MONOTONICITY OF THE OVER-ROTATION INTERVAL FOR BIMODAL MAPS

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ABSTRACT. We show that the *over-rotation interval* is a monotone function of a bimodal interval map.

1. INTRODUCTION

One-dimensional combinatorial dynamics started when O. M. Sharkovsky proved his theorem on the coexistence of periods of cycles for interval maps (in what follows by *cycles* we mean *periodic orbits*). To state it, we recall the (transitive) *Sharkovsky order* of the set natural numbers:

$$3 \succ_s 5 \succ_s 7 \succ_s \dots \succ_s 2\cdot 3 \succ_s 2\cdot 5 \succ_s 2\cdot 7 \succ_s \\ \dots \succ_s 2^2\cdot 3 \succ_s 2^2\cdot 5 \succ_s 2^2\cdot 7 \succ_s \dots \succ_s 2^2 \succ_s 2 \succ_s 1.$$

In what follows by the *period* we mean the *minimal* period. Below I always denotes a closed interval.

Theorem 1.1 ([12]; [13] for English translation)**.** *If $g : I \to I$ is continuous, $m \succ_s n$ and m is the period of a cycle of g then n is also the period of a cycle of g.*

Theorem 1.1 inspired a lot of developments. One of them is the discovery of *over-rotation numbers* [8], a combinatorial tool used for classifying interval cycles and invariant under topological conjugacy of interval maps. It can be defined as follows.

Let I be the unit interval, let $f : I \to I$ be a continuous interval map, let P be a cycle of f of period $q > 1$, and let m be the number of points $x \in P$ such that $f(x) - x$ and $f^2(x) - f(x)$ have different signs. Call the pair $(\frac{m}{2}, q) = \text{orp}(P)$ the *over-rotation pair* of P and denote it by *orp*(P); call $\frac{m}{2q} = \rho(P)$ the *over-rotation number* of P. An over-rotation pair (p, q) is *coprime* if p and q are coprime. The set of *over-rotation pairs* of all cycles of f is denoted by $ORP(f)$. Since the number $0 < m \leqslant q$ is even, then, in an *over-rotation pair* (p, q), p and q are integers and $0 < \frac{p}{q} \leq \frac{1}{2}$. The number p can be interpreted as the number of times $f(x)$ goes around x as we move along the orbit of the "vector" $\overrightarrow{xf(x)}$. If f has a unique fixed point a, then $f(x) > x$ if $x < a$ and $f(x) < x$ if $x > a$, and p is the number of points to the right of a which are mapped to the left of a (or vice versa).

Date: January 8, 2022.

2010 *Mathematics Subject Classification.* Primary 37E05, 37E15; Secondary 37E45.

Key words and phrases. Over-rotation pairs and numbers, pattern, cycle.

Theorem 1.2 ([8]). *Suppose that* (p, q) *and* (r, s) *are* over-rotation pairs. *Moreover, suppose that one of the following holds.*

(1) $\frac{p}{q} < \frac{r}{s}$.
(2) $\frac{p}{q} = \frac{r}{s}$ *so that for a coprime over-rotation pair* (k, l) *we have* $p/k = q/l = u$ *and* $r/k = s/l = v$ *are integers, and* $u \succ_s v$.

Then any interval map with a cycle of over-rotation pair (p, q) *has a cycle of over-rotation pair* (r, s).

Definition 1.3 is based upon Theorem 1.2; to rule out trivial cases from now on we will consider only maps with non-fixed periodic points (otherwise the orbit of any point converges to a fixed point).

Definition 1.3. Given an interval map f, denote by I_f the closure of the union of over-rotation numbers of f-periodic points, and call I_f the *over-rotation interval* of f.

By Theorem 1.1, any map f with non-fixed periodic points has a cycle of period 2 with over-rotation number $\frac{1}{2}$; by Theorem 1.2 if $\rho(P) = \frac{p}{q}$ for a cycle P then $[\frac{p}{q}, \frac{1}{2}] \subset I_f$; hence for any interval map f there exists a number $r_f, 0 \leq r_f < \frac{1}{2}$, such that $I_f = [r_f, \frac{1}{2}]$.

A continuous self-mapping f of an interval is said to have a *horseshoe* if there exist subintervals I and J of the domain of f disjoint except perhaps a common endpoint, such that $f(I) \cap f(J) \supset I \cup J$. By [8], if f has a horseshoe then $I_f = [0, \frac{1}{2}]$.

A map f is *piecewise-monotone* if the domain of I can be partitioned into finitely many intervals (*laps*) on which f is *strictly monotone*. If the smallest number of such intervals is 2, then f is said to be *unimodal*; in this case f has one turning point. If the smallest number of such intervals is 3, then f is said to be *bimodal*; in this case f has two turning point. In this paper we *always* consider *only* piecewise-monotone maps that are increasing on their leftmost laps.

Every cycle P of a map f induces a cyclic permutation Π obtained by looking at how the map f acts on the points of P ordered from the left to the right. One can introduce a relation $\sim$ on the family of all cycles such that for two cycles P and Q, $P \sim Q$ iff P and Q induce the same permutation; $\sim$ is an equivalence relation whose equivalence classes are called *patterns*. If an interval map f has a cycle P from a *pattern* π associated with permutation Π, say that P is a *representative* of π (Π) (in f) and f *exhibits* π (Π) (on P). A pattern π (a permutation Π) forces a pattern θ (a permutation Θ) if any continuous interval map f which exhibits π also exhibits θ. By [2], *forcing* is a partial ordering. In what follows we will interchangeably talk about patterns and permutations.

A useful algorithm allows one to describe all patterns forced by a pattern π. Consider a cycle P of pattern π, and denote the leftmost point of P by a and the rightmost point of P by b. Every component of $[a, b] - P$ is called a *P-basic interval*. Extend this map from P to $[a, b]$ by defining it linearly on each P-basic interval and call the resultant map f_P the *P-linear map*. Then, the patterns of all cycles of f_P are exactly the patterns forced by the pattern of P (see [1] and [2]).

Over-rotation pairs and numbers for patterns are defined just like for cycles. Denote the *over-rotation pair* and *over-rotation number* of a pattern π by $\text{orp}(\pi)$ and $\rho(\pi)$, resp. If P is a cycle of the pattern π, call the over-rotation interval $I_\pi = [r_\pi, \frac{1}{2}]$ of the P-linear map f_P the *over-rotation interval of* π.

Definition 1.4. A *pattern* π is called *over-twist* if it does not *force* any other *pattern* of the same *over-rotation number.*

By Theorem 1.2, an over-twist pattern has a *coprime over-rotation pair* (i.e., an over-rotation pair (p, q) where p and q are coprime); in particular, there is a unique over-twist pattern of over-rotation number $\frac{1}{2}$ associated with a unique cyclic permutation of period 2, so from now on we consider *over-twists* of *over-rotation number* distinct from $\frac{1}{2}$. By [8] and by properties of forcing relation, for any $\frac{p}{q} \in (r_f, \frac{1}{2})$, an over-twist pattern of over-rotation number $\frac{p}{q}$ is exhibited by a cycle of f; over-twists are patterns that are *guaranteed* to be exhibited by a map f if the interior of I_f contains the appropriate *over-rotation number.* This can be sharpened if f is piecewise-monotone.

Theorem 1.5 ([5], [6], [7]). *If $f : [0, 1] \to [0, 1]$ is a piecewise monotone continuous map with over-rotation interval $[r_f, \frac{1}{2}]$, then for any $\frac{p}{q} \in [r_f, \frac{1}{2}]$, there exists a cycle P which exhibits over-twist pattern of over-rotation number $\frac{p}{q}$.*

The over-rotation interval of an over-twist pattern π is $[\rho(\pi), \frac{1}{2}]$ so that $r_\pi = \rho(\pi)$ holds. In [3] the following version of the opposite statement is proven: a pattern π with *coprime over-rotation pair* is *over-twist* if and only if $r_\pi = \rho(\pi)$.

