QUADRATIC-LIKE DYNAMICS OF CUBIC POLYNOMIALS

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Abstract. A small perturbation of a quadratic polynomial \( f \) with a non-repelling fixed point gives a polynomial \( g \) with an attracting fixed point and a Jordan curve Julia set, on which \( g \) acts like angle doubling. However, there are cubic polynomials with a non-repelling fixed point, for which no perturbation results into a polynomial with Jordan curve Julia set. Motivated by the study of the closure of the Cubic Principal Hyperbolic Domain, we describe such polynomials in terms of their quadratic-like restrictions.

1. Introduction

In this paper, we study topological dynamics of complex cubic polynomials. We denote the Julia set of a polynomial \( f \) by \( J(f) \) and the filled Julia set of \( f \) by \( K(f) \). Let us recall classical facts about quadratic polynomials. The Mandelbrot set \( \mathcal{M}_2 \), perhaps the most well-known mathematical set outside of the mathematical community, can be defined as the set of all complex numbers \( c \) such that the sequence

\[
c, \quad c^2 + c, \quad (c^2 + c)^2 + c, \ldots
\]

is bounded. The numbers \( c \) label polynomials \( z^2 + c \). Every quadratic polynomial can be reduced to this form by an affine coordinate change.

By definition, \( c \in \mathcal{M}_2 \) if the orbit of 0 under \( z \rightarrow z^2 + c \) is bounded. What is so special about the point 0? It is the only critical point of the polynomial \( z^2 + c \) in \( \mathbb{C} \). A critical point of a complex polynomial has a meaning in the realm of topological dynamics. Namely, this is a point that does not have a neighborhood, on which the map is one-to-one. Generally, the behavior of critical orbits to a large extent determines the dynamics of other orbits. For example, by a classical theorem of...
Fatou and Julia, $c \in \mathcal{M}_2$ if and only if the filled Julia set of $z^2 + c$

$$K(z^2 + c) = \{ z \in \mathbb{C} | z, z^2 + c, (z^2 + c)^2 + c, \ldots \not\rightarrow \infty \}$$

is connected. If $c \notin \mathcal{M}_2$, then the set $K(z^2 + c)$ is a Cantor set.

The Mandelbrot set has a complicated fractal shape. Yet one can see many components of the interior of $\mathcal{M}_2$ bounded by real analytic curves (in fact, ovals of real algebraic curves). The central part of the Mandelbrot set, the Principal Hyperbolic Domain $\text{PHD}_2$, is bounded by a cardioid (a curve, whose shape resembles that of a heart). This cardioid is called the Main Cardioid. By definition, the Principal Hyperbolic Domain $\text{PHD}_2$ consists of all parameter values $c$ such that the polynomial $z^2 + c$ is hyperbolic, and the set $K(z^2 + c)$ is a Jordan disk (a polynomial of any degree is said to be hyperbolic if the orbits of all its critical points converge to attracting cycles). Equivalently, $c \in \text{PHD}_2$ if and only if $z^2 + c$ has an attracting fixed point.

The closure of $\text{PHD}_2$ consists of all parameter values $c$ such that $z^2 + c$ has a non-repelling fixed point. As follows from the Douady–Hubbard–Sullivan–Yoccoz landing theorem [DH84, Hub93], the Mandelbrot set itself can be thought of as the union of the main cardioid and limbs (connected components of $\mathcal{M}_2 \setminus \text{PHD}_2$) parameterized by reduced rational fractions $p/q \in (0, 1)$. This motivates our study of cubic analogs of $\text{PHD}_2$ started in [BOPT14] and continued in [BOPT14a]. We begin our discussion by describing some results of these two papers.

Complex numbers $c$ are in one-to-one correspondence with affine conjugacy classes of quadratic polynomials (throughout we call affine conjugacy classes of polynomials classes of polynomials). Thus, a higher-degree analog of the set $\mathcal{M}_2$ is the degree $d$ connectedness locus $\mathcal{M}_d$, i.e., the set of classes of degree $d$ polynomials $f$, all of whose critical points do not escape (equivalently, whose Julia set $J(f) = \text{Bd}(K(f))$ is connected). The Principal Hyperbolic Domain $\text{PHD}_d$ of $\mathcal{M}_d$ is the set of classes of hyperbolic degree $d$ polynomials with Jordan curve Julia sets; the class $[f]$ of a degree $d$ polynomial $f$ belongs to $\text{PHD}_d$ if and only if all critical points of $f$ are in the immediate attracting basin of the same attracting (or super-attracting) fixed point. In [BOPT14] we describe properties of polynomials $f$ such that $[f] \in \text{PHD}_d$.

**Theorem 1.1** (Theorem A [BOPT14]). If $[f] \in \text{PHD}_d$, then $f$ has a fixed non-repelling point, no repelling periodic cutpoints in $J(f)$, and all its non-repelling periodic points, except at most one fixed point, have multiplier 1.

Actually, in [BOPT14] we claim that all non-repelling periodic cutpoints in the Julia set $J(f)$, except perhaps 0, have multiplier 1; still,
literally repeating the same arguments one proves the version of Theorem A [BOPT14] given by Theorem 1.1. This motivates Definition 1.2 in which we define a special set \( \text{CU} \) such that \( \text{PHD}_3 \subseteq \text{CU} \).

**Definition 1.2** ([BOPT14]). Let \( \text{CU} \) be the family of classes of cubic polynomials \( f \) with connected \( J(f) \) such that \( f \) has a non-repelling fixed point, no repelling periodic cutpoints in \( J(f) \), and all its non-repelling periodic points, except at most one fixed point, have multiplier 1. The family \( \text{CU} \) is called the *Main Cubioid*, and a polynomial \( f \) with \([f] \in \text{CU}\) is said to be *cubioidal*.

Let \( \mathcal{F} \) be the space of polynomials

\[
  f_{\lambda,b}(z) = \lambda z + bz^2 + z^3, \quad \lambda \in \mathbb{C}, \quad b \in \mathbb{C}
\]

parameterized by pairs \((\lambda,b)\) of complex numbers. An affine change of variables reduces any cubic polynomial \( f \) to the form \( f_{\lambda,b} \). The point 0 is fixed for every polynomial in \( \mathcal{F} \). The set of all polynomials \( f \in \mathcal{F} \) such that 0 is non-repelling for \( f \) is denoted by \( \mathcal{F}_{nr} \). Define the \( \lambda \)-*slice* \( \mathcal{F}_\lambda \) of \( \mathcal{F} \) as the space of all polynomials \( g \in \mathcal{F} \) with \( g'(0) = \lambda \). The space \( \mathcal{F} \) maps onto the space of classes of all cubic polynomials with a fixed point of multiplier \( \lambda \) as a finite branched covering. This branched covering is equivalent to the map \( b \mapsto a = b^2 \), i.e., classes of polynomials \( f_{\lambda,b} \in \mathcal{F}_\lambda \) are in one-to-one correspondence with the values of \( a \). Thus, if we talk about, say, points \([f]\) of \( \mathcal{M}_3 \), then it suffices to take \( f \in \mathcal{F}_\lambda \) for some \( \lambda \).

