A MODEL OF THE CUBIC CONNECTEDNESS LOCUS

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Dedicated to the memory of Yuri Ilyich Lyubich

ABSTRACT. We construct a locally connected model of the cubic connectedness locus. The model is obtained by constructing a decomposition of the space of critical portraits and collapsing elements of the decomposition into points. This model is similar to a quotient of the combinatorial quadratic Mandelbrot set in which all baby Mandelbrot sets, as well as the filled Main Cardioid, are collapsed to points. All fibers of the model, possibly except one, are connected. The authors are not aware of other known models of the cubic connectedness locus.

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1. Introduction

We assume familiarity with complex polynomial dynamics and its standard notation (K_f is the filled Julia set, J_f is the Julia set of f, etc).

Let Poly_d be the space of all monic centered polynomials of degree $d > 1$, i.e. maps $f(z) = z^d + a_{d-2}z^{d-2} + \cdots + a_0 : \mathbb{C} \to \mathbb{C}$ (e.g., Poly_2 consists of polynomials $Q_c(z) = z^2 + c$). The *connectedness locus* $\mathcal{C}_d$ of Poly_d is the set of all $f \in \mathrm{Poly}_d$ with connected K_f. The *principal hyperbolic domain* PHD_d of Poly_d consists of all $f \in \mathrm{Poly}_d$ with an attracting fixed point a and all critical points in the immediate basin of a. In this case, the immediate basin of a is an invariant Fatou domain U homeomorphic to the unit disk; $\overline{U}$ is a closed Jordan disk coinciding with K_f. For example, $\mathcal{C}_2$ identifies with the *Mandelbrot set* $\mathcal{M} = \{c \in \mathbb{C} \mid Q_c^n(0) \not\to \infty\}$, and the boundary of PHD_2 is called the *Main Cardioid*.

The study of the structure of $\mathcal{M}$, initiated by Douady and Hubbard, culminated in Thurston's paper [67] (although officially published in 2009, it has circulated as a preprint since 1985) where a detailed locally connected *combinatorial* model $\mathcal{M}^{comb}$ of the quadratic connectedness locus $\mathcal{C}_2 = \mathcal{M}$ is given (it is combinatorial in the sense that its construction is independent of polynomials). The continuum $\mathcal{M}^{comb}$ is a certain quotient of the closed unit circle $\mathbb{S}$. There exists a *monotone* map $\pi_2 : \mathrm{Bd}(\mathcal{M}) \to \mathcal{M}^{comb}$ (a continuous map is *monotone* if point-preimages are connected; the boundary of a set A in a topological space is denoted by $\mathrm{Bd}(A)$). Hence, $\mathrm{Bd}(\mathcal{M})$ can be understood as the continuum $\mathcal{M}^{comb}$ with (possibly) some points in $\mathcal{M}^{comb}$ "blown up" into continua (the MLC conjecture states that $\mathcal{M}$ is homeomorphic to $\mathcal{M}^{comb}$ and the map π_2 is a homeomorphism).

The cubic case is much harder. A global view on the parameter space Poly_3 of cubic polynomials was first given by Branner and Hubbard in papers [24, 26, 27], where they described the topology of $\mathrm{Poly}_3 \setminus \mathcal{C}_3$ including that of the bifurcation locus. Fine structure of $\mathrm{Poly}_3 \setminus \mathcal{C}_3$ and, in particular, that of the *shift locus* (the subset of Poly_3 formed by polynomials whose *both* critical orbits escape) was further studied (and modeled by means of various analytic or combinatorial constructions) in [35, 34, 32, 33, 30, 49].

By the *real part* of Poly_3 we mean all polynomials with real coefficients. Milnor [57, 58] studied the real part of Poly_3, classified and parameterized hyperbolic components of Poly_3, and formulated a further research program focused on dynamically defined slices $\mathrm{Per}_k(\lambda)$ of Poly_3. Here, $\mathrm{Per}_k(\lambda)$ consists of cubic polynomials with a k-periodic point of multiplier λ. Slices $\mathrm{Per}_1(\lambda)$ were then studied in a number of papers. The case of attracting of parabolic λ was investigated, e.g., in [40, 62, 63, 64, 23, 69, 71, 72]. For the case of irrationally neutral λ (with certain arithmetic conditions imposed on the argument) see, e.g., [73, 6, 21]. For general non-repelling values of λ, slices $\mathrm{Per}_1(\lambda)$ are studied in [28, 14, 22]; corresponding spaces of invariant laminations are described in [15]. Algebraic geometry and arithmetic aspects of Poly_3 and sets $\mathrm{Per}_k(0)$ are investigated in [50, 3, 41, 29, 1].

Inou and Kiwi [48] developed a generalization of the Douady–Hubbard renormalization scheme [38] for polynomials of degree > 2. Further developments of this renormalization theory, covering cubic polynomials, are in [66]. Laminational models for specific subsets of $\mathcal{C}_3$ are given in [17, 19].

Globally, it is known that $\mathcal{C}_3$ is not locally connected [56], contains copies of many non-locally connected quadratic Julia sets [28], and has non-locally connected real part [39, 54]. The combinatorial rigidity conjecture fails for $\mathcal{C}_3$ [46]. The study of the structure of the principal hyperbolic component of $\mathcal{C}_3$ and its closure can be found in [60, 14]. Intersections between boundaries of bounded hyperbolic components are described in [65]. Important open parts of $\mathcal{C}_3$ can be described in terms of *intertwining surgery* [39]; these parts show a direct product structure. An earlier surgery scheme of Branner–Douady [25] describes certain dynamically defined slices of $\mathcal{C}_3$ in terms of parts of the quadratic Mandelbrot set. *Core entropy* [42, 43, 68] is also a useful tool in studying geography of $\mathcal{C}_3$.

As one can see from the brief survey above, even though a lot of work is devoted to the studying of Poly_3, not much is known in terms of the global structure of $\mathcal{C}_3$. This can be explained by the fact that $\mathcal{C}_3$ is complex 2-dimensional while $\mathcal{C}_2 = \mathcal{M}$ is complex 1-dimensional. Also, cubic polynomials are richer dynamically than quadratic ones: critical points are essential for the dynamics of polynomials, and cubic polynomials generically have two critical points which makes the cubic case highly intricate combinatorially [14, 18] and results in a breakdown of crucial steps of [67] (e.g., cubic invariant laminations admit wandering triangles [11, 12]).

A rare text dealing with combinatorics the entire connectedness loci and their models in degree $d \geqslant 3$ is a recent publication [68]. Thurston and his collaborators devote a significant portion of [68] to a detailed discussion of the cubic case. Yet [68] does not aim at a model of the connectedness loci. While [68] uses laminations as an important tool, this is done in a non-dynamical fashion, and the main focus of [68] is different from ours.

In the present paper we aim at constructing an explicit (combinatorial) locally connected continuum X_3 and a map from $\mathcal{C}_3$ to X_3. This can be viewed as a step towards uncovering the structure of the cubic connectedness locus $\mathcal{C}_3$. However we use a simpler map than the map π_2 mentioned above. Let us briefly explain our approach.

Douady and Hubbard [38] used subtle analytic tools to show that $\mathcal{M}$ contains infinitely many *baby Mandelbrot sets*, i.e. homeomorphic copies of $\mathcal{M}$. There is a hierarchy among baby Mandelbrot sets each of which is contained in a unique *maximal* baby Mandelbrot set. One can construct a model continuum X_2 of $\mathcal{M}$ by adjusting the map π_2, collapsing all maximal baby Mandelbrot sets to distinct points, and collapsing PHD_2 to a point, too. The resulting continuous *monotone* map $\eta_2 : \mathcal{M} \to X_2$ reveals the *macro-structure* of $\mathcal{M}$, i.e. the mutual disposition of the maximal baby Mandelbrot sets and the closure $\overline{\mathrm{PHD}_2}$ of PHD_2.

Any continuous map $h : \mathcal{M} \to Y$ that collapses all baby Mandelbrot sets and PHD_2 to points can be expressed as $h = \tilde{h} \circ \eta_2$, and so η_2 can be viewed as the *finest* among all such maps. Since η_2 is obtained by adjusting π_2 through additional collapsing of connected sets, one can expect that extending the construction of η_2 and X_2 to higher degree cases may be simpler than constructing a more detailed model.

In the present paper we construct a map $\eta_3 : \mathcal{C}_3 \to X_3$ for a certain combinatorially defined continuum X_3 so that η_3 is continuous, collapses $\overline{\mathrm{PHD}}_3$ to a point, and sends each of the subsets similar to maximal baby Mandelbrot sets to distinct points. The map η_3 solves the problem of finding a map with these special properties. Since we consider polynomials of higher degrees, we modify some concepts heuristically introduced above.

Let us now describe our main results. Denote a dynamical external ray of a polynomial f of argument α by $R_f(\alpha)$ (we call dynamical external rays simply *external rays*). An external ray $R_f(\alpha)$ has *impression* $I(R_f(\alpha)) = I_f(\alpha)$ (see, e.g., [59]) which is a subcontinuum of J_f. It reflects how points of external rays with arguments close to α accumulate in J_f (we always assume that J_f is connected). It turns out that using impressions one can define a certain equivalence relation $\sim_f$ on $\overline{\mathbb{D}}$ so that the quotient space $\overline{\mathbb{D}}/\sim_f$ is a locally connected monotone model of the filled Julia set K_f. Basically, $\overline{\mathbb{D}}/\sim_f$ is a generalization of Douady's pinched disk model [36, 37]. Notice also that the concept of a T-class introduced below is related to the concept of a *renormalization domain* [48].

Definition 1.1 ([7, 53])**.** Write $\alpha \sim_f \beta$ if $I_f(\alpha) \cap I_f(\beta) \neq \varnothing$. Extend $\sim_f$ by transitivity and pass to the closure of it. Then extend $\sim_f$ over $\overline{\mathbb{D}}$ as the smallest equivalence relation containing $\sim_f$ with the property that all classes are convex. Keep the notation $\sim_f$ for this new relation on $\overline{\mathbb{D}}$ and call it the *full laminational equivalence relation of f*. A geometric interpretation of $\sim_f$ is given by the collection $\mathcal{L}_f$ of edges of convex hulls of $\sim_f$-classes called the *lamination of f*, and the chords are called *leaves* of $\mathcal{L}_f$. The closed unit disk $\overline{\mathbb{D}}$ with $\mathcal{L}_f$ is a visual counterpart of $\sim_f$.

Laminations and laminational equivalence relations give a combinatorial description of complex polynomials from the connectedness loci. Thus, to describe $\mathcal{C}_d$ one can describe the space of laminations of degree d and use this space as a model of $\mathcal{C}_d$. Thurston successfully used this approach in [67]. We adjust it as we plan to collapse all sets of polynomials similar to baby Mandelbrot sets, and need to describe such sets. This is done below. However first we need to consider two extreme cases which produce the same picture. First, set $f(z) = z^3$; then all $\sim_f$-classes are points and $\overline{\mathbb{D}}$ with $\mathcal{L}_f$ is the closed unit disk without leaves. The second case is when the entire unit circle forms one $\sim_f$-class. Then $\overline{\mathbb{D}}$ with $\mathcal{L}_f$ is the closed unit disk without leaves (even though here $\overline{\mathbb{D}}$ is the convex hull of one $\sim_f$-class). In what follows these two laminational equivalence relations are called *trivial*.

Definition 1.2 (T-classes)**.** The *trivial T-class* consists of all polynomials $P \in \mathcal{C}_d$ with trivial $\sim_P$. Let $f, g \in \mathcal{C}_d$ generate non-trivial $\sim_f$ and $\sim_g$. If $\alpha \sim_f \beta$ always implies $\alpha \sim_g \beta$ (i.e., if any $\sim_f$-class is contained in a $\sim_g$-class), then g is said to *tune* f. If one of two polynomials tunes the other one, they are said to be *tuning related*. Extending this relation by transitivity we talk about *T-classes*. Thus, two polynomials P, Q belong to the same T-class if there exists a finite chain of polynomials $f_1 = P$, $f_2, \dots,$ $f_n = Q$ such that f_i and f_{i+1} are tuning related. The same terminology will be used for laminational equivalence relations $\sim$ and their laminations $\mathcal{L}_\sim$ so that we can talk about *T-classes of laminational equivalence relations* and *T-classes of laminations*.

Now we define maps similar to the map η_2 from the Introduction.

Definition 1.3. A map $\phi : \mathcal{C}_d \to Y$ is *T-stable* if $\phi(f) = \phi(g)$ whenever f and g belong to the same T-class, and all polynomials with a fixed non-repelling point belong to the same point preimage of ϕ.

By Definition 1.3, any T-stable map must collapse the union of T-classes of all polynomials with a non-repelling fixed point to one point. Notice that by Theorem 8.5 this union equals the union of T-classes of all polynomials with a neutral fixed point (a priori the latter union can be smaller).

In this paper we solve the following problem in the cubic case $d = 3$..

Main Problem. *Describe a T-stable map $\tilde{\pi}_d : \mathcal{C}_d \to X_d$ such that any other T-stable map ϕ is the composition of $\tilde{\pi}_d$ and a map from X_d to $\phi(\mathcal{C}_d)$. Give a combinatorial (i.e., independent of polynomials) description of X_d.*

Our Main Theorem solves this problem. However first we discuss the quadratic case. Based upon the structure of $\mathcal{M}$ uncovered by Douady-Hubbard and Thurston [36, 37, 67], it is easy to see that the map η_2 described in the Introduction, solves the Main Problem for $d = 2$. Indeed, all maximal baby Mandelbrot sets must collapse to points as all polynomials from them tune the corresponding root polynomial. Also, the *central fiber* of η_2, defined as the point preimage of η_2 containing $z \mapsto z^2$, is the union of $\overline{\mathrm{PHD}}_2$ and the maximal Mandelbrot sets non-disjoint from $\overline{\mathrm{PHD}}_2$. Evidently, η_2 is the finest (the least collapsing) among all maps with the properties from the Main Problem, and any map ϕ satisfying the properties from the Main Problem can be represented as the composition of η_2 with a map from X_2 to $\phi(\mathcal{M})$. Thus, $\eta_2 = \tilde{\pi}_2$.

To describe our main results, we use notation and terminology introduced in the next section. Say that a chord $\overline{ab}$ of $\overline{\mathbb{D}}$ is *(σ_3-)critical* if $\sigma_3(a) = \sigma_3(b)$. To describe a *combinatorial* model X_3 of $\mathcal{C}_3$, we use the space CrP_3 of *cubic critical portraits* whose elements are pairs of distinct *compatible* (not intersecting inside $\mathbb{D}$) critical chords (Section 3). This is natural: polynomials are associated to their critical portraits because critical portraits are a combinatorial counterpart of critical points of polynomials, and behavior of critical points essentially defines the dynamics of the entire polynomial. For a family $\mathcal{T}$ of polynomials let $\mathrm{CrP}(\mathcal{T})$, the *set of critical portraits of* $\mathcal{T}$, be the family of critical portraits compatible with $\mathcal{L}_P$ for at least one polynomial $P \in \mathcal{T}$ such that $\mathcal{L}_P$ is non-trivial.

Let us now define a map η_3. Let $\mathfrak{F}_3 \subset \mathcal{C}_3$ be the union of T-classes of cubic polynomials with a non-repelling fixed point. Since $z \mapsto z^3$ has a super-attracting fixed point 0, the entire trivial T-class is contained in $\mathfrak{F}_3$. For every $P \in \mathfrak{F}_3$, set $\eta_3(P) = \mathrm{CrP}(\mathfrak{F}_3)$.

Now, let $P \in \mathcal{C}_3 \setminus \mathfrak{F}_3$ be such that $\mathcal{L}_P$ is minimal (by definition of $\mathfrak{F}_3$, the lamination $\mathcal{L}_P$ is not trivial). Consider the family $\mathcal{F}_P$ of all polynomials that tune P; if $f \in \mathcal{F}_P$, set $\eta_3(f) = \mathrm{CrP}(\mathcal{F}_P)$.

Main Theorem. *The map η_3 is well-defined and continuous. The point preimage of η_3 containing $z \mapsto z^3$ is $\mathfrak{F}_3$ and coincides with the union of T-classes of cubic polynomials with a non-repelling (equivalently, neutral) fixed point. All other point preimages of η_3 are connected. Any T-stable map $\phi : \mathcal{C}_3 \to \phi(\mathcal{C}_3)$ is the composition of η_3 and a map from $\eta_3(\mathcal{C}_3)$ to $\phi(\mathcal{C}_3)$, so that η_3 is the map $\tilde{\pi}_3$ from the Main Problem.*

In the quadratic case, the fibers of η_2 not-containing $z \mapsto z^2$ are maximal baby Mandelbrot sets disjoint from $\overline{\mathrm{PHD}}_2$; observe that they are pairwise disjoint. In the cubic case, point preimages of η_3 are more complicated. Still, by the Main Theorem, all point preimages distinct from $\mathfrak{F}_3$ are connected (we do not know whether $\mathfrak{F}_3$ is connected). The point preimages of η_3 are closely related to the *combinatorial renormalization domains* defined in [48]. The relevant papers are [47, 57, 48, 58, 66, 55, 70].