In [9], it was proven that for a given rational number $\frac{p}{q}$ there exists a unique unimodal over-twist pattern $\gamma_{\frac{p}{q}}$ of over-rotation number $\frac{p}{q}$; moreover, the dynamics of $\gamma_{\frac{p}{q}}$ was described. It was also shown that the over-rotation interval is a monotone function of a map considered on a wide variety of one-parameter families of unimodal maps.

The goal of the present paper is to prove a similar result for the family of *bimodal interval maps.* To this end let us recall the following result from [11].

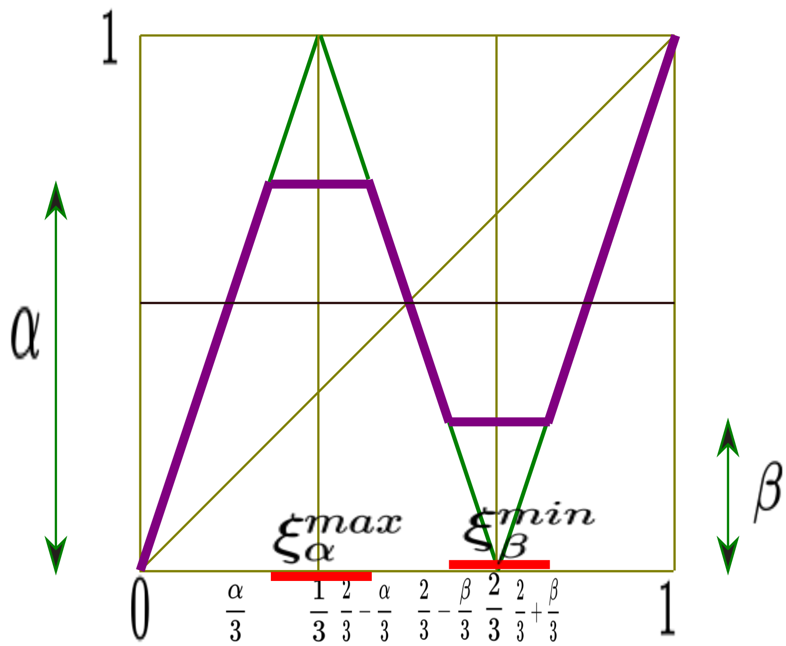
Theorem 1.6 ([11]). *To any bimodal map $f : [0, 1] \to [0, 1]$ there is associated a canonical truncation of bimodal horseshoe map $H_P : [0, 1] \to [0, 1]$ which has exactly the same kneading data. Furthermore, there exists a monotone but discontinuous correspondence $\theta : [0, 1] \to [0, 1]$ which semi-conjugates f to H_P.*

From Theorem 1.6 it follows that any *bimodal interval map* can be "*modeled*" by a *truncation of a bimodal horseshoe map.* So, we parameterize the family of all *truncations of a bimodal horseshoe map* and then show that the set of the *parameters* which correspond to a *fixed over-rotation interval* is a *connected subset* of the *parameter space.* For this we will use the explicit description of all *bimodal over-twist patterns* obtained in a recent paper [4]. Observe that our results do not imply monotonicity of the over-rotation interval in any given family of bimodal maps increasing on the leftmost lap, nor do they imply its monotonicity in any natural space of smooth bimodal maps. This issue is non-trivial even in the unimodal case (we are not aware of the relevant results).

It is natural to find an efficient measure of dynamical complexity of maps which could be effectively computed. Hopefully, *over-rotation numbers* can serve this end. Not only can they be easily computed, our work provides a way to partition the parameter space of bimodal maps into areas associated with a fixed over-rotation interval which in turn encapsulates the limiting dynamical behavior of points. The generalization of the monotonicity of over-rotation intervals from unimodal to bimodal maps suggests that future generalizations to maps of higher modality are also possible.

We discuss our plans for this paper in detail below.

FIGURE 1. The graphs of the maps H_2 and $H_{\alpha,\beta}$ with the latter shown in bolder lines



A *bimodal saw map* $H_2 : [0,1] \to [0,1]$ has cycles of all bimodal patterns; it is defined by

$$H_2(x) = \begin{cases} 3x & \text{if } 0 \leq x \leq \frac{1}{3} \\ 2 - 3x & \text{if } \frac{1}{3} \leq x \leq \frac{2}{3} \\ 3x - 2 & \text{if } \frac{2}{3} \leq x \leq 1 \end{cases}$$

For $\alpha, \beta \in [0,1]$ with $\alpha \geqslant \beta$, we call the interval $\xi_\alpha^{max} = [\frac{\alpha}{3}, \frac{2}{3} - \frac{\alpha}{3}]$ where $H_2(x) \geqslant \alpha$ the *level* α *max flat spot* and the interval $\xi_\beta^{min} = [\frac{2}{3} - \frac{\beta}{3}, \frac{2}{3} + \frac{\beta}{3}]$ where $H_2(x) \leqslant \beta$ the *level* β *min flat spot*. The map $H_{\alpha,\beta} : [0,1] \to [0,1]$ defined by:

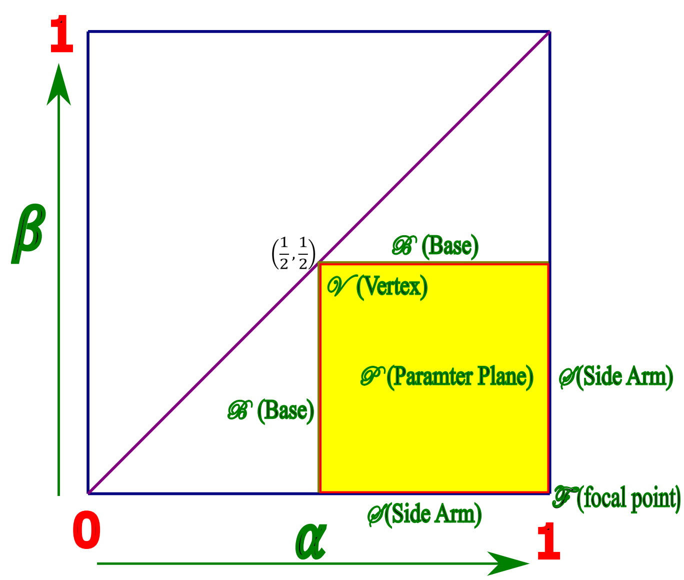
$$H_{\alpha,\beta}(x) = \begin{cases} H(x) & \text{if } x \notin \xi_\alpha^{max} \cup \xi_\beta^{min} \\ \alpha & \text{if } x \in \xi_\alpha^{max} \\ \beta & \text{if } x \in \xi_\beta^{min} \end{cases}$$

is called the *bimodal truncation* of H with parameters α and β. Figure 1 shows the graphs of H_2 and $H_{\alpha,\beta}$.
Recall that by [10] any bimodal map can be modeled by some $H_{\alpha,\beta}$. Moreover, there is an obvious hierarchy among truncations which is used in our paper. Namely, suppose that $\alpha \leqslant \alpha'$ and $\beta \geqslant \beta'$ and compare two truncations, $H_{\alpha,\beta}$ and $H_{\alpha',\beta'}$. Then any orbit of $H_{\alpha,\beta}$ that does not enter the interior of its flat spots is also an orbit of $H_{\alpha',\beta'}$. In other words, *the dynamics of the map in the parameter space of truncations becomes richer as the first coordinate in the parameter space increases and the second coordinate decreases*. This relates to all orbits that do not enter the interior of the flat spots of $H_{\alpha,\beta}$, but this does not complicate the situation since a lot of characteristics of the map do not depend on just one orbit.