Assume that \( J(f) \) is connected. In [Lyu83, MSS83], the notion of \( J \)-*stability* was introduced for any holomorphic family of rational functions: a map is \( J \)-stable if its Julia set admits an equivariant holomorphic motion over some neighborhood of the map in the given family. We say that \( f \in \mathcal{F}_\lambda \) is stable if it is \( J \)-stable with respect to \( \mathcal{F}_\lambda \) with \( \lambda = f'(0) \), otherwise we say that \( f \) is unstable. The set \( \mathcal{F}^s_\lambda \) of all stable polynomials \( f \in \mathcal{F}_\lambda \) is an open subset of \( \mathcal{F}_\lambda \). A component of \( \mathcal{F}^s_\lambda \) is called a \((\lambda-)\)-*stable component* or a *domain of \((\lambda-)\)-stability*. It is easy to see that, given \( \lambda \), the polynomial \( f_{\lambda,b} \) has a disconnected Julia set if \( |b| \) is sufficiently big. Hence, if \( f = f_{\lambda,b} \) is stable and \( J(f) \) is connected, then its domain of stability is bounded. For any subset \( \mathcal{U} \subseteq \mathcal{F} \), we let \([\mathcal{U}]\) stand for the set of classes \([f]\) of all polynomials \( f \in \mathcal{U} \). If \(|\lambda| \leq 1\), then we write \( \mathcal{P}_\lambda \) for the set of all polynomials \( f \in \mathcal{F}_\lambda \) such that \([f] \in \text{PHD}_3 \).

**Theorem 1.3** ([BOPT14]). Suppose that \( \mathcal{U} \) is a bounded stable component in \( \mathcal{F}_\lambda \), \(|\lambda| \leq 1\), such that \([\text{Bd}(\mathcal{U})]\) \( \subseteq \) \( \text{PHD}_3 \). Then \([\mathcal{U}] \subseteq \text{CU} \). Thus, if \( \mathcal{T}_\lambda \) is the union of \( \mathcal{P}_\lambda \) and all \( \lambda \)-stable components, whose boundaries are contained in \( \mathcal{P}_\lambda \), then \([\mathcal{T}_\lambda]\) \( \subseteq \) \( \text{CU} \).
Given a compact set \( X \subset \mathbb{C} \), define the topological hull \( \text{TH}(X) \) of \( X \) as the union of \( X \) with all bounded components of \( \mathbb{C} \setminus X \). In Lemma 1.4, which follows from Corollary 5.2, we show that \( \mathcal{T}_\lambda \) can be described easier than in Theorem 1.3 which implies that many properties of polynomials, whose classes belong to \( \overline{\text{PHD}_3} \), are inherited by polynomials from \( \text{TH}(P_\lambda) \).

**Lemma 1.4.** Any component of \( \text{TH}(P_\lambda) \setminus P_\lambda \) consists of stable maps. Moreover, \( \mathcal{T}_\lambda = \text{TH}(P_\lambda) \), and therefore, \( \text{TH}(P_\lambda) \subset \text{CU} \).

In [BOPT14a], we study properties of components of \( \text{TH}(P_\lambda) \setminus P_\lambda \), where \( |\lambda| \leq 1 \). Note that these are the same as bounded components of \( \mathcal{F}_\lambda \setminus P_\lambda \). Let \( \mathcal{A} \) be the set of all maps \( f_{\lambda,b} \) with \( |\lambda| < 1 \). For each such map \( f \), let \( A(f) \) be the basin of immediate attraction of 0. In Section 2, we show that if \( f \in \mathcal{F}_\lambda \setminus P_\lambda \), then \( f \) has two distinct critical points. A critical point \( c \) of \( f \) is said to be principal if there is a neighborhood \( \mathcal{U} \) of \( f \) in \( \mathcal{F} \) and a holomorphic function \( \omega_1 : \mathcal{U} \to \mathbb{C} \) with \( c = \omega_1(f) \), and such that, for every \( g \in \mathcal{U} \cap \mathcal{A} \), the point \( \omega_1(g) \) is a critical point of \( g \) contained in \( A(g) \). By Section 2, the point \( \omega_1(f) \) is well-defined; denote the other critical point of \( f \) by \( \omega_2(f) \).

**Definition 1.5 ([BOPT14a]).** Let \( \mathcal{U} \) be a component of \( \text{TH}(P_\lambda) \setminus P_\lambda \). If, for every \( f \in \mathcal{U} \), the point 0 belongs to a Siegel domain \( U \) of \( f \) in \( \mathcal{F} \) and a holomorphic function \( \omega_1 : \mathcal{U} \to \mathbb{C} \) with \( c = \omega_1(f) \), and such that, for every \( g \in \mathcal{U} \cap \mathcal{A} \), the point \( \omega_1(g) \) is a critical point of \( g \) contained in \( A(g) \), then \( \mathcal{U} \) is said to be of Siegel capture type.

Theorem 1.6 relies upon the tools developed in [BOPT14].

**Theorem 1.6 ([BOPT14a]).** Suppose that \( |\lambda| \leq 1 \). Then any component of \( \text{TH}(P_\lambda) \setminus P_\lambda \) is either of Siegel capture type or of queer type.

Let us now describe the results of the present paper. If we perturb a cubic polynomial \( f \) with a non-repelling fixed point to a polynomial \( g \) with an attracting fixed point, then \( g \) restricted to the basin of attraction \( A(g) \) of that point is either two-to-one or three-to-one. Here, we study polynomials \( f \) with a non-repelling fixed point such that \( [f] \not\in \overline{\text{PHD}_3} \), i.e., for all cubic polynomials \( g \) sufficiently close to \( f \), if \( g \) has a fixed attracting point, then \( g \) is two-to-one on its basin. While interesting by itself, this together with [BOPT14] also allows us to learn more about the structure of \( \overline{\text{PHD}_3} \).

We need the definition of hybrid equivalence: two polynomial-like maps are said to be hybrid equivalent if their restrictions to sufficiently
small neighborhoods of their filled Julia sets are quasi-conformally con-
jugate, and a quasi-conformal conjugacy \( \varphi \) can be chosen to satisfy \( \partial \varphi = 0 \) on the filled Julia set. The map \( \varphi \) is called a straightening map. It is easy to see that under hybrid equivalence repelling periodic points cannot correspond to non-repelling periodic points.

Say that a cubic polynomial \( f \in \mathcal{F} \) is immediately renormalizable if there are Jordan domains \( U^* \ni 0 \) and \( V^* \) such that \( f^* = f : U^* \to V^* \) is a quadratic-like map (we will use the notation \( f^* \) at several occasions in the future when we talk about immediately renormalizable maps). If \( f \in \mathcal{F}_{nr} \) is immediately renormalizable, then the quadratic-like Julia set \( J(f^*) = J^* \) is connected. Indeed, \( f^* \) is hybrid equivalent to a quadratic polynomial \( g^* \). Since \( 0 \in J(f^*) \) is a non-repelling \( f \)-fixed point, it corresponds to a non-repelling fixed point of \( g^* \). Hence, \( J(g^*) \) and \( J(f^*) \) are connected, and \( g^* = z^2 + c \) with \( c \in \text{PHD}_2 \).

Our main result concerns some sufficient conditions for being immediately renormalizable. Note that, if \( [f] \in \text{PHD}_3 \), then \( f \) is not immediately renormalizable. Indeed, if \( f : U^* \to V^* \) is quadratic-like, then \( g : U^* \to V^* \) is quadratic-like for all \( g \) sufficiently close to \( f \). If \( f, g \in \mathcal{F}_{nr} \) and \( 0 \in U^* \), then the quadratic-like Julia sets of both \( f \) and \( g \) are connected. Thus, \( [g] \) cannot belong to \( \text{PHD}_3 \). If \( f \in \mathcal{F}_{nr} \) but \( [f] \notin \text{PHD}_3 \), then \( f \) is said to be potentially renormalizable. Clearly, the set of all potentially renormalizable polynomials is open in \( \mathcal{F} \).

**Theorem A.** Let \( |\lambda| \leq 1 \). Then the following facts hold:

1. If a polynomial \( f \) belongs to the unbounded component of \( \mathcal{F}_\lambda \setminus \mathcal{P}_\lambda \) then \( f \) is immediately renormalizable;
2. If a polynomial \( f \) belongs to a bounded component of \( \mathcal{F}_\lambda \setminus \mathcal{P}_\lambda \) of Siegel capture type, then \( f \) is not immediately renormalizable.

