ON CRITICAL RENORMALIZATION OF COMPLEX POLYNOMIALS

ALEXANDER BLOKH, PETER HAÏSSINSKY, LEX OVERSTEEGEN, AND VLADLEN TIMORIN

Abstract. Holomorphic renormalization plays an important role in complex polynomial dynamics. We consider certain conditions guaranteeing that a polynomial which does not admit a polynomial-like connected Julia set still admits an invariant continuum on which it is topologically conjugate to a lower degree polynomial. This invariant continuum may contain extra critical points of the original polynomial that are not visible in the dynamical plane of the conjugate polynomial. Thus, we extend the notions of holomorphic renormalization and polynomial-like maps and describe a setup where new generalized versions of these notions are applicable and yield useful topological conjugacies.

1. Introduction

Throughout, let $P : \mathbb{C} \to \mathbb{C}$ be a polynomial of degree $d > 1$ with connected filled Julia set K_P. Clearly, P acts on the Riemann sphere $\overline{\mathbb{C}} = \mathbb{C}P^1$ so that $P(\infty) = \infty$. In contrast to rational dynamics, the point at infinity plays a special role in the dynamics of P. A classical theorem of Böttcher states that P is conjugate to $z \mapsto z^d$ near infinity. Since K_P is connected, the conjugacy can be defined on $\overline{\mathbb{C}} \setminus K_P$ as follows. We will write $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ for the open unit disk in $\mathbb{C}$ and $\overline{\mathbb{D}}$ for its closure. Without loss of generality we may assume that P is *monic*, i.e., the highest term of P is z^d. Let $\psi_P : \mathbb{D} \to \overline{\mathbb{C}} \setminus K_P$ be a conformal isomorphism normalized so that $\psi_P(0) = \infty$ and $\psi_P'(0) > 0$. Then $\psi_P^{-1} \circ P \circ \psi_P$ is a degree d holomorphic self-covering of $\mathbb{D}$. The only option for such a holomorphic self-covering is $z \mapsto \lambda z^d$ with $|\lambda| = 1$.

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By the chosen normalization of P and ψ_P, the coefficient λ must be equal to 1. Thus, $P(\psi_P(z)) = \psi_P(z^d)$ for any $z \in \mathbb{C}$.

Thus, if we use the polar coordinates (θ, ρ) on $\mathbb{D}$ and identify $\mathbb{D}$ with $\overline{\mathbb{C}} \setminus K_P$ by P, then the action of P will look like $(\theta, \rho) \mapsto (d\theta, \rho^d)$. Here θ is the angular coordinate; it takes values in $\mathbb{R}/\mathbb{Z}$ (elements of $\mathbb{R}/\mathbb{Z}$ are called *angles*). The coordinate ρ, the radial coordinate, is the distance to the origin. On $\mathbb{D} \setminus \{0\}$ (hence, after the transfer, on $\mathbb{C} \setminus K_P$), it takes values in $(0, 1)$. *External rays* of P are defined as the ψ_P-images of radial straight intervals in $\mathbb{D}$. More details are given in Section 2.1; the external ray of P with argument $\theta \in \mathbb{R}/\mathbb{Z}$ is denoted by $R_P(\theta)$.

1.1. **Generalized renormalization.** If external rays R and L land at the same point a, the union $\Gamma = R \cup L \cup \{a\}$ is called a *cut*. The point a is called the *root point* of Γ. The cut Γ is *degenerate* if $R = L$ and *nondegenerate* otherwise. A subarc of a degenerate cut that contains its landing point is called a *terminal segment* of the cut. Nondegenerate cuts separate K_P. A *wedge* is a complementary component of a cut in $\mathbb{C}$; the root point of a wedge is the root point of the corresponding cut. We assume that cuts are oriented from R to L so that every cut Γ bounds a unique wedge $W = W_\Gamma$ where Γ is the *oriented* boundary of W. If Γ is degenerate, then we set $W_\Gamma = \varnothing$. For a finite collection of cuts $\mathcal{Z}$, set $\bigcup \mathcal{W}_\mathcal{Z} = \bigcup_{\Gamma \in \mathcal{Z}} W_\Gamma$.