Let $\mathcal{H} = \{H_{\alpha,\beta} : \alpha, \beta \in [0,1] \ \& \ \alpha \geqslant \beta\}$ be the family of all truncations of H_2. Denote the *over-rotation interval* of $H_{\alpha,\beta}$ by $I_{\alpha,\beta}$. It is easy to see that if $\alpha = \beta$, $\alpha \leqslant \frac{1}{2}$, or $\beta \geqslant \frac{1}{2}$, then all periodic points of $H_{\alpha,\beta}$ are fixed. Hence, the ω-limit set $\omega(x)$ is a fixed point for every x, and we are not considering such maps. Excluding these parameters, we define the *parameter space* as $\mathcal{P} = \{(\alpha,\beta) \in [0,1] \times [0,1] : \alpha \geqslant \frac{1}{2} \geqslant \beta\}$; $\mathcal{P}$ is a *square* with vertices $(1,0), (1,\frac{1}{2}), (\frac{1}{2},\frac{1}{2})$ and $(\frac{1}{2},0)$.

Let us use the following terminology to describe $\mathcal{P}$ throughout the paper. The *points* $(1,0)$ and $(\frac{1}{2},\frac{1}{2})$ in the *parameter plane* $\mathcal{P}$ are called the *focal point* $\mathcal{F}$ and the *vertex* $\mathcal{V}$ respectively. Let us call the set $\mathcal{B} = \{(x,\frac{1}{2}) | \frac{1}{2} \leq x \leqslant 1\} \cup \{(\frac{1}{2},y) | 0 \leqslant y \leqslant \frac{1}{2}\}$ the *base set*. Similarly, call the set $\mathcal{S} = \{(x,0) | \frac{1}{2} \leqslant x \leqslant 1\} \cup \{(1,y) | 0 \leqslant y \leqslant \frac{1}{2}\}$ the *side arm*.

FIGURE 2. The Parameter space $\mathcal{P}$



Define a map $\psi : \mathcal{P} \to [0, \frac{1}{2}]$ with $\psi(\alpha, \beta) = \rho(\alpha, \beta)$ where $I_{\alpha,\beta} = [\rho(\alpha, \beta), \frac{1}{2}]$. Observe that at the *focal point* the value of ψ is the *least* (namely, 0), while at any point in the *base* $\mathcal{B}$, the value of ψ is the *greatest* (namely, $\frac{1}{2}$). We will call the set $\psi^{-1}(\nu) = \psi_\nu = \{(\alpha, \beta) \in \mathcal{P} : \psi((\alpha, \beta) = \nu\}$ the *Bimodal Iso-over-rotation-tract* corresponding to ν or the ν*-Bimodal Iso-over-rotation-tract.* The main objective of our paper is to show that for any $\nu \in [0, \frac{1}{2}]$ the ν*-Bimodal Iso-over-rotation-tract* is a *connected set.* In other words the map $\psi : \mathcal{P} \to [0, \frac{1}{2}]$ is a *monotone map.*

We divide our paper into three sections. Section 1 is our Introduction. Section 2 contains some preliminary ideas. Section 3 is the main section of the paper where we prove that *Bimodal Iso-over-rotation-tracts* are connected.

The results of the paper were a topic of the talk given by the first named author at 2021 Spring Topology and Dynamics Conference organized by Murray State University, KY in May, 2021.

Acknowledgements. The authors are indebted to the referee for a detailed report and numerous useful suggestions and remarks concerning the results of the article.

2. PRELIMINARIES

2.1. **Kneading sequences.** Let $f : [0, 1] \to [0, 1]$ be a bimodal map with turning points $0 < c_1 < c_2 < 1$. Call $\mathcal{S} = \{I_0, C_1, I_1, C_2, I_2\}$ where $I_0 = [0, c_1)$, $C_1 = \{c_1\}$, $I_1 = (c_1, c_2)$, $C_2 = \{c_2\}$ and $I_2 = (c_2, 1]$ the *symbolic set* of f. Order its elements as $I_0 < C_1 < I_1 < C_2 < I_2$; we will write $\mathcal{J}' \leqslant \mathcal{J}''$ if $\mathcal{J}' = \mathcal{J}''$ or $\mathcal{J}' < \mathcal{J}''$. Define a *location function* $\mathcal{J} : [0, 1] \to \mathcal{S}$ which assigns to each point $x \in [0, 1]$, its unique *location* $\mathcal{J}(x) \in \mathcal{S}$ defined by $x \in \mathcal{J}(x)$. Observe that if $x < y$, then $\mathcal{J}(x) \leqslant \mathcal{J}(y)$. Let $\mathcal{S}^{\mathbb{N}}$ be the set of all infinite sequences of symbols $(\mathcal{J}_0, \mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3, \dots)$ where $\mathcal{J}_i \in \mathcal{S}$ for every i.

Define a function $\mathcal{I} : [0, 1] \to \mathcal{S}^{\mathbb{N}}$ which associates with each point $x \in [0, 1]$, the unique sequence $\mathcal{I}(x) = (\mathcal{J}(x), \mathcal{J}(f(x)), \mathcal{J}(f^2(x)), \dots) \in \mathcal{S}^{\mathbb{N}}$. We call $\mathcal{I}(x)$, the *itinerary* of the point x. Let $\sigma : \mathcal{S}^{\mathbb{N}} \to \mathcal{S}^{\mathbb{N}}$ be the *shift map* defined by $\sigma(\mathcal{J}_0, \mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3, \dots) = (\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3, \mathcal{J}_4, \dots)$. It follows that $\mathcal{I} : [0, 1] \to \mathcal{S}^{\mathbb{N}}$ conjugates the map $f : [0, 1] \to [0, 1]$ to the *shift map* $\sigma : \mathcal{S}^{\mathbb{N}} \to \mathcal{S}^{\mathbb{N}}$. This allows one to model the dynamics of the map f using the properties of the *itineraries.*

Define a *sign* function $\Theta : \mathcal{S} \to \{-1, 0, 1\}$ as follows: for any j, $\Theta(I_j) = +1$ if f is *increasing* on I_j, $\Theta(I_j) = -1$ if f is *decreasing* on I_j, and $\Theta(C_j) = 0$ for any j.

We define a partial order $\succ$ on all *itineraries* as follows. Let $\mathcal{J} = (\mathcal{J}_0, \mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3, \dots)$ and $\mathcal{K} = (\mathcal{K}_0, \mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3, \dots)$ be two *itineraries* such that $\mathcal{J} \neq \mathcal{K}$. . Let k be the first $i \in \mathbb{N}$ such that $\mathcal{J}_i \neq \mathcal{K}_i$ and assume without loss of generality that $\mathcal{J}_k > \mathcal{K}_k$. We set $\Lambda_k = \Pi_{i=0}^{k-1} \Theta(\mathcal{J}_i) = \Pi_{i=0}^{k-1} \Theta(\mathcal{K}_i)$. Clearly $\Lambda_k \in \{0, 1, -1\}$. It is easy to see that if the itineraries are associated to points under the action of the same map, say, f, then $\Lambda_k \neq 0$. Indeed, if $\Lambda_k = 0$, then $\mathcal{J}_i = C_j$ for some $i \in \{0, 1, \dots k - 1\}$ and $j \in \{1, 2\}$. Since we are considering itineraries of points under the same map f, then $\mathcal{J} = \mathcal{K}$, a contradiction with the assumption that $\mathcal{J} \neq \mathcal{K}$. Hence there are two cases (in general if $\Lambda_k = 0$, then the ordering is not defined).