2. Laminations

We parameterize the external rays of a polynomial $f \in \mathrm{Poly}_d$ by *angles*, i.e., elements of $\mathbb{R}/\mathbb{Z}$. The external ray of argument $\theta \in \mathbb{R}/\mathbb{Z}$ is denoted by $R_f(\theta)$. Clearly, f maps $R_f(\theta)$ to $R_f(d\theta)$.

A *chord* $\overline{ab}$ is a closed segment connecting points a, b of the unit circle $\mathbb{S} = \{z \in \mathbb{C} \,|\, |z| = 1\}$, often represented by their arguments (thus, we may write $\overline{\frac{1}{3}\frac{2}{3}}$ meaning the chord connecting the points $e^{2\pi i/3}$ and $e^{4\pi i/3}$). If $a = b$, then $\overline{ab}$ is *degenerate*. *Distinct* chords *cross* if they intersect in $\mathbb{D}$ (alternatively, they are called *linked*). Chords that do not cross are said to be *unlinked*. Sets of chords are *compatible* if chords from distinct sets do not cross. Write σ_d for the self-map of $\mathbb{S}$ that takes z to z^d. A chord $\overline{ab}$ is (σ_d-) *critical* if $\sigma_d(a) = \sigma_d(b)$.

2.1. **Laminational equivalence relations.** For $f \in \mathcal{C}_d$ with locally connected J_f, let $\psi(e^{2\pi i\theta})$ be the landing point of $R_f(\theta)$; then $\psi : \mathbb{S} \to J_f$ is a semi-conjugacy between $\sigma_d : \mathbb{S} \to \mathbb{S}$ and $f : J_f \to J_f$ called the *Caratheodory loop*. Define an equivalence relation $\sim_f$ on $\mathbb{S}$ as follows: $x \sim_f y$ if and only if $\psi(x) = \psi(y)$ and call $\sim_f$ the *laminational equivalence relation (generated by f)*. The relation $\sim_f$ is σ_d-invariant; $\sim_f$-classes have pairwise disjoint convex hulls. The quotient space $\mathbb{S}/\sim_f = J_{\sim_f}$ is called a *topological Julia set*. Clearly, $J_{\sim_f}$ is homeomorphic to J_f. The map $f_{\sim_f} : J_{\sim_f} \to J_{\sim_f}$, induced by σ_d and called a *topological polynomial*, is topologically conjugate to $f|_{J_f}$.

Equivalence relations analogous to $\sim_f$ can be introduced with no reference to polynomials [9]. Let $\sim$ be an equivalence relation on $\mathbb{S}$. Equivalence classes of $\sim$ will be called *($\sim$-)classes*. Also, given a closed set $A \subset \mathbb{C}$, let $\mathrm{CH}(A)$ denote the convex hull of the set A in $\mathbb{C}$.

Definition 2.1. An equivalence relation $\sim$ is a *(σ_d-)invariant laminational equivalence relation* if it is:

(E1) *closed*: the graph of $\sim$ is a closed set in $\mathbb{S} \times \mathbb{S}$;
(E2) *unlinked*: if $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are distinct $\sim$-classes, then their convex hulls $\mathrm{CH}(\mathfrak{g}_1), \mathrm{CH}(\mathfrak{g}_2)$ in the unit disk $\mathbb{D}$ are disjoint;
(E3) *finite*: all $\sim$-classes are finite;
(D1) *forward invariant*: for a class $\mathfrak{g}$, the set $\sigma_d(\mathfrak{g})$ is a class too;
(D2) *backward invariant*: for a class $\mathfrak{g}$, its preimage $\sigma_d^{-1}(\mathfrak{g}) = \{x \in \mathbb{S} : \sigma_d(x) \in \mathfrak{g}\}$ is a union of classes;
(D3) *orientation preserving*: for any $\sim$-class $\mathfrak{g}$ with more than two points, the map $\sigma_d|_{\mathfrak{g}} : \mathfrak{g} \to \sigma_d(\mathfrak{g})$ is a *covering map with positive orientation*, i.e., for every connected component (s,t) of $\mathbb{S} \setminus \mathfrak{g}$ the arc in the circle $(\sigma_d(s), \sigma_d(t))$ is a connected component of $\mathbb{S} \setminus \sigma_d(\mathfrak{g})$.

Here, conditions (E1) – (E3) use only an **E**quivalence relation, while conditions (D1) – (D3) deal also with the **D**ynamics of σ_d. Note that (D1) implies (D2). If $\sim$ has all the properties from above except (E3) (i.e., some $\sim$-classes can be infinite), then $\sim$ is called a *(σ_d-)invariant laminational$^\infty$ equivalence relation.*

Invariant laminational$^\infty$ equivalence relations have visual counterparts.

Definition 2.2. For a laminational$^\infty$ equivalence relation $\sim$, consider the family of all edges of convex hulls of $\sim$-classes. This set of chords, together with $\mathbb{S}$, is called the *(invariant) q-lamination (generated by $\sim$)* and is denoted by $\mathcal{L}_\sim$; chords of $\mathcal{L}_\sim$ are said to be the *leaves* of $\mathcal{L}_\sim$.

Note that q-laminations generated by invariant laminational$^\infty$ equivalence relations and q-laminations generated by invariant laminational equivalence relations (i.e., without infinite classes) form the same class of sets. Indeed, let $\mathcal{L}_\sim$ be generated by an invariant laminational$^\infty$ equivalence relation $\sim$. It is easy to see that infinite classes of $\sim$, if any, are Cantor sets. Hence we can declare a new equivalence $\hat{\sim}$ which keeps all finite classes of $\sim$ and breaks all infinite $\sim$-classes into classes as follows: points x, y from an infinite $\sim$-class A are $\hat{\sim}$-equivalent if they are connected with an edge of the convex hull of A. Then it is easy to check that $\hat{\sim}$ is an invariant laminational equivalence without infinite classes. However by construction $\mathcal{L}_{\hat{\sim}} = \mathcal{L}_\sim$. This is why we use the term q-laminations without $^\infty$.

2.2. **General properties of laminations.** Thurston [67] defined *invariant laminations* as families of chords with dynamical properties resembling those of $\mathcal{L}_f$ but without invoking polynomials. We use a bit different approach [10], largely borrowing terminology (and inspiration) from [67].

Definition 2.3 (Laminations)**.** A *prelamination* is a family $\mathcal{L}$ of chords called *leaves* such that distinct leaves are unlinked and all points of $\mathbb{S}$ are leaves. If the set $\mathcal{L}^+ = \bigcup_{\ell\in\mathcal{L}} \ell$ is compact, then $\mathcal{L}$ is called a *lamination*. Two (pre)laminations are *compatible* if their leaves do not cross (thus, the union of two compatible (pre)laminations is a (pre)lamination).

From now on, $\mathcal{L}$ denotes a lamination.

Definition 2.4 (Gaps and edges)**.** *Gaps* of $\mathcal{L}$ are the closures of components of $\mathbb{D} \setminus \mathcal{L}^+$. A gap G is *countable* (*finite, uncountable*) if $G \cap \mathbb{S}$ is countable infinite (finite, uncountable). Uncountable gaps are called *Fatou* gaps. For a closed set $H \subset \mathbb{C}$, *edges* of H are maximal straight segments in $\mathrm{Bd}(H)$.

Convergence of (pre)laminations $\mathcal{L}_i$ to a set of chords $\mathcal{E}$ is understood as the Hausdorff convergence of leaves of $\mathcal{L}_i$ to chords from $\mathcal{E}$; evidently, $\mathcal{E}$ is a lamination. A lamination $\mathcal{L}$ is *nonempty* if it has nondegenerate leaves and *empty* otherwise (the empty lamination is denoted by $\mathcal{L}_\varnothing$; note that it is not the empty set as it contains all points of $\mathbb{S}$). Say that $\mathcal{L}$ is *countable* if it has countably many nondegenerate leaves and *uncountable* otherwise; $\mathcal{L}$ is *perfect* if it has no isolated leaves.

If $G \subset \overline{\mathbb{D}}$ is the convex hull of $G \cap \mathbb{S}$, define $\sigma_d(G)$ as the convex hull of $\sigma_d(G \cap \mathbb{S})$. A *sibling of a leaf* $\ell \in \mathcal{L}$ is a leaf $\ell' \in \mathcal{L}$ different from ℓ with $\sigma_d(\ell') = \sigma_d(\ell)$. Call a leaf ℓ^* such that $\sigma_d(\ell^*) = \ell$ a *pullback* of ℓ. The map σ_d can be extended continuously over $\mathcal{L}^+$ so that it acts linearly on every leaf of $\mathcal{L}$. We also denote this extended map by σ_d.

Definition 2.5 (Definition 3.1 of [10])**.** A (pre)lamination $\mathcal{L}$ is *sibling (σ_d)-invariant* if

(1) for each $\ell \in \mathcal{L}$, we have $\sigma_d(\ell) \in \mathcal{L}$,
(2) for each $\ell \in \mathcal{L}$ there exists $\ell^* \in \mathcal{L}$ with $\sigma_d(\ell^*) = \ell$,
(3) for each non-critical $\ell \in \mathcal{L}$ there exist d **pairwise disjoint** leaves ℓ_1, $\dots$, ℓ_d in $\mathcal{L}$ such that $\ell_1 = \ell$ and $\sigma_d(\ell_1) = \dots = \sigma_d(\ell_d)$.

Leaves from (3) above form *full sibling collections*. Their elements cannot intersect even on $\mathbb{S}$. Here is a useful property of such collections.

Lemma 2.6. *The following properties hold.*

(1) *Let $\ell_1, \dots, \ell_d$ be the limit of a sequence of full sibling collections. If ℓ_1 is not critical, then $\ell_1, \dots, \ell_d$ is a full sibling collection.*
(2) *The family of all non-isolated leaves of a sibling invariant lamination is a sibling invariant lamination.*

Proof. (1) We claim that ℓ_1 and ℓ_2 are disjoint. If $\ell_1 = \overline{ab}$ and $\ell_2 = \overline{bc}$, then $\sigma_d(a) = \sigma_d(c) \neq \sigma_d(b)$. A full sibling collection approximating the given one has a pair of leaves $\overline{a'b'}$ and $\overline{b''c'}$ with b', b'' close to b and $\sigma_d(\overline{a'b'}) = \sigma_d(\overline{b''c'})$, a contradiction.

(2) All non-isolated leaves in $\mathcal{L}$ form a forward invariant closed family of leaves. If ℓ is non-isolated, choose a sequence of leaves $\overline{q}_i \to \overline{q}$ with $\sigma_d(\overline{q}_i) \to \ell$ so that $\sigma_d(\overline{q}) = \ell$. Now, let ℓ be non-isolated and non-critical. Choose $\ell_i \to \ell$ so that ℓ_i's belong to their full sibling collections. We may assume that these collections of leaves converge; by (1) they converge to a full sibling collection that includes ℓ. This completes the proof. □

The next lemma is proven in [10].

Lemma 2.7 (Lemma 3.1 of [10])**.** *Any q-lamination is sibling invariant.*

Sibling invariant laminations have the following properties.

Theorem 2.8 (Theorem 3.2 of [10])**.** *If G is a gap of $\mathcal{L}$, then $H = \sigma_d(G)$ is a leaf of $\mathcal{L}$ (possibly degenerate), or a gap of $\mathcal{L}$, and, in the latter case, the* *map* $\sigma_d|_{\mathrm{Bd}(G)} : \mathrm{Bd}(G) \to \mathrm{Bd}(H)$ *is an orientation preserving composition of a monotone map and a covering map.*

Gap invariance is a part of [67], thus, by Theorem 3.2 of [10], sibling invariant laminations are invariant in the sense of Thurston [67].

A motivation for introducing sibling invariant laminations was that it is easier to deal with leaves and their sibling collections than with gaps. As a consequence, studying families of laminations became more transparent. In particular, the following theorem holds (recall that we always equip spaces of compact sets with the Hausdorff distance topology).

Theorem 2.9 (Corollary 3.20 and Theorem 3.21 of [10])**.** *The closure of a sibling invariant prelamination is a sibling invariant lamination. The space of all sibling invariant laminations of degree* d *is compact.*

From now on, by “invariant” laminations we mean “sibling invariant” laminations, and, unless stated otherwise, all laminations are σ_d-invariant for some $d \geqslant 2$ (sometimes, but not always, we emphasize this fact).

Definition 2.10 (Space of laminations)**.** The space of all invariant laminations of degree d is denoted by $\mathfrak{Lam}_d$.

Since (pre)laminations are collections of chords, the concept of compatibility applies to them, and we can talk about compatible laminations. Then a useful fact that follows from Definition 2.5 is the following lemma.

Lemma 2.11. *If invariant laminations* $\mathcal{L}, \mathcal{L}'$ *are compatible, then* $\mathcal{L} \cup \mathcal{L}'$ *is an invariant lamination. If* $\mathcal{L}, \mathcal{L}'$ *are in addition perfect, then so is* $\mathcal{L} \cup \mathcal{L}'$.

There is an important connection between laminations and (invariant laminational) equivalence relations.

Definition 2.12. Let $\mathcal{L}$ be a lamination. Define the equivalence relation $\approx_{\mathcal{L}}$ by declaring that $x \approx_{\mathcal{L}} y$ if and only if there exists a finite concatenation of leaves of $\mathcal{L}$ joining x to y.

2.3. **Gaps and gap-leaves of arbitrary laminations.** A chord ℓ is *inside* a gap G if, except for the endpoints, ℓ is in the interior of G; if $\ell \subset G$, say that ℓ is *contained in* G. A gap G of $\mathcal{L}$ is *critical* if all edges of G are critical, or there is a critical chord *inside* G. A *critical set* of $\mathcal{L}$ is a critical leaf or a critical gap. A *gap-leaf* of $\mathcal{L}$ is either a finite gap of $\mathcal{L}$ or a nondegenerate leaf of $\mathcal{L}$ not on the boundary of a finite gap. By the *period* we mean the *minimal* period. For a chord $\ell = \overline{ab}$, let $|\ell|$ be the length of the smaller circle arc with endpoints a and b (computed with respect to the Lebesgue measure on $\mathbb{S}$ normalized so that the total length of $\mathbb{S}$ is 1); call $|\ell|$ the *length* of ℓ.

Lemma 2.13. *If $0 < |\ell| < \frac{1}{d+1}$, then $\sigma_d(\ell) > |\ell|$. The forward σ_d-orbit of any chord contains a chord of length $\geqslant \frac{1}{d+1}$. In any nonempty σ_d-invariant lamination, there are leaves of length $\geqslant \frac{1}{d+1}$.*

The proof of Lemma 2.13 is straightforward; it is left to the reader.

Lemma 2.14. *Any edge of an infinite gap is (pre)periodic or (pre)critical.*

Proof. By Kiwi [52] we may assume that our infinite gap is periodic. Then the lemma follows from the fact that lengths of edges of any infinite gap converge to 0 and from Lemma 2.13. □

It is known that Fatou gaps of σ_d-invariant laminations are (pre)periodic ([52, Theorem 1.1] or [9, Theorem B]). If U is a σ_d-periodic Fatou gap of period n and the map $\sigma_d^n : \mathrm{Bd}(U) \to \mathrm{Bd}(U)$ has topological degree $k \geqslant 1$, then U is called a *periodic gap of degree k*. If $k > 1$, then a folklore result claims the existence of a monotone map from $\mathrm{Bd}(U)$ to $\mathbb{S}$ collapsing all edges of U also semi-conjugating $\sigma_d^n|_{\mathrm{Bd}(U)}$ with σ_k.

The case of infinite gaps of degree one is more delicate.

Lemma 2.15. *Suppose that $\ell = \overline{xy}$ is a chord such that x is σ_d-periodic and $\sigma_d(x) = \sigma_d(y)$. Then ℓ is not the limit of leaves of σ_d-invariant laminations not equal to ℓ and unlinked with ℓ. In particular, if ℓ is a critical leaf of a σ_d-invariant lamination $\mathcal{L}$, then ℓ is isolated in $\mathcal{L}$.*

Proof. Let x be fixed. If a leaf ℓ' is close to ℓ, unlinked with ℓ, and $x \notin \ell'$, then $\sigma_d(\ell')$ crosses ℓ', a contradiction. Let a leaf $\overline{xz}$ be very close to $\overline{xy}$; then the points $\sigma_d(z)$ and z are separated in $\overline{\mathbb{D}}$ by $\overline{xy}$. By [10, Lemma 3.8], the leaf $\overline{x\sigma_d(z)}$ has a sibling leaf $\overline{yt}$ where t and $\sigma_d(z)$ are separated in $\overline{\mathbb{D}}$ by $\overline{xy}$ which implies that $\overline{xt}$ crosses $\overline{xz}$, a contradiction (notice that the short circle arc from x to $\sigma_d(z)$ is longer than the short circle arc from y to z). □

Definition 2.16 (Major)**.** Let G be an invariant gap of a cubic lamination. An edge $M = \overline{ab}$ of G is called a *major (of G)* if the open circle arc with endpoints a and b disjoint from G is of length $\frac{1}{3}$ or longer. We will call the length of this arc the *length of M*.