Definition 1.1. A finite set $\mathcal{Z}$ of cuts is *admissible* if it is P-invariant, and all $\Gamma \in \mathcal{Z}$ are in the same component of $\mathbb{C} \setminus \bigcup \mathcal{W}_\mathcal{Z}$ called the *principal component* (of $\mathbb{C} \setminus \bigcup \mathcal{W}_\mathcal{Z}$). Let $A_P(\mathcal{Z})$ be the set of all $x \in K_P$ such that $f^n(x)$ is in the principal component, for all $n \geqslant 0$. Equivalently, $x \in K_P$ belongs to $A_P(\mathcal{Z})$ if $f^n(x) \notin \bigcup \mathcal{W}_\mathcal{Z}$ for $n \geqslant 0$. The set $A_P(\mathcal{Z})$ is called the *avoiding set* of $\mathcal{Z}$.

By definition $\Gamma \cap K_P \subset A_P(\mathcal{Z})$ for every $\Gamma \in \mathcal{Z}$. Formally, the definition of $A_P(\mathcal{Z})$ is applicable to the case $\mathcal{Z} = \varnothing$. In this case we have $A_P(\varnothing) = K_P$. Otherwise, $A_P(\mathcal{Z})$ is a proper subset of K_P. A root point a of a cut $\Gamma \in \mathcal{Z}$ is called *outward parabolic* if a is a parabolic periodic point, and there is a Fatou component in $K_P \setminus A_P(\mathcal{Z})$ containing an attracting petal of a. If a periodic root point a of a cut $\Gamma \in \mathcal{Z}$ is not outward parabolic, then it is said to be *outward repelling*. Observe that an outward repelling periodic root point a of a cut $\Gamma \in \mathcal{Z}$ may be parabolic; in that case Fatou components containing attracting petals of a are all contained in $A_P(\mathcal{Z})$. Define $\mathrm{Rt}_\mathcal{Z}$ as the collection of root points of all cuts from $\mathcal{Z}$. Classical arguments yield Theorem 1.2. We write $U \Subset V$ if $\overline{U} \subset V$. For polynomial-like maps and hybrid equivalence [DH85] see Definitions 3.1 and 3.2.

Theorem 1.2. *Let $\mathcal{Z}$ be admissible, and suppose that $A_P(\mathcal{Z})$ is connected. If there are no critical or outward parabolic points in* $\mathrm{Rt}_{\mathcal{Z}}$*, then there exist Jordan domains $U \Subset V$ such that $P : U \to V$ is polynomial-like, and $A_P(\mathcal{Z})$ is the filled Julia set of this polynomial-like map. In particular, $P|_U$ is hybrid equivalent to a polynomial Q restricted to a neighborhood of K_Q. If $\mathcal{Z} \neq \varnothing$, then* $\deg(Q) < d$.

We want to generalize Theorem 1.2. Indeed, it is useful to know if $P|_{A_P(\mathcal{Z})}$ is *topologically conjugate* to a polynomial on its connected filled Julia set. Since this conclusion is weaker than that of Theorem 1.2, it is natural to expect that it can be achieved with weaker assumptions than those of Theorem 1.2. In this paper we meet these expectations allowing for certain critical points in $\mathrm{Rt}_{\mathcal{Z}}$. For the notion of a quasi-symmetric homeomorphism, see Definition 2.3.

Main Theorem. *Consider an admissible collection of cuts $\mathcal{Z}$. Suppose that $A_P(\mathcal{Z})$ is connected, every critical point of* $\mathrm{Rt}_{\mathcal{Z}}$ *is eventually mapped to a repelling periodic orbit, and no point of* $\mathrm{Rt}_{\mathcal{Z}}$ *is outward parabolic. Then either $A_P(\mathcal{Z})$ is a singleton, or $P|_{A_P(\mathcal{Z})}$ is quasi-symmetrically conjugate to $Q|_{K_Q}$, where Q is a polynomial of degree greater than one. If $\mathcal{Z} \neq \varnothing$, then the degree of Q is less than d. Moreover, the conjugacy can be arranged to preserve the complex structure almost everywhere on $A_P(\mathcal{Z})$.*

If $P|_{A_P(\mathcal{Z})}$ as in the Main Theorem is injective, then $A_P(\mathcal{Z})$ is a singleton by Theorem 4.11. Note that a quasi-symmetric conjugacy is in particular a topological conjugacy. A statement similar to the Main Theorem first appeared in [Haï98, Prop. 1, Ch. 5]. It was made in a more general context of *polynomial figures*; it is straightforward to verify that the assumptions on the germ of P near $A_P(\mathcal{Z})$ made in [Haï98] are satisfied in our setting.

