and elegant as it is less structured and requires going back and forth between different methods. This explains the choice of techniques.

The paper is organized as follows. We discuss laminations in Section 2. In Section 3 we consider general results and concepts concerning symmetric laminations introduced in [BOSTV1]. In Section 4 we state a few specific properties of the space C_sCL and use them, and additional arguments, to verify the Fatou conjecture about the density of hyperbolicity for symmetric cubic laminations. More precisely, observe that co-periodic comajors correspond to periodic majors; in Section 4 we associate them with q-laminations with periodic Fatou gaps of degree greater than 1 and show that these are dense. Finally, in Section 5 we describe and justify the L-algorithm similar to the Lavaurs algorithm.

2. Laminations: classical definitions

Identify $\mathbb{S}$ with $\mathbb{R}/\mathbb{Z}$ and define the map $\sigma_d : \mathbb{S} \to \mathbb{S}$ for $d \geq 2$ as $\sigma_d(z) = dz \mod 1$; clearly, σ_d is locally one-to-one on $\mathbb{S}$. A monic (i.e., with leading coefficient 1) complex polynomial P with locally connected Julia set J_P gives rise to an equivalence relation $\sim_P$ on $\mathbb{S}$ so that $x \sim_P y$ if and only if the external rays of arguments x and y land at the same point of J_P. Equivalence classes of $\sim_P$ have pairwise disjoint convex hulls. The *topological Julia set* $\mathbb{S}/\sim_P = J_{\sim_P}$ is homeomorphic to J_P, and the *topological polynomial* $f_{\sim_P} : J_{\sim_P} \to J_{\sim_P}$, induced by σ_d, is topologically conjugate to $P|_{J_P}$.

$$\begin{array}{ccc} \mathbb{S}/\sim_P & \xrightarrow{f_{\sim_P}} & \mathbb{S}/\sim_P \\ \downarrow \psi & & \downarrow \psi \\ J(P) & \xrightarrow{P} & J(P) \end{array}$$

An equivalence relation $\sim$ on $\mathbb{S}$, with similar properties to those of $\sim_P$ above, can be introduced with no reference to complex polynomials.

Definition 2.1 (Laminational equivalence relation)**.** An equivalence relation $\sim$ on the unit circle $\mathbb{S}$ is called a *laminational equivalence relation* if it has the following properties:
(E1) the graph of $\sim$ is a closed subset in $\mathbb{S} \times \mathbb{S}$;
(E2) convex hulls of distinct equivalence classes are disjoint;
(E3) each equivalence class of $\sim$ is finite.

For a closed set $A \subset \mathbb{S}$ we denote its convex hull by $\mathrm{CH}(A)$. An *edge* of $\mathrm{CH}(A)$ is a chord of $\mathbb{S}$ contained in the boundary of $\mathrm{CH}(A)$. Given points a, $b \in \mathbb{S}$, let (a, b) be the positively oriented arc in $\mathbb{S}$ from a to b and let $\overline{ab}$ be the chord with endpoints a and b.

Definition 2.2 (Invariance)**.** A laminational equivalence relation $\sim$ is (σ_d-)*invariant* if:
(I1) $\sim$ is *forward invariant*: for a class $\mathbf{g}$, the set $\sigma_d(\mathbf{g})$ is a class too;
(I2) $\sim$ is *backward invariant*: for a class $\mathbf{g}$, its pre-image $\sigma_d^{-1}(\mathbf{g}) = \{x \in \mathbb{S} : \sigma_d(x) \in \mathbf{g}\}$ is a union of classes;
(I3) for any $\sim$-class $\mathbf{g}$ with more than two points, the map $\sigma_d|_{\mathbf{g}} : \mathbf{g} \to \sigma_d(\mathbf{g})$ is a *covering map with positive orientation*, i.e., for every connected component (s, t) of $\mathbb{S} \setminus \mathbf{g}$ the arc in the circle $(\sigma_d(s), \sigma_d(t))$ is a connected component of $\mathbb{S} \setminus \sigma_d(\mathbf{g})$;

Definition 2.3. A *lamination* $\mathcal{L}$ is a set of chords in the closed unit disk $\overline{\mathbb{D}}$, called *leaves* of $\mathcal{L}$, if it satisfies the following conditions:
(L1) leaves of $\mathcal{L}$ do not cross; (L2) the set $\mathcal{L}^* = \cup_{\ell \in \mathcal{L}} \ell$ is closed.
If (L2) is not assumed then $\mathcal{L}$ is called a *prelamination.*

A degenerate leaf is a point of $\mathbb{S}$. Given a leaf $\ell = \overline{ab} \in \mathcal{L}$, let $\sigma_d(\ell)$ be the chord with endpoints $\sigma_d(a)$ and $\sigma_d(b)$; then ℓ is called a *pullback* of $\sigma_d(\ell)$. If $a \neq b$ but $\sigma_d(a) = \sigma_d(b)$, call ℓ a *critical leaf.* Let $\sigma_d^* : \mathcal{L}^* \to \overline{\mathbb{D}}$ be the linear extension of σ_d over all the leaves in $\mathcal{L}$. Then σ_d^* is continuous and σ_d^* is one-to-one on any non-critical leaf. If $\mathcal{L}$ includes all points of $\mathbb{S}$ as degenerate leaves, then $\mathcal{L}^*$ is a continuum.

Definition 2.4 (Gap)**.** A *gap* G of a lamination $\mathcal{L}$ is the closure of a component of $\mathbb{D} \setminus \mathcal{L}^*$; its boundary leaves are called *edges (of the gap).*

Figure 3 illustrates these notions. If G is a leaf/gap of $\mathcal{L}$, then G equals the convex hull of $G \cap \mathbb{S}$. If G is a leaf or a gap of $\mathcal{L}$ we let $\sigma_d(G)$ be the convex hull of $\sigma_d(G \cap \mathbb{S})$. Notice that $\mathrm{Bd}(G) \cap \mathbb{S} = G \cap \mathbb{S}$. Points of $G \cap \mathbb{S}$ are called the *vertices* of G. A gap G is called *infinite (finite)* if and only if $G \cap \mathbb{S}$ is infinite (finite). A gap G is called *triangular gap* if $G \cap \mathbb{S}$ consists of three points.

Definition 2.5. Let $\mathcal{L}$ be a lamination. The equivalence relation $\sim_{\mathcal{L}}$ on $\mathbb{S}$ induced by $\mathcal{L}$ is defined by declaring that $x \sim_{\mathcal{L}} y$ if and only if there exists a finite concatenation of leaves of $\mathcal{L}$ joining x to y.

Definition 2.6 (Invariant (pre)laminations)**.** A (pre)lamination $\mathcal{L}$ is *(σ_d-)invariant* if,
(D1) $\mathcal{L}$ is *forward invariant.* For each $\ell \in \mathcal{L}$ either $\sigma_d(\ell) \in \mathcal{L}$ or $\sigma_d(\ell)$ is a point in $\mathbb{S}$ and
(D2) $\mathcal{L}$ is *backward invariant.*

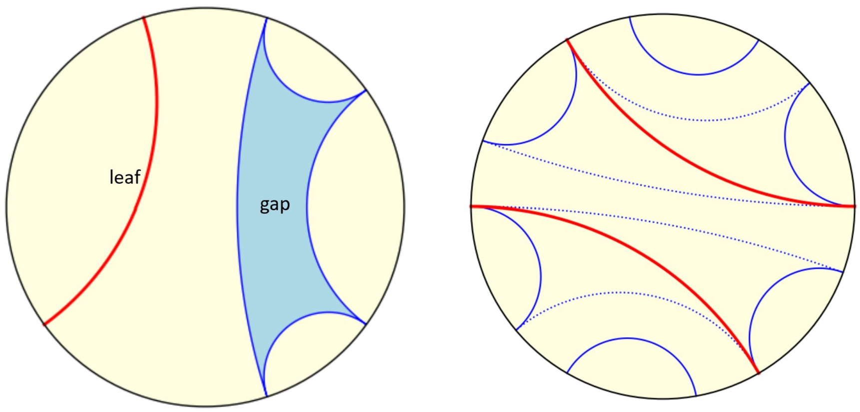

FIGURE 3. Left: a gap and a leaf. Right: long pullbacks (dotted) vs short pullbacks (solid).

(1) For each $\ell \in \mathcal{L}$ there exists a leaf $\ell' \in \mathcal{L}$ such that $\sigma_d(\ell') = \ell$.
(2) For each $\ell \in \mathcal{L}$ such that $\sigma_d(\ell)$ is a non-degenerate leaf, there exists **d disjoint** leaves $\ell_1, \dots, \ell_d$ in $\mathcal{L}$ such that $\ell = \ell_1$ and $\sigma_d(\ell_i) = \sigma_d(\ell)$ for all i.

Definition 2.7 (q-lamination)**.** A lamination $\mathcal{L}$ is called a *q-lamination* if the equivalence relation $\sim_{\mathcal{L}}$ is laminational and $\mathcal{L}$ consists of the edges of the convex hulls of $\sim_{\mathcal{L}}$-classes.

Remark 2.8. Since a *q-lamination* $\mathcal{L}$ consists of edges of the $\sim_{\mathcal{L}}$ classes, two leaves of $\mathcal{L}$ sharing an endpoint must be edges of a finite gap. Also, given a laminational equivalence relation $\approx$ we may consider the family of edges of convex hulls of $\approx$-classes; this family is denoted by $\mathcal{L}_{\approx}$ and is called the *lamination generated by* $\approx$*.* Evidently, such $\mathcal{L}_{\approx}$ is a q-lamination.

Definition 2.9 (Siblings)**.** Two chords are called *siblings* if they have the same image. Any d disjoint chords with the same non-degenerate image are called a *sibling collection*.

Definition 2.10 (Monotone Map)**.** Let X, Y be topological spaces and $f : X \to Y$ be continuous. Then f is said to be *monotone* if $f^{-1}(y)$ is connected for each $y \in Y$. It is known that if f is monotone and X is a continuum then $f^{-1}(Z)$ is connected for every connected $Z \subset f(X)$.

Definition 2.11 (Gap-invariance)**.** A lamination $\mathcal{L}$ is *gap invariant* if for each gap G, the set $\sigma_d(G)$ is a gap, or a leaf, or a single point. In the first case we also require that $\sigma_d^*|_{\text{Bd}(G)} : \text{Bd}(G) \to \text{Bd}(\sigma_d(G))$ maps as the composition of a monotone map and a covering map to the boundary of the image gap, with positive orientation (i.e., as you move through the vertices of G in clockwise direction around $\mathrm{Bd}(G)$, their corresponding images in $\sigma_d(G)$ must also be aligned clockwise in $\mathrm{Bd}(\sigma_d(G))$).