In Lemma 2.17, and in what follows, we often consider the linear extension of σ_d over leaves of laminations described before Definition 2.5.

Lemma 2.17. *Suppose that G is a degree one k-periodic infinite gap of a σ_d-invariant lamination $\mathcal{L}$ for some $d \geqslant 2$. Then some gaps from the orbit of G have critical edges. Moreover, there are two possibilities.*

(1) *There is a monotone semi-conjugacy between $\sigma_d^k|_{\mathrm{Bd}(G)}$ and an irrational rotation of $\mathbb{S}$ that collapses all edges of G to points; moreover, if there are concatenations of edges of G, then each concatenation consists of at most $d - 1$ leaves.*

(2) *There are periodic (possibly, degenerate) edges of G; for some minimal q all periodic edges of G are σ_d^{kq}-fixed. Moreover, each arc I located between two maximal adjacent concatenations of σ_d^{kq}-fixed edges of G has the following properties: I is σ_d^{kq}-invariant, at exactly one endpoint, say x, of I a σ_d^{kq}-critical edge $\ell \subset I$ is located, and all points of I map towards x by σ_d^{kq}.*

Also, $\mathcal{L}$ has isolated leaves with both endpoints non-preperiodic, or with one periodic and one non-periodic endpoint. In particular, (a) the lamination $\mathcal{L}$ is not perfect, and (b) it cannot have a dense subset of (pre)periodic leaves whose endpoints have equal preperiods.

Proof. Consider the case when G has periodic edges. Definitions and the fact that the degree of $\sigma_d^k|_{\mathrm{Bd}(G)}$ is one imply that there exists a number q such that $\sigma_d^{kq}|_{\mathrm{Bd}(G)}$ has non-zero (finite) number of fixed edges, closures of all arcs between them are σ_d^{kq} invariant, and points on each such arc map in the same direction (clockwise or counterclockwise). If an arc I like that contains no σ_d^{kq}-critical edges, then one of the endpoints of I attracting for $\sigma_d^k|_{\mathrm{Bd}(G)}$, contradicting the fact that σ_d^k is expanding. Hence each such arc contains a σ_d^{kq}-critical edge that shares a σ_d^{kq}-fixed endpoint with a σ_d^{kq}-invariant edge of G. It follows that some gaps from the orbit of G have critical edges as claimed. On the other hand, if there are no periodic edges of G then there must exist critical edges of certain images of G (recall that every edge of G eventually maps to a critical or a periodic leaf). This proves the first claim of the lemma and completes case (2) of the lemma. Observe that in the second case there are critical leaves of gaps from the orbit of G that connect a periodic and a non-periodic point.

Suppose now that there are no periodic edges of G. Then the map σ_d^{kq} : $\mathrm{Bd}(G) \to \mathrm{Bd}(G)$ has no periodic points, and one can apply results from one-dimensional dynamics. Namely, by [2] in the case of the circle, and by [4, 5] in the case of any one-dimensional branched manifold, the following holds: the map $\sigma_d^{kq} : \mathrm{Bd}(G) \to \mathrm{Bd}(G)$ is semiconjugate to an irrational rotation of the circle by means of a monotone map. This corresponds to case (1) of the lemma. Consider maximal concatenations of edges of G. Note that the image of a concatenation is a (possibly degenerate) concatenation of edges. Because the rotation number is irrational, it follows that any maximal concatenation is wandering in the strong sense: all images of a maximal concatenation are pairwise disjoint. Now, let A be a concatenation like that. Then $\overline{A}$ has well-defined endpoints, say, a and b. Connect them with a chord; then the resulting gap is such that all its images have pairwise disjoint interiors which implies by [52, Theorem 1.1] that the concatenation A can consist of at most $d-1$ edges of G as claimed.

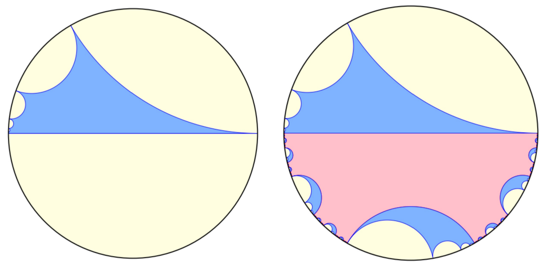

FIGURE 1. Left: a degree 1 invariant caterpillar gap of σ_3 with periodic edge $\overline{0\frac{1}{2}}$ and critical edge $\overline{0\frac{1}{3}}$. Right: a quadratic invariant gap with periodic major $\overline{0\frac{1}{2}}$ and iterated pullbacks of the above caterpillar gap attached to its edges. The union of all these gaps is a degree 2 invariant caterpillar gap.

Consider now claim (a) stated in the end of the lemma. As we observed in both cases some gaps from the orbit of G are critical, and connect either periodic and non-periodic point, or two non-periodic points. In either case it is well-known (and follows from properties of laminations) that an infinite gap H from the orbit of G that has a critical edge ℓ must have another gap H' attached to H at ℓ; then ℓ is isolated which proves claim (a) and (b) in the end of the lemma. □

Gaps in case (1) are called *Siegel gaps* and, in case (2), they are called *caterpillar gaps*. If a critical edge ℓ of a gap G from Lemma 2.17 has a σ_d^k-fixed point, then there is a countable concatenation of edges of G consisting of ℓ and its consecutive σ_d^k-pullbacks. The name "caterpillar" refers to these countable concatenations of edges. Observe that this phenomenon (having a countable concatenation of leaves that begins with a critical leaf with a periodic endpoint) is not confined to gaps of degree one. For example, a gap of degree greater than one may have, say, a periodic concatenation like that mapping onto itself under an appropriate iterate of the map. In these cases we will still refer to such gaps as *caterpillar gaps*.

A critical edge ℓ of a Fatou gap U is isolated, and there is a gap U' on the other side of ℓ with $\sigma_d(U) = \sigma_d(U')$. Since cycles of Siegel (caterpillar) gaps include gaps with critical edges, then, in a lamination, such gaps share edges with other infinite gaps attached on the opposite side.

Lemma 2.18. *Suppose that $\mathcal{L}_i \to \mathcal{L}$ are σ_d-invariant laminations, and let G be a periodic gap-leaf of $\mathcal{L}$. Then G is also a gap-leaf of $\mathcal{L}_i$ for all sufficiently large i.*

Proof. Let ℓ be a k-periodic edge of G; $\mathcal{L}_i$, for large i, must have a gap-leaf G_i with $G_i \to G$. Choose an edge ℓ_i of G_i so that $\ell_i \to \ell$. Then ℓ_i does not cross ℓ for large i as otherwise the leaves $\sigma_d^k(\ell_i)$ and ℓ_i cross. Moreover, ℓ_i is disjoint from the interior of G for large i as otherwise $\sigma_d^k(\ell_i)$ intersect the interior of G_i (note that ℓ_i is repelled away from ℓ by σ_d^k). By way of contradiction assume that $\mathcal{L}_i$ do not contain G. Then $G_i \supsetneqq G$ and $\ell_i \neq \ell$ for at least one edge ℓ of G. It follows that $\sigma_d^k(G_i) \supsetneqq G_i$, a contradiction. □

2.4. **Minimal laminations.** A lamination $\mathcal{L}$ is *clean* [67] if any pair of distinct non-disjoint leaves of $\mathcal{L}$ is on the boundary of a finite gap. Clean laminations give rise to equivalence relations: $a \sim_{\mathcal{L}} b$ if $a = b$ or a, b are in the same gap-leaf of $\mathcal{L}$. If $\mathcal{L}$ is clean and σ_d-invariant, then the quotient $\mathbb{S}/\sim_{\mathcal{L}} = J_{\mathcal{L}}$ is called a *topological Julia set* and the map $f_{\mathcal{L}} : J_{\mathcal{L}} \to J_{\mathcal{L}}$, induced by σ_d, is called a *topological polynomial*. By [15, Lemma 3.16], a clean lamination has the following property: if one endpoint of a leaf is periodic, then the other endpoint is also periodic with the same period. It is easy to see that clean laminations $\mathcal{L}$ are the same as q-laminations $\mathcal{L}_\sim$ associated with laminational equivalence relations $\sim=\sim_{\mathcal{L}}$.

Definition 2.19 (Minimal laminations)**.** A minimal, by inclusion, nonempty lamination is called a *minimal lamination*. A *minimal sublamination of* $\mathcal{L} \neq \mathcal{L}_\varnothing$ is a sublamination of $\mathcal{L}$ that is minimal (and hence nonempty).

Consider two examples of quadratic minimal laminations. First, take the lamination $\mathcal{L}_P$ associated with a polynomial $P(z) = e^{2\pi i\alpha}z + z^2$ with $\alpha \in \mathbb{Q}$. Then J_P is locally connected, and $\mathcal{L}_P$ looks as explained in Subsection 2.1, with an invariant gap-leaf representing the fixed parabolic point 0. Moreover, $\mathcal{L}_P$ is minimal as it consists of the grand orbit of one leaf (namely, of any edge of the nondegenerate invariant gap-leaf of P).

As another quadratic example, take a non-horizontal σ_2-critical leaf $\overline{xy}$, where $\sigma_2^N(x) = \sigma_2^N(y) = a_0$ is the unique σ_2-fixed point of $\mathbb{S}$ (i.e., the point with argument 0) for some positive integer N, consider iterated pullbacks of $\overline{xy}$ compatible with $\overline{xy}$, and close this set of chords to obtain a lamination $\mathcal{L}$ (the construction of a *pullback lamination* is due to Thurston [67]).

Assume that there exists a nonempty $\mathcal{L}' \subsetneqq \mathcal{L}$; then $\overline{xy}$ is contained in a gap G of $\mathcal{L}'$ as otherwise $\mathcal{L}'$ contains all pullbacks of $\overline{xy}$ and the closure of their union, i.e. $\mathcal{L}' = \mathcal{L}$, a contradiction. The gap G cannot contain a_0 as then it must be an invariant gap of $\mathcal{L}'$ that maps onto itself two-to-one which implies that $G = \overline{\mathbb{D}}$ and that $\mathcal{L}'$ is degenerate, a contradiction.

Consider $\sigma_2^N(G)$. Observe that $\sigma_2^N(x) = a_0 \in \sigma_2^N(G) \cap \mathbb{S}$. Also, $\sigma_2(G)$ is not a point, and since no further image of $\sigma_2(G)$ equals $\overline{xy}$ (recall that $\overline{xy}$ is inside a gap G of $\mathcal{L}'$ and, hence, is not a leaf of $\mathcal{L}'$) then $\sigma_2^N(G)$ is not a point, and $a_0 \in \sigma_2^N(G)$. Since $\overline{xy}$ is a leaf of $\mathcal{L}$, all points of $\sigma_2^N(G) \cap \mathbb{S}$ have orbits contained in the half-circle S_0 with endpoints x, y containing a_0, and the only such point is a_0 itself, a contradiction. Hence $\mathcal{L}$ is minimal.

We claim that $\mathcal{L}$ is perfect. Indeed, repeatedly pulling $\overline{xy}$ back towards a_0 one can find a leaf ℓ of $\mathcal{L}$ that is arbitrarily close to a_0 and separates a_0 from $\overline{xy}$. Since $\sigma_2^N(\overline{xy}) = a_0$, we can pull ℓ back N steps along the backward orbit of a_0 that leads to $\overline{xy}$. In this way, one obtains two leaves with the same images enclosing $\overline{xy}$ in a narrow strip. Thus, $\overline{xy}$ is not isolated in $\mathcal{L}$. Hence pullbacks of $\overline{xy}$ are not isolated in $\mathcal{L}$ either. Since by definition any leaf of $\mathcal{L}$ is either a pullback of $\overline{xy}$ or a limit of such pullbacks, $\mathcal{L}$ is perfect.

A maximal, by inclusion, perfect sublamination $\mathcal{L}^p$ of $\mathcal{L}$ is called the *perfect part* of $\mathcal{L}$; it is the set of all leaves $\ell \in \mathcal{L}$ such that, arbitrarily close to ℓ, there are uncountably many leaves of $\mathcal{L}$. Perfect laminations are clean.

Lemma 2.20. *If $\mathcal{L}$ is an invariant lamination, then so it $\mathcal{L}^p$. If $\mathcal{L}$ is uncountable, then $\mathcal{L}^p \subset \mathcal{L}$ is nonempty. A minimal lamination is either perfect or countable. In the latter case all its nondegenerate leaves are isolated.*

Proof. By [20, Lemma 3.12], the set $\mathcal{L}^p$ is an invariant lamination. If $\mathcal{L}$ is uncountable, then it is easy to see that $\mathcal{L}^p \subset \mathcal{L}$ is nonempty. The last claim holds by part (2) of Lemma 2.6. □

Lemma 2.21. *If $\mathcal{L}$ is nonempty, then $\mathcal{L}$ contains a minimal lamination.*

Proof. Let $\mathcal{L}_\alpha$ be a nested family of laminations. Definition 2.5 implies that then $\bigcap \mathcal{L}_\alpha$ is a sibling invariant lamination too. If all $\mathcal{L}_\alpha$ are nonempty, then by Lemma 2.13 each of them has a leaf of length at least $\frac{1}{d+1}$ and so $\bigcap \mathcal{L}_\alpha$ is nonempty. Now the desired statement follows from Zorn's lemma. □

For $\mathcal{L}$ and a nondegenerate leaf $\ell \in \mathcal{L}$, let $\mathcal{G}(\ell) \subset \mathcal{L}$ be the set of all iterated pullbacks of ℓ and of all its nondegenerate iterated images. Lemma 2.20 and compactness of invariant laminations imply Lemma 2.22.

Lemma 2.22. *Let $\mathcal{L}$ be a minimal lamination. If $\ell \in \mathcal{L}$ is a nondegenerate leaf, then all nondegenerate leaves of $\mathcal{L}$ are in the closure of $\mathcal{G}(\ell)$. In particular, if $\mathcal{L}$ is minimal and countable, then for any nondegenerate leaf ℓ of $\mathcal{L}$ we have that $\mathcal{G}(\ell) = \mathcal{L}$.*

2.5. **Invariant objects.** Let $\Delta \subset \mathbb{C}$ be an open Jordan disk. Recall [44, Definition 3.6] that a continuous map $f : \overline{\Delta} \to \mathbb{C}$ is *weakly polynomial-like* (*weakly PL* for short) of degree d if $f(\mathrm{Bd}(\Delta)) \cap \Delta = \varnothing$, and the induced map on integer homology

$$f_* : H_2(\overline{\Delta}, \mathrm{Bd}(\Delta)) \cong \mathbb{Z} \ \to \ H_2(\mathbb{C}, \mathbb{C} \setminus \{z_0\}) \cong \mathbb{Z}$$

is the multiplication by d, where z_0 is any base point in Δ.

Lemma 2.23 (Lemma 3.7 of [44]). *If $f : \overline{\mathbb{D}} \to \mathbb{C}$ is weakly PL with isolated fixed points, then the degree of f equals the sum of the Lefschetz indices over all fixed points of f in $\overline{\mathbb{D}}$.*

The notion of the *Lefschetz index*, adapted to the present context, is discussed in [44]. A gap G of a lamination is *invariant* if $\sigma_d(G) = G$ (with "$=$" rather than "$\subset$").

Lemma 2.24. *A σ_d-invariant lamination has an invariant gap-leaf or an invariant infinite gap.*

Proof. Consider a σ_d-invariant lamination $\mathcal{L}$. Extend σ_d linearly over each simplex in the barycentric subdivision of $\mathcal{L}$ (cf. [67]), and denote the extended map by $f : \overline{\mathbb{D}} \to \mathbb{C}$; f is weakly PL. If there are invariant nondegenerate leaves of $\mathcal{L}$ or invariant gaps with fixed points on the boundary, we are done. Otherwise, f has isolated fixed points, and there are $d-1$ fixed points on $\mathbb{S}$, all with Lefschetz index one. Thus, there is a fixed point a inside $\mathbb{D}$. If G is a gap of $\mathcal{L}$ containing a, then G is clearly invariant. □

3. Invariant gaps, their canonical laminations, and flower-like sets

By *cubic* (resp., *quadratic*) laminations, we always mean *sibling* σ_3-(resp., σ_2-) *invariant* laminations. For brevity we denote the horizontal diameter of the unit disk by $\overline{\mathrm{Hdi}}$. ***From now on $\mathcal{L}$ (possibly with sub- and superscripts) denotes a cubic sibling invariant (pre)lamination.*** This section is based upon [15] but contains some further developments, too.