The next Corollary follows immediately from the Main Theorem and, in the irrational neutral case, results by Perez-Marco [P-M97].

Corollary 1.3. *Assuming the conditions of the Main Theorem, let φ be a topological conjugacy between $P|_{A_P(\mathcal{Z})}$ and $Q|_{K_Q}$, and let C be a periodic cycle in $A_P(\mathcal{Z})$. Then C is attracting (resp., repelling, neutral) if and only if $\varphi(C)$ is attracting (resp., repelling, neutral). Moreover, if C is non-repelling, then it has the same multiplier as $\varphi(C)$.*

Figure 1 shows $\mathcal{W}_{\mathcal{Z}}$ and $A_P(\mathcal{Z})$ for a specific cubic polynomial P.

1.2. **Analogs and extensions.** Branner and Douady [BD88] consider the space $\mathcal{F}^+$ of cubic polynomials $P_a(z) = z(z-a)^2$ (in a different coordinate) such that $R_{P_a}(0) = 0$. They suggested a surgery that relates

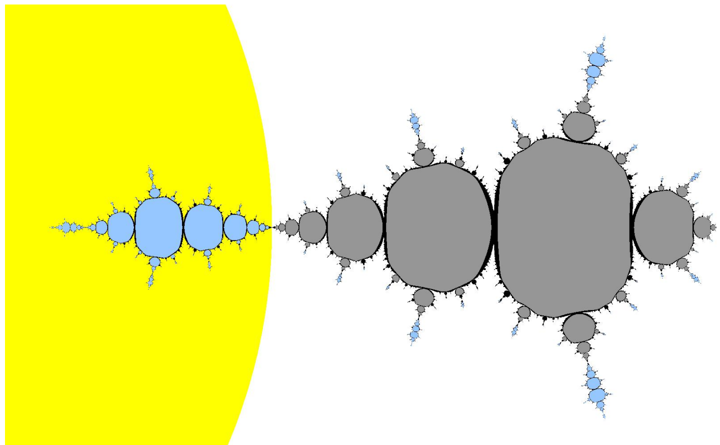

FIGURE 1. The set $A_P(\mathcal{Z})$ for $P(z) = z(z+2)^2$ is shown in dark grey. Here $\mathcal{W}_{\mathcal{Z}}$ consists of a single wedge W (highlighted on the left) whose boundary is mapped to $R_P(0)$. The boundary rays of W are $R_P(1/3)$ and $R_P(2/3)$, and the root point $a = -2$ of W maps to the fixed point 0. By the Main Theorem, the filled Julia set K_P consists of a copy of K_Q, where $Q(z) = -z + z^2$, and countably many decorations. The parabolic point 0 of Q corresponds to the parabolic point -1 of P of the same multiplier. The maps $P|_{A_P(\mathcal{Z})}$ and $Q|_{K_Q}$ are topologically conjugate, but $A_P(\mathcal{Z})$ is not a PL filled Julia set.

cubic polynomials from $\mathcal{F}^+$ to quadratic polynomials from the 1/2-limb of the Mandelbrot set $\mathcal{M}_2$. There is a connection with our Main Theorem in the special case considered in [BD88]. Given $P \in \mathcal{F}^+$ and $\mathcal{Z} = \{R_P(1/3) \cup R_P(2/3) \cup \{a\}\}$, we produce a quadratic polynomial Q^*. In [BD88], the first return map to the side of $\overline{R_Q(1/3) \cup R_Q(2/3)}$ containing $R_Q(0)$, where Q is a quadratic polynomial in the 1/2-limb of $\mathcal{M}_2$, is modeled on the same P and $\mathcal{Z}$. Our Q^* is a renormalization of Q that exists if $A_P(\mathcal{Z})$ is connected. Methods employed in the proof of the Main Theorem generalize those of Branner and Douady. A recent extension in a different direction is given in [DLS20].