Definition 2.12 (Degree)**.** The *degree* of the map $\sigma_d^*|_{\mathrm{Bd}(G)} : \mathrm{Bd}(G) \to \mathrm{Bd}(\sigma_d(G))$, or of the gap G, is defined as the number of components of $(\sigma_d^*)^{-1}(x)$ in $\mathrm{Bd}(G)$, for any $x \in \mathrm{Bd}(\sigma_d(G))$. In other words, if every leaf of $\sigma_d(G)$ has k disjoint pre-image leaves in G, then the degree of the map σ_d^* is k. A gap G is called a *critical* gap if $k > 1$.

The following results are proved in [BMOV13].

Theorem 2.13. *Every* (σ_d-)invariant *lamination is gap invariant.*

Theorem 2.14. *The closure of an invariant prelamination is an invariant lamination. The space of all σ_d-invariant laminations is compact.*

3. Symmetric Cubic Comajor Lamination: preliminaries

This section describes results of [BOSTV1]. From now on normalize the circle so that its length is 1; the length of arcs and angles are measured accordingly. Given a chord $\ell = \overline{ab}$ denote by $-\ell$ the chord obtained by rotating ℓ by the angle $\frac{1}{2}$. Define the *length* $\|\overline{ab}\|$ of a chord $\overline{ab}$ as the shorter of the lengths of the arcs in $\mathbb{S} = \mathbb{R}/\mathbb{Z}$ with the endpoints a and b. The maximum length of a chord is $\frac{1}{2}$. Divide leaves into four categories by their length.

Definition 3.1. A *short leaf* is a leaf ℓ such that $0 < \|\ell\| < \frac{1}{6}$, a *medium leaf* is a leaf ℓ such that $\frac{1}{6} \le \|\ell\| < \frac{1}{3}$ and a *long leaf* is a leaf ℓ such that $\frac{1}{3} < \|\ell\| \le \frac{1}{2}$. *Critical leaves* are leaves of length exactly $\frac{1}{3}$.

For brevity we call a leaf ℓ *long/medium* if $\|\ell\| \ge \frac{1}{6}$.

By Definition 2.9 we cannot talk about sibling collections that include critical leaves. On the other hand, it is clear that a non-critical leaf ℓ has *siblings*. Consider this issue in detail.

Lemma 3.2 (Lemma 3.4 [BOSTV1])**.** *The possibilities for leaves in a sibling collection are*

- **(sss):** *all leaves are short;*
- **(mmm):** *all leaves are medium;*
- **(sml):** *one leaf is short, one medium, and one long.*

A sibling collection is completely determined by its type and one leaf.

These are general facts; let us now become more specific.

Definition 3.3 (Cubic symmetric lamination)**.** A σ_3-invariant lamination $\mathcal{L}$ is called a *cubic symmetric lamination* if:
(D3) for each $\ell \in \mathcal{L}$ we have $-\ell \in \mathcal{L}$.

Here, $-\ell$ denotes the result of the half-turn (rotation by 180 degrees) about the origin of $\mathbb{C}$ applied to ℓ. Note: if the endpoints of ℓ have arguments α and β, then the endpoints of $-\ell$ have arguments $\alpha + \frac{1}{2}$ and $\beta + \frac{1}{2}$. Unless otherwise stated, let $\mathcal{L}$ *be a cubic symmetric lamination.*

Definition 3.4. Suppose that $\ell = \overline{ab}$ is a non-critical chord which is not a diameter and the arc (a, b) is shorter than the arc (b, a). Denote the chord $\overline{(a + \frac{1}{3})(b - \frac{1}{3})}$ by ℓ' and the chord $\overline{(a + \frac{2}{3})(b - \frac{2}{3})}$ by ℓ''.

As $\sigma_3(\ell') = \sigma_3(\ell'') = \sigma_3(\ell)$, the chords ℓ, ℓ', ℓ'' form a sibling collection. For a long/medium non-critical leaf $\ell \in \mathcal{L}$, it follows that ℓ' is long/medium and ℓ'' is short; if, moreover, $\ell \in \mathcal{L}$ where $\mathcal{L}$ is a cubic symmetric lamination, then its sibling collection is $\{\ell, \ell', \ell''\}$ (all other possibilities lead to crossings with ℓ or $-\ell$). So, for a symmetric lamination $\mathcal{L}$ a sibling collection of type (mmm) is impossible.

Definition 3.5. Given two chords $\ell, \hat{\ell}$ that do not cross let $\mathcal{S}(\ell, \hat{\ell})$ be a component of $\mathbb{D} \setminus [\ell \cup \hat{\ell}]$ with boundary containing ℓ and $\hat{\ell}$; call $\mathcal{S}(\ell, \hat{\ell})$ the *strip between* ℓ *and* $\hat{\ell}$.

The above notation is convenient when dealing with laminations.

Definition 3.6 (Short strips)**.** For a sibling collection $\{\ell, \ell', \ell''\}$ of type (sml), with ℓ and ℓ' long/medium, let $C(\ell) = \overline{\mathcal{S}(\ell, \ell')}$, (the short leaf ℓ'' cannot lie in $C(\ell)$). The set $C(\ell)$ has two boundary circle arcs of length $|\frac{1}{3} - \|\ell\||$ (and so does $-C(\ell)$). Given a long/medium chord $\ell \in \mathcal{L}$, call the region $\mathrm{SH}(\ell) = C(\ell) \cup -C(\ell)$ the *short strips (of* ℓ*)* and each of $C(\ell)$ and $-C(\ell)$ a *short strip (of* ℓ*).* Call $|\frac{1}{3} - \|\ell\|| = w(C(\ell)) = w(-C(\ell)) = w(\mathrm{SH}(\ell))$ the *width* of $C(\ell)$ (or of $-C(\ell)$, or of $\mathrm{SH}(\ell)$). Note that $-C(\ell) = C(-\ell)$.

Definition 3.7. A leaf ℓ is *closer to criticality* than a leaf $\hat{\ell}$ if $\|\ell\|$ is closer to $\frac{1}{3}$ than $\|\hat{\ell}\|$. A chord ℓ is *closest to criticality (in a family of chords* $\mathcal{A}$*)* if its length is the closest to criticality among lengths of chords from $\mathcal{A}$.

The next two facts established in [BOSTV1] are similar to important results proven in [Thu85]. The first one is somewhat technical.

Proposition 3.8 (Lemma 3.7 [BOSTV1])**.** *If* $\ell \in \mathcal{L}$, $\|\ell\| > \frac{1}{6}$, *and* $k \in \mathbb{N}$ *is minimal such that* $\ell_k = \sigma_3^k(\ell)$ *intersects the interior of* $\mathrm{SH}(\ell)$, *then* $\|\ell_k\| > \frac{1}{6}$*, and* ℓ_k *is closer to criticality than* ℓ*. A leaf* ℓ *that is the closest to criticality in its forward orbit is medium/long, and no forward image of* ℓ *enters the interior of* $\mathrm{SH}(\ell)$*.*

Proposition 3.8 implies Theorem 3.9.

Theorem 3.9 (Theorem 3.8 [BOSTV1])**.** *Let* $\mathcal{L}$ *be a symmetric lamination and* G *be a gap of* $\mathcal{L}$*. Then* G *is preperiodic unless an eventual forward image of* G *is a leaf or a point.*

Call a finite periodic gap of $\mathcal{L}$ a *periodic polygon*.

Lemma 3.10 (Lemma 4.5 [BOSTV1])**.** *Let* G *be a periodic polygon, and let* g *be the first return map of* G*. One of the following is true.*

(*a*) *The edges of* G *are permuted transitively under* g *as a combinatorial rotation, i.e., preserving their cyclic order.*

(*b*) *The edges of* G *form two disjoint periodic cycles, and* G *eventually maps to the gap* $-G$*. If* ℓ *and* $\hat{\ell}$ *are two adjacent edges of* G*, then the leaf* ℓ *eventually maps to the edge* $-\hat{\ell}$ *of* $-G$*.*

Definition 3.11. If case (a) from Lemma 3.10 holds, we call a gap G a *1-rotational gap*. If case (b) from Lemma 3.10 holds we call such a gap *a 2-rotational gap*.

If c is a short chord, then there are two long/medium chords with the same image as c. We will denote them by M_c and M'_c. Also, denote by Q_c the convex hull of $M_c \cup M'_c$. This applies in the degenerate case, too: if $c \in \mathbb{S}$ is just a point, then $M_c = M'_c = Q_c$ is a critical leaf ℓ disjoint from c such that $\sigma_3(c) = \sigma_3(M_c)$.

Definition 3.12 (Major)**.** A leaf $M \in \mathcal{L}$ closest to criticality in $\mathcal{L}$ is called a *major* of $\mathcal{L}$.

If M is a major of $\mathcal{L}$, then the medium/long sibling M' of M is also a major of $\mathcal{L}$, as well as the leaves $-M$ and $-M'$. A lamination has either exactly 4 non-critical majors or 2 critical majors.

Definition 3.13 (Comajor)**.** The short siblings of major leaves of $\mathcal{L}$ are called *comajors*; we also say that they form a *comajor pair*. A pair of symmetric chords is called a *symmetric pair*. If the chords are degenerate, their symmetric pair is called *degenerate*, too.

A symmetric lamination has a symmetric pair of comajors $\{c, -c\}$.

Definition 3.14 (Minor)**.** Images of majors (equivalently, comajors) are called *minors* of a symmetric lamination. Similarly to comajors, every symmetric lamination has two symmetric minors $\{m, -m\}$.

Critical majors of a lamination have no siblings, and we define *degenerate* comajors and minors as corresponding points on $\mathbb{S}$. If majors M and $-M$ are non-critical, then there is a critical gap, say, G with edges M and M', and a critical gap $-G$ with edges $-M$ and $-M'$.

Lemma 3.15 (Lemma 5.4 [BOSTV1]). *Let $\{m, -m\}$ be the minors of $\mathcal{L}$, and let ℓ be a leaf of $\mathcal{L}$. Then no forward image of ℓ is shorter than $\min(\|\ell\|, \|m\|)$.*

Definition 3.16. For a family $\mathcal{A}$ of chords, ℓ is a *two sided limit leaf* of $\mathcal{A}$ if ℓ is approximated by chords of $\mathcal{A}$ from both sides.

Lemma 3.17 (Lemma 5.5 [BOSTV1]). *Let c be a comajor and M be a major of $\mathcal{L}$ such that $\sigma_3(c) = \sigma_3(M)$.*

(1) *If c is non-degenerate, then one of the following holds:*
   (a) *the endpoints of c are both strictly preperiodic with the same preperiod and period;*
   (b) *the endpoints of c are both not preperiodic, and c is approximated from both sides by leaves of $\mathcal{L}$ that have no common endpoints with c.*

(2) *If M is non-critical, then its endpoints are either both periodic or both strictly preperiodic with the same preperiod and period, or both not preperiodic.*

In particular, a non-degenerate comajor is not periodic.

Comajors can be described more explicitly.

Definition 3.18 (Legal pairs, Definition 5.6 [BOSTV1]). Let a symmetric pair $\{c, -c\}$ be either degenerate or satisfy the following:

(a) no two iterated forward images of c and $-c$ cross, and
(b) no forward image of c crosses the interior of $\mathrm{SH}(M_c)$.