3.1. Invariant gaps.

An *invariant gap* is an invariant gap of a cubic lamination (the latter may be unspecified). Various types of infinite invariant gaps are described in [15].

3.1.1. *Quadratic invariant gaps.* An infinite invariant gap is *quadratic* if it has degree 2. A critical chord $\overline{c}$ gives rise to an open circle arc $L(\overline{c})$ of length $2/3$ with the same endpoints as $\overline{c}$; let $I(\overline{c})$ be the complement of $\overline{L(\overline{c})}$. Clearly, the set $\Pi(\overline{c})$ of all points with orbits in $\overline{L(\overline{c})}$ is nonempty, closed and forward invariant. Let $\Pi'(\overline{c})$ be the maximal perfect subset of $\Pi(\overline{c})$.

Lemma 3.1 (Lemmas 3.3, 3.6, and 3.9 of [15])**.** *The convex hulls $G(\overline{c})$ of $\Pi(\overline{c})$ and $G'(\overline{c})$ of $\Pi'(\overline{c})$ are invariant quadratic gaps, and any invariant quadratic gap is like that. If $\overline{c}$ has non-periodic endpoints, or both its endpoints eventually map to $I(\overline{c})$, then $\Pi(\overline{c}) = \Pi'(\overline{c})$ and $G(\overline{c}) = G'(\overline{c})$.*

Moreover, by [15, Subsection 3.1, Lemmas 3.3, 3.6, 3.9], quadratic invariant gaps can be classified as in Definition 3.2.

Definition 3.2. If the σ_3-orbit of $\sigma_3(\overline{c})$ is contained in $L(\overline{c})$, then $\overline{c}$ is an edge of $G(\overline{c})$, and $G(\overline{c})$ is said to be of *regular critical type*. If an endpoint of $\overline{c}$ is periodic, and the orbit of $\overline{c}$ is contained in $\overline{L(\overline{c})}$, then $\Pi'(\overline{c}) \subsetneqq \Pi(\overline{c})$, the gap $G(\overline{c})$ is said to be of *caterpillar* type, and $G'(\overline{c})$ is said to be of *periodic* type.

Quadratic invariant gaps give rise to *canonical* laminations [15, Lemmas 3.11, 3.12 and 3.13]. Defining canonical laminations is easy for quadratic invariant gaps of regular critical and caterpillar type as in those cases (i.e., if the major of the quadratic invariant gap U is critical) there is a *unique* lamination $\mathcal{L}_U$ that has the gap U. Indeed, under the given assumptions, we already have two critical sets of $\mathcal{L}_U$, namely U and its (critical) major, therefore, all iterated pullbacks of U that are gaps of $\mathcal{L}_U$ are well-defined; these pullbacks of U "tile" the entire $\overline{\mathbb{D}}$ and give rise to $\mathcal{L}_U$.

For a gap U of periodic type, $\mathcal{L}_U$ is as follows. Add to U a critical quadrilateral $Q_U = Q$, which is the convex hull of the major $M_U = M$ of U and its sibling $M'_U = M'$ located outside of U. Form the pullback lamination with critical sets U and Q; this lamination is now well defined. After removing the edges ℓ and ℓ' of Q distinct from M, M', and all iterated pullbacks of ℓ and ℓ', we obtain the canonical lamination $\mathcal{L}_U$ which, unlike before, has *two* cycles of Fatou gaps. Indeed, $\mathcal{L}_U$ has a gap $V \supset Q$ with edges M and M', and $\sigma_3|_{\mathrm{Bd}(V)}$ is two-to-one. The entire $\overline{\mathbb{D}}$ is "tiled" by concatenated pullbacks of U and V. In this context, U is said to be the *senior (gap)*, and V is called the *vassal (gap)*, cf. [15, Lemma 3.4].

3.1.2. *Invariant gaps of degree one.* Finite rotational sets (under the name of fixed point portraits) are classified in [44]; in [15, Subsection 4.1], we specify the picture for the cubic case. Infinite invariant gaps of degree 1 are studied in [15, Lemma 4.6]. Lemma 3.3 follows from these results.

Lemma 3.3. *A degree one invariant gap G of a cubic lamination has one or two majors; every edge of G eventually maps to a major and, if G is infinite, at least one of its majors is critical.*

An invariant gap G is *rotational* if σ_3 acts on $G \cap \mathbb{S}$ as a combinatorial rotation different from the identity. A chord is *compatible* with a finite collection of gaps if it does not cross edges of these gaps.

3.2. **Flower-like sets.** Let us introduce the following useful concept.

Definition 3.4 (Flower-like sets)**.** Suppose that $\mathcal{L}$ has an infinite invariant gap U or an invariant gap-leaf G and an infinite gap U that shares an edge with G (in the latter case, it follows that U is periodic). Then $\{U\}$ (in the former case) or the set consisting of G and all periodic Fatou gaps attached to it (in the latter case) is said to be a *flower-like set*. Thus, a flower-like set is a certain set of gaps or gap-leaves. Flower-like sets can be viewed as standing alone (i.e., without specifying a lamination but with the understanding that such a lamination exists).

By saying that $\mathcal{L}$ *has* a flower-like set F, we mean that all gaps/gap-leaves from F are gaps/gap-leaves of $\mathcal{L}$. Say that $\mathcal{L}$ is *compatible* with a flower-like set F if no leaf of $\mathcal{L}$ crosses an edge of a gap from F. Flower-like sets represent dynamics of polynomials with a non-repelling fixed point.

Lemma 3.5. *If sets $F_i, i = 1, 2, \dots$ are flower-like, and $F_i \to F$, then F contains a flower-like set.*

Proof. If periodic gaps/gap-leaves H_i of period m converge to a gap/gap-leaf H, then H is periodic of period m. If all H_i's are infinite, then, by Lemma 2.13, for any integer $N > 0$, there is $\varepsilon > 0$ such that N edges of H_i are longer than ε for any i. Choosing a subsequence, we see that H has at least N nondegenerate edges, and H is an infinite periodic gap.

Thus, we may assume that each F_i consists of a finite invariant gap-leaf G_i and the cycle of an infinite gap U_i sharing an edge ℓ_i with G_i, and that G_i's converge to an invariant gap G. If G is infinite, we are done. Assume that G is finite. Then $G_i = G$ for large i (by Lemma 2.18), and by the above $U_i \to U$ where U is infinite and periodic, which completes the proof. □

4. Two types of cubic minimal laminations

We classify cubic minimal laminations into central and non-central ones.

Definition 4.1. A minimal lamination is *central* if it is compatible with a flower-like set; a minimal lamination is *non-central* otherwise.

If $A \subset \mathbb{S}$ is a closed set and $a, b \in A$ then a chord $\overline{ab}$ of $\mathbb{S}$ is called a *diagonal* of the convex hull $\mathrm{CH}(A)$ if intersects the interior of $\mathrm{CH}(A)$.

Lemma 4.2. *Let $\mathcal{L}$ be a cubic countable minimal lamination. Then:*

(1) *all nondegenerate leaves of $\mathcal{L}$ are isolated;*
(2) *for any nondegenerate leaf $\ell \in \mathcal{L}$, the set of all nondegenerate leaves in $\mathcal{L}$ coincides with $\mathcal{G}(\ell)$;*
(3) *the lamination $\mathcal{L}$ has a flower-like set.*

Proof. (1) This claim follows from Lemma 2.20.

(2) By Lemma 2.22, the set $\mathcal{G}(\ell)$ is dense in $\mathcal{L}$. Since each leaf is of $\mathcal{L}$ is isolated, then $\mathcal{G}(\ell)$ equals the set of all nondegenerate leaves in $\mathcal{L}$.

(3) By Lemma 2.24, find an invariant gap-leaf or infinite gap G of $\mathcal{L}$. If G is infinite, then by definition $\mathcal{L}$ has a flower-like set. Assume that G is finite. Let ℓ be an edge of G; it is isolated by (1). Let H be a gap of $\mathcal{L}$ attached to G along ℓ. If H is infinite, then, again by definition, $\mathcal{L}$ has a flower-like set. Assume that H is finite. If n is the period of ℓ, then there are two cases: $\sigma_3^n(H) = H$ and $\sigma_3^n(H) = \ell$. The former case contradicts (2), hence we may assume that $H \cap G = \ell$ is a leaf and $\sigma_3(H) = \sigma_3(\ell)$, i.e. that H is a collapsing gap that shares an edge with G.

Then there are several cases: a) the gap H is a collapsing quadrilateral with a sibling gap of G attached to H at the edge of H that is a sibling of ℓ; b) the gap H is a collapsing hexagon with two more edges ℓ_1 and ℓ_2 so that ℓ, ℓ_1 and ℓ_2 are pairwise disjoint and at ℓ_1 and ℓ_2 sibling gaps of G are attached to H (the fact that ℓ, ℓ_1 and ℓ_2 are pairwise disjoint uniquely defines ℓ_1 and ℓ_2 since ℓ is given); c) a modified version of the second case is possible when H is a hexagon from the second case but there is also a diagonal of this H that creates two collapsing quadrilaterals inside H and is the only common edge of these two quadrilaterals. Consider case a); then there are edges $\overline{q}_1$ and $\overline{q}_2$ of H that share endpoints with ℓ. They are isolated in $\mathcal{L}$. It follows by Definition 2.5 of a sibling invariant lamination that we can remove these leaves and their pullbacks and still have a smaller sibling invariant lamination than $\mathcal{L}$, a contradiction with the fact that $\mathcal{L}$ is a minimal lamination. Similarly in cases b) and c) the lamination $\mathcal{L}$ is minimal. □

There are also uncountable (hence, by Lemma 2.20, perfect) minimal laminations compatible with flower-like sets, but not having them. Here is a heuristic example. Take a perfect non-renormalizable quadratic lamination $\mathcal{L}_2$ with critical diameter ℓ. Choose a non-periodic non-precritical leaf $\overline{xy} \in \mathcal{L}_2$ and blow up y to create a regular σ_3-critical major $\overline{y}$ which results into a triangle contained in a quadratic invariant gap U of σ_3, then reflect this triangle with respect to $\overline{y}$ and erase $\overline{y}$ to create a critical quadrilateral Q. Together with the leaf $\hat{\ell}$ that used to be ℓ before the transformations, we have two critical sets Q and $\hat{\ell}$ that define a cubic lamination $\mathcal{L}_3$. We claim that this is a perfect minimal lamination. Indeed, suppose that there is a nonempty lamination $\widehat{\mathcal{L}} \subsetneqq \mathcal{L}_3$. If $\hat{\ell} \notin \widehat{\mathcal{L}}$, then $\hat{\ell}$ is inside a gap G of $\widehat{\mathcal{L}}$. By the assumptions pullbacks of $\hat{\ell}$ in $\mathcal{L}_3$ approach $\hat{\ell}$ from both sides; hence G cannot be finite and so G must be infinite, a contradiction with the fact that $\mathcal{L}$ is non-renormalizable. Since pullbacks of $\hat{\ell}$ approximate $\overline{xy}$, the set Q survives, too, and in the end $\mathcal{L}_3 = \widehat{\mathcal{L}}$, a contradiction. By construction, $\mathcal{L}_3$ is compatible with U while edges of U are not leaves of $\mathcal{L}_3$.

Lemma 4.3. *A non-central minimal lamination $\mathcal{L}$ is perfect and has infinitely many periodic gap-leaves. Given any periodic leaf of $\mathcal{L}$, the family of all iterated pullbacks of it is dense in $\mathcal{L}$.*

Proof. By Lemma 2.20, a minimal lamination is countable or perfect. By Lemma 4.2 and since $\mathcal{L}$ is non-central, $\mathcal{L}$ is perfect, and, hence, clean. Consider the topological polynomial $f_{\mathcal{L}} : J_{\mathcal{L}} \to J_{\mathcal{L}}$ to which σ_3 is semiconjugate by a map φ. Since $\mathcal{L}$ is perfect, there are uncountably many grand orbits of nondegenerate leaves of $\mathcal{L}$ containing no leaves of critical sets of $\mathcal{L}$. If ℓ is a leaf from such grand orbit, then $\varphi(\ell) = x$ is a cutpoint of $J_{\mathcal{L}}$, and all points of the $f_{\mathcal{L}}$-orbit of x are cutpoints of $J_{\mathcal{L}}$. Such dynamics was studied in [13] where, in Theorem 3.8, it was proven that $f_{\mathcal{L}}$ has infinitely many periodic cutpoints. Taking their φ-preimages, we see that $\mathcal{L}$ has infinitely many periodic gap-leaves. The last claim holds by Lemma 2.22. □

Recall several notions from [10]. Two leaves with a common endpoint v and the same image which is a leaf (and not a point) are said to form a *critical wedge* (the point v then is said to be its *vertex*). An invariant lamination is *proper* if it has neither a critical leaf with a periodic endpoint nor a critical wedge with a periodic vertex. A laminational equivalence relation is *proper* if its lamination is proper.

By definition and by Lemma 2.17, a proper lamination has no caterpillar gaps. Recall that the equivalence relation $\approx_{\mathcal{L}}$ associated with a lamination $\mathcal{L}$ is constructed by declaring points connected with a chain of leaves equivalent, and then closing this equivalence relation (see Definition 2.12). Recall also that q-laminations are introduced in Definition 2.2.

Theorem 4.4 (Theorem 4.9 of [10]). *Let $\mathcal{L}$ be a proper invariant lamination. Then $\approx_{\mathcal{L}}$ is an invariant laminational equivalence relation (so that all $\approx_{\mathcal{L}}$-classes are finite). In particular, a concatenation of leaves of a proper invariant lamination cannot connect a periodic and a non-periodic point. Conversely, if $\mathcal{L}$ is a q-lamination, then it is proper.*

Observe that, normally, an invariant laminational equivalence relation $\approx$ is visualized as the union of edges of convex hulls of $\approx$-classes. However, in the setting of Theorem 4.4, this is not necessarily the case. For example, a proper lamination $\mathcal{L}$ may include convex hulls of $\approx_{\mathcal{L}}$ classes *and* some diagonals of those convex hulls.

Let us now consider central minimal perfect laminations. Observe that perfect laminations are proper (it is easy to see that if a lamination is not proper, then it must have isolated leaves). Thus, perfect laminations have no caterpillar gaps. It is also clear that they have no infinite gaps with critical edges (in particular they have no Siegel gaps).

Lemma 4.5. *If a perfect minimal lamination $\mathcal{L}^{min}$ is central, then there exists a flower-like set E without caterpillar or Siegel gaps compatible with $\mathcal{L}^{min}$. Moreover, there exists a minimal proper lamination compatible with $\mathcal{L}^{min}$ that has the flower-like set E and has no Siegel or caterpillar gaps.*

Proof. By definition $\mathcal{L}^{min}$ is compatible with a flower-like set F. Consider cases. Let F be an invariant gap U. Since $\mathcal{L}^{min}$ is perfect, the existence of a diagonal of U that is a leaf of $\mathcal{L}^{min}$ implies the existence of uncountably many such diagonals. If U is a Siegel gap, then eventual images of these diagonals will cross one another, a contradiction. On the other hand, if U

is a caterpillar gap of degree one, then it has only countably many vertices, a contradiction. Hence an infinite invariant gap U of degree one is always contained in an infinite invariant gap V of $\mathcal{L}^{min}$. By the remark before the lemma, V is a quadratic non-caterpillar gap. We can set $E = \{V\}$.

Let F be a flower-like set F but not an invariant infinite gap. It suffices to consider the case when F involves periodic Fatou gaps whose boundaries include maximal (countable) concatenations of edges. A concatenation A like that always connects two points a and b. We claim that then $\overline{ab}$ is compatible with $\mathcal{L}^{min}$. Indeed, otherwise there must exist a leaf ℓ of $\mathcal{L}^{min}$ that crosses $\overline{ab}$. However, this contradicts the assumption that $\mathcal{L}^{min}$ is perfect. Now, each such concatenation A of edges can be replaced with the corresponding chord $\overline{ab}$. In this way, one obtains a new flower-like set E with no nontrivial concatenations of edges, which implies that E is a flower-like set that does not include caterpillar gaps.

Consider all pullbacks of E consistent with its critical sets and the critical sets of $\mathcal{L}^{min}$. Since E is forward invariant (in fact, it maps onto itself under σ_3), the closure $\mathcal{L}_E$ of this system of pullbacks is an invariant lamination, and, by construction, it is compatible with $\mathcal{L}^{min}$ (the details about the pullback laminations can be found in [67]). Moreover, since the construction already uses the critical sets of $\mathcal{L}^{min}$ and critical sets contained in E, it cannot give rise to critical leaves with periodic endpoints or critical wedges with periodic endpoints (if a lamination is not proper, then it has isolated leaves). Thus, $\mathcal{L}_E$ is proper. Moreover, it has no Siegel or caterpillar gaps as these would have isolated edges. It remains to take a minimal sublamination $\widehat{\mathcal{L}}$ of $\mathcal{L}_E$. Evidently, $\widehat{\mathcal{L}}$ will contain a flower-like set, has no caterpillar or Siegel gaps, and is compatible with the lamination $\mathcal{L}^{min}$. □

The next lemma deals with unions of laminations.