If $\mathrm{Rt}_{\mathcal{Z}}$ is allowed to include outward parabolic points, then the situation appears to be more involved. One cannot hope to extend the Main Theorem in its present form, as the QS geometry of the pieces of $A_P(\mathcal{Z})$ "squeezed" in cusps of parabolic domains is different from the QS geometry of a polynomial filled Julia set near a repelling periodic point. On the other hand, a topological rather than QS conjugacy may still exist. David maps can be used for a transition between parabolic and repelling periodic points (see [Haï98a, BF14]). Lomonaco in [Lom15] introduced the theory of parabolic-like maps. The corresponding straightening theorem is applicable to an admissible collection $\mathcal{Z} = \{\Gamma\}$ of just one cut Γ with a fixed parabolic root point; it replaces the complement of $A_P(\mathcal{Z})$ with a single parabolic domain. However, the latter surgery does not change parabolic dynamics to repelling one.

1.3. **Invariant continua similar to $A_P(\mathcal{Z})$.** We now suggest a generalization of invariant continua $A_P(\mathcal{Z})$ not referring to a choice of $\mathcal{Z}$. Theorem 1.4 is proved in [BOPT16a].

Theorem 1.4 (Theorem B [BOPT16a]). *Let $P : \mathbb{C} \to \mathbb{C}$ be a polynomial, and $Y \subset \mathbb{C}$ be a non-separating P-invariant continuum. The following assertions are equivalent:*

(1) *the set Y is the filled Julia set of some polynomial-like map $P : U \to V$ of degree k,*
(2) *the set Y is a component of the set $P^{-1}(P(Y))$ and, for every attracting or parabolic point y of P in Y, the attracting basin of y or the union of all parabolic domains at y is a subset of Y.*

We will show that a slightly weaker consequence than (1) holds under more general assumptions than (2).

Consider a polynomial $P : \mathbb{C} \to \mathbb{C}$ and a full P-invariant continuum $Y \subset \mathbb{C}$. Say that $P : Y \to Y$ is a *degree k branched covering* if there is a degree k branched covering $\tilde{P} : U \to \tilde{P}(U)$ where U is a neighborhood of Y, the restriction of $\tilde{P}$ to Y is $P|_Y$, and Y is a component of $\tilde{P}^{-1}(Y)$. Evidently, U can be chosen so tight that all points of Y but critical values of $\tilde{P}$ have exactly k preimages in U.

Points of $\overline{P^{-1}(Y) \setminus Y} \cap Y$ are called *irregular points* of Y. A point $y \in Y$ is irregular if arbitrarily close to y there are points y' that do not belong to Y but map into Y. Since $P|_Y$ is locally onto, for each such y' there is a point $y'' \in Y$ close to y' and such that $P(y'') = P(y')$. It follows that y is critical. Thus, all irregular points of Y are critical; the converse is not true in general.

Theorem 1.5. *Let $P : \mathbb{C} \to \mathbb{C}$ be a polynomial. Consider a full P-invariant continuum $Y \subset \mathbb{C}$ and an integer $k > 1$ such that:*

(1) *the map $P : Y \to Y$ is a degree k branched covering;*
(2) *all irregular points are eventually mapped to repelling periodic points;*
(3) *the immediate basins of all attracting or parabolic points in Y are subsets of Y.*

Under these assumptions, $P : Y \to Y$ *is topologically conjugate to* $Q|_{K(Q)}$*, where* Q *is a polynomial of degree* k*.*

It is easy to see that if K_P is connected, $Y = A_P(\mathcal{Z})$ is also connected for some admissible $\mathcal{Z}$, and $P|_{A_P(\mathcal{Z})}$ is not injective, then Y satisfies the assumptions of Theorem 1.5. Thus Theorem 1.5 generalizes the Main Theorem. Moreover, the former applies to polynomials P with disconnected Julia sets.

Theorem 1.5 applies to the *planar fibers* (the notion is due to Schleicher [Sch99] and was studied in other papers, e.g., in [BCLOS16]). Let P be a polynomial with connected Julia set. Call a periodic repelling or parabolic point, or a preimage thereof, a *valuable* point. If z is a valuable point at which more than one external ray lands, call the union Cu_z of z with all external rays landing at z the *star cut (at z).* The set Cu_z partitions $\mathbb{C}$ into finitely many open wedges. A *planar fiber (of P)* is a non-empty intersection of the closures of open wedges chosen at every valuable point with a star cut. It follows that a planar fiber is the union of a full subcontinuum of K_P and various rays, and that planar fibers map onto planar fibers. Theorem 1.5 implies Corollary 1.6 (applicable, e.g., if Y contains a periodic Cremer or Siegel point).