Then $\{c, -c\}$ is said to be a *legal pair*.

We will also need an important concept of a *pullback* of a set.

Definition 3.19 (Pullbacks, Definition 5.7 [BOSTV1]). Suppose that a family $\mathcal{A}$ of chords is given and ℓ is a chord. A *pullback chord of ℓ generated by $\mathcal{A}$* is a chord ℓ' with $\sigma_3(\ell') = \ell$ such that ℓ' that does not cross chords from $\mathcal{A}$. An *iterated pullback chord of ℓ generated by $\mathcal{A}$* is a pullback chord of an (iterated) pullback chord of ℓ.

Lemma 3.20 considers two specific cases (see Figure 4).

Lemma 3.20 (Lemma 5.8 [BOSTV1]). *There are only two symmetric cubic laminations $\mathcal{L}_1$, $\mathcal{L}_2$ with comajors of length $\frac{1}{6}$, namely:*

(1) *The lamination $\mathcal{L}_1$ has comajors $\overline{\frac{1}{6}\frac{1}{3}}$, $\overline{\frac{2}{3}\frac{5}{6}}$ and invariant critical Fatou gaps U_1', U_1'', where $U_1' \cap \mathbb{S}$ consists of all $\gamma \in \mathbb{S}$ such that $\sigma_3^n(\gamma) \in [0, \frac{1}{2}]$ (for all n), and $U_1' \cap \mathbb{S}$ consists of all $\gamma \in \mathbb{S}$ such that $\sigma_3^n(\gamma) \in [\frac{1}{2}, 0]$. The gaps U_1', U_1'' share the edge $\overline{0\frac{1}{2}}$; their other edges are the appropriate iterated pullbacks of $\overline{0\frac{1}{2}}$ that never separate $\overline{\frac{1}{6}\frac{1}{3}}$, $\overline{\frac{2}{3}\frac{5}{6}}$, and $\overline{0\frac{1}{2}}$.*
(2) *The lamination $\mathcal{L}_2$ has comajors $\overline{\frac{11}{12}\frac{1}{12}}$, $\overline{\frac{5}{12}\frac{7}{12}}$ and critical Fatou gaps U_2', U_2'' that form a 2-cycle, where the set $(U_2' \cup U_2'') \cap \mathbb{S}$ consists of all $\gamma \in \mathbb{S}$ such that $\sigma_3^n(\gamma) \in [\frac{1}{12}, \frac{5}{12}] \cup [\frac{7}{12}, \frac{11}{12}]$. The gaps U_2', U_2'' share the edge $\overline{\frac{1}{4}\frac{3}{4}}$; their other edges are the appropriate iterated pullbacks of $\overline{\frac{1}{4}\frac{3}{4}}$ that never separate $\overline{\frac{11}{12}\frac{1}{12}}$, $\overline{\frac{5}{12}\frac{7}{12}}$ and $\overline{\frac{1}{4}\frac{3}{4}}$.*

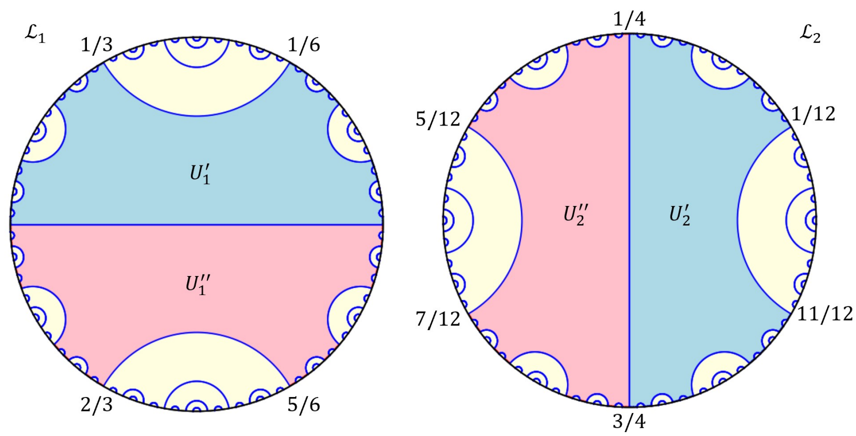

FIGURE 4. Laminations $\mathcal{L}_1$ and $\mathcal{L}_2$ from Lemma 3.20.

Though the laminations from Lemma 3.20 are not the pullback laminations described below, knowing them allows us to consider only legal pairs with comajors of length less than $\frac{1}{6}$ and streamline the proofs.

The construction below repeats the one from [BOSTV1]; we add it here for the sake of completeness and convenience of the reader.

Construction of a symmetric pullback lamination $\mathcal{L}(c)$ for a legal pair $\{c, -c\}$.

Degenerate case. For $c \in \mathbb{S}$, let $\pm\ell = \pm M_c$. (call ℓ, $-\ell$ and their pullbacks "leaves" even though we apply this term to existing laminations, and we are only constructing one). Consider two cases.

(a) If ℓ and $-\ell$ do not have periodic endpoints, then the family of all iterated pullbacks of $\ell, -\ell$ generated by $\ell, -\ell$ is denoted by $\mathcal{C}_c$ (see Figure 5).

(b) Suppose that ℓ and $-\ell$ have periodic endpoints p and $-p$. Then there are two similar cases. First, the orbits of p and $-p$ may be distinct

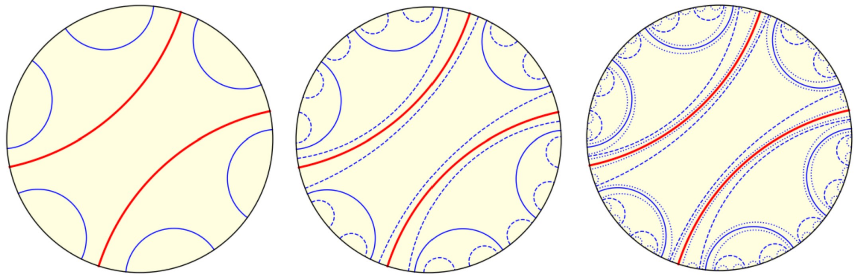

FIGURE 5. This figure illustrates the pullback construction in the degenerate non-periodic case. The two critical leaves are shown in boldface, their first pullbacks are in normal, second pullbacks are dashed and third pullbacks are dotted.

(and hence disjoint). Then iterated pullbacks of ℓ generated by ℓ, $-\ell$ are well-defined (unique) until the n-th step (n equals the period of p *and* the period of $-p$), when there are two iterated pullbacks of ℓ that have a common endpoint x and share other endpoints with ℓ. Two other iterated pullbacks of ℓ have a common endpoint $y \neq 0$ and share other endpoints with ℓ. These four iterated pullbacks of ℓ form a collapsing quadrilateral Q with diagonal ℓ; moreover, $\sigma_3(x) = \sigma_3(y)$ and $\sigma_3^n(x) = \sigma_3^n(y) = z$ is the non-periodic endpoint of ℓ. Evidently, $\sigma_3(Q) = \overline{\sigma_3(p)\sigma_3(x)}$ is the $(n-1)$-st iterated pullback of ℓ. Then in the pullback lamination $\mathcal{L}(c)$ that we are defining we postulate the choice of *only the short pullbacks* among the above listed iterated pullbacks of ℓ (see Figure 3, right). So, only two short edges of Q are included in the set of pullbacks $\mathcal{C}_c$. A similar situation holds for $-\ell$ and its iterated pullbacks.

In general, the choice of pullbacks of the already constructed leaf $\hat{\ell}$ is ambiguous only if $\hat{\ell}$ has an endpoint $\sigma_3(\pm\ell)$. In this case we *always* choose short pullbacks of $\hat{\ell}$. Evidently, this defines a set $\mathcal{C}_c$ of chords in a unique way.

We claim that $\mathcal{C}_c$ is an invariant prelamination. To show that $\mathcal{C}_c$ is a prelamination we need to show that its leaves do not cross. Suppose otherwise and choose the minimal n such that $\hat{\ell}$ and $\tilde{\ell}$ are pullbacks of ℓ or $-\ell$ under at most the n-th iterate of σ_3 that cross. By construction, $\hat{\ell}, \tilde{\ell}$ are not critical. Hence their images $\sigma_3(\hat{\ell})$, $\sigma_3(\tilde{\ell})$ are not degenerate and do not cross. It is only possible if $\hat{\ell}, \tilde{\ell}$ come out of the endpoints of a critical leaf of $\mathcal{L}$. We may assume that $\|\hat{\ell}\| \geq \frac{1}{6}$ (if $\hat{\ell}$ and $\tilde{\ell}$ are shorter than $\frac{1}{6}$ then they cannot cross). However by construction this is impossible. Hence $\mathcal{C}_c$ is a prelamination. The claim that $\mathcal{C}_c$ is invariant is straightforward; its verification is left to the reader. By Theorem 2.14, the closure of $\mathcal{C}_c$ is an invariant lamination denoted $\mathcal{L}(c)$. Moreover, by construction $\mathcal{C}_c$ is symmetric (this can be easily proven using induction on the number of steps in the process of pulling back ℓ and $-\ell$). Hence $\mathcal{L}(c)$ is a symmetric invariant lamination. See Figure 7, left, for an illustration of $\mathcal{L}(c)$.

Non-degenerate case. As in the degenerate case, we will talk about pullback leaves even though we are still constructing a lamination. By Lemma 3.20, we may assume that $|c| < \frac{1}{6}$. Set $\pm M = \pm M_c, \pm Q = \pm Q_c$. If d is an iterated forward image of c or $-c$, then, by Definition 3.18(b), it cannot intersect the interior Q or $-Q$. Consider the set of leaves $\mathcal{D}$ formed by the edges of $\pm Q$ and $\bigcup_{m=0}^{\infty}\{\sigma_3^m(c), \sigma_3^m(-c)\}$. It follows that leaves of $\mathcal{D}$ do not cross among themselves. The idea is to construct pullbacks of leaves of $\mathcal{D}$ in a step-by-step fashion and show that this results in an invariant prelamination $\mathcal{C}_c$ as in the degenerate case.

More precisely, we proceed by induction. Set $\mathcal{D} = \mathcal{C}_c^0$. Construct sets of leaves $\mathcal{C}_c^{n+1}$ by collecting pullbacks of leaves of $\mathcal{C}_c^n$ generated by Q and $-Q$ (the step of induction is based upon Definition 3.18 and Definition 3.19). The claim is that except for the property (D2)(1) from Definition 2.6 (a part of what it means for a lamination to be backward invariant), the set $\mathcal{C}_c^n$ has all the properties of invariant laminations listed in Definition 2.6. Let us verify this property for $\mathcal{C}_c^1$. Let $\ell \in \mathcal{C}_c^1$. Then $\sigma_3(\ell) \in \mathcal{D}$, so property (D1) from Definition 2.6 is satisfied. Property (D2)(2) is, evidently, satisfied for edges of Q and $-Q$. If ℓ is not an edge of $\pm Q$, then, since leaves $\pm\sigma_3(Q) = \sigma_3(\pm c)$ do not cross $\sigma(\ell)$, and since on the closure of each component of $\mathbb{S} \setminus [Q \cup -Q]$ the map is one-to-one, then ℓ will have two sibling leaves in $\mathcal{C}_c^1$ as desired. Literally the same argument works for $\ell \in \mathcal{C}_c^{n+1}$ and proves that each set $\mathcal{C}_c^{n+1}$ has properties (D1) and (D2)(2) from Definition 2.6. This implies that $\bigcup_{i\geq 0} \mathcal{C}_c^i = \mathcal{C}_c$ has all properties from Definition 2.6 and is, therefore, an invariant prelamination. By Theorem 2.14, its closure $\mathcal{L}(c)$ is an invariant lamination.