Lemma 4.6. *Let $\sim_1$ and $\sim_2$ be σ_3-invariant laminational equivalence relations with associated laminations $\mathcal{L}_1$ and $\mathcal{L}_2$ such that $\mathcal{L}_1$ and $\mathcal{L}_2$ are compatible. Then the following holds.*

1. *$\mathcal{L}_1 \cup \mathcal{L}_2 = \mathcal{L}$ is a proper lamination.*
2. *If $\mathcal{L}_1$ is perfect, then $\mathcal{L}$ has no Siegel or caterpillar gaps.*

In particular, $\mathcal{L}_1, \mathcal{L}_2$ and $\mathcal{L}$ belong to the same T-class of laminations.

Proof. Note that $\mathcal{L}_1$ and $\mathcal{L}_2$ are proper laminations. Claim (1) of the lemma follows from the definitions and Theorem 4.4. Indeed, it is clear that $\mathcal{L}_1 \cup \mathcal{L}_2$ has no critical leaves with periodic endpoints. Suppose that $\mathcal{L}_1 \cup \mathcal{L}_2$ has a a critical wedge $\overline{xp} \cup py$ with an n-periodic vertex p. Then at least one of the points x, y is non-periodic, a contradiction with Theorem 4.4.

Consider now claim (2). Neither $\mathcal{L}_1$ nor $\mathcal{L}_2$ has a Siegel gap. Suppose that $\mathcal{L}_1 \cup \mathcal{L}_2$ has a periodic Siegel gap H with a critical leaf $\hat{\ell}$; then H

must be the intersection of a periodic infinite gap U_1 of $\mathcal{L}_1$ and a periodic infinite gap U_2 of $\mathcal{L}_2$. We may assume that an edge ℓ_1 of U_1 is an edge of H and a diagonal of U_2, and that an edge ℓ_2 of U_2 is an edge of H and a diagonal of U_1. Suppose that $\mathcal{L}_1$ is perfect. Since every edge of a Siegel map eventually maps to a critical leaf it follows that an image of U_1 has a critical edge $\ell = \sigma_3^n(\ell_1)$ for some $n \geqslant 0$. However, ℓ is then a critical edge of an infinite gap $\sigma_3^n(U_1)$ of $\mathcal{L}_1$, which implies that ℓ is isolated in $\mathcal{L}_1$, a contradiction. Also, since $\mathcal{L}$ is proper, it has no caterpillar gaps. The last claim of the lemma follows from the definitions. □

Lemma 4.7 deals with perfect minimal laminations with flower-like set.

Lemma 4.7. *A perfect minimal laminations with flower-like set must have a quadratic invariant gap.*

Proof. Suppose that $\mathcal{L}$ is a perfect minimal lamination with a flower-like set F. If F includes an invariant finite gap G and an infinite periodic gap U attached to G then it (and, hence, $\mathcal{L}$) has isolated leaves, a contradiction. Suppose that F includes an invariant gap U; since $\mathcal{L}$ is perfect, U must be quadratic and not caterpillar. □

5. Fibers and alliances

For sets A, B, let $A \vee B$ be the set of all unordered pairs $\{a, b\}$ with $a \in A, b \in B$. Let CCh be the set of all σ_3-critical chords with the natural topology; CCh is homeomorphic to $\mathbb{S}$. Consider cubic critical portraits.

Definition 5.1 (Cubic critical portraits)**.** A (cubic) *critical portrait* is a pair $\{\overline{c}, \overline{y}\} \in \mathrm{CCh} \vee \mathrm{CCh}$ such that $\overline{c}$ and $\overline{y}$ do not cross. Let $\mathrm{CrP} = \mathrm{CrP}_3$ be the space of all cubic critical portraits. For an invariant cubic $\mathcal{L}$, let $\mathrm{CrP}(\mathcal{L})$ be the family of all critical portraits compatible with $\mathcal{L}$; if $\mathcal{K} \in \mathrm{CrP}(\mathcal{L})$, call $\mathcal{K}$ a critical portrait *of* $\mathcal{L}$.

By [68], the set CrP is homeomorphic to the Möbius band [68]. Recall that *distinct* chords *cross* if they have common points in $\mathbb{D}$.

Lemma 5.2 (Lemma 3.53 of [20])**.** *If there is a critical portrait compatible with cubic laminations $\mathcal{L}$ and $\mathcal{L}'$, then any leaf of $\mathcal{L}$ crosses at most countably many leaves of $\mathcal{L}'$, and vice versa.*

We are ready to construct upper-semicontinuous partitions of $\mathcal{C}_3$ and CrP into subsets resulting in two homeomorphic quotient spaces. This gives rise to a model of $\mathcal{C}_3$. The elements of these partitions will be called *fibers*. The collections of laminations associated with such fibers are instrumental in the arguments, but are not used in the Main Theorem. To emphasize that, we call such collections of laminations *alliances* rather than fibers. Recall that, unless stated otherwise, we consider cubic invariant laminations.

Definition 5.3 (Fibers of critical portraits and alliances of laminations)**.**
Below, we define *fibers* of critical portraits and *alliances* of laminations.

(1) The set $\mathrm{CrP}(\mathcal{L})$ of all critical portraits compatible with a non-central minimal lamination $\mathcal{L}$ is called a *non-central fiber of critical portraits (generated by $\mathcal{L}$)*. Critical portraits from non-central fibers of critical portraits are said to be *non-central*.
(2) The set $\mathcal{F}_0$ of all critical portraits compatible with central minimal laminations is called the *central fiber of critical portraits*. Critical portraits from $\mathcal{F}_0$ are called *central*.
(3) All laminations whose minimal laminations are non-central are said to be *non-central*. Non-central laminations with the same minimal lamination form a *non-central alliance of laminations*. The empty lamination and all laminations with central minimal laminations are called *central*. All central laminations form the *central alliance of laminations* $\mathcal{A}_0$.

Lemma 5.4. *If $\mathcal{L}$ is non-central (i.e., non-compatible with a flower-like set) and minimal then, for any minimal lamination $\mathcal{L}'$, we have that* $\mathrm{CrP}(\mathcal{L}) \cap \mathrm{CrP}(\mathcal{L}') \neq \varnothing$ *implies $\mathcal{L}' = \mathcal{L}$.*

Proof. Let $\mathrm{CrP}(\mathcal{L}) \cap \mathrm{CrP}(\mathcal{L}') \neq \varnothing$ for a minimal lamination $\mathcal{L}'$. Since $\mathcal{L}$ is non-central, $\mathcal{L}$ is perfect by Lemma 4.3. Let $\mathcal{K} = \{\overline{c}, \overline{y}\}$ be a critical portrait compatible with $\mathcal{L}$ and $\mathcal{L}'$. Since $\mathcal{L}$ is perfect, $\mathcal{L}$ and $\mathcal{L}'$ are compatible by Lemma 5.2. If $\mathcal{L}'$ has a flower-like set, then $\mathcal{L}$ is compatible with it, a contradiction with $\mathcal{L}$ being non-central. Hence $\mathcal{L}'$ has no flower-like sets and $\mathcal{L}'$ is perfect (by Lemma 4.2). Also, by Lemma 2.24, there is an invariant gap-leaf or infinite gap G' of $\mathcal{L}'$, and G' is a gap-leaf, again because $\mathcal{L}$ is non-central and compatible with G'. If a leaf $\ell \in \mathcal{L}$ is inside G', then, since $\mathcal{L}$ is perfect, other leaves of $\mathcal{L}$ approximate ℓ and cross leaves of $\mathcal{L}'$, a contradiction. Hence $G' \subset G$, where G is an invariant gap-leaf of $\mathcal{L}$ (since $\mathcal{L}$ is non-central, G is finite). Since no leaf of $\mathcal{L}'$ can be inside G, then $G = G'$.

We claim that each edge of G is a limit of leaves of $\mathcal{L}$. By way of contradiction, this claim holds if G is a gap or a "flipping" leaf (whose endpoints map to each other under σ_3), since $\mathcal{L}$ is perfect. Suppose that $G = \overline{\mathrm{Hdi}}$ is isolated in $\mathcal{L}$ from below. Notice that perfect laminations cannot have collapsing quadrilaterals with a periodic edge; hence the only gap of $\mathcal{L}$ that can be attached to G from below is the (infinite) gap FG_b, a contradiction with $\mathcal{L}$ being non-central. Thus, each edge of G is a limit of leaves of $\mathcal{L}$. Similarly, each edge of $G' = G$ is a limit of leaves of $\mathcal{L}'$.

If iterated images of $\overline{c}$ and $\overline{y}$ avoid G, then iterated $\mathcal{L}$-pullbacks of G and iterated $\mathcal{L}'$-pullbacks of G' are the same. Hence $\mathcal{L} = \mathcal{L}'$ since the iterated pullbacks of G are dense in both $\mathcal{L}$ and $\mathcal{L}'$ by Lemma 2.22. Let for some minimal $n \geqslant 0$ the point $\sigma_3^n(\overline{c})$ be a vertex of G. Let C, C' be the critical sets of $\mathcal{L}$, $\mathcal{L}'$, resp., containing $\overline{c}$. By the previous paragraph, the only way this can happen is when $C = C'$ is the appropriate σ_3^n-pullback of $G = G'$ (so, neither C nor C' can be a collapsing quadrilateral because both laminations are perfect). Similarly, either $\overline{y}$ never maps to G or the critical sets of $\mathcal{L}$, $\mathcal{L}'$ containing $\overline{y}$ coincide. Thus, iterated pullbacks of G in $\mathcal{L}$ are the same as iterated pullbacks of G in $\mathcal{L}'$, and $\mathcal{L} = \mathcal{L}'$. □

Theorem 5.5 shows that fibers of critical portraits are pairwise disjoint. Moreover, alliances of laminations are pairwise disjoint, too.

Theorem 5.5. *Let $\mathcal{L} \neq \mathcal{L}'$ be minimal laminations. Then*

$$\mathrm{CrP}(\mathcal{L}) \cap \mathrm{CrP}(\mathcal{L}') = \varnothing$$

unless $\mathcal{L}$ and $\mathcal{L}'$ are central (i.e., a minimal non-central lamination is not compatible with any other minimal lamination). Thus, distinct fibers of critical portraits are disjoint, and distinct alliances of laminations are disjoint. Moreover, any non-central alliance of laminations is a T-class of laminations while the central alliance of laminations is the union of of all T-classes of central laminations.

Proof. Consider two distinct fibers of critical portraits. As they are distinct, one of them is generated by a minimal non-central lamination $\mathcal{L}$. By Lemma 5.4, if the other fiber intersects $\mathrm{CrP}(\mathcal{L})$, then it must coincide with $\mathrm{CrP}(\mathcal{L})$, a contradiction. Thus, the fibers in question are disjoint.

Consider two distinct alliances $\mathcal{A}_1$ and $\mathcal{A}_2$ of laminations. Since they are distinct, we may assume that $\mathcal{A}_1$ is generated by a non-central minimal lamination $\mathcal{L}_1$ and consists of all laminations that tune $\mathcal{L}_1$. Suppose that $\widehat{\mathcal{L}}$ is a lamination that belongs to both $\mathcal{A}_1$ and $\mathcal{A}_2$. Choose a minimal sublamination $\mathcal{L}_2$ of $\widehat{\mathcal{L}}$. Then there is a critical portrait compatible with both $\mathcal{L}_1$ and $\mathcal{L}_2$. By Lemma 5.4, this implies that $\mathcal{L}_1 = \mathcal{L}_2$ and, hence, $\mathcal{A}_1 = \mathcal{A}_2$.

Now we prove that alliances of non-central laminations are the same as and their T-classes. Indeed, let $\mathcal{L}$ be a minimal non-central lamination. Its alliance $\mathcal{A}(\mathcal{L})$ consists of all laminations $\widehat{\mathcal{L}}$ such that $\mathcal{L}$ is a sublamination of $\widehat{\mathcal{L}}$. Hence $\mathcal{A}(\mathcal{L})$ is contained in the T-class of $\mathcal{L}$. On the other hand, if two laminations are tuning related then they have a common critical portrait compatible with them both. Hence in any chain of tuning related laminations the consecutive sets of compatible critical portraits have nonempty intersections. By Lemma 5.4, it follows that any lamination from the T-class of $\mathcal{L}$ comes from $\mathcal{A}(\mathcal{L})$. This proves the last claim of the theorem. □

Recall (see Definition 2.10) that $\mathfrak{Lam}_3$ is the space of all invariant cubic laminations. By Theorem 5.5, the following sets are well defined.

Definition 5.6. For $\mathcal{L} \in \mathfrak{Lam}_3$, let $\mathcal{A}(\mathcal{L})$ be the alliance of laminations to which $\mathcal{L}$ belongs; for a critical portrait $\mathcal{K}$, let $\mathcal{F}(\mathcal{K})$ be the fiber of critical portraits to which $\mathcal{K}$ belongs.

Theorem 5.7 is the key theorem of this section.

Theorem 5.7. *Alliances of laminations form a USC-partition of* $\mathfrak{Lam}_3$. *Fibers of critical portraits form a USC-partition* X_3 *of* CrP. *The map* $\Psi : \mathfrak{Lam}_3 \to X_3$ *that associates to each lamination* $\mathcal{L}$ *the fiber of critical portraits compatible with minimal lamination(s) from* $\mathcal{A}(\mathcal{L})$ *is well defined and continuous. The union of non-central fibers of critical portraits is open and dense in* CrP.

Proof. By Theorem 5.5, alliances of laminations partition $\mathfrak{Lam}_3$ and fibers of critical portraits partition CrP. Clearly, a non-central alliance of laminations is closed in $\mathfrak{Lam}_3$, and a non-central fiber of critical portraits $\mathrm{CrP}(\mathcal{L})$, with $\mathcal{L}$ a non-central minimal lamination, is closed in CrP.

We claim that the central alliance of laminations is closed. Let $\mathcal{L}_i \to \mathcal{L}$ where $\mathcal{L}_i$ are central laminations with central minimal laminations $\mathcal{L}'_i$; by Theorem 2.9 we may assume that $\mathcal{L}'_i$ converge to an invariant lamination $\mathcal{L}' \subset \mathcal{L}$. Choose flower-like sets F'_i compatible with $\mathcal{L}'_i$ for every i. By Lemma 3.5 we may assume that $F'_i \to F'$ with F' containing a flower-like set F' compatible with $\mathcal{L}'$. Hence minimal laminations of $\mathcal{L}'$ are central as desired. This implies that the central alliance of critical portraits is closed.

To show that alliances of laminations form a USC-partition of $\mathfrak{Lam}_3$, let $\mathcal{L}_i \to \mathcal{L}$ and $\mathcal{L}'_i \to \mathcal{L}'$ be two sequences of laminations where $\mathcal{L}_i$ and $\mathcal{L}'_i$ belong to the same non-central alliance of laminations with a minimal lamination $\mathcal{L}''_i$ for every i. Assume that $\mathcal{L}''_i \to \mathcal{L}''$; then $\mathcal{L}'' \subset \mathcal{L} \cap \mathcal{L}'$ and, hence, that $\mathcal{L}$ and $\mathcal{L}'$ have a common minimal lamination, and belong to the same alliance (non-central or central).

To show that fibers of critical portraits form a USC-partition of CrP, let $\mathcal{K}_i \to \mathcal{K}$ and $\mathcal{K}'_i \to \mathcal{K}'$ be two sequences of critical portraits, where $\mathcal{K}_i$ and $\mathcal{K}'_i$ belong to the same non-central fiber of critical portraits with minimal laminations $\mathcal{L}''_i$ compatible with $\mathcal{K}_i$ and $\mathcal{K}'_i$ for every i. Assume that $\mathcal{L}''_i \to \mathcal{L}''$. Then $\mathcal{L}''$ is compatible with both $\mathcal{K}$ and $\mathcal{K}'$, and $\mathcal{K}$ and $\mathcal{K}'$ belong to the same fiber of critical portraits (non-central or central). It is easy to see that the map $\Psi : \mathfrak{Lam}_3 \to X_3$ is well defined and continuous.