(1) If $\Lambda_k = 1$, then we say that $\mathcal{J}$ is stronger than $\mathcal{K}$ and write $\mathcal{J} \succ \mathcal{K}$.
(2) If $\Lambda_k = -1$, then we say that $\mathcal{K}$ is stronger than $\mathcal{J}$ and write $\mathcal{K} \succ \mathcal{J}$.

Also, we will write $\mathcal{J} \succeq \mathcal{K}$ if $\mathcal{J} \succ \mathcal{K}$ or $\mathcal{J} = \mathcal{K}$.

Theorem 2.1 ([10])**.** *Let $x, y \in [0,1]$ with $x > y$. Then $\mathcal{I}(x) \succeq \mathcal{I}(y)$. Conversely, if for $x, y \in [0,1]$, we have $\mathcal{I}(x) \succ \mathcal{I}(y)$, then $x > y$.*

The *itineraries* $\mathcal{K}_j = \mathcal{I}(f(c_j)) \in \mathcal{S}^{\mathbb{N}}$ of the *critical points* c_j for $j = 1, 2$ are called the *kneading sequences* of the map f. The *vector* $\overrightarrow{\mathcal{K}(f)} = (\mathcal{K}_1, \mathcal{K}_2)$ is called the *kneading vector* of the map f.

In this paper we will use Theorem 2.1 to pinpoint the distribution of the cycles corresponding to different bimodal over-twist patterns under the action of a bimodal horseshoe map H_2 by comparing the itineraries of the points of absolute maxima and minima of these orbits.

2.2. **Some results on degree one circle maps.** Recall some results on circle maps that we need (see [1] for detail). Consider the unit circle $\mathbb{S}$ normalized so that the circumference equals 1 and identify $[0, 1)$ with it. Define the natural projection $\pi : \mathbb{R} \to \mathbb{S}$ associating with $t \in \mathbb{R}$ its fractional part $\pi(t)$ considered as a point of $\mathbb{S}$. If $f : \mathbb{S} \to \mathbb{S}$ is continuous, then there is a continuous map $F : \mathbb{R} \to \mathbb{R}$ such that $F \circ \pi = \pi \circ f$. Such a map F is called a *lifting* of f. It is unique up to translation by an integer. An integer d with $F(x+1) = F(x)+d$ for all $x \in \mathbb{R}$ is called the *degree* of the map f and is independent of the choice of F. Conversely, maps $G : \mathbb{R} \to \mathbb{R}$ such that $G(x + 1) = G(x) + d$ for every real x are said to be *maps of the real line of degree* d, can be defined independently, and are semiconjugate by the same map π to the circle maps of degree d. In this paper we consider both maps of the circle and maps of the real line *of degree one*.

Denote by $\mathcal{L}_1$ the set of all liftings of continuous degree one self-mappings of $\mathbb{S}$ endowed with the *sup norm*. Let $F \in \mathcal{L}_1$. Define *upper and lower rotation numbers* of $x \in \mathbb{R}$ for $F \in \mathcal{L}_1$ as $\overline{\rho_F}(x) = \limsup_{n\to\infty} \frac{F^n(x) - x}{n}$ and $\underline{\rho_F}(x) = \liminf_{n\to\infty} \frac{F^n(x) - x}{n}$ respectively. If $\overline{\rho_F}(x) = \underline{\rho_F}(x)$, this number is called the *rotation number of* x for F and is denoted by $\rho_F(x)$. Let $\mathcal{L}'_1$ be the space of all non-decreasing elements of $\mathcal{L}_1$.

Theorem 2.2 ([1])**.** *If $F \in \mathcal{L}'_1$ is a lifting of a circle map f then $\rho_F(x)$ exists for all $x \in \mathbb{R}$ and is independent of x. Moreover, it is rational if and only if f has a periodic point.*

In the situation of Theorem 2.2, set $\rho(F) = \rho_F(x)$ and call it the *rotation number* of F. The following lemma is left to the reader.

Lemma 2.3. *The function $\rho : \mathcal{L}'_1 \to \mathbb{R}$ is continuous.*

For $F \in \mathcal{L}_1$ we define maps $F_l, F_u \in \mathcal{L}_1'$ as follows: $F_l(x) = \inf\{F(y) : y \geqslant x\}$ and $F_u(x) = \sup\{F(y) : y \leqslant x\}$.

Theorem 2.4 ([1]). *The following statements are true.*

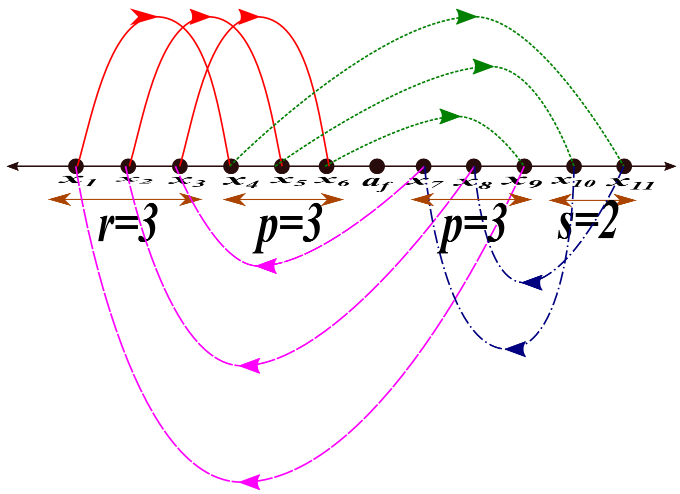
(1) Let $F \in \mathcal{L}_1$. Then, $F_l(x) \leqslant F(x) \leqslant F_u(x)$ for every $x \in \mathbb{R}$.
(2) If $F, G \in \mathcal{L}_1$ with $F \leqslant G$, then $F_l \leqslant G_l$ and $F_u \leqslant G_u$.
(3) Let $F \in \mathcal{L}_1'$. Then, $F_l = F_u = F$.
(4) The maps $F \mapsto F_l$ and $F \mapsto F_u$ are Lipschitz continuous with constant 1 in the sup norm.
(5) The maps $F \mapsto \rho(F_l)$ and $F \mapsto \rho(F_u)$ are continuous.
(6) If $F_l(x) \neq F(x)$, then $x \in Const(F_l)$ where $Const(F_l)$ denotes the union of all open intervals on which F_l is constant.
(7) Let $F \in \mathcal{L}_1$ be a lifting of a circle map f. Then the set of all rotation numbers of points is equal to the interval $[\rho(F_l), \rho(F_u)]$. Moreover, for each rational a from this interval there is a point x such that $\pi(x)$ is periodic for f and $\rho_F(x) = a$.
(8) The points $\rho(F_l), \rho(F_u)$ depend continuously on $F \in \mathcal{L}_1$.

The interval $[\rho(F_l), \rho(F_u)]$ from Theorem 2.4(7) is called the *rotation interval of* F and is denoted by $\mathrm{Rot}(F)$.