Corollary 1.6. *Let* Y *be an* m*-periodic non-degenerate planar fiber of* P *containing no parabolic points. Then* $P^m|_Y$ *is topologically conjugate to a certain polynomial* Q *of degree greater than* 1 *restricted to* K_Q*; moreover,* K_Q *has no valuable cutpoints (and, hence, a unique non-repelling fixed point).*

1.4. **Plan of the paper.** Section 2 provides some background from complex polynomial dynamics and quasi-conformal geometry. We also discuss the notion of transversality and its relationship with quasi-symmetric maps. In Section 3 we reduce the Main Theorem to the case when $\mathcal{Z}$ has specific properties (e.g., one may assume that all periodic cuts in $\mathcal{Z}$ are degenerate); this is done with the help of classical theory of polynomial-like maps of Douady and Hubbard. The proof of the Main Theorem is given in Section 4, where P is replaced with a quasi-regular map f such that $P = f$ on K_P, and f repels points off K_P. Straightening the map f using a result of [SW20] (a generalization of the Douady–Hubbard straightening theorem) yields the Main Theorem. Section 5 contains the proof of Theorem 1.5 and Corollary 1.6.

2. Straightening and transversality

2.1. **Background in complex polynomial dynamics.** Consider a straight radial interval $\mathsf{R}(\theta) = \{e^{2\pi i\theta}\rho \mid \rho \in (0,1)\}$ from 0 to the point

$e^{2\pi i\theta}$. Let P be a degree $d > 1$ polynomial *with connected Julia set.* The *external ray of* P *of argument* $\theta \in \mathbb{R}/\mathbb{Z}$ is the set $R_P(\theta) = \psi_P(\mathsf{R}(\theta))$. External rays are useful for studying the dynamics of P. In particular, it is important to know when different rays land at the same point.

Definition 2.1 (Ray landing)**.** A ray $R_P(\theta)$ *lands* at $a \in K_P$ if $a = \lim_{\rho\to 1^-} \psi_P(e^{2\pi i\theta}\rho)$ is the only accumulation point of $R_P(a)$ in $\mathbb{C}$.

By the Douady–Hubbard–Sullivan landing theorem, if θ is rational, then $R_P(\theta)$ lands at a (pre)periodic point that is eventually mapped to a repelling or parabolic periodic point. Conversely, any point that eventually maps to a repelling or parabolic periodic point is the landing point of at least one and at most finitely many external rays with rational arguments.

An *equipotential curve* of P (or simply an *equipotential*) is the ψ_P-image of a circle $\{z \in \mathbb{C} \,|\, |z| = \rho\}$ of radius $\rho \in (0, 1)$ centered at 0. External rays and equipotentials form a net that is the ψ_P-image of the polar coordinate net.

2.2. **Quasi-regular and quasi-symmetric maps.** Let us recall the definition of quasi-regular [Ric93] and quasi-conformal maps [Ahl66].

Definition 2.2 (Quasi-regular maps)**.** Let U and V be open subsets of $\mathbb{C}$, and let $\kappa \geqslant 1$ be a real number. A map $f : U \to V$ is said to be κ*-quasi-regular* if it has distributional partial derivatives in L^2_{loc}, and $||df||^2 \leqslant \kappa \operatorname{Jac}_f$ in L^1_{loc}. Here df is the first differential of f, and Jac_f is the Jacobian determinant of f. Note that any holomorphic map is κ-quasi-regular with $\kappa = 1$. We say that f is *quasi-regular* if it is κ-quasi-regular for some $\kappa \geqslant 1$. A *quasi-conformal map* is by definition a quasi-regular homeomorphism.

The inverse of a (κ-)quasi-conformal map is (κ-)quasi-conformal. Quasi-conformal maps admit a number of analytic and geometric characterizations. They can be characterized in terms of Beltrami differentials and in terms of moduli of annuli or similar conformal invariants. See 4.1.1 and 4.5.16 — 4.5.18 in [Hub06]. A metric characterization of quasi-conformal maps is based on the following notion applicable to general metric spaces, cf. [TV80].