We claim that the union $\mathcal{U}$ of non-central fibers of critical portraits is open and dense in CrP. The set $\mathcal{U}$ is open since its complement is the central fiber of critical portraits which is closed. Let $\mathcal{K} = \{\overline{c}, \overline{y}\}$ be a critical portrait such that the orbits of $\sigma_3(\overline{c})$ and $\sigma_3(\overline{y})$ are dense in $\mathbb{S}$. We claim that $\mathcal{K}$ is non-central. Indeed, if it is central, then there is a central minimal lamination $\mathcal{L}$ compatible with $\mathcal{K}$. Let $C \supset \overline{c}$ and $Y \supset \overline{y}$ be the critical sets of $\mathcal{L}$. If C is infinite, then C is a (pre)periodic gap. This follows from Theorem

1.1 of [52], which is stated for polynomials but uses only combinatorial arguments, hence it is true also for invariant laminations. Now, we obtained a contradiction with the density of $\sigma_3(\overline{c})$. Thus, C (and Y) are finite. On the other hand, $\mathcal{L}$ is compatible with a flower-like set F. In particular, there is a cycle of infinite gaps compatible with $\mathcal{L}$. Let G be a gap from this cycle that contains a critical chord. Since endpoints of $\overline{c}$ and $\overline{y}$ cannot be vertices of G, we may assume that $\overline{c} = \overline{xy}$ where x, y belong to distinct components I, J of $\mathbb{S} \setminus G$. Since $\mathcal{L}$ and F are compatible, the finite concatenation of edges of C that connects the endpoints of $\overline{c}$ must pass through an endpoint of, say, I. As $\sigma_3(\overline{c})$ visits I infinitely often, each time the corresponding image of a vertex of C coincides with endpoint of I. The fact that C has finitely many vertices implies now that a vertex of C is preperiodic. Together with the density of the orbit of $\sigma_3(\overline{c})$ this yields a contradiction. □

6. THE MODEL

The connection between abstractly defined laminational equivalence relations and the laminational equivalence relations generated by polynomials is established in the following fundamental result of Kiwi.

Theorem 6.1 (Theorem 1 of [53]). *Let Q be a polynomial of degree d without Cremer or Siegel cycles such that J_Q is connected. Then all $\sim_Q$-classes are finite, and there exists a monotone map $p : J_Q \to \mathbb{S}/\sim_Q$ that semiconjugates $Q|_{J_Q}$ with the induced map $f_{\sim_Q} : \mathbb{S}/\sim_Q \to \mathbb{S}/\sim_Q$; the map p is one-to-one on all (pre)periodic points of Q in J_Q. Moreover, if $\sim$ is a laminational equivalence such that $f_\sim : \mathbb{S}/\sim \to \mathbb{S}/\sim$ has no Siegel gaps, then there exists a polynomial Q such that $\sim = \sim_Q$.*

The next theorem complements Theorem 6.1.

Theorem 6.2 (Theorem 2 and Lemma 37 of [7]). *Suppose that Q is a polynomial of degree d such that J_Q is connected. There exists a monotone map $p : J_Q \to \mathbb{S}/\sim_Q$ that semiconjugates $Q|_{J_Q}$ with the induced map $f_{\sim_Q} : \mathbb{S}/\sim_Q \to \mathbb{S}/\sim_Q$. If Q has a parabolic or attracting periodic Fatou domain, then its boundary is not one $\sim_Q$-class. If $\mathbf{h}$ is a finite periodic $\sim_Q$-class, then the impressions of all angles from $\mathbf{h}$ coincide and equal a periodic repelling or parabolic point of Q.*

Theorem 6.2 is weaker than Theorem 6.1 as it does not claim the finiteness of $\sim_Q$-classes. Still, it can be helpful as it applies to all polynomials with connected Julia sets. By Theorem 6.2, each $\sim_P$-class $\mathbf{h}$ corresponds to the (connected and closed) union of impressions of rays whose arguments are elements of $\mathbf{h}$. We will call this union the *impression* of $\mathbf{h}$.

Now we can move on to describing our approach. A point x is *(pre)repelling* if it eventually maps to a *repelling* periodic point. An unordered pair of rational angles $\{\alpha, \beta\} \subset \mathbb{Q}/\mathbb{Z}$ is *(pre)repelling* if the external rays with arguments α and β land at the same (pre)repelling point. Let Rep_P be the set of all (pre)repelling pairs of angles. Observe that this set (and related to it concepts defined later) can be considered for a polynomial P of any degree if its Julia set is connected.

Definition 6.3. Let $\bowtie_P$ be the equivalence relation on $\mathbb{Q}/\mathbb{Z}$ given by $\alpha \bowtie_P \beta$ if $\{\alpha, \beta\} \in \mathrm{Rep}_P$ or $\alpha = \beta$. Let $\mathcal{L}_P^{rep}$ be the set of all edges of the convex hulls in $\overline{\mathbb{D}}$ of all $\bowtie_P$-classes *and the limits of these edges*. By Theorem 2.9, the set $\mathcal{L}_P^{rep}$ is a lamination (cf. [10]), and it is easy to see that it is clean.

By definition $\bowtie_P$ deals with *repelling* cycles but *not* with parabolic cycles. Yet, $\mathcal{L}_P^{rep}$ may have periodic gap-leaves associated with parabolic points of P because limits of edges of $\bowtie_P$-classes are also leaves of $\mathcal{L}_P^{rep}$.

The lamination $\mathcal{L}_P^{rep}$ is associated with an equivalence relation $\sim_{\mathcal{L}_P^{rep}}$ on $\mathbb{S}$ so that all gap-leaves of $\mathcal{L}_P^{rep}$ are convex hulls of $\sim_{\mathcal{L}_P^{rep}}$-classes. Since $\sim_{\mathcal{L}_P^{rep}}$ is the closure of $\bowtie_P$, in what follows we simply denote it by $\bowtie_P$. Clearly, if $\alpha \bowtie_P \beta$ then $\alpha \sim_P \beta$. Therefore $\sim_P$ tunes $\bowtie_P$ (in other words, $\sim_P$ may add more connections among arguments of external rays compared to $\bowtie_P$). Thus, if a polynomial g is such that $\sim_g = \bowtie_P$, then P tunes g; therefore, if g belongs to a T-class then so does P. We use this observation all the time without additional explanations because it allows us to replace, in a lot of arguments, P and $\sim_P$ by g such that $\sim_g = \bowtie_P$.

Definition 6.4. For $P \in \mathcal{C}_3$, the *fiber of critical portraits $\mathcal{F}_P$ generated by P* is defined as follows.

(1) If $\mathcal{L}_P^{rep}$ is non-central, then, by Theorem 5.5, it has a unique non-central minimal lamination denoted by $\mathcal{L}_P^{min}$; denote by $\mathcal{F}_P$ the fiber $\mathrm{CrP}(\mathcal{L}_P^{min})$ of critical portraits compatible with $\mathcal{L}_P^{min}$.
(2) If $\mathcal{L}_P^{rep}$ is central (e.g., if $\mathcal{L}_P^{rep}$ is trivial), we let $\mathcal{F}_P = \mathcal{F}_0$ be the central fiber of critical portraits (e.g., $\mathcal{F}_0$ serves all polynomials P with empty $\mathcal{L}_P^{rep}$).

By Theorem 5.7, the sets $\mathcal{F}_P$ and $\mathcal{F}_0$ are closed.

Lemma 6.5. *For $P \in \mathcal{C}_3$ with non-central fiber $\mathcal{F}_P$ of critical portraits, $\mathcal{F}_P = \mathrm{CrP}(\mathcal{L}_Q^{rep})$ for a polynomial Q (possibly $Q \neq P$); here $\mathcal{L}_Q^{rep} = \mathcal{L}_P^{min}$.*

Proof. By Lemma 4.3, the lamination $\mathcal{L}_P^{min}$ is perfect. Then $\mathcal{L}_P^{min}$ has no infinite periodic gaps of degree 1 by Lemma 2.17. By Theorem 6.1, there exists a polynomial Q such that $\sim_Q = \sim_{\mathcal{L}_P^{min}}$ and $\mathcal{L}_{\sim_Q} = \mathcal{L}_P^{min}$. By Lemma 4.3, the lamination $\mathcal{L}_P^{min}$ has infinitely many periodic gap-leaves, and, for any such gap-leaf, the iterated pullbacks of its leaves are dense in $\mathcal{L}_P^{min}$. We can choose a periodic gap-leaf like that so that, in terms of Q, it is associated with a repelling periodic point. It follows that $\mathcal{L}^{rep}_Q = \mathcal{L}_{\sim_Q} = \mathcal{L}^{min}_P$. Then, by definition, $\mathcal{F}_P = \mathrm{CrP}(\mathcal{L}) = \mathrm{CrP}(\mathcal{L}^{rep}_Q)$, as desired. □

A point x is *(P-)stable* if its forward orbit is finite and contains no critical points and no non-repelling periodic points. Lemma 6.6 follows from [37] (cf. [44, Lemma B.1]).

Lemma 6.6. [37, 44] *Let $g \in \mathcal{C}_3$ be a polynomial, and z be a g-stable point. If an external ray $R_g(\theta)$ with rational argument θ lands at z, then, for every polynomial $\tilde{g}$ sufficiently close to g, the external ray $R_{\tilde{g}}(\theta)$ lands at a stable point $\tilde{z}$ close to z. Moreover, $\tilde{z}$ depends holomorphically on $\tilde{g}$.*

Suppose that $\{\mathcal{A}_i\}$, $i = 1, 2, \dots$ is a sequence of compacta in the plane, and that $\mathcal{A}$ is another set of compacta. We write $\mathcal{A}_i \hookrightarrow \mathcal{A}$ if the limit of every convergent sequence $C_i \in \mathcal{A}_i$ in the Hausdorff metric is a compactum from $\mathcal{A}$. Note that $\mathcal{A}_i \hookrightarrow \mathcal{A}$ implies $\mathcal{A}_i \hookrightarrow \mathcal{A}'$ for every $\mathcal{A}' \supset \mathcal{A}$.

Lemma 6.7. *Consider a sequence of polynomials $P_i \in \mathcal{C}_3$ converging to a polynomial $P \in \mathcal{C}_3$. If $\mathcal{L}^{rep}_P$ is non-central and $\mathcal{L}^{rep}_{P_i}$ converge to some lamination $\mathcal{L}'$, then $\mathcal{L}^{rep}_P$ and $\mathcal{L}'$ belong to the same non-central alliance of laminations, $\mathcal{L}^{rep}_{P_i}$ are non-central for all sufficiently large i, and $\mathcal{F}_{P_i} \hookrightarrow \mathcal{F}_P$.*

Proof. By Lemma 4.3, the lamination $\mathcal{L}^{min}_P$ is perfect, has infinitely many periodic gap-leaves, and all iterated pullbacks of any periodic leaf of $\mathcal{L}^{min}_P$ are dense in $\mathcal{L}^{min}_P$. Choose a repelling P-periodic point x that is not an eventual image of a critical point of P. Take any iterated P-preimage y of x and write $A_y(P)$ for the set of arguments of all P-external rays landing at y. By Lemma 6.6, there exists N_y such that for every $i > N_y$, there is a point y_i with $A_{y_i}(P_i) = A_y(P)$. It follows that $\mathcal{L}^{min}_P \subset \mathcal{L}'$. This implies the first claim of the lemma which, by Theorem 5.7, implies the other claims. □

Recall: X_3 is the partition of CrP formed by the fibers of critical portraits.

Definition 6.8. Define the map $\eta_3 : \mathcal{C}_3 \to X_3$ by the formula $\eta_3(P) = \mathcal{F}_P$.

We may talk of central or non-central polynomials, as well of central or non-central fibers of polynomials, as is specified in the following definition.

Definition 6.9. A polynomial P is a *non-central polynomial* if $\mathcal{L}^{rep}_P$ is non-central. A polynomial P is *central* if $\mathcal{L}^{rep}_P$ is central. A minimal non-central lamination $\mathcal{L}^{min}$ defines the corresponding *non-central fiber of polynomials* which consists of polynomials f such that $\mathcal{L}^{rep}_f \supset \mathcal{L}^{min}$. The *central fiber of polynomials* is the family of polynomials g such that $\mathcal{L}^{rep}_g$ is trivial or contains a minimal lamination $\mathcal{L}^{min}_g$ compatible with a flower-like set.

Theorem 6.10. *The map η_3 is continuous.*

Proof. Consider a sequence $P_i \to P$ of polynomials, and set $\mathcal{F}_i = \mathcal{F}_{P_i}$. If $\mathcal{F}_i \not\to \mathcal{F}_P$, then by Theorem 5.7 we may assume that $\mathcal{F}_i \hookrightarrow \mathcal{F}' \neq \mathcal{F}_P$ where (1) the limit $\mathcal{F}'$ is a fiber of critical portraits, and (2) all fibers of critical portraits $\mathcal{F}_i$ are non-central, or $\mathcal{F}_i = \mathcal{F}_0$ for every i. By Lemma 6.7, this situation is impossible if $\mathcal{F}_P$ is non-central, or if $\mathcal{F}_i = \mathcal{F}_P = \mathcal{F}_0$. It remains to assume that all fibers of critical portraits $\mathcal{F}_i$ are non-central, $\mathcal{F}'$ is non-central, $\mathcal{F}_P = \mathcal{F}_0$ is central, and bring this to a contradiction. Let $\mathcal{L}'$ be the non-central minimal lamination such that $\mathcal{F}' = \mathrm{CrP}(\mathcal{L}')$.

If $\mathcal{L}^{rep}_{P_i} \to \mathcal{L}$ for a non-central lamination $\mathcal{L}$, then $\mathcal{L}' \subset \mathcal{L}$ by Theorem 5.7. By Lemma 4.3, there are *infinitely* many periodic gap-leaves of $\mathcal{L}'$; since $\mathcal{L}' \subset \mathcal{L}$, it follows from Lemma 2.18 that any such gap-leaf is a gap-leaf of $\mathcal{L}$. Let B_P be the set of vertices of gap-leaves of $\mathcal{L}^{rep}_P$ associated with parabolic points. Choose a periodic gap-leaf G of $\mathcal{L}'$ so that no vertex of G belongs to B_P (this is always possible as B_P is *finite*). By Lemma 2.18, the set G is a gap-leaf of $\mathcal{L}_i$ for sufficiently large i. By Theorem 6.1 and by our assumptions, G is associated with a repelling periodic point, say, y_i of P_i.

Evidently, G is a gap-leaf of $\mathcal{L}^{rep}_P$, too. Indeed, P-external rays corresponding to the vertices of G land at repelling points by the choice of G. By the above, P_i-external rays with the same arguments all land at y_i. If the P-external rays mentioned above do not land at the same point, then by continuity (Lemma 6.6) neither do the corresponding P_i-external rays, a contradiction. So, G is a gap-leaf of $\mathcal{L}^{rep}_P$, too. Let y be the repelling periodic point of P associated with G. By the choice of G, no critical point of P ever maps to y, hence, by Lemma 6.6, all pullbacks of G in $\mathcal{L}^{rep}_P$ eventually become gap-leaves of $\mathcal{L}_i$, and, therefore, of $\mathcal{L}'$. By the properties of non-central minimal laminations listed in Lemma 4.3, it follows that $\mathcal{L}' \subset \mathcal{L}^{rep}_P$. This contradicts the assumption that $\mathcal{F}_P = \mathcal{F}_0$ and completes the proof. □

7. CONNECTEDNESS OF THE FIBERS OF POLYNOMIALS

Let $P \in \mathcal{C}_3$. Recall [51] that $\lambda(P)$ is an equivalence relation on $\mathbb{Q}/\mathbb{Z}$ such that $(\alpha, \beta) \in \lambda(P)$ if and only if the external rays $R_P(\alpha)$, $R_P(\beta)$ land at the same point. Let $\overline{\lambda(P)}$ be the closure of the equivalence relation $\lambda(P)$.

Lemma 7.1 (Lemma 3.9 of [51])**.** *For a polynomial P, the relation $\lambda(P)$ is closed as a subset of $\mathbb{Q} \times \mathbb{Q}$. If $(x, y) \in \overline{\lambda(P)}$ is (pre)periodic then $(x, y) \in \lambda(P)$ and the external rays $R_P(x)$ and $R_P(y)$ form a cut.*

For $f_0 \in \mathrm{Poly}_d$, the *combinatorial renormalization domain* $\mathcal{C}(f_0)$ is defined [48] as $\{f \in \mathrm{Poly}_d \mid \lambda(f) \supset \lambda(f_0)\}$ (i.e., if two angles α and β are $\lambda(f_0)$-equivalent, then they are $\lambda(f)$-equivalent).

Lemma 7.2. *Assume that P, $P_0 \in \mathcal{C}_3$, every nondegenerate leaf of $\mathcal{L}^{rep}_{P_0}$ is in a gap-leaf of $\mathcal{L}^{rep}_P$, and P_0 has no neutral cycles. Then $P \in \mathcal{C}(P_0)$.*

Note that Lemma 7.2 is applicable in the case $\mathcal{L}_P^{rep} \supset \mathcal{L}_{P_0}^{rep}$.