2.3. **Results on over-rotation numbers of bimodal maps.** In what follows when talking about (over-)rotation numbers $\frac{p}{q}$ and the like we assume (unless specified otherwise) that the numerator and the denominator are coprime. The article [4] relates the *over-rotation numbers* of bimodal interval maps and the classical rotation numbers of degree one maps of the circle and of the real line. This yields a description of *bimodal over-twist patterns*. Namely, given an over-rotation number $\frac{p}{q}$ there are $q - 2p + 1$ *bimodal over-twist patterns* $\Gamma_{r,\frac{p}{q}}$ with $r \in \{0, 1, 2, \dots, q - 2p\}$ of over-rotation number $\frac{p}{q}$. The patterns with $r = 0$ and $r = q - 2p$ are *unimodal* while the remaining $q - 2p - 1$ patterns are *strictly bimodal*. Denote the permutation associated to the over-twist pattern $\Gamma_{r,\frac{p}{q}}$ by $\Pi_{r,\frac{p}{q}}$. By [4] $\Pi_{r,\frac{p}{q}}$ is as follows.

$$\Pi_{r,\frac{p}{q}}(j) = \begin{cases} j + p & \text{if } 1 \leqslant j \leqslant r \\ q - j + r + 1 & \text{if } r + 1 \leqslant j \leqslant r + p \\ 2p - j + r + 1 & \text{if } r + p + 1 \leqslant j \leqslant r + 2p \\ j - p & \text{if } r + 2p + 1 \leqslant j \leqslant q \end{cases} \tag{2.1}$$

FIGURE 3. The Bimodal over-twist pattern $\Gamma_{3,\frac{3}{11}}$



Let $P_{r,\frac{p}{q}}$ be a cycle of a map f which exhibits the pattern $\Gamma_{r,\frac{p}{q}}$. Number its q points using *spatial labelling* as $x_1 < x_2 < \cdots < x_q$ and partition them into 4 disjoint parts. Strictly speaking, in the notation that we introduce now, the dependence upon the cycle at hand should be reflected. However we omit it to make notation lighter.

The first r points $x_1, x_2, \ldots, x_r$ are called *red points* and are denoted by $\mathcal{R}_1^r, \mathcal{R}_2^r, \mathcal{R}_3^r, \ldots, \mathcal{R}_r^r$. Under the action of f the *red points* are shifted to the right by p points.

The next p points $x_{r+1}, x_{r+2}, \ldots, x_{r+p}$ are called *green points* and are denoted by $\mathcal{G}_1^r, \mathcal{G}_2^r, \mathcal{G}_3^r, \ldots, \mathcal{G}_p^r$. The *green points* map onto the last p points $x_{q-p+1}, \ldots, x_{q-1}, x_q$ of the orbit with a *flip*, that is orientation is reversed but without any expansion.

The next p points $x_{r+p+1}, x_{r+p+2}, \ldots, x_{r+2p}$ are called *pink points* and are denoted by $\mathcal{P}_1^r, \mathcal{P}_2^r, \mathcal{P}_3^r, \ldots, \mathcal{P}_p^r$. The *pink points* map onto the first p points $x_1, x_2, \ldots, x_p$ of the orbit with a *flip* but with no expansion.

The last $q - 2p - r$ points are called *blue points* and are denoted by $\mathcal{B}_1^r, \mathcal{B}_2^r, \mathcal{B}_3^r, \ldots, \mathcal{B}_{q-2p-r}^r$. They are shifted to the *left* by p points.

If $r = 0$, we have no *red points*; if $r = q - 2p$, we have no *blue points*. For all other values of r, we have points of all colors.

The dynamics of $P_{3,\frac{3}{11}}$ is depicted in the Figure 3.

Remark 2.5. We will need an interpretation of the dynamics of $P_{r,\frac{p}{q}}$ (or $\Gamma_{r,\frac{p}{q}}$, or $\Pi_{r,\frac{p}{q}}$) that relates this dynamics (or combinatorics) and that of the rotation of the circle by the rational angle $\frac{p}{q}$. This relation is based upon a special discontinuous lifting of a bimodal interval map to a discontinuous degree one map of the real line which yields a relation between $P_{r,\frac{p}{q}}$ and the rotation of the circle by the rational angle $\frac{p}{q}$. In fact, we will consider the combinatorial rotation by $\frac{p}{q}$ understood as a rotation with the combinatorial rotation number $\frac{p}{q}$ of q circularly ordered points. One can visualize points $y_0 < y_1 < \cdots < y_{q-1} < y_0$ sitting on the unit circle with the order understood in terms of counterclockwise order; then the combinatorial rotation in question is defined as $\varphi_{\frac{p}{q}}(y_i) = y_{i+p \mod q}$.

To relate $P_{r,\frac{p}{q}}$ and $\varphi_{\frac{p}{q}}$, do the following. Place all r red points on the circle in their normal order. Then place all p green points on the circle right after the red points in the normal order. Then put the blue points on the circle in the reversed order. Finally, put all the pink points on the circle in the reversed order. The above given description of $P_{r,\frac{p}{q}}$ (i.e., the main result of [4]) simply means that the just described map of $P_{r,\frac{p}{q}}$ to q circularly ordered points conjugates $f|_{P_{r,\frac{p}{q}}}$ and the combinatorial rotation of these q circularly ordered points by $\frac{p}{q}$; we leave an easy verification of this fact to the reader.

3. CONNECTEDNESS OF BIMODAL ISO-OVER-ROTATION-TRACTS

Recall that $\mathcal{H} = \{H_{\alpha,\beta} : \alpha, \beta \in [0,1] \ \& \ \alpha \geqslant \beta\}$ is the family of all truncations of H_2, $I_{\alpha,\beta}$ is the *over-rotation interval* of $H_{\alpha,\beta}$ and $\psi : \mathcal{P} \to [0, \frac{1}{2}]$ is defined by $\psi(\alpha, \beta) = \rho(\alpha, \beta)$ where $\rho(\alpha, \beta)$ is the *left end point* of $I_{\alpha,\beta}$. For $\alpha, \beta \in [0, 1]$ with $\alpha \geqslant \beta$, we call the interval $\xi^{max}_{\alpha} = [\frac{\alpha}{3}, \frac{2}{3} - \frac{\alpha}{3}]$ where $H_2(x) \geqslant \alpha$ the *level α max flat spot* and the interval $\xi^{min}_{\beta} = [\frac{2}{3} - \frac{\beta}{3}, \frac{2}{3} + \frac{\beta}{3}]$ where $H_2(x) \leqslant \beta$ the *level β min flat spot*. For two sets $U, V \subset \mathbb{R}^2$, $d(U, V) = \inf\{d(u, v) : u \in U \ \& \ v \in V\}$. Also, we set $I_0 = [0, \frac{1}{3}], I_1 = [\frac{1}{3}, \frac{2}{3}], I_2 = [\frac{2}{3}, 1]$.

For any rational $\frac{p}{q}$ with *coprime* p and q, there are $q-2p+1$ bimodal over-twist patterns $\Gamma_{r,\frac{p}{q}}$ for $r \in \{0, 1, 2, \dots, q-2p\}$. Out of these the patterns $\Gamma_{0,\frac{p}{q}}$ and $\Gamma_{q-2p,\frac{p}{q}}$ are *unimodal* and the remaining $q-2p-1$ patterns are strictly *bimodal*. Suppose that a cycle P represents a pattern that forces a unique fixed point. Then we denote this fixed point a and use the following *colors* introduced in Section 2.3: *red* points are to the left of a and stay to the left of a under the map, *green* points are to the left of a but map to the right of a, *pink* points are to the right of a but map to the left of a and *blue* points are to the right of a and stay to the right of a under the action of the map. When talking about the corresponding pattern itself, we use the same colors. This coloring scheme applies to cycles exhibiting patterns $\Gamma_{r,\frac{p}{q}}$.

For a given cycle P let us denote its *left endpoint* by $le(P)$ and *right endpoint* by $ri(P)$. Let P be a cycle exhibiting one of the patterns $\Gamma_{r,\frac{p}{q}}$. We will need the next lemma describing the monotonicity of the P-linear map $f : [le(P), ri(P)] \to [le(P), ri(P)]$ on certain P-basic intervals.