Moreover, if \( f \) is immediately renormalizable, the corresponding quadratic-like map is hybrid equivalent to a quadratic polynomial \( z^2 + c \) with \( c \in \text{PHD}_2 \).

If \( f \) belongs to a bounded component of \( \mathcal{F}_\lambda \setminus \mathcal{P}_\lambda \) of queer type, we cannot conclusively say if \( f \) is immediately renormalizable or not; in Section 5 we prove a few lemmas describing properties of \( f \) in that case.

There is no loss of generality in that we consider only perturbations of \( f \) in \( \mathcal{F} \): instead, we could consider small perturbations \( g \) of \( f \) such that, arbitrarily close to 0, the map \( g \) has an attracting fixed point. Theorem A generalizes some results from [BuHe01, Zak99].

**Notation and Preliminaries:** we write \( \overline{A} \) for the closure of a subset \( A \) of a topological space and \( \text{Bd}(A) \) for the boundary of \( A \); the \( n \)-th iterate of a map \( f \) is denoted by \( f^n \). Let \( \mathbb{C} \) stand for the complex plane, \( \mathbb{C}^* \) for the Riemann sphere,
\[ \mathbb{D} \] for the open unit disk consisting of all complex numbers \( z \) with \( |z| < 1 \), and \( S = \text{Bd}(\mathbb{D}) \) for the unit circle which is identified with \( \mathbb{R}/\mathbb{Z} \). The d-tupling map of the unit circle is denoted by \( \sigma_d \). We assume knowledge of basic notions from complex dynamics, such as Green function, dynamic rays (of specific argument), Böttcher coordinate, Fatou domain, repelling, attracting, neutral periodic points, parabolic, Siegel, Cremer periodic points etc (see, e.g., [McM94]).

2. Potentially renormalizable polynomials

Throughout Section 2, we consider a potentially renormalizable cubic polynomial \( f \). We want to see when \( f \) is immediately renormalizable. Below we outline our strategy; to motivate our approach, assume for the moment that \( f \) is already immediately renormalizable. The main idea is to observe that some points and sets related to \( f \) (including \( J = J(f) \)) can actually be defined independently of the fact that \( f \) is immediately renormalizable. This allows us to define them for all potentially renormalizable maps \( f \); in other words, we define a potential quadratic-like Julia set and then prove that in some cases the potential of being a quadratic-like Julia set is realized.

Recall that \( \mathcal{A} \) is the set of all cubic polynomials \( g \in \mathcal{F} \) with \( |g'(0)| < 1 \), and, for \( g \in \mathcal{A} \), we write \( A(g) \) for the immediate basin of attraction of 0 with respect to \( g \). If \( g \in \mathcal{A} \), we let \( \omega_1(g) \) be its critical point in \( A(g) \) and let \( \omega_2(g) \) be the other critical point of \( g \). As a tool, we consistently approximate \( f \) by polynomials from \( \mathcal{A} \). If \( g \in \mathcal{A} \), then \( f \) itself serves as its own approximation. By our assumption, there is a neighborhood of \( f \) in \( \mathcal{F} \), in which there is no polynomial \( g \in \mathcal{A} \) with \( |g| \in \text{PHD}_3 \).

Our first step will be to define the critical points \( \omega_1(f) \) and \( \omega_2(f) \) for all immediately renormalizable maps, and to show that these points depend holomorphically on \( f \in \mathcal{F} \setminus \mathcal{P}_\lambda \). We next consider a countable set \( Z(f) \) of iterated \( f \)-preimages of \( \omega_1(f) \), each of which depends holomorphically on \( f \in \mathcal{F} \setminus \mathcal{P}_\lambda \). For \( g \in \mathcal{A} \), the set \( Z(g) \) is simply the set of all iterated \( g \)-preimages of \( \omega_1(g) \) contained in \( A(g) \). Finally, the potential quadratic-like Julia set of \( f \) can be defined as the set of all non-isolated points in \( \overline{Z(f)} \). We will show that this set moves holomorphically with \( f \).

2.1. The principal critical point of \( f \). Fix \( f \in \mathcal{F} \setminus \mathcal{P}_\lambda \) as above.

Lemma 2.1. The polynomial \( f \) has two distinct critical points.

Proof. Assume that \( \omega(f) \) is the only critical point of \( f \) (then it has multiplicity two). Let \( \mathcal{C} \) be the space of all polynomials \( g \in \mathcal{F} \) with a multiple critical point \( \omega(g) \). This is an algebraic curve in \( \mathcal{F} \) passing
By Lemma 2.1, there are two critical points of \( f \). A critical point \( c \) of \( f \) is said to be principal if there is a neighborhood \( U \) of \( f \) in \( F \) and a holomorphic function \( \omega_1 : U \rightarrow \mathbb{C} \) defined on this neighborhood such that \( c = \omega_1(f) \), and, for every \( g \in U \cap A \), the point \( \omega_1(g) \) is a critical point of \( g \) contained in \( A(g) \).

**Theorem 2.2.** There exists a unique principal critical point of \( f \).

*Proof.* By Lemma 2.1, the two critical points of \( f \) are different. Then there are two holomorphic functions, \( \omega_1 \) and \( \omega_2 \), defined on a convex neighborhood \( U \) of \( f \) in \( F \), such that \( \omega_1(g) \) and \( \omega_2(g) \) are the critical points of \( g \) for all \( g \in U \). Suppose that neither \( \omega_1(f) \), nor \( \omega_2(f) \) is principal. Then, arbitrarily close to \( f \), there are cubic polynomials \( g_1 \) and \( g_2 \in A \) with \( \omega_2(g_1) \notin A(g_1) \) and \( \omega_1(g_2) \notin A(g_2) \). Since \( A(g_i) \) contains a critical point for \( i = 1, 2 \), we must have that \( \omega_1(g_i) \in A(g_i) \).

The set \( A \) is convex. Therefore, the intersection \( U \cap A \) is also convex, hence connected. Let \( O_i \), \( i = 1, 2 \), be the subset of \( U \cap A \) consisting of all polynomials \( g \) with \( \omega_i(g) \in A(g) \). By the preceding paragraph, \( g_1 \in O_1 \) and \( g_2 \in O_2 \). We claim that \( O_i \) is open. Indeed, if \( g \in O_i \) then there exists a Jordan disk \( U \subset A(g) \) with \( g(U) \) compactly contained in \( U \), and \( \omega_i(g) \in U \). If \( \tilde{g} \in U \cap A \) is sufficiently close to \( g \), then \( \tilde{g}(U) \) is still compactly contained in \( U \), and \( \omega_i(\tilde{g}) \) is still in \( U \), by continuity. It follows that \( U \subset A(\tilde{g}) \), in particular, \( \omega_i(\tilde{g}) \in A(\tilde{g}) \). Thus, \( O_i \) is open. Since \( O_1,O_2 \) are open and non-empty, the set \( U \cap A \) is connected, and

\[ U \cap A = O_1 \cup O_2, \]

the intersection \( O_1 \cap O_2 \) is nonempty. Note that \( O_1 \cap O_2 \) consists of polynomials, whose classes are in \( \text{PHD}_3 \). Since \( U \) can be chosen arbitrarily small, it follows that \( f \) can be approximated by maps \( g \in A \) with \( [g] \in \text{PHD}_3 \), a contradiction.

The existence of a principal critical point of \( f \) is thus proved. The uniqueness follows immediately from our assumption on \( f \). \( \square \)

Denote by \( \omega_1(f) \) the principal critical point of \( f \). For \( g \in F_{nr} \) sufficiently close to \( f \), the point \( \omega_1(g) \) is a holomorphic function of \( g \).