Proof. Since P_0 has no neutral cycles, then $\lambda(P_0) = \sim_{P_0}$ on rational angles. If now α is $\lambda(P_0)$-equivalent to β, then $\alpha \sim_{P_0} \beta$. By the assumptions, if $\alpha \sim_{P_0} \beta$ then $\alpha \sim_P \beta$; by Lemma 7.1, the angle α is $\lambda(P)$-equivalent to β. Thus, $\lambda(P_0) \subset \lambda(P)$ as desired. □

Theorem 7.3 easily follows from much stronger results of Shen–Wang [66], Wang [70], and Kozlovsky–van Strien [55].

Theorem 7.3. *Suppose that $\mathcal{L}_P^{rep}$ is perfect for some $P \in \mathcal{C}_3$. If there exists no infinite sequence $P_n \in \mathcal{C}_3$ such that $\mathcal{C}(P_1) \supsetneq \mathcal{C}(P_2) \supsetneq \dots$ and $\bigcap_n \mathcal{C}(P_n) \supset \mathcal{C}(P)$ then the set $\mathcal{C}(P)$ is connected. In particular, $\mathcal{C}(P)$ is connected provided that $\mathcal{L}_P^{rep}$ is a perfect minimal lamination.*

Proof. By Theorem 6.1, there is $P_0 \in \mathcal{C}_3$ without neutral cycles such that $\mathcal{L}_P^{rep} = \mathcal{L}_{P_0}^{rep}$. By Lemma 7.2, $\mathcal{C}(P) = \mathcal{C}(P_0)$. So, we may assume that P has no neutral cycles. If all periodic points of P are repelling, then $\mathcal{C}(P)$ is a singleton, by [55, Theorems 1.1 and 1.2] and the assumptions. Assume that P has (super)attracting domains. If P is hyperbolic, the result follows from the main result of Shen and Wang [66]. Finally, if one critical point c_1 of P lies in a (super)attracting periodic basin, the other critical point c_2 is in the Julia set, and P has the properties from the statement of the theorem, then $\mathcal{C}(P)$ is also connected by [70, Theorem A]. Note: only the nonrenormalizable case is considered in [70] but the extension to the finitely renormalizable case is straightforward; the connectedness of $\mathcal{C}(P)$ follows from the bijectivity of the straightening map in the same way as in [66]. □

Let $\mathcal{F}$ be a non-central fiber of critical portraits, and consider the corresponding non-central fiber $\eta_3^{-1}(\mathcal{F})$ of polynomials. Lemma 7.4 completes the proof of the fact that non-central fibers of polynomials are connected. To avoid cumbersome notation, set $\mathcal{L}_{\overline{\lambda(P)}} = \mathcal{L}_P^{rat}$.

Lemma 7.4. *If $\mathcal{F}$ is a non-central fiber of critical portraits, then $\eta_3^{-1}(\mathcal{F}) = \mathcal{C}(P)$ for some $P \in \mathcal{C}_3$ without neutral cycles but with infinitely many periodic cuts. Any non-central fiber of polynomials is connected, and if $\phi : \mathcal{C}_3 \to Y$ is T-stable, then the ϕ-image of a non-central fiber is a point.*

Proof. By Lemma 6.5, $\mathcal{F} = \mathrm{CrP}(\mathcal{L}_P^{rep})$ for $P \in \mathcal{C}_3$ without neutral cycles such that $\mathcal{L}_P^{rep}$ is a perfect minimal lamination. By definition, in this case $\mathcal{L}_P^{rep} = \mathcal{L}_P^{rat}$. By Lemma 7.1, every periodic gap-leaf of $\mathcal{L}_P^{rep}$ is associated with a periodic cut in $\lambda(P)$. Since, by Lemma 4.3, the lamination $\mathcal{L}_P^{rep}$ has infinitely many periodic finite gap-leaves, P has infinitely many periodic cuts. As $\eta_3^{-1}(\mathcal{F}) = \mathcal{C}(P)$ by definition, a non-central fiber of polynomials is connected, by Theorem 7.3. The last claim of the lemma is immediate. □

Observe that for $P \in \mathcal{C}_3$ such that $\mathcal{C}(P)$ is a non-central fiber of polynomials, the periodic Fatou domains of P have pairwise disjoint closures.

8. MAIN THEOREM

We have constructed the map $\eta_3 : \mathcal{C}_3 \to X_3$ by the formula $\eta_3(P) = \mathcal{F}_P$. This models $\mathcal{C}_3$ by the quotient space X_3 of the space of all cubic critical portraits constructed without invoking polynomials so that the modeling space is combinatorial. In the rest of the paper, we prove that η_3 solves the Main Problem. Corollary 8.1 follows from Theorem 5.5.

Corollary 8.1. *A non-central fiber of polynomials is a T-class.*

The central fiber of polynomials $\eta_3^{-1}(\mathcal{F}_0)$ is the set of polynomials f such that $\mathcal{L}_f^{rep}$ is either empty or central (i.e., it contains a minimal lamination $\mathcal{L}_f^{min}$ compatible with a flower-like set). To complete the proof of the Main Theorem, we need to show that it equals the union of T-classes of all polynomials with a non-repelling fixed point which equals the union of T-classes of all polynomials with a neutral fixed point. Recall that $\mathcal{L}_f^{rep}$ is introduced in Definition 6.3) while $\mathcal{L}_f$ is introduced in Definition 1.1.

Lemma 8.2. *A polynomial $h \in \mathcal{C}_3$ belongs to the central fiber of polynomials $\eta_3^{-1}(\mathcal{F}_0)$ if and only $\mathcal{L}_h$ is central.*

Recall that the empty lamination is central (see Definition 5.3).

Proof. If $\mathcal{L}_h$ is non-central, then it contains a unique non-central minimal lamination $\widehat{\mathcal{L}}_h$. By Lemma 4.3, $\widehat{\mathcal{L}}_h$ has infinitely periodic gap-leaves and the pullbacks of any periodic leaf of $\widehat{\mathcal{L}}_h$ is dense in $\widehat{\mathcal{L}}_h$. Hence, by Theorem 6.2 there exists a periodic gap-leaf G which is associated with a repelling cut so that $\widehat{\mathcal{L}}_h \subset \mathcal{L}_h^{rep}$. By Theorem 5.5, non-central alliances of laminations are disjoint from the central alliance of laminations. Hence, $\mathcal{L}_h^{rep}$ is non-central.

Let $\mathcal{L}_h^{rep}$ be non-central and $\mathcal{L}_h$ be not empty. Then, by definition, $\mathcal{L}_h \supset \mathcal{L}_h^{rep}$, which by Theorem 5.5 implies that $\mathcal{L}_h$ is non-central. It remains to rule out the case when $\mathcal{L}_h$ is empty. Choose a unique minimal sublamination $\mathcal{L}_h^{min}$ of $\mathcal{L}_h^{rep}$. By Lemma 4.3, we can choose a periodic gap-leaf G in $\mathcal{L}_h^{min}$ with dense pullbacks in $\mathcal{L}_h^{min}$. By definition of $\mathcal{L}_h$ (Definition 1.1) it follows that all gap-leaves of $\mathcal{L}_h$ are associated with separate $\sim_h$-classes ($\sim_h$ is also introduced in Definition 1.1), which implies that $\mathcal{L}_h$ is nonempty. □

The next lemma is immediate and is left to the reader.

Lemma 8.3. *If $\mathcal{L}_f^{rep}$ is empty, then f has a non-repelling fixed point and, therefore, belongs to the union of T-classes of polynomials with a non-repelling fixed point. If $\mathcal{L}_f$ is empty, then f belongs to the union of T-classes of polynomials with a non-repelling fixed point.*

The next lemma complements Lemma 8.3.

Lemma 8.4. *Suppose that both $\mathcal{L}_f^{rep}$ and $\mathcal{L}_f$ are nonempty, and that $\mathcal{L}_f$ is central. Then f belongs to the T-class of some polynomial P such that either P has a non-repelling fixed point or $\mathcal{L}_P$ is minimal, has no Siegel or caterpillar gaps, but has a flower-like set.*

Proof. If $\mathcal{L}_f$ has an invariant Siegel gap, it follows from Theorem 6.2 that f has a non-repelling fixed point, which implies the desired. From now on, we assume that $\mathcal{L}_f$ does not have an invariant Siegel gap.

For each of the periodic Siegel gaps of $\mathcal{L}_f$ (if any), remove the critical edge and all iterated σ_3-pullbacks of it. Since all edges of Siegel gaps are isolated, the set $\widehat{\mathcal{L}}_f$ of remaining leaves is closed. Moreover, $\widehat{\mathcal{L}}_f$ is an invariant lamination, by Definition 2.5. Observe that, since $\mathcal{L}_f^{rep}$ is nonempty, $\mathcal{L}_f$ has (pre)periodic gap-leaves and, therefore, $\widehat{\mathcal{L}}_f$ is nonempty. Moreover, by construction, $\widehat{\mathcal{L}}_f$ has no Siegel gaps.

Choose a minimal sublamination $\widehat{\mathcal{L}}_f^{min}$ of $\widehat{\mathcal{L}}_f$ compatible with a flower-like set F (this is possible because $\mathcal{L}_f$ is central). By construction, $\widehat{\mathcal{L}}_f^{min}$ has no Siegel gaps. Moreover, since $\mathcal{L}_f$ has no caterpillar gaps, neither does $\widehat{\mathcal{L}}_f^{min}$. By Theorem 6.1, there exists a polynomial h such that $\mathcal{L}_h = \widehat{\mathcal{L}}_f^{min}$. By construction, f belongs to the T-class of h. Consider cases.

First suppose that $\mathcal{L}_h = \widehat{\mathcal{L}}_f^{min}$ is countable. Then by Lemma 4.2 it *has* a flower-like set which does not involve Siegel gaps or caterpillar gaps. If $\mathcal{L}_h = \widehat{\mathcal{L}}_f^{min}$ has a quadratic invariant gap, then P has a non-repelling fixed point; otherwise, it has a flower-like set, which consists of a gap-leaf and Fatou non-caterpillar gaps rotating around it. This proves the lemma when $\mathcal{L}_h = \widehat{\mathcal{L}}_f^{min}$ is countable (we can set $P = h$).

Consider the case when $\mathcal{L}_h = \widehat{\mathcal{L}}_f^{min}$ is perfect (minimal laminations are either countable of perfect). Then, by Lemma 4.5, there exists a minimal proper lamination $\mathcal{L}'$ compatible with $\mathcal{L}_h$ that has a flower-like set E' and has no Siegel or caterpillar gaps. By Theorem 6.1, there exists a polynomial P such that $\mathcal{L}_P = \mathcal{L}'$. Next consider the union $\mathcal{L}_h \cup \mathcal{L}' = \widetilde{\mathcal{L}}$. The lamination $\widetilde{\mathcal{L}}$ is proper and has no Siegel gaps, by Lemma 4.6. By Theorem 6.1, there exists a polynomial g such that $\mathcal{L}_g = \widetilde{\mathcal{L}}$. It follows that h, P, and g belong to the same T-class of polynomials. Since P is exactly of the type mentioned in the lemma, this completes the proof. □

Theorem 8.5 describes the central fiber of polynomials.

Theorem 8.5. *The central fiber of polynomials is the union of T-classes of polynomials with a non-repelling fixed point which coincides with the union of T-classes of polynomials with a neutral fixed point.*

Recall that a polynomial $g \in \mathcal{C}_3$ belongs to the central fiber of polynomials if and only if $\mathcal{L}_g$ is central.

Lemma 8.6. *If $P \in \mathcal{C}_3$ has a non-repelling fixed point, then either $\mathcal{L}_P$ is empty or it has a flower-like set so that P belongs to the central fiber of polynomials. If $f \in \mathcal{C}_3$ belongs to the T-class of P then f also belongs to the central fiber of polynomials. Thus, the union of T-classes of polynomials with a non-repelling fixed point is contained in the central fiber of polynomials.*

Proof. By Theorem 6.2, we may assume that P has a Cremer or Siegel fixed point a; assume also that $\mathcal{L}_P$ is not empty. By Theorem 6.2, each $\sim_P$-class $\mathbf{h}$ corresponds to the (connected and closed) union of impressions of rays with arguments from $\mathbf{h}$. We call this union the *impression* of $\mathbf{h}$. Suppose that a is a Cremer fixed point. Consider an angle θ such that $a \in I_P(\theta)$ (recall: $I_P(\theta)$ is the impression of angle θ in the dynamical plane of P) and then all the angles from the $\sim_P$-class $\mathbf{h}$ of θ. Since a is fixed, $\sigma_3(\theta) \in \mathbf{h}$, which implies that $\mathbf{h}$ is a σ_3-invariant $\sim_P$-class. By Theorem 6.2, the convex hull of $\mathbf{h}$ is an infinite gap, i.e. a flower-like set.

Assume that a is a Siegel point. Let B_a be its invariant Siegel domain. Suppose that there is an angle α such that $I_P(\alpha) \supset \mathrm{Bd}(B_a)$ and consider the $\sim_P$-class $\mathbf{h}$ of α. As above, it follows that $\mathbf{h}$ is an infinite invariant $\sim_P$-class, and its convex hull is the desired flower-like set of $\mathcal{L}_P$.

Now, suppose that no impression of an external ray contains $\mathrm{Bd}(B_a)$ and $\mathcal{L}_P$ has no flower-like sets. Choose a finite invariant gap-leaf G of $\mathcal{L}_P$; by Theorem 6.2 the corresponding external rays land on the same point, say, a, and have impressions coinciding with $\{a\}$. Close all rational gap-leaves of $\mathcal{L}_P$ to obtain a lamination $\widehat{\mathcal{L}}$. Let $\mathcal{B}$ be the family of all $\sim_P$-classes with impressions that intersect $\mathrm{Bd}(B_a)$; $\mathcal{B}$ is a closed infinite invariant family of angles, and no two of them are separated by the convex hull of a class from $\widehat{\mathcal{L}}$. Hence $\mathcal{B}$ is a subset of an invariant infinite gap U of $\widehat{\mathcal{L}}$.

The gap U cannot be Siegel or caterpillar as then it will have an isolated in $\widehat{\mathcal{L}}$ critical edge which cannot belong to the edges of rational gap-leaves of $\mathcal{L}_P$. Hence U is a quadratic invariant non-caterpillar gap which is tuned by leaves of $\mathcal{L}_P$. By Lemma 2.23 there must exist an invariant gap-leaf of $\mathcal{L}_P$ inside U or an infinite invariant gap $V \subset U$ of $\mathcal{L}_P$; since the former is impossible by construction, then $\mathcal{L}_P$ has an infinite invariant gap V which completes the proof of the first claim of the lemma.

Now let f belong to the T-class of P. We claim that f belongs to the central fiber of polynomials. Assume that $\mathcal{L}_f$ and $\mathcal{L}_P$ are nonempty. By definition and by Theorem 5.5 $\mathcal{L}_f$ belongs to the same alliance (equivalently, the same T-class of laminations) as $\mathcal{L}_P$; by Theorem 5.5 and by the first claim of the lemma, the alliance of laminations of $\mathcal{L}_P$ is contained in the central alliance of laminations. Hence $\mathcal{L}_f$ is central. By Lemma 8.2, this implies that f belongs to the central fiber of polynomials. □

To deal with converse claim, let $\approx$ be a laminational equivalence relation such that $\mathcal{L}_\approx$ has no Siegel gaps but has a flower-like set. By Theorem 6.1, there are polynomials f with $\approx = \sim_f$ ($\sim_f$ is an equivalence relation among the arguments of external rays defined by the non-disjointness of impressions of these rays). We claim that if $\mathcal{L}_\approx$ is as above and minimal, then f can be chosen to have a neutral fixed point. Let $F_\approx$ be the flower-like set of $\mathcal{L}_\approx$. Its *combinatorial rotation number* is 0 unless $F_\approx$ consists of a gap-leaf G and a cycle (or two cycles) of more than one quadratic Fatou gaps attached to G. In the latter case, the combinatorial rotation number of $F_\approx$ is the same as that of G.

Theorem 8.7. *Let $\approx$ be a laminational equivalence relation such that $\mathcal{L}_\approx$ is minimal, has no Siegel gaps, but* has *a flower-like set $F_\approx$. Set $\mu = e^{2\pi i\rho}$, where ρ is the combinatorial rotation number of $F_\approx$. There exists a cubic polynomial $P(z) = \mu z + bz^2 + z^3$ with $\sim_P = \approx$ and parabolic fixed point 0.*

The following lemma is a special case of Theorem 8.7, assuming that all critical sets of $\approx$ have finite forward orbits.

Lemma 8.8. *Let $\approx$ and μ be as in Theorem 8.7 except that $\mathcal{L}_\approx$ does not have to be minimal, and assume that all critical gaps or leaves of $\mathcal{L}_\approx$ are periodic or preperiodic. Then there is a geometrically finite $f_*(z) = \mu z + bz^2 + z^3$ with $\sim_{f_*} = \approx$.*

A polynomial P is *geometrically finite* if all critical points of P in J_P have finite forward orbits.