Lemma 3.1. *Let P be a cycle exhibiting one of the patterns $\Gamma_{r,\frac{p}{q}}$. Consider the P-linear map $f_P = f$ with a unique fixed point a. Let $u \in P$ be such that $f(u) = ri(P)$; let $v \in P$ be such that $f(v) = le(P)$. Choose a small interval $I = (u, u + \varepsilon)$ and consider $f^j|_I$ for some j. Then $f^j|_I$ is decreasing if $f^j(u) > a$ and increasing if $f^j(u) < a$.*

Proof. The map f is decreasing on an iterated image K of I if and only if $K \subset (u, v)$. The structure of the pattern of P implies that the orbit of I can be viewed as a concatenation of a few segments each of which looks as either (1) an iterated image of I contained in (u, a)

and then several (maybe none) images of I contained in $(v, ri(P))$, or (2) an iterated image of I contained in (a, v) and then several (maybe none) images of I contained in $(le(P), u)$. This immediately implies the claim of the lemma. □

3.1. **Weakest Cycles.** Clearly, for each $r \in \{0, 1, 2, \dots, q-2p\}$ there exist cycles of H_2 that exhibit $\Gamma_{r,\frac{p}{q}}$. Each such cycle P gives rise to the corresponding truncation T_P defined as follows: choose the point $u_P = u \in P$ such that $H_2(u)$ is the rightmost point of P. Clearly $u \in (0, \frac{1}{2})$; draw a horizontal line at the point $(u, H_2(u))$ on the graph of H_2 creating the max flat spot of T_P. Likewise, take $v_P = v \in P$ such that $H_2(v)$ is the left most point of P, and use it to create the min flat spot of T_P. Evidently, P remains a cycle of T_P. We will use the notation u_P and v_P in what follows.

Definition 3.2. We say that a *cycle* P of H_2 is a *weakest* cycle exhibiting $\Gamma_{r,\frac{p}{q}}$ if P is the *unique* cycle of T_P exhibiting $\Gamma_{r,\frac{p}{q}}$.

Lemma 3.3. *Let P be a weakest cycle of H_2 which exhibits $\Gamma_{r,\frac{p}{q}}$ for some r. Then all pink and green points of P lie in the decreasing lap I_1 and P is unique.*

Proof. We first prove that all *green* and *pink* points of P lie in the decreasing lap I_1. By way of contradiction suppose that $x = u_P \notin I_1$. Set $f = T_P$. Consider the maximal interval $T_1 = [x, z], z \in P$, on which f is non-strictly decreasing. Let $T_0 = [0, x]$ and $T_2 = [z, 1]$, and call T_0, T_1 and T_2 T*-intervals*. Let X be the set of all points $s \in [0, 1]$ such that for any i the points $f^i(s)$ and $f^i(x)$ belong to the same T-interval, additionally requiring that if $f^i(x) \in T_1$ then $f^i(s) \in T_1$, too (the additional requirement resolves the ambiguous situation one faces if $f^i(x)$ is the common endpoint of T_1 and T_0, or a common endpoint of T_1 and T_2). It is well-known (and not hard to observe) that X is a closed interval of the form $[x, y]$. Consider the point y.

The definition implies that $f^q(X) \subset X$ (because $f^q(x) = x$) and that $f^q|_X$ is (non-strictly) monotone. If $y_0 \in I_1$ is such that $f(x) = f(y_0)$, then $y_0 \le y$. Moreover, by Lemma 3.1 a small interval $[y_0, y_0 + \varepsilon]$ has the f^q-image contained in $[x, y_0]$; thus, $[y_0, y_0 + \varepsilon] \subset X$ and $y_0 < y$. We claim that $f^q(y) = y$. Indeed, the fact that $y_0 < y$ and the definitions imply that $x < f^q(y)$. If now $f^q(y) < y$ then by continuity a small interval $[y, y + \delta]$ is contained in X, a contradiction with the fact that $X = [x, y]$.

The structure of the pattern $\Gamma_{r,\frac{p}{q}}$ implies that X cannot contain points of P other than x itself (in particular, y *cannot* be a point of P). Thus, all points of X at every moment of time are mapped into the same P-basic interval as x. Hence the orbit of y exhibits the same pattern as that of x, i.e. the orbit of y exhibits $\Gamma_{r,\frac{p}{q}}$, a contradiction with the assumption that P is weakest. Thus, $u_P \in I_1$; similarly one shows that $v_P \in I_1$.

Now, we show that P is unique. Indeed otherwise there exists another weakest cycle Q which exhibits the pattern $\Gamma_{r,\frac{p}{q}}$. Then $u_P \neq u_Q$.

On the other hand, by the previous paragraph and because the orbits of u_P and u_Q exhibit the same pattern it follows that for each k the points $H_2^k(u_P)$ and $H_2^k(u_Q)$ belong to the same lap of H_2, be it I_0, I_1 or I_2. In other words, u_P and u_Q have the same itinerary under H_2 which implies that $u_P = u_Q$, a contradiction. □

Observe that an alternative proof of Lemma 3.3 is based upon the techniques of so-called *admissible* intervals introduced in [8]; we choose our arguments to make the paper shorter and more self-contained.

From now on $P_{r,\frac{p}{q}}$ shall denote the *weakest cycle* of H_2 which exhibits $\Gamma_{r,\frac{p}{q}}$. Let $\mathcal{S}_{\frac{p}{q}}$ be the collection of these $q-2p+1$ cycles $P_{r,\frac{p}{q}}$ for $r \in \{0,1,2,\dots,q-2p\}$.

3.2. **Order among weakest cycles.** Interestingly, there is an order among these $q-2p+1$ orbits on H_2 which we study now. For $P_{r,\frac{p}{q}} \in \mathcal{S}_{\frac{p}{q}}$, denote its left endpoint, right endpoint, point of absolute maxima and point of absolute minima by $le_r^{\frac{p}{q}}$, $ri_r^{\frac{p}{q}}$, $M_r^{\frac{p}{q}}$ and $m_r^{\frac{p}{q}}$ respectively. Clearly, $H_2(m_r^{\frac{p}{q}}) = le_r^{\frac{p}{q}}$ and $H_2(M_r^{\frac{p}{q}}) = ri_r^{\frac{p}{q}}$.

Lemma 3.4. *For any rational $\frac{p}{q}$ where p and q are coprime we have*

$$le_{q-2p}^{\frac{p}{q}} < le_{q-2p-1}^{\frac{p}{q}} < le_{q-2p-2}^{\frac{p}{q}} < le_{q-2p-3}^{\frac{p}{q}} < \dots < le_0^{\frac{p}{q}}$$

Proof. Our proof is based upon the interpretation of the results of [4] given in Remark 2.5. According to it, let us place q points $Y = \{y_0 < \dots < y_{q-1} < y_0\}$ on the unit circle and consider the map $\varphi_{\frac{p}{q}} : Y \to Y$. Moreover, let us associate points of $P_{r,\frac{p}{q}}$ and points of Y as explained in Remark 2.5. Then in the case of $P_{r,\frac{p}{q}}$ we have the partition, say, L' of Y into sets $\mathcal{R}', \mathcal{G}', \mathcal{P}'$ and $\mathcal{B}'$ of $r, p, p, q-2p-r$ points respectively of *colors* : *red*, *green* , *pink* and *blue*. In the case of $P_{r+1,\frac{p}{q}}$ we have the partition L'' of Y into sets $\mathcal{R}'', \mathcal{G}'', \mathcal{P}''$ and $\mathcal{B}''$ of $r+1, p, p, q-2p-r-1$ points respectively of the same *colors*. We can view the transformation from L' to L'' as follows: (i) the set $\mathcal{G}'$ of p *green points* in L' moves one click away from y_0 (in other words, $\mathcal{G}'$ occupies points from y_r through y_{r+p-1} while $\mathcal{G}''$ occupies points from y_{r+1} through y_{r+p}) (ii) the set $\mathcal{R}'$ adds the point : y_r while the set $\mathcal{B}'$ loses the point : y_{r+p}.