The lamination $\mathcal{L}(c)$ is called the *pullback lamination (of c);* we often use c as the argument, instead of the less discriminatory $\{c, -c\}$.

Lemma 3.21 (Lemma 5.9 [BOSTV1])**.** *A legal pair $\{c, -c\}$ is the comajor pair of the lamination $\mathcal{L}(c)$. A symmetric pair $\{c, -c\}$ is a comajor pair if and only if it is legal.*

Theorem 3.22 contains the main results of [BOSTV1]. *Co-periodic comajors* are defined as preperiodic comajors of preperiod 1. The name is due to the fact that a co-periodic comajor is a sibling of a periodic major.

Theorem 3.22. *The set of all comajors of cubic symmetric laminations is a q-lamination. Co-periodic comajors are disjoint from all other comajors.*

Based upon this theorem we define the main object of our interest.

Definition 3.23. All comajors of cubic symmetric laminations form a lamination C_sCL called the *Cubic symmetric Comajor Lamination.*

The following useful notation is justified by Theorem 3.22.

Definition 3.24. For a non-diameter chord $n = \overline{ab}$, the smaller of the two arcs into which n divides $\mathbb{S}$, is denoted by $H(n)$. Denote the closed subset of $\overline{\mathbb{D}}$ bounded by n and $H(n)$ by $R(n)$. Given two comajors m and n, write $m \prec n$ if $m \subset R(n)$, and say that m is *under* n.

Lemma 3.25 (Lemma 5.14 [BOSTV1])**.** *Let $\{c, -c\}$ and $\{d, -d\}$ be legal pairs, where c is degenerate and $c \prec d$. Suppose that c is not an endpoint of d, or $\sigma_3(c)$ is not periodic. Then $d \in \mathcal{L}(c)$. In addition, the following holds.*

1. *Majors D, D' of $\mathcal{L}(d)$ are leaves of $\mathcal{L}(c)$ unless $\mathcal{L}(c)$ has two finite gaps G, G' that contain D, D' as their diagonals, share a critical leaf M of $\mathcal{L}(c)$ as a common edge, and are such that $\sigma_3(G) = \sigma_3(G')$ is a preperiodic gap.*
2. *If majors of $\mathcal{L}(d)$ are leaves of $\mathcal{L}(c)$ and $\ell \in \mathcal{L}(d)$ is a leaf that never maps to a short side of a collapsing quadrilateral of $\mathcal{L}(d)$, then $\ell \in \mathcal{L}(c)$.*

4. Combinatorial analog of the Fatou conjecture on density of hyperbolicity

We begin by stating a few results that will be used in what follows.

Definition 4.1. If a symmetric lamination $\mathcal{L}$ has a periodic Fatou gap of degree greater than 1 (i.e., if it has properties listed in Lemma 4.10), then $\mathcal{L}$ is called *hyperbolic.*

We need a result of [BMOV13]. Recall that, as in Definition 2.5, a lamination $\mathcal{L}$ generates an equivalence relation $\sim_{\mathcal{L}}$ on $\mathbb{S}$ by declaring that $a \sim_{\mathcal{L}} b$ if and only if a finite concatenation of leaves of $\mathcal{L}$ connects points $a \in \mathbb{S}$ and $b \in \mathbb{S}$.

Definition 4.2 (Proper lamination, Definition 4.1 [BMOV13])**.** Two leaves with a common endpoint v and the same image which is a leaf (and not a point) are said to form a *critical wedge* (the point v then is said to be its vertex). A lamination $\mathcal{L}$ is *proper* if it contains no critical leaf with periodic endpoint and no critical wedge with periodic vertex.

Proper laminations generate laminational equivalence relations.

Theorem 4.3 (Theorem 4.9 [BMOV13]). *Let $\mathcal{L}$ be a proper invariant lamination. Then $\sim_{\mathcal{L}}$ is an invariant laminational equivalence relation.*

We also need a nice result due to Jan Kiwi [Kiw02].

Theorem 4.4 ([Kiw02]). *Let $\mathcal{L}$ be a σ_d-invariant lamination. Then any infinite gap of $\mathcal{L}$ is (pre)periodic. For any finite periodic gap G of $\mathcal{L}$ its vertices belong to at most $d-1$ distinct cycles except when G is a fixed return d-gon. In particular, a cubic lamination cannot have a fixed return n-gon for $n > 3$. Moreover, if all images of a k-gon G with $k > d$ have at least $d+1$ vertices then G is preperiodic.*

Finally, here are several important claims from [BOSTV1].

Corollary 4.5 (Corollary 4.8 [BOSTV1]). *If E is a preperiodic polygon of a symmetric lamination such that E is not precritical, then no diagonal of E can be a leaf of a symmetric lamination.*

Lemma 4.6 (Lemma 6.2 [BOSTV1]). *Let $c \in C_sCL$ be a non-degenerate comajor. If $\ell \in \mathcal{L}(c)$, $\ell \prec c$ and $\|\ell\| > \frac{\|c\|}{3}$, then $\ell \in C_sCL$. In particular, if $c_i \in \mathcal{L}(c)$, $c_i \prec c$ and $c_i \to c$, then $c_n \in C_sCL$ for sufficiently large n.*

Lemma 4.7 (Lemma 6.6 [BOSTV1]). *Let $c \in C_sCL$ be a non-degenerate comajor such that $\sigma_3(c)$ is not periodic. If there exists a sequence of leaves $c_i \in \mathcal{L}(c)$ with $c \prec c_i$ and $c_i \to c$, then c is the limit of co-periodic comajors $\hat{c}_j \in \mathcal{L}(c)$ with $c \prec \hat{c}_j$ for all j.*

Corollary 4.8 (Corollary 6.7 [BOSTV1]). *Every not eventually periodic comajor c is a two sided limit leaf in the Cubic Symmetric Comajor Lamination C_sCL.*

Lemma 4.9 (Lemma 6.8 [BOSTV1]). *A co-major which is non-degenerate and preperiodic of preperiod at least 2 is a two sided limit leaf of C_sCL or an edge of a finite gap H of C_sCL all of whose edges are limits of comajors of C_sCL disjoint from H.*

Let us now describe laminations related to co-periodic comajors.

Lemma 4.10. *Let $\mathcal{L}$ be a symmetric lamination with a periodic Fatou gap of degree greater than 1. Then $\mathcal{L}$ has two critical Fatou gaps of degree greater than 1. Moreover, $\mathcal{L}$ is a q-lamination.*

Proof. Because of the symmetry, a hyperbolic symmetric lamination $\mathcal{L}$ has two critical Fatou gaps of degree greater than 1. These gaps either belong to the same cycle of Fatou gaps, or belong to two distinct cycles of Fatou gaps. Moreover, by Theorem 4.3 the equivalence $\sim_{\mathcal{L}}$ is laminational. We claim that $\mathcal{L}$ coincides with the q-lamination $\widehat{\mathcal{L}}$ generated by $\sim_{\mathcal{L}}$. We need to show that any leaf of $\mathcal{L}$ is a leaf of $\widehat{\mathcal{L}}$.

In general, edges of a Fatou gap U may form a finite concatenation in which case U is not a gap of the corresponding q-lamination (by definition, in the q-lamination we add one more leaf to the concatenation to make it into a finite gap; this extra leaf will be an edge of a new, smaller Fatou gap of the q-lamination). However this cannot happen in our case: if it did it would yield a symmetric q-lamination with fixed return finite gaps contradicting Lemma 3.10. Hence the Fatou gaps of $\mathcal{L}$ are gaps of $\widehat{\mathcal{L}}$. Otherwise, if $\ell \in \mathcal{L}$ is not a leaf of $\widehat{\mathcal{L}}$ then ℓ must be a diagonal of a finite gap G of $\widehat{\mathcal{L}}$. However by Corollary 4.5 this is impossible. Hence $\mathcal{L} = \widehat{\mathcal{L}}$ is a q-lamination as desired. □

Hyperbolic laminations are constructed in Theorem 4.11.

Theorem 4.11. *A preperiodic point $q \in \mathbb{S}$ of preperiod 1 and period k is an endpoint of a non-degenerate co-periodic comajor c of period k of a cubic symmetric lamination. Take the short edges of $\pm Q_c$, and remove their backward orbits from $\mathcal{L}(c)$. Then the resulting lamination $\widehat{\mathcal{L}}(c)$ is a hyperbolic q-lamination with comajor pair $\{c, -c\}$.*

Proof. Let $\ell = \overline{x_0 p}$ be the critical leaf with $\sigma_3(\ell) = \sigma_3(q)$ and a k-periodic endpoint p. Consider the pullback lamination $\mathcal{L}(q)$. Let G be the central symmetric gap or leaf of $\mathcal{L}(q)$ located between ℓ and $-\ell$. Then G contains the origin and has leaves $\pm M$ closest to criticality. Let M' and d be the medium and short siblings of M. Clearly, the short siblings $\pm d$ of leaves $\pm M$ form a legal pair. Hence if ℓ shares an endpoint with M, then, by Lemma 3.21, we can set $c = d$. Assume now that leaves $\pm\ell$ are disjoint from $\pm M$.

If the orbits of p and $-p$ are disjoint, let $n = k$. Otherwise, $k = 2n$ for some n, while $\sigma_3^n(p) = -p$ and $\sigma_3^n(-p) = p$. We will assume in the rest of the proof that $k = n$; the case $k = 2n$ is similar. Observe that $\mathcal{L}(q)$ contains the critical leaf $\ell = \overline{x_0 p}$ with periodic endpoint p which in fact implies that $\mathcal{L}(q)$ contains caterpillar gaps (see Figure 6). Indeed, consider the strip S between M and M'. Let $s = \overline{x_0 x_1}$ be the short pullback of ℓ under σ_3^k which is included in $\mathcal{L}(q)$ by construction; then $\sigma_3^k(s) = \overline{x_0 p}$. Hence there is another leaf $\overline{x_1 x_2}$ such that $\sigma_3^k(\overline{x_1 x_2}) = \overline{x_0 x_1}$. The leaf $\overline{x_1 x_2}$ is short as if $\overline{x_1 x_2}$ is long/medium, then its k-th image s is short and non-disjoint from the interior of its short strips, contradicting Lemma 3.8. Repeating this, we get a concatenation A of pullbacks of ℓ under powers of σ_3^k; A consists of short leaves of $\mathcal{L}(q)$, begins with $\ell \cup \overline{x_0x_1} \cup \overline{x_1x_2}$, converges to a point $t \in \mathbb{S}$ of period k, and points $x_0, x_1, \dots$ belong to the short circular arc I that bounds S and does *not* contain p. Since t and p belong to distinct circle arcs on the boundary of S, then $t \neq p$.