Proof. We will use the *pinching deformation* of Guizhen Cui and Tan Lei [31] (instead of [31], one can use a result of Haissinsky [45] that specifically deals with polynomials). Let f be a cubic polynomial such that $\approx = \sim_f$ (such f exists by Theorem 6.1). By a suitable affine conjugacy, arrange that 0 is the fixed point of f corresponding to G (in the sense that the external rays whose arguments represent the vertices of G land on 0). If this point is parabolic for f, then set $f_* = f$, and we are done. Otherwise, the fixed point 0 of f is repelling, and we will make it into a parabolic point via a pinching surgery.

By [31, Theorem 1.3], there exists a continuous one-parameter family $\{\phi_t\}$, $t \in [0, 1)$ of quasiconformal maps $\phi_t : \mathbb{C} \to \mathbb{C}$ with the following properties:

- for each $t \in [0, 1)$, the map ϕ_t conjugates f with a cubic monic polynomial $f_t = \phi_t \circ f \circ \phi_t^{-1}$;
- on $\mathbb{C} \setminus K_{f_t}$, the map ϕ_t is holomorphic; $\phi_t(0) = 0$;

- as $t \to 1$, the polynomial f_t converges uniformly to a cubic polynomial f_* such that 0 is a parabolic fixed point of f_*;
- the maps ϕ_t converge uniformly to a continuous map ϕ;
- restricting ϕ to J_f, we obtain a topological conjugacy between $f|_{J_f}$ and $f_*|_{J_{f_*}}$.

Let $\psi_t : \mathbb{C} \setminus K_{f_t} \to \mathbb{C} \setminus \overline{\mathbb{D}}$ be the Böttcher map, and similarly with $\psi_* : \mathbb{C} \setminus K_{f_*} \to \mathbb{C} \setminus \overline{\mathbb{D}}$. Note that $\phi_t = \psi_t^{-1} \circ \psi_0$ and $\phi = \psi_*^{-1} \circ \psi_0$ on $\mathbb{C} \setminus K_f$. It follows easily that $\phi : \mathbb{C} \setminus K_f \to \mathbb{C} \setminus K_{f_*}$ is a homeomorphism that conjugates f with f_*. Since by the above the restriction of ϕ on J_f topologically conjugates $f|_{J_f}$ and $f_*|_{J_{f_*}}$ then external rays $R_f(\alpha)$ and $R_f(\beta)$ land on a common point if and only if $R_{f_*}(\alpha)$ and $R_{f_*}(\beta)$ land on a common point.

The maps f and f_* are topologically conjugate on their Julia sets; it follows that the parabolic fixed point 0 has the same combinatorial rotation number as the corresponding fixed point a of f. We see that $f_*'(0) = \mu$. A linear change of variables then takes f_* to the form $f_*(z) = \mu z + bz^2 + z^3$ for some $b \in \mathbb{C}$. □

Recall that a cubic polynomial P with *degenerate parabolic* fixed point 0 has two cycles of petals at 0.

Lemma 8.9. *Consider cubic polynomials from $\mathcal{C}_3$. Suppose that P has no parabolic fixed points, or two distinct parabolic fixed points, or one degenerate parabolic point. If α is σ_3-periodic, $P_n \to P$, $R_{P_n}(\alpha)$ lands on a point z_n, and z_n converge to a P-fixed point z, then $R_P(\alpha)$ lands on z.*

Proof. By Lemma 6.6 we may assume that P has parabolic points. Moreover, without loss of generality we may assume that P has parabolic fixed points 0 and z so that either $z \neq 0$, or $z = 0$ is a degenerate parabolic fixed point. Set $R_n(\alpha) = R_{P_n}(\alpha)$. Passing to a subsequence if necessary, assume that there is a limit of $R_n(\alpha) \cup \{z\}$ in the Hausdorff metric associated with the spherical metric on $\mathbb{C} \cup \{\infty\}$. Denote this limit by $R^{\lim}(\alpha)$. Clearly, $R_P(\alpha) \subset R^{\lim}(\alpha)$, but the difference $R^{\lim}(\alpha) \setminus R_P(\alpha)$ may a priori be larger than just the landing point of $R_P(\alpha)$. Theorem A of Petersen–Zakeri [61] gives a description of such limits $R^{\lim}(\alpha)$. As follows from this theorem and the Basic Structure Lemma [61] (note that different critical points of P are in different parabolic Fatou domains), the limit set $R^{\lim}(\alpha)$ is the union of $R_P(\alpha)$ and an at most countable set of loops based on the landing point of $R_P(\alpha)$. Since $z \in R^{\lim}(\alpha)$, then $R_P(\alpha)$ lands on z as claimed. □

Let prove Theorem 8.7 under an additional assumption of "rationality".

Lemma 8.10. *The conclusion of Theorem 8.7 holds under the additional assumption that $F_\approx$ is not a quadratic invariant gap of regular critical type.*

Observe that $F_\approx$ cannot be a caterpillar gap because it is a gap of $\mathcal{L}_\approx$.

Proof. If $\mathcal{L}_\approx$ is countable, then by Lemma 4.2 and by the assumptions $\mathcal{L}_\approx$ satisfies all the conditions of Lemma 8.8 which implies the desired.

If $\mathcal{L}_\approx$ is perfect, then, by Lemma 4.7, $F_\approx = V$ is a quadratic invariant gap with major $M = \overline{uv}$; by the assumptions M is periodic. Set $\mathcal{L} := \mathcal{L}_\approx$. As $\mathcal{L}$ is minimal, by Lemma 2.22 it is the closure of the set of all iterated pullbacks of M. Let $C \neq V$ be the other critical set of $\mathcal{L}$. If C is periodic or preperiodic, then the desired holds by Lemma 8.8. If C is a Fatou gap then C is periodic or preperiodic. Hence we may assume that C is a critical gap-leaf of $\mathcal{L}$ with infinite orbit; since $\mathcal{L}$ is minimal, edges of C can be approximated by iterated pullbacks of M. Thus, $\mathcal{L}$ has only one orbit of Fatou gaps, namely $\{V\}$, and all infinite gaps of $\mathcal{L}$ are pullbacks of V.

Form the sequence of laminations $\mathcal{L}_n$ as follows. First, take all iterated pullbacks M in $\mathcal{L}$ of level at most n. Let $\mathcal{L}'_n$ be the thus obtained forward invariant lamination. Next, take a critical chord c_n compatible with $\mathcal{L}'_n$ and mapping eventually to an endpoint of M. Set $\mathcal{L}''_n = \mathcal{L}'_n \cup \{c_n\}$; this is a forward invariant cubic lamination. There is only one way of taking iterated pullbacks of leaves of $\mathcal{L}''_n$ so that they do not cross the interior of V or the chord $\{c_n\}$. Adding all these iterated pullbacks and their limits to $\mathcal{L}''_n$ yields a cubic invariant lamination $\mathcal{L}'''_n$.

By construction $\mathcal{L}'''_n$ is proper and V is a gap of $\mathcal{L}'''_n$. Define the laminational equivalence relation $\approx_{\mathcal{L}'''_n} = \approx_n$ (see Definition 2.12); by Theorem 4.4 $\approx_n$ is a laminational equivalence relation. Set $\mathcal{L}_n = \mathcal{L}_{\approx_n}$. All critical sets of $\mathcal{L}_n$ are periodic or preperiodic: one critical set is V, and the other one is the finite gap-leaf $C_n \supset c_n$ (since $\approx_n$ a laminational equivalence relation, $\approx_n$-classes are finite, and C_n is the convex hull of such a class). By Lemma 8.8, choose a polynomial $P_n(z) = \mu z + b_n z^2 + z^3$ with $\lambda(P_n) = \approx_n$ (in our case $\mu = 1$). Note that every P_n has a critical point ω_n (associated to C_n) that eventually maps to a periodic point of P_n.

Passing to a subsequence if necessary, assume that P_n converge to a cubic polynomial $P = \mu z + bz^2 + z^3$. Consider cases; set $\mathcal{L}_P := \mathcal{L}_{\sim_P}$

(1) Let 0 be a nondegenerate parabolic point of P. We claim that M is in $\mathcal{L}_P$, i.e. that $R_P(u)$ and $R_P(v)$ land on the same point. Let $R_{P_n}(u)$ and $R_{P_n}(v)$ land on a point x_n (since M is a leaf of each $\mathcal{L}_n$ then these rays land on the same point); let $R_P(u)$ land on a point y while $R_P(v)$ land on a point z. We need to show that $y = z$.

By Lemma 8.9 it suffices to consider the case when P has only one parabolic fixed point, and it is nondegenerate. Then the only parabolic fixed point of P is 0; assume that $z = 0$ while $y \neq 0$ is a repelling periodic point. However if v is not σ_3-fixed this is impossible (a non-fixed periodic ray cannot land on a fixed point with multiplier 1). Thus, we may assume that $v = 0$, hence $u = 1/2$, and, say, V is the quadratic gap with major

$M = \overline{\mathrm{Hdi}}$ located above M. However then by Lemma 8.8 in all polynomials P_n the leaf $M = \overline{\mathrm{Hdi}}$ of $\mathcal{L}_n$ corresponds to the parabolic point 0 which contradicts Lemma 8.9 and the assumptions about the point y.

By construction and definition of $\mathcal{L}_P$ it follows that for every n all level $\leqslant n$ pullbacks of M in $\mathcal{L}_P$ coincide with those of $\mathcal{L}_n$, hence also with those of $\mathcal{L}$. We conclude that $\mathcal{L} \subset \mathcal{L}_P$. If $\mathcal{L}_P$ is minimal, then we are done as $\mathcal{L} = \mathcal{L}_P$ in this case. Suppose $\mathcal{L}_P$ is not minimal, that is, it tunes the lamination $\mathcal{L}$ in a nontrivial way, i.e. leaves of $\mathcal{L}_P$ are diagonals of V. We claim that this is impossible. Indeed, the map on V is two-to-one. Since the fixed point 0 is parabolic and has to have a parabolic domain associated to it, this leaves no room for any tuning of V, a contradiction.

(2) Suppose that the parabolic point 0 of P is degenerate, that is, P has two cycles of parabolic petals. This implies that $\mathcal{L}_{\sim P}$ has a leaf $\overline{\mathrm{Hdi}}$ and two quadratic invariant gaps sharing the same major: the gap V above $\overline{\mathrm{Hdi}}$ and the gap U below $\overline{\mathrm{Hdi}}$. Let $\{U, \overline{\mathrm{Hdi}}, V\} = F_P$; the presence of sets from F_P in a lamination completely defines the lamination. By Lemma 8.9, $G = \overline{\mathrm{Hdi}}$. If $\mathcal{L} \neq \mathcal{L}_P$ then there must exist a distant pullback ℓ of M which is a leaf of $\mathcal{L}_n$ for all large n located close to $\overline{\mathrm{Hdi}}$ and, say, below $\overline{\mathrm{Hdi}}$ (otherwise $\mathcal{L} = \mathcal{L}_P$).

By [8, Theorem 7.5.2] there is an invariant leaf, finite gap, or infinite gap T of $\mathcal{L}$ located below M and disjoint from M. Infinite gap T is impossible as otherwise it is easy to see that T coincides with U [15], a contradiction with the existence of ℓ Hence T is a gap-leaf. Since pullbacks of M separate C and T, hence for large n, T is a gap-leaf of $\mathcal{L}_n$. By Lemma 8.9, rays whose arguments are vertices of T land on the same fixed repelling point of P which contradicts the existence of U in $\mathcal{L}_P$. □

We can now complete the proof of Theorem 8.7.

Proof of Theorem 8.7. It remains only to consider the case when $\mathcal{L} = \mathcal{L}_{\approx}$ is the canonical lamination of a quadratic invariant gap of regular critical type (that is, the only cubic lamination containing this gap). Consider the space $\mathcal{F}_1$ of all cubic polynomials of the form $f_{1,b}(z) = z + bz^2 + z^3$. Let $\mathcal{H}_1$ be the interior component of the connectedness locus in $\mathcal{F}_1$ consisting of polynomials with both critical points in the same parabolic Fatou domain attached to 0. By [71, Theorem A], the boundary of $\mathcal{H}_1$ is a Jordan curve. For every $f \in \mathrm{Bd}(\mathcal{H}_1)$, one critical point $\omega_1(f)$ lies in the immediate basin $B_f(0)$ of 0, and the other critical point $\omega_2(f)$ lies either in the boundary of $B_f(0)$ or in a parabolic basin attached to a boundary point of $B_f(0)$. This follows from [64] (Theorem 1 and Section 6) as well as the following statements: the boundary $\mathrm{Bd}(B_f(0))$ is a Jordan curve, on which f is canonically topologically conjugate with the angle doubling map of the circle. Write $\nu_2(f)$ for the image of $\omega_2(f)$ (or of the parabolic point whose immediate basin contains $\omega_2(f)$) under this conjugacy. As f loops around $\mathrm{Bd}(\mathcal{H}_1)$, the point $\nu_2(f)$ moves continuously on the circle. In fact, $\nu_2(f)$ makes at least one full loop. This follows from the Argument Principle and the fact that the function $f \mapsto \omega_1(f) - \omega_2(f)$ has at least one zero and no poles in $\mathcal{H}_1$. We may therefore choose $f \in \mathrm{Bd}(\mathcal{H}_1)$ so that to make $\nu_2(f)$ equal any prescribed point of the circle.

Now let M be the major of the invariant quadratic gap U of $\mathcal{L}$. By our assumption, M is a non-periodic critical leaf. There is a monotone projection $\psi_U : \mathbb{S} \to \mathbb{S}$ that collapses all components of $\mathbb{S} \setminus U$ and that semi-conjugates σ_3 with σ_2 on $U \cap \mathbb{S}$. In particular, $\psi_U(M)$ is a well-defined point of $\mathbb{S}$. Choose $f_* \in \mathrm{Bd}(\mathcal{H}_1)$ so that $\psi_U(M) = \{\nu_2(f_*)\}$. It is easy to see now that the full lamination of f_* coincides with $\mathcal{L}$. □

Corollary 8.11 relates various types of T-classes of polynomials.

Corollary 8.11. *The union of the T-classes of polynomials with a non-repelling fixed point coincides with the union of T-classes of polynomials with a neutral fixed point.*

Proof. It suffices to consider the T-class of a polynomial f with a (super)attracting fixed point. By Theorem 6.2, $\mathcal{L}_f$ is nonempty and has a quadratic invariant gap U. Consider a minimal sublamination $\widehat{\mathcal{L}}$ of $\mathcal{L}_f$. By definition, $\widehat{\mathcal{L}}$ is nonempty and has a gap $V \supset U$. It follows that $V = U$. Moreover, $\widehat{\mathcal{L}}$ cannot contain a Siegel gap W as otherwise we can remove W with all its pullbacks and still have a nonempty sublamination of $\mathcal{L}_f$ (because U will remain after that removal). By Theorem 8.7, there exists a polynomial g with a parabolic fixed point such that $\mathcal{L}_g = \widehat{\mathcal{L}}$, and f belongs to the T-class of g as desired. □

Proof of Theorem 8.5. Let f belong to the central fiber of polynomials. By Corollary 8.11 and Lemma 8.3 we may assume that $\mathcal{L}_f^{rep}$ is nonempty and $\mathcal{L}_f$ is nonempty. By Lemma 8.2 $\mathcal{L}_f$ is central. Hence, by Lemma 8.4, f belongs to the T-class of a polynomial P, and there are two cases.

(1) If P has a non-repelling fixed point then by Corollary 8.11 P and f belong to the T-class of a polynomial with a neutral fixed point.

(2) $\mathcal{L}_P$ is minimal, has no Siegel or caterpillar gaps, but has a flower-like set. Then by Theorem 8.7, there exists a polynomial g with a parabolic fixed point such that $\mathcal{L}_g = \widehat{\mathcal{L}}$.

Thus, the central fiber of polynomials is contained in the union of T-classes of polynomials with neutral fixed point. The opposite inclusion holds by Lemma 8.6. □

We can now prove the Main Theorem.

Proof of the Main Theorem. The map η_3 is introduced in Definition 6.8 (fibers of polynomials are η_3-point preimages). This map associates to a polynomial $f \in \mathcal{C}_3$ its fiber of critical portraits (introduced in Definition 6.4). By Theorem 6.10, the map η_3 is continuous. By Lemma 7.4, all non-central fibers of polynomials are connected. By Corollary 8.1, all non-central fibers of polynomials are T-classes. By Theorem 8.5, the central fiber of polynomials is the union of T-classes of polynomials with a non-repelling fixed point equal (by Corollary 8.11) to the union of T-classes of polynomials with a neutral fixed point. It follows that η_3 is T-stable. Since by Definition 1.3, any T-stable map $\Psi : \mathcal{C}_3 \to Y$ collapses any T-class of polynomials to a point, and the union of the T-classes of the family of all polynomials with non-repelling fixed points to one point, then $\Psi = \widehat{\Psi} \circ \eta_3$ as desired. □

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