Let us iterate $\varphi_{\frac{p}{q}}$ applying it to y_0. Then for some time the foreword iterates of the point y_0 will be located in the equally named sets of the partitions ($\mathcal{R}'$ and $\mathcal{R}''$, $\mathcal{G}'$ and $\mathcal{G}''$, etc). However, at some moment *before* y_0 is mapped back to itself, for the *first time* the sets of the two partitions containing, say, y^k will be named differently. A priori, this can happen in one of the following two ways:

(1) $y_0^k = y_r$ belongs to $\mathcal{G}'$ and to $\mathcal{R}''$;
(2) $y_0^k = y_{r+p}$ belongs to $\mathcal{B}'$ and to $\mathcal{G}''$.

However it immediately follows that if (2) takes places then already on the previous step the sets in the two partitions that contains the point will have different names, a contradiction. Thus, the first time the point is in two sets of partitions L', L'' that have different names is when $y^k = y_r$ belongs to $\mathcal{G}'$ and to $\mathcal{R}''$. It follows that the first difference between itineraries of le_r and le_{r+1} is such that the entry in the itinerary of le_r is I_1 (because all *green points* belong to I_1) while the simultaneous entry in the itinerary of le_{r+1} is I_0 (because all *red points* belong to I_0). Let us now figure out the sign associated with this according to the definition of the order among itineraries. To do so notice that on its way to y_r the point y_0 enters *green* and *pink points* in pairs because the length of the segment with *green points* and the segment with *pink points* is the same and equals the step of the map, i.e. p. Hence at the moment when we are at y_r the product of signs associated with laps of H_2 and defining the order among itineraries is positive implying that $le_{r+1} < le_r$ as desired. □

We can interpret the result of Lemma 3.4 as follows: the more *red points* our *weakest cycle* has, the more to the *left* its *left endpoint* is. Because of the *symmetry* of the map H_2 (formally this is based upon the fact that H_2 is conjugate to itself by a map that *symmetrically flips* $[0, 1]$) we can state a similar claim for the *rightmost points* of *weakest cycles*: the more *blue points* our *weakest cycl*e has, the more to the *right* its *right endpoint* is. This can be summarized in the next theorem stated with proof.

Theorem 3.5. *For any rational number $\frac{p}{q}$ where p and q are coprime we have*
$$le_{q-2p} < le_{q-2p-1} < \cdots < le_0 < ri_{q-2p} < ri_{q-2p-1} < \cdots < ri_0.$$

Example 3.6. Let us illustrate Theorem 3.4 using the particular case: $\frac{p}{q} = \frac{3}{11}$. In this case $\mathcal{S}_{\frac{3}{11}}$ has $q - 2p + 1 = 6$ elements : $P_{r,\frac{3}{11}}$, $r = 0, 1, 2, \ldots 5$. The *itineraries* of $H_2(m_r)$ are :

(1) $\mathcal{I}(H_2(m_0)) = \{I_1, I_2, I_2, I_1, I_1, I_2, I_2, I_1, I_1, I_2, I_1, \ldots\}$ (Fig 4).
(2) $\mathcal{I}(H_2(m_1)) = \{I_0, I_1, I_2, I_1, I_1, I_2, I_2, I_1, I_1, I_2, I_1, \ldots\}$ (Fig 5).
(3) $\mathcal{I}(H_2(m_2)) = \{I_0, I_1, I_2, I_1, I_0, I_1, I_2, I_1, I_1, I_2, I_1, \ldots\}$ (Fig 6).
(4) $\mathcal{I}(H_2(m_3)) = \{I_0, I_1, I_2, I_1, I_0, I_1, I_2, I_1, I_0, I_1, I_1, \ldots\}$ (Fig 7).
(5) $\mathcal{I}(H_2(m_4)) = \{I_0, I_0, I_1, I_1, I_0, I_1, I_2, I_1, I_0, I_1, I_1, \ldots\}$ (Fig 8).
(6) $\mathcal{I}(H_2(m_5)) = \{I_0, I_0, I_1, I_1, I_0, I_0, I_1, I_1, I_0, I_1, I_1, \ldots\}$ (Fig 9).

Comparing the itineraries it is easy to see that
$$\mathcal{I}(H_2(m_0)) \succ \mathcal{I}(H_2(m_1)) \succ \cdots \succ \mathcal{I}(H_2(m_4)) \succ \mathcal{I}(H_2(m_5));$$
thus, by Theorem 2.1, we have
$$H_2(m_0) > H_2(m_1) > H_2(m_2) > H_2(m_3) > H_2(m_4) > H_2(m_5)$$
and hence
$$le_5^{\frac{3}{11}} < le_4^{\frac{3}{11}} < le_3^{\frac{3}{11}} < le_2^{\frac{3}{11}} < le_1^{\frac{3}{11}} < le_0^{\frac{3}{11}}$$
.

FIGURE 4. $P_{0,\frac{3}{11}}$

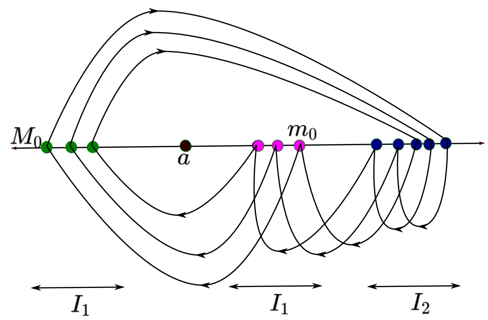


FIGURE 5. $P_{1,\frac{3}{11}}$

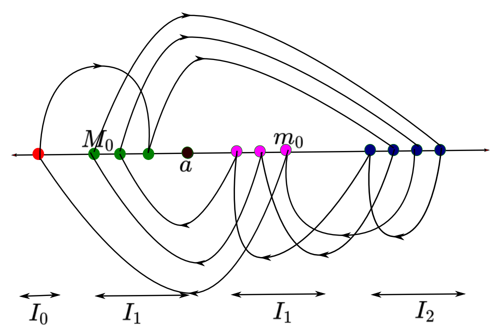


FIGURE 6. $P_{2,\frac{3}{11}}$

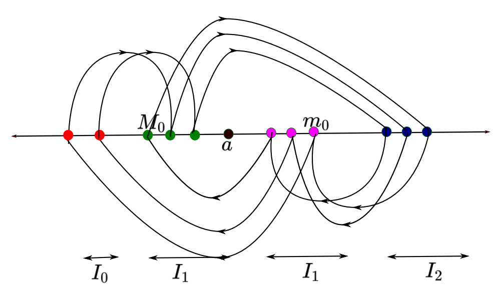


FIGURE 7. $P_{3,\frac{3}{11}}$

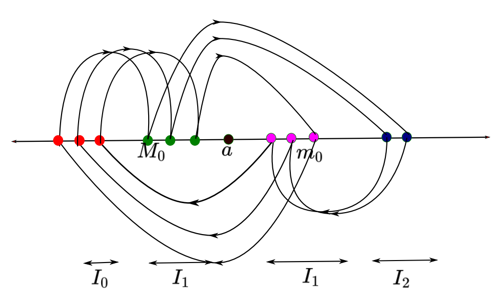


FIGURE 8. $P_{4,\frac{3}{11}}$

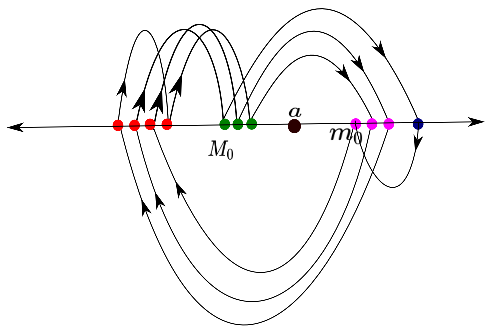
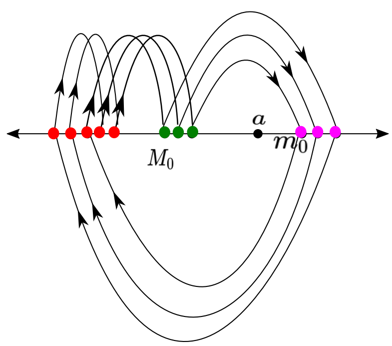