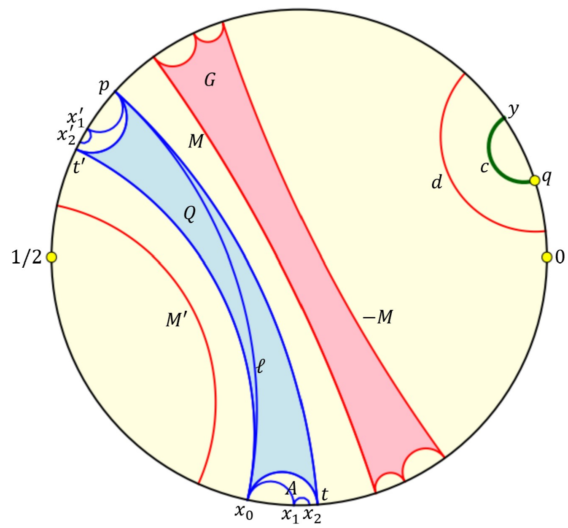

FIGURE 6. Illustration of the proof of Theorem 4.11

Clearly, an infinite periodic gap U of $\mathcal{L}(q)$ contains A in its boundary, and there is a gap U' with the same image as U that shares an edge ℓ with U. Consider the chord $\overline{pt}$; it is periodic of period k, and there is another chord $\overline{x_0t'}$ with the same image as $\overline{pt}$. The chord $\overline{pt}$ is compatible with $\mathcal{L}(q)$ because by construction its images stay inside images of U and never cross leaves of $\mathcal{L}(q)$. Moreover, the iterated images of $\overline{pt}$ do not cross as for this to happen some leaves from the concatenation A must cross, and this is not the case. We claim that then $\overline{pt}$ never enters the strip between itself and $\overline{x_0t'}$. Indeed, if it does then, by Lemma 3.8 it will have to cross ℓ, a contradiction. Likewise, images of $\overline{pt}$ never cross $-\ell$. By definition this implies that the short sibling $\overline{qy}$ of $\overline{pt}$, together with $-\overline{qy}$, form a legal pair. Thus, $\overline{qy} = c$ is a comajor of a symmetric lamination as desired.

The leaf $\sigma_3(c) = \sigma_3(\overline{pt})$ is a k-periodic leaf of $\mathcal{L}(q)$. By Proposition 3.8, the leaf $\sigma_3^k(c) = \overline{pt}$ is a major of $\mathcal{L}(c)$. Let $Q_c = Q$ be the critical quadrilateral of $\mathcal{L}(c)$ with edge $\overline{pt}$ and $\bar{x}$ and $\bar{y}$ be the two short edges of Q. Removing them and their backward orbits from $\mathcal{L}(c)$ yields the family of chords $\widehat{\mathcal{L}}$; we claim that $\widehat{\mathcal{L}}$ is an invariant lamination, too.

Indeed, by definition $\mathcal{L}(c)$ has two quadrilaterals X and Y attached to Q at $\bar{x}$ and $\bar{y}$, respectively. This implies that both $\bar{x}$ and $\bar{y}$ are isolated in $\mathcal{L}(c)$. So, $\widehat{\mathcal{L}}$ is obtained by removing a countable family of isolated leaves from $\mathcal{L}$; hence, $\widehat{\mathcal{L}}$ is closed. The other properties of invariant laminations for $\widehat{\mathcal{L}}$ are immediate. Thus, $\widehat{\mathcal{L}}$ is an invariant lamination. Evidently, $\widehat{\mathcal{L}}$ is symmetric.

Consider the gap U of $\widehat{\mathcal{L}}$ with $U \supset Q$. Countably many pullbacks of Q are consecutively attached to one another and contained in U. Hence U is an infinite periodic gap that maps forward 2-to-1, that is, U is a Fatou gap of degree two. By definition, $\widehat{\mathcal{L}}$ is hyperbolic. Moreover, by the construction c remains a leaf of $\widehat{\mathcal{L}}$. Hence $\{c, -c\}$ is the comajor pair of $\widehat{\mathcal{L}}$. □

We now consider preperiodic points of preperiod greater than 1 or periodic points (by Lemma 3.17, there are no non-degenerate periodic comajors). Recall that a *dendrite* is a locally connected continuum that contains no Jordan curves. A q-lamination with no infinite gaps gives rise to a topological Julia set which is a dendrite; we call such q-laminations *dendritic* (see [BOPT17, BOPT19]). We will also need Theorem 2.19 from [BOSTV1]. This theorem coincides with Lemma 2.31 of [BOPT20] except for two extra claims proven in [BOSTV1]

Theorem 4.12 (Lemma 2.31[BOPT20], Theorem 2.19[BOSTV1])**.** *Let G be an infinite n-periodic gap and $K = \mathrm{Bd}(G)$. Then $\sigma_d^n|_K$ is the composition of a covering map and a monotone map of K. If $\sigma_d^n|_K$ is of degree one, then either statement* (1) *or statement* (2) *below holds.*

(1) *The gap G has countably many vertices, finitely many of which are periodic and the rest are preperiodic. All non-periodic edges of G are* (*pre*)*critical and isolated. There is a critical edge with a periodic endpoint among edges of gaps from the orbit of G.*
(2) *The map $\sigma_d^n|_K$ is monotonically semi-conjugate to an irrational circle rotation so that each fiber is a finite concatenation of* (*pre*)*critical edges of G. Thus, there are critical leaves (edges of some images of G) with non-preperiodic endpoints.*

In particular, if all critical gaps/leaves of a lamination are finite and have strictly preperiodic vertices then the lamination has no infinite gaps.

Consider now the preperiodic case of preperiod greater than 1.

Lemma 4.13. *Let $x \in \mathbb{S}$ be preperiodic of preperiod $n > 1$, let $\mathcal{L}(x) = \mathcal{L}$ be the corresponding pullback lamination. Then $\mathcal{L}$ is proper and $\sim_{\mathcal{L}}$ is a laminational equivalence relation that defines a q-lamination*

$\widehat{\mathcal{L}}$. The lamination $\widehat{\mathcal{L}}$ is symmetric, dendritic and coincides with the family of limit leaves of iterated pullbacks of the critical leaves $\pm M_x$ of $\mathcal{L}$. Let $\pm G$ be the critical gaps/leaves of $\widehat{\mathcal{L}}$. Then $\pm G$ are preperiodic of preperiod n, finite, and contain $\pm M_x$. Let T be the sibling gap/leaf of G in $\widehat{\mathcal{L}}$. Then $x \in T$ and T is a gap/leaf of C_sCL or a singleton disjoint from gaps/leaves of C_sCL. Moreover,

1. *if T is degenerate, then there are no non-degenerate comajors containing x,*
2. *if T is a non-degenerate leaf, then T is a comajor which is the limit from both sides of comajors disjoint from T,*
3. *if T is a gap then all edges of T are comajors that are limits of comajors disjoint from T.*

Proof. The critical leaves $\pm M_x$ of $\mathcal{L}$ have preperiodic endpoints. Hence by Definition 4.2 $\mathcal{L}$ is a proper lamination. By Theorem 4.3 the equivalence relation $\sim_{\mathcal{L}}$ is laminational. Let $\widehat{\mathcal{L}}$ be the q-lamination generated by $\sim_{\mathcal{L}}$.

We claim that all gaps of $\widehat{\mathcal{L}}$ are finite. Indeed, suppose that U is an infinite gap of $\widehat{\mathcal{L}}$. Since $\widehat{\mathcal{L}}$ is a q-lamination generated by $\sim_{\mathcal{L}}$ and by definition, then U must be an infinite gap of $\mathcal{L}$. However, by Theorem 4.12, the lamination $\mathcal{L}$ has no infinite gaps. Hence the topological Julia set $J_{\sim_{\mathcal{L}}}$ is a *dendrite*, and there are no isolated leaves in $\widehat{\mathcal{L}}$. Clearly, $\widehat{\mathcal{L}}$ is symmetric, with critical sets $G \supset \mathcal{M}_x$, $-G \supset -M_x$, and there is a convex hull T of a $\sim_{\mathcal{L}}$-class, with $\sigma_3(T) = \sigma_3(G)$. It follows that $\pm G$ are preperiodic of preperiod n and finite.

Take a leaf ℓ of $\widehat{\mathcal{L}}$. Suppose that it is not a limit leaf for pullbacks of $\pm M_x$. Then, since by Theorem 4.12 $\mathcal{L}$ has only finite gaps, it follows that either ℓ is a diagonal of a finite gap of $\mathcal{L}$, or ℓ is a pullback of M_x or $-M_x$, and on either side of ℓ there is a finite gap of $\mathcal{L}$. In either case we arrive at a contradiction with the assumption that ℓ is a leaf of $\widehat{\mathcal{L}}$. Hence every leaf of $\widehat{\mathcal{L}}$ is a limit leaf for pullbacks of $\pm M_x$.

To prove (1), assume that $T = \{x\}$; then $\widehat{\mathcal{L}}$ has critical leaves $\pm G = \pm M_x$. Suppose that there is a sequence of $\widehat{\mathcal{L}}$-gaps H_i that converges to M_x. By Theorem 3.9 all of them are (pre)periodic. We may assume that $H_1 = H$ has an edge c that separates the interior of H from M_x, with endpoints close to the endpoints of M_x. We may follow the orbit of H and c and choose the closest to criticality iterated image d of c (it is always possible since the orbit of c is finite and c never maps to $\pm M_x$). By Proposition 3.8, the leaf d never enters its short strips $\mathrm{SH}(d)$. Hence the short sibling d'' of d, together with $-d''$, forms a legal pair. Evidently, d'' separates a short circle arc containing x from the rest of the circle. Since by Theorem 3.22 comajors form a q-lamination, non-degenerate comajors cannot contain x as claimed (see Figure 7).

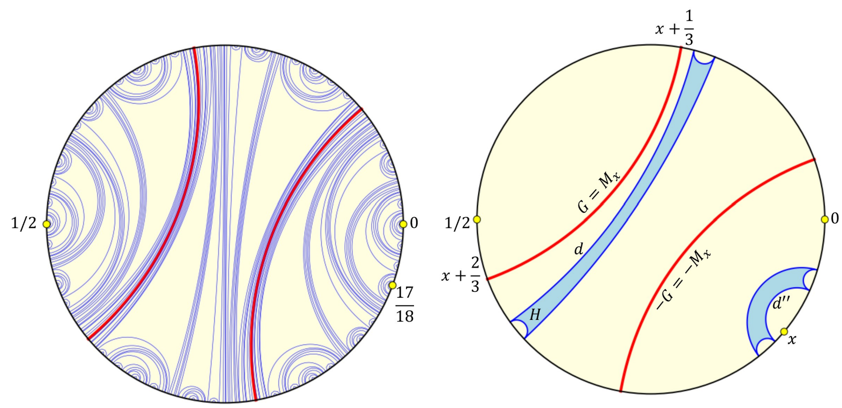

FIGURE 7. Illustration of the proof of Lemma 4.13. The left figure shows the pullback lamination for $x = \frac{17}{18}$ (i.e., of preperiod 2 and period 1). The right figure illustrates the proof in case (1) of the lemma.