Definition 2.3 (Quasi-symmetric maps)**.** Let (X, d_X) and (Y, d_Y) be metric spaces, and let $\eta : [0, \infty) \to [0, \infty)$ be an increasing onto homeomorphism. A continuous embedding $f : X \to Y$ is said to be *quasi-symmetric of modulus* η (or η*-quasi-symmetric*) if

$$\frac{d_Y(f(x), f(y))}{d_Y(f(x), f(z))} \leqslant \eta\left(\frac{d_X(x, y)}{d_X(x, z)}\right) \tag{1}$$

for all $x \neq y \neq z$ that are sufficiently close to each other. We will sometimes abbreviate quasi-symmetric as QS. The inverse of a QS embedding (defined on $f(X)$) is η'-QS, where $\eta'(t) = 1/\eta^{-1}(1/t)$. The composition of QS embeddings is also QS. A continuous embedding $f : X \to Y$ is *κ-weakly QS* for some $\kappa > 0$ if

$$d_X(x, y) \leqslant d_X(x, z) \quad \Longrightarrow \quad d_Y(f(x), f(y)) \leqslant \kappa\, d_Y(f(x), f(z)).$$

Weakly QS embeddings are κ-weakly QS for some $\kappa > 0$. Clearly, QS embeddings are weakly QS. The converse is not true in general, however, by Theorem 10.19 of [Hei01], weakly QS embeddings are QS in a lot of cases. In particular, a weakly QS embedding of a connected subset of $\mathbb{R}^n$ to $\mathbb{R}^n$ is QS. Occasionally we will talk about "QS maps" which will always mean "QS embeddings".

The following theorem establishes a relationship between QS embeddings and quasi-conformal maps.

Theorem 2.4 (A special case of Theorems 2.3 and 2.4 of [Väi81])**.** *An η-QS embedding between domains in $\mathbb{C}$ is κ-quasi-conformal ($\kappa \geqslant 1$ is a constant depending only on η). Conversely, consider a κ-quasi-conformal map $f : U \to V$, where U, $V \subset \mathbb{R}^2$ are open. Then, for any $z \in U$ and $\varepsilon > 0$ such that the 2ε-neighborhood of z lies in U, the map f is η-QS on the ε-neighborhood of z, where η depends only on κ.*

Quasi-conformal images of circle arcs, circles, and disks can be described explicitly.

Definition 2.5 (Quasi-arc, quasi-circle, quasi-disk)**.** A *simple arc* in $\mathbb{C}$ is a homeomorphic image of $[0, 1]$ under a map $\xi : [0, 1] \to \mathbb{C}$. A simple arc I is a *quasi-arc* if for any such ξ and any $x \leqslant y \leqslant z$ we have

$$|\xi(x) - \xi(z)| \geqslant C|\xi(x) - \xi(y)|, \tag{2}$$

where $C > 0$ is a constant independent of x, y, z and ξ. A *quasi-circle* is a Jordan curve such that any arc of it is a quasi-arc with a uniform constant C. For quasi-arcs and quasi-circles in the *Riemann sphere* $\overline{\mathbb{C}} = \mathbb{C}P^1$, we use the spherical distance between a and b instead of $|a - b|$, etc. A *quasi-disk* is a Jordan disk bounded by a quasi-circle. A *quasi-conformal reflection* in a Jordan curve is an orientation-reversing quasi-conformal involution of the sphere which switches the inside and the outside of the curve fixing points on the curve.

The following theorem is due to L. Ahlfors, see [Ahl66] or 4.9.8, 4.9.12, and 4.9.15 in [Hub06]:

Theorem 2.6. *Properties* $(1)-(3)$ *of a Jordan curve S are equivalent:*

(1) *the curve S is a quasi-circle;*
(2) *there is a bi-Lipschitz reflection in S;*
(3) *there is a quasi-conformal map $h : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ such that $S = h(\overline{\mathbb{R}})$.*

Observe also that QS embeddings of quasi-arcs are quasi-arcs; moreover, preimages of quasi-arcs under QS-embeddings are quasi-arcs too.

Quasi-symmetric maps between quasi-circles can be extended inside the corresponding quasi-disks as quasi-conformal maps.