FIGURE 9. $P_{5,\frac{3}{11}}$



3.3. **Leading Sets.** Let us fix an over-rotation number $\frac{p}{q}$. Set $le_i = \beta_i$, $ri_i = \alpha_i$, $0 \leqslant i \leqslant q-2p$. By Theorem 3.5 $\beta_{i+1} < \beta_i$ and $\alpha_{i+1} < \alpha_i$ for $i = 0, 1, \dots, q-2p-1$. Plot points $C_i = (\alpha_i, \beta_i)$ that characterize the weakest cycles of over-rotation number $\frac{p}{q}$ in the parameter square. It follows that as r increases from 0 to $(q-2p)$, the points C_i line up in the "south-west" direction, i.e. in the direction in which α and β coordinates decrease. The *leading set* of over-rotation number $\frac{p}{q}$ is a staircase that connects C_i's as follows. Draw a vertical segment down from C_i until we reach the level β_{i+1}; then draw a horizontal segment until we reach C_{i+1}. Also, at C_0 draw a horizontal segment until we meet the right (the vertical) side arm, and at C_{q-2p} draw a vertical segment until we meet the bottom (the horizontal) side arm. Acting in this fashion, we will connect C_0 and C_{q-2p} with a "staircase" $Z_{\frac{p}{q}}$ called the *leading set of over-rotation number* $\frac{p}{q}$. Observe that leading sets connect two side arms of the parameter square.

Lemma 3.7. *For any* $(\alpha, \beta) \in Z_{\frac{p}{q}}$ *we have* $\psi((\alpha, \beta)) = \frac{p}{q}$.

Proof. The claim is immediate for points $C_i = (\alpha_i, \beta_i)$. Indeed, if $\psi((\alpha_i, \beta_i)) < \frac{p}{q}$ then the map H_{α_i,β_i} has a cycle of rational over-rotation number $\rho < \frac{p}{q}$ that in turn forces a pattern of over-rotation number $\frac{p}{q}$. In other words, H_{α_i,β_i} has a cycle of over-rotation number $\frac{p}{q}$ which is distinct from the cycle P_i containing α_i and β_i, a contradiction with the fact that the cycle P_i is a weakest cycle.

Let us now consider the horizontal segment in $Z_{\frac{p}{q}}$ with the left endpoint C_i. Then its horizontal coordinate varies from α_i to α_{i-1} while its vertical coordinate equals β_i. Suppose that the map H_{α,β_i} with $\alpha_i \leqslant \alpha \leqslant \alpha_{i-1}$ is such that $\psi((\alpha,\beta_i)) < \frac{p}{q}$. Then it must have a cycle Q of over-rotation number $\rho < \frac{p}{q}$. The cycle Q has its left endpoint $le(Q)$ and the point $m(Q) \in Q$ such that $H_{\alpha,\beta_i}(m(Q)) = le(Q)$. Evidently, $m(Q)$ does not belong to the level β_i min flat spot as its orbit does not coincide with P_i. Hence the truncation of H_2 defined by Q does not have P_i as its cycle. On the other hand, the max value of Q is less than α_{i-1} by construction. Hence the next (in the "northeast" direction) weakest cycle of over-rotation number $\frac{p}{q}$ (i.e., the cycle that passes through points α_{i-1} and β_{i-1}) is not forced by Q either. It follows that *no weakest cycle of over-rotation number $\frac{p}{q}$ is forced by Q*, a contradiction with properties of over-rotation numbers (see Theorem 1.2). The arguments dealing with vertical segments in $Z_{\frac{p}{q}}$ are completely analogous. This implies the claim of the lemma. □

We see that there entire parameter square is cut by countably many pairwise disjoint leading sets $Z_{\frac{p}{q}}$ associated to the rational numbers ν from $(0,\frac{1}{2})$. To describe their mutual location we need the next lemma.

Lemma 3.8. *If (α,β) and (γ,δ) are points in $\mathcal{P}$ co-linear with the focal point $\mathcal{F}$ such that $d(\mathcal{F},(\alpha,\beta)) \geqslant d(\mathcal{F},(\gamma,\delta))$ then $\psi((\alpha,\beta)) \geqslant \psi((\gamma,\delta))$.*

Proof. Since, (α,β) and (γ,δ) are points in $\mathcal{P}$ co-linear with the focal point $\mathcal{F}$ with $d(\mathcal{F},(\alpha,\beta)) \geqslant d(\mathcal{F},(\gamma,\delta))$, it follows that $\gamma \geqslant \alpha$ and $\delta \leqslant \beta$. Thus, as we move from (α,β) to (γ,δ), the first coordinate increases and the second coordinate decreases which implies that $I_{(\alpha,\beta)} \subseteq I_{(\gamma,\delta)}$ and $\psi((\alpha,\beta)) \geqslant \psi((\gamma,\delta))$ as desired. □

This lemma immediately implies the next corollary.

Corollary 3.9. *Consider rational numbers $0 < \nu < \mu \leqslant \frac{1}{2}$. Then Z_ν separates Z_μ from the focal point inside the parameter square $\mathcal{P}$.*

Corollary 3.9 allows us to define the notion of a *leading strip* $S(\mu,\nu)$, $0 < \nu < \mu \leqslant \frac{1}{2}$. Namely the *leading strip* $S(\mu,\nu), 0 < \nu < \mu \leqslant \frac{1}{2}$ is the closed set squeezed in the parameter square between Z_μ and Z_ν.

Theorem 3.10. *For any number $\gamma, 0 < \gamma \leqslant \frac{1}{2}$ the* γ-bimodal iso-over-rotation-tract *$\mathcal{T}_\gamma$ is the intersection of all leading strips $S(\mu,\nu)$ with $0 < \nu < \gamma < \mu \leqslant \frac{1}{2}$. The set $\mathcal{T}_\gamma$ is a simply connected continuum. The map $\psi : \mathcal{P} \to [0,\frac{1}{2}]$ is continuous and monotone.*

Proof. By Corollary 3.9, the containment $[\mu,\nu] \subset [\mu',\nu']$ implies the containment $S(\mu,\nu) \subset S(\mu',\nu')$. Since all leading strips are simply connected continua, then so is the intersection of their nested family. Hence the intersection X of all leading strips $S(\mu,\nu)$ with $0 < \nu < \gamma < \mu \leqslant \frac{1}{2}$ is a simply connected continuum. Let us show that in fact X coincides with the *γ-bimodal iso-over-rotation-tract $\mathcal{T}_\gamma$*. Indeed, choose a point $(\alpha,\beta) \notin X$. Then by construction there is a leading set Z_ν separating (α,β) from X such that $\nu \neq \gamma$. Assume for the sake of definiteness that $\nu < \gamma$. Then by Lemma 3.8 $\psi((\alpha, \beta)) \leqslant \nu < \gamma$. On the other hand, suppose that a point $(\alpha, \beta) \in X$. Suppose that, contrary to the desired, $\psi((\alpha, \beta)) < \gamma$ (the case of $\psi((\alpha, \beta)) > \gamma$ can be considered similarly). Choose the number ν so that $\psi((\alpha, \beta)) < \nu < \gamma$. Then, again by Lemma 3.8, the leading set Z_ν separates (α, β) from X, a contradiction. □

Acknowledgments

The authors would like to thank the referees for their valuable comments and suggestions towards the improvement of the paper in the present form.

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