If there are no gaps located close to M_x then, since σ_3-periodic points are dense in $\mathbb{S}$, we can choose a sequence of periodic leaves of $\widehat{\mathcal{L}}$ converging to M_x, and repeat for them the above argument. So, the case when $T = \{x\}$ is a singleton is considered. If T is a leaf/gap, then it is easy to check that any leaf on the boundary of T is legal. Hence in this case T is a gap or leaf of C_sCL as desired.

To prove (2), observe that if T is a non-degenerate leaf then by the above it is the limit of leaves of $\widehat{\mathcal{L}}$ from both sides. Therefore it is the limit of pullbacks of $\pm M_x$ from both sides. Since under our assumptions critical leaves $\pm M_x$ are contained in the critical quadrilaterals $\pm G$ then in fact pullbacks of $\pm G$ accumulate on T from both sides. By definition it means that leaves of the pullback lamination $\mathcal{L}(T)$ approach T from both sides. It now follows from Lemmas 4.6 and 4.7, that T is the limit from both sides of comajors disjoint from T as desired.

To prove (3) observe that by definition T has edges that are comajors; then the desired follows from Lemma 4.9. □

Definition 4.14. A preperiodic comajor c of preperiod greater than 1 or a periodic comajor (necessarily degenerate) is called a *Misiurewicz comajor*, and any symmetric lamination with a Misiurewicz comajor pair is said to be a *Misiurewicz* symmetric lamination.

We are ready to prove the density of hyperbolicity (Fatou conjecture) for symmetric laminations.

Theorem 4.15. *Co-periodic comajors are dense in C_sCL.*

Proof. Consider a non-degenerate comajor $c \in C_sCL$ that is not co-periodic. We have two cases here.

(a) *There is a sequence of leaves $c_i \in \mathcal{L}(c)$ with $c \prec c_i$ and $c_i \to c$.* Then, by Lemma 4.7, the comajor c is the limit of co-periodic comajors $\hat{c}_i$ such that $c \prec \hat{c}_i$.

(b) *A sequence of leaves $c_i \in \mathcal{L}(c)$ converging to c with $c \prec c_i$ does not exist.* Then c is an edge of a gap G of $\mathcal{L}(c)$ with all vertices of G outside of $H(c)$. The lamination $\mathcal{L}(c)$ has critical quadrilaterals $\pm Q_c = \pm Q$. If $\sigma_3(c)$ eventually maps to an edge of Q, then this edge is periodic which shows that c is co-periodic, a contradiction with our assumption. Hence $\sigma_3(c)$ never maps to an edge of Q, and, therefore, G never maps to a leaf or point. By Theorem 3.9, this implies that G and c are preperiodic of preperiod greater than 1 (recall that c is not periodic by Lemma 3.17).

We claim that all edges of G are comajors. Properties of laminations imply that there are two gaps, L and R, attached to Q_c at the appropriate majors of $\mathcal{L}(c)$ and such that $\sigma_3(L) = \sigma_3(R) = \sigma_3(G)$. Now, choose among the edges of G the edge ℓ with the greatest length. Then, clearly, $G \cap \mathbb{S} \subset \overline{H(\ell)}$. Set $M = M_\ell, M' = M'_\ell$. Then M (or M') cannot enter the strip S between M and M' as otherwise, by Proposition 3.8, their images would have to cross edges of L, R, or Q_c. This implies that in fact any edge d of G is a comajor because $\{d, -d\}$ is legal.

It follows now that this is exactly the situation described in Lemma 4.13 and that $\mathcal{L}(c)$ gives rise to a laminational equivalence relation $\sim_{\mathcal{L}(c)}$ which, in turn, gives rise to a dendritic q-lamination $\widehat{\mathcal{L}}$ such that G is a gap of $\widehat{\mathcal{L}}$ (the last claim follows, e.g., from the fact that, by Theorem 3.22, comajors form a q-lamination). Since there are no isolated leaves in $\widehat{\mathcal{L}}$, the comajor c is approximated by uncountably many leaves $\hat{\ell}$ of $\widehat{\mathcal{L}}$ such that $\hat{\ell} \prec c$. By Lemma 4.6, we may assume that all these leaves of $\widehat{\mathcal{L}}$ are comajors. Now, choose a sequence of them that converge to c and satisfy the conditions of case (a) of this proof. By (a) these leaves are all limits of co-periodic comajors, hence so is c as desired. □

5. L-algorithm

In this section, we provide an algorithm for constructing all co-periodic comajor leaves. By Theorem 4.15, they are dense in C_sCL, hence this renders the entire C_sCL. The algorithm is similar to the famous *Lavaurs* algorithm for Thurston's Quadratic Minor Lamination QML [Lav86, Lav89] (see [Sou21, BBS21] for an extension of this algorithm to the degree d unicritical case). We call it the *L-algorithm*.

5.1. **Preliminaries.**

Lemma 5.1 (Lemma 6.1 [BOSTV1]). *A co-periodic comajor leaf is disjoint from all other comajors in C_sCL.*

The following is Definition 6.4 from [BOSTV1].

Definition 5.2. Let ℓ be a leaf of a symmetric lamination $\mathcal{L}$ and $k > 0$ be such that $\sigma_3^k(\ell) \neq \ell$ (in particular, the leaf ℓ is not a diameter). If the leaf $\sigma_3^k(\ell)$ is under ℓ, then we say that the leaf ℓ *moves in* by σ_3^k; if $\sigma_3^k(\ell)$ is not under ℓ, then we say that the leaf ℓ *moves out* by σ_3^k. If two leaves ℓ and $\hat{\ell}$ with $\ell \prec \hat{\ell}$ of the same lamination both move in or both move out by the map σ_3^k, then we say that the leaves *move in the same direction*. If one of the leaves ℓ, $\hat{\ell}$ moves in and the other moves out, then we say that the leaves *move in the opposite directions*. There are two ways of moving in the opposite directions: if ℓ moves out and $\hat{\ell}$ moves in, we say they *move towards each other*; if ℓ moves in and $\hat{\ell}$ moves out, we say that they *move away from each other*.

The strip $\mathcal{S}(\ell, \hat{\ell})$ between non-crossing chords ℓ, $\hat{\ell}$ was introduced in Definition 3.5.

Lemma 5.3 (Lemma 6.5 [BOSTV1]). *Let $\hat{\ell} \neq \ell$ be non-periodic leaves of a symmetric lamination $\mathcal{L}$ with $\hat{\ell} \succ \ell$. Given an integer $k > 0$, let $h : \mathbb{S} \to \mathbb{S}$ be either the map σ_3^k or the map $-\sigma_3^k$. Suppose that the leaves ℓ and $\hat{\ell}$ move towards each other by the map h and neither the leaves ℓ and $\hat{\ell}$, nor any leaf separating them, can eventually map into a leaf (including degenerate) with both endpoints in one of the boundary arcs of the strip $\mathcal{S}(\ell, \hat{\ell})$. Then there exists a σ_3-periodic leaf $y \in \mathcal{L}$ that separates ℓ and $\hat{\ell}$.*

For the notion of *two-sided limit leaves*, see Definition 3.16.

Definition 5.4. We say a gap G *weakly separates* two leaves ℓ_1 and ℓ_2 if $\ell_1 \setminus G$ and $\ell_2 \setminus G$ are nonempty sets in two different components of $\overline{\mathbb{D}} \setminus G$. Similarly we say a leaf ℓ *weakly separates* two leaves ℓ_1 and ℓ_2 if $\ell_1 \setminus \ell$ and $\ell_2 \setminus \ell$ are nonempty sets in two different components of $\overline{\mathbb{D}} \setminus \ell$.

Lemma 5.5. *Let $\ell' \neq \ell$ be two leaves in a cubic symmetric lamination $\mathcal{L}$ such that $\ell \prec \ell'$. Suppose that:*

(1) *the leaves ℓ and ℓ' move away from each other under σ_3^k,*

(2) *no leaf weakly separating* ℓ *and* ℓ' *maps to a critical chord of* $\mathbb{S}$ *under the map* σ_3^i *for* $i < k$.

Then, there exists a periodic leaf $y = \overline{ab}$ *with* $\sigma_3^k(a) = a$, $\sigma_3^k(b) = b$ *that weakly separates* ℓ *and* ℓ'.

Proof. A gap G of $\mathcal{L}$ with edges ℓ, ℓ' does not exist as otherwise the gap $\sigma_3^k(G)$ would strictly cover the gap G. Hence the family of leaves $\mathcal{C} \subset \mathcal{L}$ that consists of ℓ, ℓ', and the leaves that weakly separate ℓ and ℓ' has at least one leaf that weakly separates ℓ and ℓ'. Clearly, $\mathcal{C}$ is closed.

Let A be the set of leaves of $\mathcal{C}$ that move in under σ_3^k such that for every leaf $m \in A$, if a leaf n weakly separates ℓ and m, then n also moves in under the map σ_3^k. So, all the leaves in A move in under σ_3^k. Then the closure $\overline{A}$ of A (with respect to the Hausdorff metric) is a family of leaves, too; let $y \in \overline{A}$ be the leaf of $\overline{A}$ farthest from ℓ (i.e., every leaf in $A \setminus \{\ell, y\}$ weakly separates ℓ from y). By continuity, either $y \in A$, or $\sigma_3^k(y) = y$. We claim that $\sigma_3^k(y) = y$. Indeed, suppose that y moves in under σ_3^k. There are two cases. First, it can be that y is approximated by leaves with endpoints outside $H(y)$ (see Definition 3.24). However, this contradicts the choice of y. Second, y can be an edge of a gap G with vertices outside of $H(y)$ while all vertices of $\sigma_3^k(G)$ belong to $\overline{H(y)}$. If now $\hat{\ell}$ is the edge of G with $y \prec \hat{\ell}$, then $\hat{\ell} \in A$, a contradiction.

Thus, $y = \overline{ab} = \sigma_3^k(y)$. We claim that σ_3^k fixes the endpoints of the leaf y. Assume that σ_3^k flips y, and consider cases. If y is a two sided limit leaf and $t \in A$ is close to y, then the leaf t would move out under σ_3^k, a contradiction. If y is an edge of a gap G, then y is an edge of the gap $G' = \sigma_3^k(G)$, the gaps G and G' are on both sides of the leaf y, and σ_3^k maps one gap to the other. Hence there is an edge $t \prec y$ of G or G' that belongs to A but moves out under σ_3^k, a contradiction. Finally, $\sigma_3^k(y) = y$ is non-degenerate. □

Lemma 5.6 (Lemma 6.3 [BOSTV1])**.** *Suppose that* $\mathcal{L}$ *is a cubic symmetric lamination with comajor pair* $\{c, -c\}$ *and that a short leaf* $\ell_s \in \mathcal{L}$ *with* $c \prec \ell_s$ *is such that the leaf* $\ell_m = \sigma_3(\ell_s)$ *never maps under* $\pm\ell_m$. *Then, there exists a cubic symmetric lamination* $\mathcal{L}(\ell_s)$ *with comajor pair* $\{\ell_s, -\ell_s\}$.