**2.2 Holomorphic motion.** Let \( \Lambda \) be a Riemann surface, and \( Z \subset \mathbb{C}^* \) any (!) subset. A holomorphic motion of the set \( Z \) is a map \( \mu : Z \times \Lambda \rightarrow \mathbb{C}^* \) with the following properties:
for every $z \in Z$, the map $\mu(z, \cdot) : \{z\} \times \Lambda \to \mathbb{C}^*$ is holomorphic;

- for $z \neq z'$ and every $\nu \in \Lambda$, we have $\mu(z, \nu) \neq \mu(z', \nu)$;

- there is a point $\nu_0$ such that $\mu(z, \nu_0) = z$ for all $z \in Z$.

We will use the following crucial $\lambda$-lemma of Mañé, Sad and Sullivan [MSS83]: a holomorphic motion of a set $Z$ extends to a unique holomorphic motion of the closure $\overline{Z}$; moreover, this extension is a continuous function in two variables such that, for every $\nu \in \Lambda$, the map $\varphi : \overline{Z} \to \mathbb{C}^*$ defined as $\varphi(z) = \mu(z, \nu)$ is quasi-symmetric. There have been useful generalizations of this result, but we will only need the original version.

We will now define a countable set $Z(f)$ of iterated preimages of the principal critical point $\omega_1(f)$. By definition, a point $z \in \mathbb{C}$ belongs to $Z(f)$ if there exists an open convex neighborhood $U_z$ of $f$ in $F$ and a holomorphic function $g : U_z \to \mathbb{C}$ with the following properties:

- $\zeta(f) = z$;
- we have $g^m(\zeta(g)) = \omega_1(g)$ for all $g \in U_z$ and for some $n \geq 0$ independent of $g$;
- we have $\zeta(g) \in A(g)$ for all $g \in U_z \cap A$.

A holomorphic function $\zeta : U \to \mathbb{C}$ like above is called a deformation of $z \in Z(f)$. As it is always clear what kind of deformation we consider, in what follows we will suppress the subscript in the notation for $U$.

A connected component of the set of potentially renormalizable polynomials in $F$ is called a potentially renormalizable component. For any $f \in F \setminus P$, let $W_f$ be a potentially renormalizable component containing $f$; clearly, $W_f$ is open.

Lemma 2.3. Let $f$ be as above.

1. The critical point $\omega_1(f)$ is not eventually mapped to $\omega_2(f)$.
2. The set $Z(f)$ contains no critical values of $f$.

Proof. Suppose first that $\omega_1(f)$ is eventually mapped to $\omega_2(f)$, say, $f^m(\omega_1(f)) = \omega_2(f)$, and the number $m$ is the minimal positive integer with this property. Consider the set $C$ of all $g \in U$ such that $g^m(\omega_1(g)) = \omega_2(g)$. This set is an open part (not necessarily connected) of an algebraic curve. The function $g \mapsto g'(0)$ is a complex analytic function on $C$. Since the value of this function at $f$ lies in $\mathbb{D}$, there are maps $g \in C$ arbitrarily close to $f$ such that $|g'(0)| < 1$. The class of any such $g$ must belong to PHD. Indeed, the attracting basin $A(g)$ must contain the principal critical point $\omega_1(g)$ by definition of the principal critical point. Since $\omega_1(g)$ is eventually mapped to $\omega_2(g)$, the critical point $\omega_2(g)$ is also contained in $A(g)$. We arrive at a contradiction with our assumption on $f$. 
Suppose now that \( v \in Z(f) \) is a critical value. Let \( \zeta : U \to \mathbb{C} \) be a deformation of \( v \). Consider the set \( C \) of all \( g \in U \) such that \( \zeta(g) \) is a critical value. This set is a part of an algebraic curve. Take a sequence \( g_n \in C \cap A \) that converges to \( f \). Since \( \zeta(g_n) \in A(g_n) \) is a critical value with at least two \( g_n \)-preimages in \( A(g_n) \), counting multiplicities, the set \( A(g_n) \) must contain a critical point \( d_n \) with \( g_n(d_n) = \zeta(g_n) \). The fact that \( \omega_1(g_n) \) is not periodic implies that \( d_n \neq \omega_1(g_n) \). Thus, both critical points of \( g_n \) are contained in \( A(g_n) \), and so \([g_n] \in \text{PHD}_3 \). We again arrive at a contradiction with our assumption on \( f \).

**Lemma 2.4.** For every \( z \in Z(f) \), there are exactly two points of \( Z(f) \) that are mapped to \( z \) under \( f \).

**Proof.** The proof is similar to that of Theorem 2.2. Let \( \zeta : U \to \mathbb{C} \) be a deformation of \( z \). Since the set \( Z(f) \) cannot contain a critical value of \( f \), there are three holomorphic functions \( \zeta_1, \zeta_2, \zeta_3 \) defined on \( U \) and such that \( g(\zeta_i(g)) = \zeta(g) \) (we may need to pass to a smaller neighborhood \( U \) to arrange this).

The intersection \( U \cap A \) is convex, hence connected. For any 2-element subset \( \{i, j\} \subset \{1, 2, 3\} \), define a subset \( O_{ij} \subset U \cap A \) as the set of all polynomials \( g \in U \cap A \) such that \( \zeta_i(g) \in A(g) \) and \( \zeta_j(g) \in A(g) \). All three sets \( O_{12}, O_{23} \) and \( O_{13} \) are open (cf. the proof of Theorem 2.2). On the other hand, we have

\[
A \cap U = O_{12} \cup O_{23} \cup O_{13}.
\]

Hence either only one of the sets \( O_{ij} \) is nonempty, or at least two of the sets \( O_{ij} \) intersect. In the latter case, \( \zeta_i(g) \in A(g) \) for some \( g \in A \cap U \) and all \( i = 1, 2, 3 \). It follows that \([g] \in \text{PHD}_3 \). Since the neighborhood \( U \) can be chosen to be arbitrarily small, it follows that \( f \) can be approximated by polynomials in \( A \), whose classes are in \( \text{PHD}_3 \), a contradiction. The contradiction shows that only one of the sets \( O_{ij} \) is nonempty, for a suitable choice of the neighborhood \( U \). Assume that \( i = 1 \) and \( j = 2 \); then \( \zeta_1(f), \zeta_2(f) \in Z(f) \) but \( \zeta_3(f) \notin Z(f) \).

The proof of Lemma 2.4 implies a stronger claim below.

**Corollary 2.5.** Let \( \zeta_i \) be holomorphic functions introduced in the proof of Lemma 2.4. Suppose that \( \zeta_1(f), \zeta_2(f) \in Z(f) \). Then there is a neighborhood \( U \) of \( f \) in \( \mathcal{F} \) such that \( \zeta_3(g) \notin A(g) \) for all \( g \in U \cap A \).

**Proposition 2.6.** For every \( z \in Z(f) \), there is a holomorphic function \( \zeta : W_f \to \mathbb{C} \) such that \( \zeta(h) \in Z(h) \) for all \( h \in W_f \) and \( \zeta(f) = z \).

**Proof.** The function \( \zeta \) with these properties is defined at least on some open neighborhood of \( f \) in \( W_f \), by definition of the set \( Z(f) \). Assume
by induction that the statement of the proposition holds for the point \( f(z) \), i.e., there is a holomorphic function \( \eta : \mathcal{W}_f \to \mathbb{C} \) such that \( \eta(h) \in Z(h) \) for all \( h \in \mathcal{W}_f \) and \( \eta(f) = f(z) \). It follows that there is an integer \( n \) such that \( h^{\circ(n-1)}(\eta(h)) = \omega_1(h) \) for all \( h \in \mathcal{W}_f \). Consider the multivalued analytic function \( h \mapsto h^{-1}(\eta(h)) \). If this function has no branch points in \( \mathcal{W}_f \), then we can define the holomorphic function \( \zeta \) as the branch of this function such that \( \zeta(f) = z \). Suppose that there is a branch point \( h_0 \) of the multivalued function \( h \mapsto h^{-1}(\eta(h)) \). Then the point \( \eta(h_0) \) is a critical value of \( h_0 \), a contradiction with Lemma 2.3.