Theorem 2.7. *If U and V are quasi-disks in $\overline{\mathbb{C}}$, and $f : \mathrm{Bd}(U) \to \mathrm{Bd}(V)$ is a quasi-symmetric map, then there is a continuous map $F : \overline{U} \to \overline{V}$ such that $F = f$ on $\mathrm{Bd}(U)$, and F is quasi-conformal in U.*

Proof. By Theorem 2.6, there are quasi-conformal maps h_U, $h_V : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ that take the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} \,|\, \mathrm{Im} z > 0\}$ onto U, V, respectively. Then the map $\varphi = h_V^{-1} \circ f \circ h_U : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ is quasi-symmetric as a composition of quasi-symmetric maps. Pre-composing h_U and h_V with suitable real fractional linear maps, arrange that $\varphi(\infty) = \infty$. Let η be a modulus of φ (so that φ is η-quasi-symmetric). Setting $y = x+t$ and $z = x - t$ in the definition of an η-quasi-symmetric map, we see that

$$M^{-1} \leqslant \frac{\varphi(x+t) - \varphi(x)}{\varphi(x) - \varphi(x-t)} \leqslant M,$$

where $M = \eta(1)$. Maps φ that satisfy the above condition for some $M > 0$ are called $\mathbb{R}$-*quasi-symmetric* in [Hub06]. The constant M is called the *modulus* of an $\mathbb{R}$-quasi-symmetric map. By a theorem of Ahlfors and Beurling [AB56] (see also [Ahl66] and 4.9.3 and 4.9.5 of [Hub06]), an $\mathbb{R}$-quasi-symmetric map of modulus M admits a κ-quasi-conformal extension in $\mathbb{H}$, where κ depends only on M. More precisely, there is a continuous map $\Phi : \overline{\mathbb{H}} \to \overline{\mathbb{H}}$ such that $\Phi = \varphi$ on $\overline{\mathbb{R}}$, and $\Phi|_{\mathbb{H}}$ is κ-quasi-conformal. Then $F = h_V \circ \Phi \circ h_U^{-1}$ has the desired property. □

2.3. **Straightening.** A quasiregular map $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is called a *quasiregular polynomial* if $f^{-1}(\infty) = \{\infty\}$, and f is holomorphic near infinity. Let us state a partial case of [SW20, Theorem 5].

Theorem 2.8. *Let $f : \mathbb{C} \to \mathbb{C}$ be a quasi-regular polynomial of degree $d \geqslant 2$ and let $A \subset \mathbb{C}$ be a Borel set such that $\overline{\partial} f = 0$ a.e. outside A. Assume that there is a positive integer T such that, for every z, the set of nonnegative integers k with $f^k(z) \in A$ has cardinality $\leqslant T$. Then there is a QC map $\Psi : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ and a rational map $F : \mathbb{C} \to \mathbb{C}$ of degree d such that $F \circ \Psi = \Psi \circ f$. Moreover, $\overline{\partial}\Psi = 0$ holds a.e. on the set $\{z \in \mathbb{C} \,|\, f^n(z) \notin A \ \forall n \geqslant 0\}$.*

This version is a rather straightforward extension of the Douady–Hubbard straightening theorem (cf. Theorem 3.3) and is similar to Shishikura's Fundamental Lemma for qc-surgery (cf. [Shi87, Lemma 3.1]), however, the general version of [SW20, Theorem 5] is much more powerful.

2.4. **Transversality.** Consider two simple arcs R, $L \subset \mathbb{C}$ sharing an endpoint a and otherwise disjoint. The arcs R, L are *transverse* at a if, for any sequences $u_n \in R$ and $v_n \in L$ converging to a,

$$\frac{u_n - a}{v_n - a} \not\to 1.$$

Transversality is related to the notion of a quasi-arc as the following lemma explicates.