5.2. **The description of the L-algorithm.** According to [Mil93, Mil09], cubic polynomials with *Fatou domains whose first return map is of degree 4* are said to be of type B (Bi-transitive) and cubic polynomials with *two cycles of Fatou domains* are said to be of type D

(Disjoint); in the latter case first return maps on periodic Fatou domains are, evidently, of degree 2. We classify co-periodic comajors of in the similar fashion below. Recall that, by Theorem 4.11, co-periodic comajors c generate hyperbolic q-laminations $\widehat{\mathcal{L}}(c)$.

The nature of cubic symmetric laminations gives rise to two notions describing two types of periodic points and related (pre)periodic objects. We give a general definition that applies to all of them. Recall that, for $x \in \mathbb{S}$, we write $-x$ for the point symmetric to x with respect to the center of $\mathbb{S}$. Under the identification between $\mathbb{S}$ and $\mathbb{R}/\mathbb{Z}$, the point $-x$ identifies with $x + \frac{1}{2}$.

Definition 5.7 (Types B and D)**.** A $2n$-periodic point x of σ_3 such that $\sigma_3^n(x) = -x$, is said to be *of type B*. All other periodic points of σ_3 are said to be *of type D*. A periodic leaf of a symmetric lamination is of type B if its endpoints are of type B, and of type D otherwise. A co-periodic leaf of a symmetric lamination is of type B if its image is a periodic leaf of type B, and of type D otherwise.

Lemma 5.8 (Corollary 3.7 [BMOV13])**.** *Suppose that ℓ and $\hat{\ell}$ are two leaves of a σ_d-invariant lamination that share an endpoint and have non-degenerate distinct images. Then the orientation of the triple of their endpoints is preserved under the map σ_d.*

To justify Definition 5.7 we need the next lemma.

Lemma 5.9. *A periodic leaf of a symmetric lamination $\mathcal{L}$ cannot have one endpoint of type B and the other endpoint of type D.*

Proof. Suppose that $\ell = \overline{xy}$ is a periodic leaf of $\mathcal{L}$ such that x is of type B while y is of type D. Then x is of period $2n$ and $\sigma_3^n(x) = -x$. It follows that y is also of period $2n$ but $\sigma_3^n(y) \neq -y$. Since $\overline{(-x)(\sigma_3^n(y))} = \sigma_3^n(\ell)$ is a leaf of $\mathcal{L}$, then the leaf $\overline{x(-\sigma_3^n(y))}$ is a leaf of $\mathcal{L}$, too. Thus, the union of leaves $\ell = \overline{xy}$ and $\overline{x(-\sigma_3^n(y))}$ is mapped by σ_3^n to the union of leaves $\overline{(-x)(\sigma_3^n(y))}$ and $\overline{(-x)(-y)}$. However it is easy to see that the orientation of the triple $(y, x, -\sigma_3^n(y))$ is opposite to the orientation of the triple $(\sigma_3^n(y), -x, -y)$. This contradicts Lemma 5.8 and completes the proof. □

Evidently, the σ_3-image of an object of type B or D is an object of the same type; co-periodic comajors can be either of type B or of type D. Also, Definition 5.7 allows us to talk about majors, comajors, and minors of types B or D. In the type B case a periodic major $M = \overline{ab}$ eventually maps to $-M$ so that a and b of M eventually map to the $-a$ and $-b$, respectively. In the type D case, the orbits of a and $-a$

(and also b and $-b$) are disjoint. Thus, if a co-periodic comajor c is of type B, then the lamination $\widehat{\mathcal{L}}(c)$ from Theorem 4.11 has a pair of symmetric Fatou gaps whose first return map is of degree 4; if c is of type D then $\widehat{\mathcal{L}}(c)$ has a pair of symmetric Fatou gaps whose first return map is of degree 2.

Definition 5.10. A periodic point (leaf) of type B and period $2n$ is said to be *of block period* n. A periodic point (leaf) of type D and period n is said to be *of block period* n. A co-periodic leaf is said to be *of block period* n if its image is of block period n.

In [BOSTV1] we considered the map τ that rotates the unit disk by 180 degrees. If $\mathcal{L}$ is a cubic symmetric lamination, then τ acts on leaves and gaps of $\mathcal{L}$. We will also interchangeably use the notation $-\ell$ for $\tau(\ell)$ and $-G$ for $\tau(G)$ where ℓ is a leaf of $\mathcal{L}$ and G is a gap of $\mathcal{L}$. Define the map $g_j = \tau \circ \sigma_3^j : \mathcal{L} \to \mathcal{L}$ for some j. Lemma 5.11 is similar to Lemma 5.5. We state it without proof.

Lemma 5.11. *Let $\ell' \neq \ell$ be two leaves in a cubic symmetric lamination $\mathcal{L}$ such that $\ell \prec \ell'$. Suppose that:*

(i) *the leaves ℓ and ℓ' move away from each other under g_k,*
(ii) *no leaf weakly separating ℓ and ℓ' maps to a critical chord of $\mathbb{S}$ under the map g_i for $i < k$.*

Then, there exists a periodic leaf y of period 1 under the map g_k that weakly separates ℓ and ℓ'.

The next lemma deals with dynamics of comajors.

Lemma 5.12. *Suppose that $c' \prec c$ are distinct co-periodic comajors that are leaves of a lamination $\mathcal{L}$. Then there is no finite gap H of $\mathcal{L}$ such that both c' and c are edges of H.*

Proof. The leaves $m' = \sigma_3(c') \prec m = \sigma_3(c)$ are periodic. By way of contradiction assume that both are edges of a periodic gap $\sigma_3(H) = G$ of $\mathcal{L}$. Then their endpoints stay in the same circular order along their periodic orbits. By Lemma 3.10, if G is *1-rotational*, then the leaf m will eventually map to the leaf m', and if G is *2-rotational*, then the leaf m will eventually map to the leaf $-m'$, in either case contradicting that m is the shortest leaf in its orbit (see Lemma 3.15). □

Now, the main theorem needed for the L-algorithm is as follows.

Theorem 5.13. *Suppose that co-periodic comajors c and c' have the following properties:*

(i) *$c' \prec c$,*

(ii) *both c and c' are either of type B or type D, and*
(iii) *c and c' have the same block period n.*

Then there exists a co-periodic comajor d with $c' \prec d \prec c$ such that d is of block period $j < n$.

Proof. Choose a preperiodic point p of preperiod bigger than 1 and period bigger than n in the arc $H(c')$. By Lemma 4.13, there exists a cubic symmetric dendritic q-lamination $\mathcal{L}$ with a pair of finite critical gaps/leaves $\{\Delta, -\Delta\}$ such that $\sigma_3(p) \in \sigma_3(\Delta)$ (i.e., the critical leaves $\pm\ell$ of $\mathcal{L}(p)$ are contained in the critical sets Δ and $-\Delta$), iterated preimages of $\pm\ell$ converge to all sides of Δ and $-\Delta$, so that pullbacks of the critical sets are dense in $\mathcal{L}$, and c and c' are leaves of $\mathcal{L}$. The leaves $m = \sigma_3(c)$ and $m' = \sigma_3(c')$ are periodic and such that $m' \prec m$. Since preimages of $\pm\Delta$ are dense in $\mathcal{L}$, then it follows from Lemma 5.12 that for a minimal k, the set Δ (or $-\Delta$) separates $\sigma_3^k(m)$ and $\sigma_3^k(m')$. Consider cases.

(i): *comajors c and c' are of type D.* Then the periodic orbits of m and $-m$ (and also m' and $-m'$) are disjoint and have period n. We claim that $k \neq n-1$. If $k = n-1$, then $\sigma_3^k(m)$ and $\sigma_3^k(m')$ are long/medium siblings of c and c', respectively. Hence they must be separated by Δ. The circular order of the four endpoints of m and m' is preserved in the leaves $\sigma_3^{n-1}(m)$ and $\sigma_3^{n-1}(m')$, but when σ_3 is applied one more time, exactly one of the leaves $\sigma_3^{n-1}(m)$ and $\sigma_3^{n-1}(m')$ flips because of the critical gap between them. Hence the order among the endpoints of $\sigma_3^n(m) = m$ and $\sigma_3^n(m') = m'$ cannot be the same as the order among the endpoints of m and m', which is absurd. Thus, $0 \leq k < n-1$.

(a): *it is Δ that separates the leaves $\sigma_3^k(m)$ and $\sigma_3^k(m')$.* Since the leaves and gaps separating m and m' map one-to-one under σ_3^k, there is a set Δ^* separating m and m' with $\sigma_3^k(\Delta^*) = \Delta$. Moreover, since $\sigma_3(p) \prec m'$, $\sigma_3^{k+1}(\Delta^*) = \sigma_3(\Delta) \prec m'$ (see Figure 8). Let ℓ^* be the side of Δ^* that separates m and m' and is closest to the leaf m. Then ℓ^* moves in under the map σ_3^{k+1}. On the other hand, the leaf $\sigma_3^{k+1}(m)$ is neither under the leaf m nor under the leaf $-m$ because the minor is the shortest leaf in its orbit. Hence the leaves m and ℓ^* move away from each other under the map σ_3^{k+1}.

Let us verify condition (2) from Lemma 5.5. Note that $\sigma_3^k(\ell^*) = M$ is a major of $\mathcal{L}$. For $i \leq k$, the map σ_3^i takes the leaves separating ℓ^* and m in the strip $\mathcal{S}(\ell^*, m)$ one-to-one to the leaves separating $\sigma_3^i(\ell^*)$ and $\sigma_3^i(m)$ in the strip $\mathcal{S}(\sigma_3^i(\ell^*), \sigma_3^i(m))$. As there are no critical chords of $\mathbb{S}$ in $\mathcal{S}(\sigma_3^i(\ell^*), \sigma_3^i(m))$ for $i < k$, no leaf separating ℓ^* and m maps to a critical chord of $\mathbb{S}$ under the map σ_3^i for $i < k$. Moreover, if $i = k$, then all such leaves are situated between Δ and M and, hence, can also not be critical. Hence, by Lemma 5.5, there is a periodic leaf $y \in \mathcal{L}$ of period $k+1 < n$ separating m and ℓ^*.