Thus we have defined the holomorphic function \( \zeta : \mathcal{W}_f \to \mathbb{C} \) such that \( h(\zeta(h)) = \eta(h) \), and \( \zeta(f) = z \). Moreover, \( \zeta(h) \in Z(h) \) for all \( h \in \mathcal{W}_f \) sufficiently close to \( f \). It suffices to prove that \( \zeta(h) \in Z(h) \) for all \( h \in \mathcal{W}_f \). To this end, we will prove that the set of polynomials \( h \in \mathcal{W}_f \) such that \( \zeta(h) \in Z(h) \) is open and closed in \( \mathcal{W}_f \). The openness is obvious. Consider a sequence \( h_n \in \mathcal{W}_f \) converging to some polynomial \( h \in \mathcal{W}_f \), and suppose that \( \zeta(h_n) \in Z(h_n) \) but \( \zeta(h) \notin Z(h) \). Therefore, there are two other holomorphic functions \( \zeta_1, \zeta_2 \) defined on some neighborhood of \( h \) such that \( \zeta_i(h) \in Z(h), i = 1, 2 \). It follows that \( \zeta_i(h_n) \in Z(h_n) \) for sufficiently large \( n \). But then all three points \( \zeta_1(h_n), \zeta_2(h_n) \) and \( \zeta(h_n) \) are preimages of \( \eta(h_n) \) in \( Z(h_n) \). This contradicts Lemma 2.4. \( \square \)

Proposition 2.6 and Lemma 2.3 imply the following theorem.

**Theorem 2.7.** There exists a holomorphic motion \( \mu : Z(f) \times \mathcal{W}_f \to \mathbb{C} \) that is equivariant in the sense that for every \( h \in \mathcal{W}_f \), and for every \( z \in Z(f) \setminus \{\omega_1(f)\} \), we have \( h(\mu(z, h)) = \mu(f(z), h) \).

By the \( \lambda \)-lemma, the holomorphic motion \( \mu \) gives rise to the holomorphic motion \( \overline{\mu} : \overline{Z(f)} \times \mathcal{W}_f \to \mathbb{C} \). Since \( \mu \) is equivariant, the holomorphic motion \( \overline{\mu} \) is equivariant too.

2.3. **The set** \( X(f) \). Let \( Z_n(f) \) be the subset of \( Z(f) \) consisting of all preimages of \( \omega_1(f) \) mapped to \( \omega_1(f) \) in \( n \) steps, in other words, \( z \in Z_n(f) \) if \( f^{\circ n}(z) = \omega_1(f) \). Define the set \( X(f) \) as the limit of the sets \( Z_n(f) \), i.e.,

\[
X(f) = \bigcap_{m \geq 0} \bigcup_{n \geq m} Z_n(f)
\]

Theorem 2.7 and the \( \lambda \)-lemma imply that the sets \( X(h) \) move holomorphically for \( h \in \mathcal{W}_f \). Clearly, \( X(h) \) is forward invariant under \( h \).

Let \( P \) be a polynomial of degree \( d \). Then \( P \) on a small neighborhood of any point \( t \) is \( k \)-to-1 (at regular points \( t \), we have \( k = 1 \), and at critical points \( k > 1 \)); \( k \) is called the multiplicity of \( t \).
Lemma 2.8. Every point \( x \in X(f) \) has at least two preimages in \( X(f) \), counting multiplicities.

Proof. This follows immediately from Lemma 2.4. \( \square \)

Lemma 2.9. The set \( X(f) \) is a subset of the Julia set \( J(f) \).

Proof. The set \( X(f) \) is contained in the accumulation set of the backward orbit of \( \omega_1(f) \). The backward orbit of a point can accumulate in the Fatou set only if the point lies in a Siegel disk. However \( \omega_1(f) \) cannot lie in a Siegel disk as a Siegel disk contains no critical points. \( \square \)

Recall that by the \( \lambda \)-lemma:
\[ \overline{Z} \times \mathcal{W}_f \to \mathbb{C} \] is continuous. In particular, if a sequence \( z_n \in \overline{Z} \) converges to \( z \in \overline{Z} \), then \( \mu(z_n, h) \) converges to \( \mu(z, h) \), for every \( h \in \Lambda \).

Lemma 2.10. The set \( X(f) \) contains no neutral periodic points different from 0.

Proof. Let \( X(f) \) contain a periodic neutral point \( x \neq 0 \) of minimal period \( k \). Since the holomorphic motion \( \overline{\mu} \) is equivariant, \( \overline{\mu}(x, h) = \overline{\mu}(f^{\omega_r}(x), h) = h^{\omega_r}(\overline{\mu}(x, h)) \) for every \( r \). This proves that \( \overline{\mu}(x, h) = x(h) \) is a periodic point of \( h \) of period \( k \), for every \( h \in \mathcal{W}_f \).

The holomorphic function \( h \mapsto (h^{\omega_k})'(x(h)) \) is non-constant on the multiplier slice \( \mathcal{F}_\lambda \). Indeed, the slice \( \mathcal{F}_\lambda \) contains polynomials with disconnected Julia sets, and such polynomials cannot have non-repelling periodic points different from 0 by the Fatou–Shishikura inequality. It follows that \( x(h) \) is an attracting periodic point with respect to \( h \), for some polynomials \( h \) in arbitrarily small neighborhood of \( f \). Now, the point \( x \in X(f) \) is the limit of some sequence \( z_n \in Z(f) \). Hence we must also have \( x(h) = \overline{\mu}(x, h) = \lim_{n \to \infty} \overline{\mu}(z_n, h) \). However, by definition of \( Z(f) \) this is impossible if \( x(h) \) is attracting. \( \square \)

Theorem 2.11 explicitly summarizes the results of this section.

Theorem 2.11. Suppose that \( f \in \mathcal{F}_\lambda \) is potentially renormalizable. Then there is an equivariant holomorphic motion \( \overline{\mu} : X(f) \times \mathcal{W}_f \to \mathbb{C} \). The set \( X(f) \) is a forward invariant subset of \( J(f) \). It contains no neutral periodic points different from 0. Every point of \( X(f) \) has at least two preimages in \( X(f) \) counting multiplicities.

In the rest of the paper, we adopt the following approach. First we establish several types of conditions on \( X(f) \) and the holomorphic motion \( \overline{\mu} \) sufficient for \( f \) being immediately renormalizable; the set \( X(f) \) plays here the role of a potential quadratic-like Julia set. Then we verify that these conditions are fulfilled for various cubic polynomials. In the end, this leads to the proofs of our results.
3. Properties of polynomial-like maps

In this section, we prove a criterion for a polynomial $P$ of any degree to have a polynomial-like restriction. Recall that, for any map $F$, by an $F$-invariant set, we mean a set $A$ such that $F(A) \subset A$ but not necessarily $F(A) = A$.

Consider a compact set $T \subset \mathbb{C}$ and a polynomial $P : \mathbb{C} \to \mathbb{C}$. Let $\nu_T(z)$ be the number of all $P$-preimages of $z$ in $T$ counted with multiplicities. It follows that there exists a neighborhood $V$ of $z$ and $r$ pullbacks $W_1, \ldots, W_r$ of $V$ each containing exactly one point of the set $P^{-1}(z) \cap T$ and such that the sum of degrees of $P$ restricted on $W_1, \ldots, W_r$ is $\nu_T(z)$. If a point $x \in P^{-1}(z) \cap T$ is not critical and belongs to $W_i$, the map $P|_{W_i}$ is a homeomorphism onto image. If a point $x \in P^{-1}(z) \cap T \cap W_i$ is critical then it is the unique critical point of $P$ in $W_i$.