Let $\mathcal{C}$ be the collection of the leaves separating m and m'. Let C_1 be the collection of all σ_3-periodic leaves in $\mathcal{C}$ of period smaller than n. Let C_2 be the collection of all fixed leaves under the maps $g_i = -\sigma_3^i$, $0 < i < n$ in $\mathcal{C}$; we associate the minimal such i with all leaves from C_2. Since $y \in C_1$, then $C_1 \neq \varnothing$, but C_2 could be empty.

Let y_1 be a leaf of the least period $j_1 \leq k+1 < n$ in C_1. Choose y_1 to be the closest to m among leaves of C_1 of period j_1. Similarly, choose a $-\sigma_3^{j_2}$-fixed leaf y_2 in C_2 such that j_2 is the smallest possible; choose y_2 to be the closest to m among $-\sigma_3^{j_2}$-fixed leaves in C_2. If $j_1 \leq j_2$, then we claim that the leaf d which is the short pullback of y_1 in $\mathcal{L}$ is the desired comajor of block period $j = j_1 < n$ (recall that y_1 is located between the minors m and m'). By Lemma 5.6, it suffices to prove that the leaf y_1 neither maps under itself nor under the leaf $-y_1$ under the map σ_3^i for any $i < j_1$.

(1) If y_1 maps under itself under σ_3^i, for some $i < j_1$, then the leaves y_1 and m move away from each other under σ_3^i. By Lemma 5.5, there is a σ_3-periodic leaf y_1' of period $i < j_1$ separating m and y_1; a contradiction with the minimality of j_1.

(2) If y_1 maps under $-y_1$ under σ_3^i for some $i < j_1$, then the leaf $g_i(y_1)$ is under the leaf y_1. Now, the leaves y_1 and m move away from each other under $g_i = -\sigma_3^i$. By Lemma 5.11, there is a $-\sigma_3^i$-fixed leaf y_1' that separates m and y_1. Clearly y_1' separates m and m', too. Then $i < j_1 \leq j_2$ is the block period associated with y_1, contradicting the choice of j_2.

Thus, the short pullback d of y_1 in $\mathcal{L}$ is the desired comajor of block period $j = j_1 < n$. Similarly if $j_2 < j_1$, then we obtain that the short pullback d of y_2 in $\mathcal{L}$ is the desired comajor of block period $j = j_2 < n$.

(b): *it is* $-\Delta$ *that separates the leaves* $\sigma_3^k(m)$ *and* $\sigma_3^k(m')$*, not* Δ. We use the arguments from case (a) and find a gap Δ^* with $\sigma_3^k(\Delta^*) = -\Delta$ separating m and m'. Then we have the gap $\sigma_3^{k+1}(\Delta^*)$ going under the leaf $-m'$. The only difference in the arguments is that we use Lemma 5.11 first to find a leaf y separating m and m' such that $g_{k+1}(y) = y$. Thus, the collection C_2 is non-empty here whereas collection C_1 could be empty. The rest of the argument follows exactly as before and we end up with a comajor d between c and c' of a block period $j < n$.

(ii): *comajors* c *and* c' *are of type B*. The leaves m and m' are now periodic of period $p = 2n$ and have symmetric orbits (the orbits of m and $-m$ are the same). Similarly, the orbits of the leaves m' and $-m'$

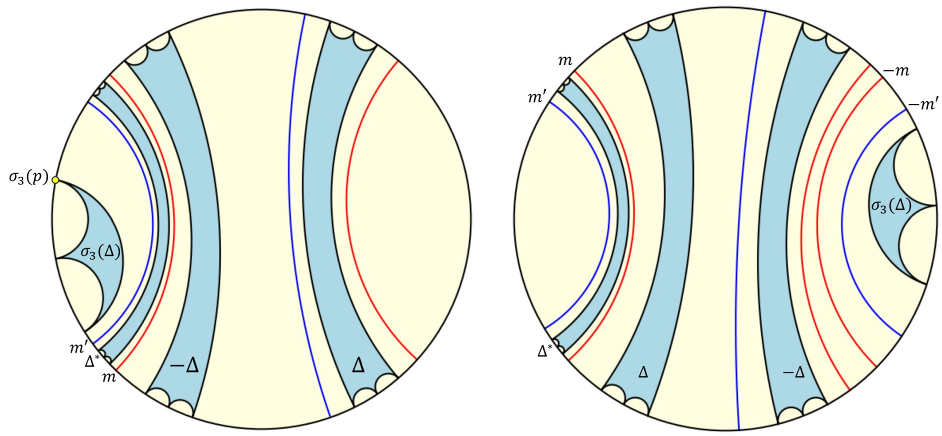

FIGURE 8. Left: cases (i)(a) and (ii)(a) of the proof of Theorem 5.13, cases (i) and (ii). Right: cases (i)(b) and (ii)(b) of the proof of Theorem 5.13.

are the same as well. In this case, the proof is very similar to that of case (i) (see Figure 8).

First, we show that there exists an integer k with $0 \leq k < n-1$ such that Δ or $-\Delta$ separates the leaves $\sigma_3^k(m)$ and $\sigma_3^k(m')$. Indeed, let k be the smallest integer between 0 and $p = 2n$ such that the leaves $\sigma_3^k(m)$ and $\sigma_3^k(m')$ are separated by a critical gap/leaf. As the orbits of both the leaves m and m' are symmetric, the strips formed by the leaves $\sigma_3^i(m)$ and $\sigma_3^i(m')$ where $0 < i \leq n-1$ are symmetric to the strips formed by the leaves $\sigma_3^r(m)$ and $\sigma_3^r(m')$ where $n \leq r < 2n$. It follows that, for the first time, the separation by one of the critical gaps/leaves Δ and $-\Delta$ happens during the first half of the cycle, i.e., $0 \leq k \leq n-1$.

To see that k cannot be equal to $n-1$, assume the contrary. Since $\sigma_3^n(m) = -m$ and $\sigma_3^n(m') = -m'$, the leaves $\sigma_3^{n-1}(m)$ and $\sigma_3^{n-1}(m')$ must be long/medium siblings of $-c$ and $-c'$, respectively. Hence they are separated by $-\Delta$. The circular order of the four endpoints of m and m' is preserved in the leaves $\sigma_3^{n-1}(m)$ and $\sigma_3^{n-1}(m')$, and exactly one of them flips under the next iteration because of a critical gap between them. Without loss of generality, assume that the leaf $\sigma_3^{n-1}(m)$ flips its endpoints when it maps to the leaf $-m = \sigma_3^n(m)$. Since no additional flip takes place under the remaining part of the orbit, it would follow that m returns to itself under σ_3^{2n} with its endpoints flipped, a contradiction. Thus, $0 < k < n-1$. We have two subcases here similar to case (i).

(a): *it is* Δ *that separates the leaves* $\sigma_3^k(m)$ *and* $\sigma_3^k(m')$. Then, following the similar arguments as in case (i)(a), we find a comajor d of block period $j < k+1 = n$ separating the leaves c and c'.

(b): *it is* $-\Delta$ *that separates the leaves* $\sigma_3^k(m)$ *and* $\sigma_3^k(m')$*, not* Δ. Then, using similar arguments to case(i) part(b), we find a comajor d of block period $j < k+1 = n$ separating the leaves c and c'. □

Theorem 5.13 allows us to describe an algorithm for finding co-periodic cubic comajors similar to the Lavaurs algorithm [Lav86, Lav89] for finding periodic quadratic minors. We call this algorithm the *L-algorithm.*

L-algorithm

(1) Draw co-periodic comajors of block period 1. It is easy to verify that type D co-periodic comajors of period 1 are $\overline{\frac{1}{6}\frac{1}{3}}$ and $\overline{\frac{2}{3}\frac{5}{6}}$. Similarly, type B co-periodic comajors of block period 1 are $\overline{\frac{5}{12}\frac{7}{12}}$ and $\overline{\frac{11}{12}\frac{1}{12}}$.

We proceed by induction. Suppose that all preperiodic comajors of block periods from 1 to k (inclusively) have been drawn. Denote the family of them by $\mathcal{F}_k$. Consider a component A of $\overline{\mathbb{D}} \setminus \bigcup_{\ell\in\mathcal{F}_n} \ell$. Then there are two cases.

(a) Suppose that there is a comajor ℓ_0 such that all points of A are located under ℓ_0. Then there may be several comajors $\ell_1, \dots, \ell_s$ located under ℓ_0 with endpoints in $A \cap \mathbb{S}$ (this collection of comajors may be empty). Consider the set $B = \{b_1 < \dots < b_t\}$ of preperiodic points of type B of preperiod 1 and block period $k+1$ that belong to $A \cap \mathbb{S}$. These points (if any) must be connected to create several comajors. By Lemma 5.1, these comajors are pairwise disjoint. By Theorem 5.13, no two comajors from that collection can be located so that one of them is under the other one. Hence $t = 2r$ is even and the comajors in question are $\overline{b_1b_2}, \dots, \overline{b_{2r-1}b_{2r}}$. We can also consider the set D of preperiodic points of type D of preperiod 1 and block period $k+1$ that belong to $A \cap \mathbb{S}$. These points should be connected similar to how points from B were connected, i.e. *consecutively.*

Do this for all components A for which there is a comajor ℓ_0 such that all points of A are located under ℓ_0.

(b) There is exactly one component C of $\overline{\mathbb{D}} \setminus \bigcup_{\ell\in\mathcal{F}_n} \ell$ for which there is no comajor ℓ_0 with all points of A located under ℓ_0. This is the "central" component left after the closures of all components described in (a) are removed from $\mathbb{D}$. Evidently, this component contains the center of $\overline{\mathbb{D}}$ and is unique.

As before, let B be the set of preperiodic points of type B of preperiod 1 and block period $k+1$ that belong to $C \cap \mathbb{S}$. However, unlike before let us divide B into four subsets: $B^1 = B \cap (\frac{1}{12}, \frac{1}{6})$, $B^2 = B \cap (\frac{1}{3}, \frac{5}{12})$, $B_3 = B \cap (\frac{7}{12}, \frac{2}{3})$, and $B_4 = B \cap (\frac{5}{6}, \frac{11}{12})$. Since comajors are short, a comajor cannot connect two points from two distinct B-sets. Hence, as in case (a), comajors connect points from B consecutively within B-sets. If, e.g., $B_1 = \{b_1 < \dots < b_t\}$, then, as in (a), $t = 2r$ is even, and the corresponding comajors are $\overline{b_1 b_2}$, $\dots$, $\overline{b_{2r-1} b_{2r}}$. Points of type D that belong to ∂C should be treated similarly.

Thus, the L-algorithm for cubic symmetric laminations is as follows. First, we make step 1 and draw the comajors $\overline{\frac{11}{12}\frac{1}{12}}$, $\overline{\frac{1}{6}\frac{1}{3}}$, $\overline{\frac{5}{12}\frac{7}{12}}$, and $\overline{\frac{2}{3}\frac{5}{6}}$. Then on each next step, say, $k+1$, we first plot all type B points of preperiod 1 and block period $k+1$ and connect them consecutively, starting from the smallest positive angle. Then we plot all type D points of preperiod 1 and block period $k+1$ and connect them consecutively, too, starting from the smallest positive angle.

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