Set $\widetilde{W} = \bigcup_{i=1}^r W_i$. By compactness, the $P$-image of $T \setminus \widetilde{W}$ is positively distant from $z$. Hence, for some smaller neighborhood $V' \subset V$ of $z$, all preimages of any point $z' \in V'$ in $T \cap P^{-1}(V')$ belong to $\widetilde{W}$ and the entire preimage of $V'$ in $T$ breaks down into $r$ pieces contained in $W_1, \ldots, W_r$. From now on let us call such $V'$ a $(T-)$suitable neighborhood of $z'$. Since any point $y \in V$ has exactly $\nu_T(z)$ preimages in $\widetilde{W}$ (not necessarily in $T$), the value of $\nu_T$ can only drop at points $z' \in V'$, and $\nu_T$ is upper-semicontinuous.

Let $\nu_T|_{P(T)}$ be continuous at $z$. Choose a suitable neighborhood $V$ of $z$ on which $\nu_T|_{P(T)}$ is a constant. Then, for every point $y \in V \cap P(T)$, the set $P^{-1}(y) \cap T$ of its preimages in $T$ consists exactly of all its preimages in $\widetilde{W}$. Indeed, by the previous paragraph if $y \in V$ then it has $\nu_T(z)$ preimages in $\widetilde{W}$. Together with the fact that $\nu_T(y) = \nu_T(z)$ this implies our claim. Hence if $x \in P(T)$ is a point of continuity of $\nu_T|_{P(T)}$, then $x$ has a neighborhood $V$ such that in the corresponding open set $\widetilde{W}$ points from $T$ and not from $T$ cannot have the same image. If $\nu_T|_{P(T)}$ is continuous at all points, it follows that there exists a neighborhood $U$ of $T$ such that for any $z \in U \setminus T$ we have $P(z) \notin P(T)$. It is equally easy to show that, conversely, if $\nu_T|_{P(T)}$ is discontinuous at $z$, then there is a preimage of $z$ in $T$, in whose arbitrarily small neighborhood “collisions” between a point from $T$ and a point not from $T$ take place. All this is summarized in Lemma 3.1.

**Lemma 3.1.** Suppose that $T$ is a compact set and $P$ is a polynomial. Then the following two properties are equivalent.

1. The function $\nu_T$ is continuous on $P(T)$. 

There exists a neighborhood $U$ of $T$ such that for any $z \in U \setminus T$ we have $P(z) \notin P(T)$.

If $T$ is connected, these conditions are equivalent to the following.

(3) $T$ is a component of $P^{-1}(P(T))$.

Proof. By the arguments right before Lemma 3.1, (1) and (2) are equivalent. Assume now that $T$ is connected. Then, clearly, (2) implies (3). Suppose that (3) holds. Then there is a neighborhood $U$ of $T$ that does not intersect other components of $P^{-1}(P(T))$. It follows for every $z \in U \setminus T$ we have $P(z) \notin P(T)$. □

There is a useful sufficient condition for (3).

Lemma 3.2. Suppose that $T$ is a continuum and $P$ is a polynomial. Set $m = 1 + \sum (d_c - 1)$, where the sum is taken over all critical points $c$ of $P$ in $T$, and $d_c$ is the multiplicity of the point $c$. If $P(c) \notin \text{TH}(P(T))$ for any critical point $c \notin \text{TH}(T)$, and, for each point $x \in P(T)$, we have $\nu_T(x) \geq m$, then all conditions from Lemma 3.1 hold.

Proof. Take all critical points of $P$ not belonging to $\text{TH}(T)$, connect their $P$-images with infinity with pairwise disjoint rays avoiding $\text{TH}(P(T))$, and pull them back to the appropriate critical points to construct a finite collection of cuts of the plane. Let $W$ be a component of the complement to this collections of cuts containing $T$. It follows that $P : W \to P(W)$ is a branched covering map. By the Riemann-Hurwitz formula, the topological degree of this branched covering is $m$. Thus points of $P(T)$ can have at most $m$ preimages in $T$. By the assumptions, this implies that they have exactly $m$ preimages in $T$ counting multiplicities. Since this condition coincides with condition (1) from Lemma 3.1, we are done. □

Theorem 3.3 is the main result of this section. Recall that a parabolic domain at a periodic parabolic point $y$ is a periodic Fatou component, whose points converge to $y$ under the iterates of the given polynomial.

Theorem 3.3. Let $P : \mathbb{C} \to \mathbb{C}$ be a polynomial, and $Y \subset \mathbb{C}$ be a non-separating $P$-invariant continuum. The following assertions are equivalent:

(1) the set $Y$ is the filled Julia set of some polynomial-like map $P : U^* \to V^*$ of degree $k$,

(2) $\nu_Y|_Y = k$, and, for every attracting or parabolic point $y$ of $P$ in $Y$, the attracting basin of $y$ or the union of all parabolic domains at $y$ is a subset of $Y$. 

The proof uses some ideas communicated by M. Lyubich to the fourth named author.

Proof. It suffices to prove \((2) \implies (1)\). Let \(\phi : D \to \mathbb{C}^* \setminus Y\) be a Riemann map. By Lemma 3.1, a point \(x \notin Y\) close to \(Y\) cannot map into \(Y\). Hence we can choose \(\varepsilon > 0\) so that the map \(F = \phi^{-1} \circ P \circ \phi\) is defined and holomorphic on the annulus \(A_{\varepsilon} = \{z : 1 - \varepsilon < |z| < 1\}\). Moreover, the map \(\phi\) induces a homeomorphism \(\hat{\phi}\) between the set of prime ends of \(\mathbb{C}^* \setminus Y\) and the unit circle. Note that \(P\) induces a continuous map \(\hat{P}\) on the prime ends of \(\mathbb{C}^* \setminus Y\). The continuous extension of \(F\) is obtained by conjugating the map \(\hat{P}\) by the homeomorphism \(\hat{\phi}\).

By the Schwarz reflection principle, we can extend the map \(F\) to a holomorphic map of the annulus \(A_{\varepsilon} = \{z : 1 - \varepsilon < |z| < 1\}\) to \(\mathbb{C}\) preserving \(S\) (hence taking this annulus to another annulus around \(S\)). By a theorem of Mañé [Mañé85], if \(F\) has no attracting or parabolic periodic points on \(S\), and no critical points on \(S\), then \(F\) is expanding, i.e., \(|(F^m)'(z)| \geq C\mu^m\) for some \(C > 0\) and \(\mu > 1\).

Since \(F\) takes \(A_{\varepsilon}\) to a subset of the disk \(|z| < 1\), it has no critical points on \(S\). Suppose that \(F\) has an attracting or a parabolic periodic point \(z\) of period \(r\) on \(S\). In both cases, there is a convex Jordan domain \(E\) such that \(F^r(E) \subset E\), the closure of \(E\) contains \(z\), and all points of \(\hat{E}\) converge to \(z\) under the iterations of \(F^r\). Since the unit circle is invariant under \(F\), by the local theory of parabolic points, we can arrange that \(\hat{E} \cap D \neq \emptyset\). Note that \(\hat{E}\) and the unit disk intersect over a convex Jordan domain \(E\). By definition, \(F^r(E) \subset E\), and all points in \(E\) converge to \(z\) under the iterations of \(F^r\).

Set \(B = \phi(E)\). Then \(P^r(B) \subset B\). By the Denjoy–Wolff theorem, all points of \(B\) converge under the iterations of \(P^r\) to a \(P^r\)-fixed point \(x \in \text{Bd}(B)\). Clearly, \(x \in Y\), and \(x\) is either attracting or parabolic (as it attracts an open set of points). However, by the assumptions, the attracting basin of \(x\) or the union of all parabolic domains at \(x\) is a subset of \(Y\), a contradiction.

Thus, \(F\) expands on \(S\), and \(\varepsilon\) can be chosen so that the \(F\)-pullback of \(A_{\varepsilon}\) is compactly contained in \(A_{\varepsilon}\). Let \(V^*\) be the Jordan domain bounded by the \(\phi\)-image of the curve \(|z| = 1 - \varepsilon\). Set \(U^*\) to be the component of \(P^{-1}(V^*)\) containing \(Y\). Then \(\hat{U}^* \subset \hat{V}^*\), and \(P : U^* \to \hat{V}^*\) is a polynomial-like map. The fact that \(Y\) is the filled Julia set of this polynomial-like map, follows easily.

Lemma 3.2 now implies Corollary 3.4 given here without proof.
Corollary 3.4. Let $P : \mathbb{C} \to \mathbb{C}$ be a polynomial, and $Y \subset \mathbb{C}$ be a non-separating $P$-invariant continuum. Suppose that

1. for any critical point $c \notin Y$, we have $P(c) \notin Y$,
2. we have $\nu_Y|_Y \geq m = 1 + \sum_{c \in Y} (d_c - 1)$, and
3. for every attracting or parabolic point $y$ of $P$ in $Y$, the attracting basin of $y$ or the union of all parabolic domains at $y$ is a subset of $Y$.

Then the set $Y$ is the filled Julia set of some polynomial-like map $P : U^* \to V^*$ of degree $k$.

Theorem 3.3 shows that, for a continuum $Y$, being the filled Julia set of a polynomial-like restriction of the map is an intrinsic topological property: it is preserved under topological conjugacies. Thus, if there is a domain of stable polynomials in $\mathcal{F}_\lambda$ and one of them admits a polynomial-like restriction with connected polynomial-like Julia set, then they all have similar restrictions.

4. Dynamics on $X(f)$ and quadratic-like maps

Recall that a stable component in $\mathcal{F}_\lambda$ is a component of the set of all stable maps in $\mathcal{F}_\lambda$. If $f \in \mathcal{F}_\lambda^s$ is stable, let $\mathcal{V}_f$ be a stable component containing $f$. Recall that a connected component of the set of potentially renormalizable polynomials in $\mathcal{F}_\lambda$ is said to be potentially renormalizable; for $f \in \mathcal{F}_\lambda \setminus \mathcal{P}_\lambda$, we write $\mathcal{W}_f$ for the potentially renormalizable component containing $f$. The main result of this section is the following theorem.

Theorem 4.1. Let $f \in \mathcal{F}_\lambda$ be a potentially renormalizable polynomial. Then the following assertions are equivalent.

1. The map $f$ is immediately renormalizable.
2. The set $X(f)$ is connected, and every point of $X(f)$ has two $f$-preimages in $X(f)$ counting multiplicities.
3. All maps $g \in \mathcal{W}_f$ are immediately renormalizable.

If $f$ is stable, then all maps $g \in \mathcal{V}_f$ do not belong to $\mathcal{P}_\lambda$ and are immediately renormalizable.

We will need Corollary 4.2 proven in [BOPT14a]. It identifies $\omega_1(f)$ in the attracting and parabolic cases.

Corollary 4.2. Let $f$ be potentially renormalizable. If 0 is a parabolic (resp., attracting) fixed point of $f$, and $c$ is a critical point of $f$ belonging to a parabolic (resp., the attracting) domain $\Omega$ of $f$ at 0, then $c = \omega_1(f)$. Thus, such $\Omega$ is unique. Moreover, $\Omega \subset \text{TH}(X(f))$. 

For convenience, we prove a part of Theorem 4.1 as a separate theorem (Theorem 4.3 below).

**Theorem 4.3.** Let \( f \in \mathcal{F}_X \) be a potentially renormalizable polynomial such that \( X(f) \) is connected. If \( \nu_X(f) = 2 \) on \( X(f) \), then \( f \) is immediately renormalizable with \( J(f^*) = X(f) \) and \( K(f^*) = \text{TH}(X(f)) \).

**Proof.** Set \( X = X(f), Y = \text{TH}(X), \vartheta(X) = f^{-1}(X) \setminus X \). We verify that the set \( Y \) satisfies the assumptions of part (2) of Theorem 3.3. Since \( X \) is a continuum, \( Y \) is also a continuum. Every component of \( Y \setminus X \) maps to a bounded subset of \( \mathbb{C} \), whose boundary is a subset of \( X \); therefore, \( Y \) is \( f \)-invariant. We have \( \nu_X = 2 \) on \( X \) by the assumption. By Lemma 3.1, there is a neighborhood \( U \) of \( X \) such that \( U \setminus f^{-1}(X) = X \). Then \( V = U \cup Y \) is a neighborhood of \( Y \) such that \( V \setminus f^{-1}(Y) = Y \). Indeed, suppose that \( x \in V \setminus Y \) is mapped to \( Y \). Since \( \text{Bd}(Y) \subset X \), we may assume that \( x \) is in some interior component \( W \) of \( f^{-1}(Y) \). We have \( \text{Bd}(W) \subset X \) or \( \text{Bd}(W) \subset \vartheta(X) \). In both cases, \( W \) must contain \( \infty \), a contradiction. Applying Lemma 3.1 again, we conclude that \( \nu_Y = 2 \) on \( Y \).

It now suffices to prove that, if \( y \) is an attracting or a parabolic point of \( Y \), then the attracting basin of \( y \) or the union of all parabolic basins at \( y \) is included into \( Y \). Note that \( y \) necessarily coincides with 0 by Theorem 2.11, and the desired statement follows by Corollary 4.2.

We are ready to prove Theorem 4.1.

**Proof of Theorem 4.1.** It is obvious that \((3) \implies (1)\). The implication \((1) \implies (2)\) follows from the fact that, if \( f \) is immediately renormalizable, then \( X(f) = J(f^*) \) is connected, see the Introduction and the beginning of Section 2. Thus, \((1)\) and \((2)\) are equivalent. Now, let \((2)\) hold for \( f \). Then by Theorem 2.11, claim \((2)\) holds for all \( g \in \mathcal{W}_f \) (as follows from the properties of equivariant holomorphic motion). Hence by the above all maps \( g \in \mathcal{W} \) satisfy \((1)\) as desired. Finally, let \( f \) be stable. By definition, any map \( g \in \mathcal{V}_f \) has an invariant set \( E \ni 0 \) corresponding to \( X(f) \) and hence satisfying all the conditions of Theorem 3.3. By Theorem 3.3, the set \( E \) is a quadratic-like Julia set of \( g^* : U^* \to V^* \) with appropriately chosen domains \( U^* \subset V^* \). The arguments from the Introduction now imply that \( [g] \notin \text{PHD}_3 \) and that \( E = X(20) \).

### 5. Proof of Theorem A

By Theorem 4.1, either all maps in a potentially renormalizable component are immediately renormalizable, or none of them are.
Corollary 5.1. The unbounded potentially renormalizable component consists of immediately renormalizable maps.

Proof. Let \( W_\infty \) be the unbounded potentially renormalizable component in \( \mathcal{F}_\lambda \). Then there is a polynomial \( f \in W_\infty \), whose Julia set is disconnected. Such a polynomial is necessarily immediately renormalizable by [BrHu, Theorem 5.3]. Since in Theorem 4.1 (1) \( \iff \) (3), all maps in \( W_\infty \) are immediately renormalizable. \( \square \)

Let us now study bounded potentially renormalizable components. We now need the notion of an active critical point introduced by McMullen in [McM00]. Set \( i = 1 \) or 2, and take \( f \in \mathcal{F}_\lambda \). The critical point \( \omega_i(f) \) is active if, for every neighborhood \( U \) of \( f \) in \( \mathcal{F}_\lambda \), the sequence of the mappings \( g \mapsto g^{\omega_i(g)} \) fails to be normal in \( U \). If the critical point \( \omega_i(f) \) is not active, then it is said to be passive.

Corollary 5.2. Let \( |\lambda| \leq 1 \). Every bounded potentially renormalizable component \( W \) in \( \mathcal{F}_\lambda \) consists of stable maps. If \( W \) contains an immediately renormalizable map, then it coincides with a stable component. The union \( \mathcal{T}_\lambda \) of \( \mathcal{P}_\lambda \) with all domains of stability, whose boundaries are contained in \( \mathcal{P}_\lambda \), equals \( \text{TH}(\mathcal{P}_\lambda) \).

Proof. By [MSS83], to prove that \( f \in W \) is stable, it suffices to show that both critical points of \( f \) are passive. Note that, if \( g \in \text{Bd}(W) \), then the \( g \)-orbits of \( \omega_1(g) \) and of \( \omega_2(g) \) are bounded uniformly with respect to \( g \). By the maximum principle, the \( f \)-orbits of \( \omega_1(f) \) and \( \omega_2(f) \) are uniformly bounded for all \( f \in W \), which implies normality. Thus both critical points are passive, and the first claim of the corollary is proved. The second claim now follows because if there exists a stable component \( V \supseteq W \) then a map \( g \in \mathcal{P}_\lambda \cap V \) exists. However, by Theorem 4.1, the map \( g \) must be immediately renormalizable, a contradiction. By Corollary 5.1, this implies that, if \( U \subseteq \mathcal{F}_\lambda \) is a domain of stability containing points of \( \mathcal{P}_\lambda \), then \( U \subseteq \text{TH}(\mathcal{P}_\lambda) \). \( \square \)

Corollary 5.2 implies Lemma 1.4. It remains to discern, in which bounded potentially renormalizable components maps are immediately renormalizable.

Definition 5.3. Assume that \( f \) is immediately renormalizable with filled quadratic-like Julia set \( K^* \). Assume that there exists the smallest \( n \) such that \( f^{on} (\omega_2(f)) \in K^* \). Then we say that \( f \) is an immediately renormalizable polynomial of capture type.

Observe that \( n \) in Definition 5.3 is greater than 0 by the definition of a quadratic-like map. Moreover, the pullback \( Q \) of \( K^* \) containing
\( \omega_2(f) \) is disjoint from \( K^* \) by the definition of a pullback. Hence we can consider all pullbacks of \( K^* \), which will form a family of pairwise disjoint subcontinua of \( K(f) \). It follows that in fact \( n > 1 \) as otherwise points of \( K^* \) will have two preimages in \( K^* \) and two more preimages in \( Q \), thus four preimages overall, a contradiction (recall that \( f \) is cubic).

**Lemma 5.4.** An immediately renormalizable polynomial of capture type cannot belong to a bounded potentially renormalizable component.

**Proof.** Let \( f \in F, |\lambda| \leq 1 \), belong to a bounded potentially renormalizable component \( \mathcal{W}_f = \mathcal{W} \). By Theorem 1.6, the component \( \mathcal{W} \) is either of Siegel capture type or of queer type; by Corollary 5.2, the map \( f \) is stable. Assume that \( f \) is of capture type. Let \( n \) be smallest number such that \( f^n(\omega_2(f)) \in K^* \); then by the above \( n > 1 \).

The set \( K^* \) and all its pullbacks contain lots of repelling periodic points and their preimages (such points are dense in the corresponding quadratic-like Julia set \( J^* \) and its pullbacks). Recall that \( Q \) is the pullback of \( K^* \) containing \( \omega_2(f) \). Obviously, we can choose two rays \( R_x, R_y \) with arguments \( \alpha \) and \( \beta \) landing at preperiodic points \( x, y \in Q \) such that \( f(x) = f(y) \) and \( f(R_x) = f(R_y) \). The union \( R_x \cup Q \cup R_y \) cuts the plane into two pieces denoted by \( L \) and \( T \). Assume that \( K^* \subset L \). Since \( f|_{K^*} \) is two-to-one, we may then assume that the arc \( (\alpha, \beta) \subset \mathbb{S} = \mathbb{R}/\mathbb{Z} \) is of length \( \frac{2}{3} \) and contains all angles, whose rays are contained in \( L \).

Let us show that \( Z = f^{n-1}(Q) \subset T \) (observe that \( f(Z) = K^* \)). Indeed, otherwise \( Z \subset L \). Choose a point \( z \in Z \) such that \( f(z) \in K^* \) is a repelling periodic point. Then \( f(z) \) has three preimages in \( L \): two of them in \( K^* \) itself and the third one is \( z \in Z \). Considering arguments of rays landing at these points we will find three distinct angles in \( (\alpha, \beta) \) which have the same image under the angle tripling map, a contradiction with \( (\alpha, \beta) \) mapping forward two-to-one.

Thus, \( Z = f^{n-1}(Q) \subset T \). By Theorem 7.5.2 of [BFMOT12], this implies that there exists a non-repelling periodic point of \( f \) in \( T \) such that its multiplier is not 1. However, this contradicts Theorem 1.6. This completes the proof of the lemma.

Clearly, Corollary 5.1 and Lemma 5.4 imply Theorem A.

Finally we obtain a few results describing additional properties of polynomials \( f \in F_{nr} \). Lemma 5.4 is an example of such a result; the other ones are listed in Lemma 5.5.

**Lemma 5.5.** Let \( f \in F_{nr} \) be potentially renormalizable and such that \( X(f) \) is connected. Then \( X(f) \) contains a critical point. If the set
$J(f) \setminus \text{TH}(X(f))$ also contains a critical point, then $f$ is immediately renormalizable.

Proof. Observe that both claims hold if $f$ is immediately renormalizable. Thus we may assume that $f$ is not immediately renormalizable. By Theorem A, the potentially renormalizable component $W_f$ containing $f$ is bounded, and so $J(f)$ is connected. Let us prove the second claim. Suppose that $c \in J(f) \setminus \text{TH}(X(f))$ is a critical point. Note that the local degree at $c$ is two. Choose a small open neighborhood $V$ of $c$ such that $V \cap \text{TH}(X(f)) = \emptyset$. We can choose $V$ so that it is stable under a natural local involution $\tau$ defined on $V$ such that $f \circ \tau = f$ and $\tau(z) \neq z$ for $z \neq c$. Finally, choose an external ray $R_\alpha$ to $J(f)$ with argument $\alpha$ which enters $V$.

Clearly, rays $R_{\alpha+\frac{1}{3}}$ and $R_{\alpha+\frac{2}{3}}$ have the same image as $R_\alpha$. We may assume that $R_{\alpha+\frac{1}{3}}$ enters $V$ as well so that $\tau$ maps points of $R_{\alpha} \cap V$ to points of $R_{\alpha+\frac{1}{3}} \cap V$ and vice versa. Choose a point $z \in R_\alpha \cap V$ and connect it to the point $\tau(z) \in R_{\alpha+\frac{1}{3}} \cap V$ with a short curve $T \subset V$ passing through $c$ and stable under $\tau$. This creates a cut $Z$ of the plane which is disjoint from $\text{TH}(X(f))$ and such that the remaining two parts of the plane map onto their images in exactly two-to-one and one-to-one fashion. Since we know from Lemma 2.8 that $f|_{X(f)}$ is at least two-to-one, it follows that $f|_{X(f)}$ is exactly two-to-one. Hence, by Theorem 4.1, the map $f$ is immediately renormalizable. This proves the second claim.

Now, suppose that $\text{TH}(X(f))$ contains no critical points. Then, by the second claim, we may assume that no critical point of $f$ belongs to $J(f)$. By Theorem 1.6, the component $W_f$ is of Siegel capture type or of queer type. In the first case, it follows from [Ma93] that $f$ has a recurrent critical point $c$ such that the boundary of an invariant Siegel disk around $0$ is contained in the $\omega$-limit set of $c$. In particular, $c \in J(f)$, a contradiction. In the second case, the fact that $W_f$ is of queer type implies by definition that $\omega_2(f) \in J(f)$, again a contradiction. \hfill $\square$

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