Lemma 2.9. *Let simple arcs R, L share an endpoint a and be otherwise disjoint. If $R \cup L = I$ is a quasi-arc, then R and L are transverse.*

Proof. By way of contradiction, suppose that

$$\frac{u_n - a}{v_n - a} \to 1.$$

for some $u_n \in R$ and $v_n \in L$ such that u_n, $v_n \to a$. It follows that

$$\frac{u_n - v_n}{v_n - a} \to 0.$$

This contradicts the inequality $|v_n - u_n| \geqslant C|v_n - a|$ with $C > 0$ from the definition of a quasi-arc. □

Proposition 2.10. *Consider a simple arc R such that the image R' of R under $w \mapsto w^k$ with $k > 1$ is a simple arc. Moreover, assume that 0 is an endpoint of R, and $\lambda R' \supset R'$ for some $\lambda \in \mathbb{C}$ with $|\lambda| > 1$. Then R is transverse to ζR at 0, for every k-th root of unity $\zeta \neq 1$.*

Proof. The map $w \mapsto w^k$ is injective on the arc R. Indeed, by our assumption, R' is a simple arc; a locally injective continuous map from an interval to an interval is injective. Thus, R and ζR share only the endpoint 0. By way of contradiction, assume that $v_n/u_n \to \zeta$, where u_n, $v_n \in R$ and u_n, $v_n \to 0$. Passing to a subsequence and choosing positive integers m_n properly, we may assume that $\lambda^{m_n} u_n^k \to u \neq 0$, where $u \in R'$ is not an endpoint of R'. Then also $\lambda^{m_n} v_n^k \to u$. Let I_n be the segment of R connecting u_n and v_n. Then the corresponding segment I'_n of R' connects u_n^k with v_n^k. Consider the arc $T_n = \lambda^{m_n} I'_n \subset R'$; its endpoints $\lambda^{m_n} u_n^k$ and $\lambda^{m_n} v_n^k$ converge to u but the arc itself has diameter bounded away from 0 as it makes one or several loops around 0 (if S_n is the union of T_n and the straight segment connecting its endpoints then, since $v_n/u_n \to \zeta$, the loop S_n has a nonzero winding number with respect to 0). Since sets T_n are subarcs of R', in the limit they converge to a nondegenerate loop in R', a contradiction. $\square$

The following is a typical application of Proposition 2.10. Let P be a polynomial; consider a repelling fixed point a of P and an invariant ray $R_P(\theta')$ landing at a. Set $\lambda = P'(a)$, that is, λ is the multiplier of the fixed point a. Then, in some local coordinate y near a, we have $y = 0$ at a, and P coincides with $y \mapsto \lambda y$. On the other hand, suppose that a ray $R_P(\theta)$ maps to $R_P(\theta')$ under P. Write b for the landing point of $R_P(\theta)$, and assume that P has local degree $k > 1$ at b (thus, b is critical). Then, in some local coordinate x near b combined with the local coordinate y near a, the map P looks like $y = x^k$, and b is the point where $x = 0$. We can now define R as an arc of $\overline{R}_P(\theta)$ connecting b with some point of $R_P(\theta)$. Set $R' = P(R)$. Then Proposition 2.10 is applicable to arcs R, R' and the chosen local coordinates. It claims that R is transverse to all other P-pullbacks of R' originating at b.

Lemma 2.11. *Let arcs R, L have the same properties as the arc R from Proposition 2.10, and let R', L' be their images under $x \mapsto x^k$. If $R' \cap L' = \{0\}$ and R', L' are transverse, then the restriction of $x \mapsto x^k$ to $R \cup L$ is QS.*

If $R' \cup L'$ is a quasi-arc, then, by Lemma 2.11, the arc $R \cup L$ is also a quasi-arc. This observation will be useful in what follows.

Proof. We will prove that $x \mapsto x^k$ restricted to $R \cup L$ is weakly QS. Assume, by way of contradiction, that there are three sequences u_n, v_n, $w_n \in R \cup L$ such that, if we set $\overline{v}_n = v_n/u_n, \overline{w}_n = w_n/u_n$, we will have

$$\Delta_n = \frac{u_n^k - v_n^k}{u_n^k - w_n^k} \to \infty, |u_n - v_n| \leqslant |u_n - w_n| \,(\text{and so}\, |1 - \overline{v}_n| \leqslant |1 - \overline{w}_n|).$$

Assume that $u_n \to u$, $v_n \to v$ and $w_n \to w$. If $u \neq w$, then Δ_n is bounded since $x \mapsto x^k$ is injective on $R \cup L$, a contradiction. Thus $u = w$. If $u \neq v$, then $|u_n - v_n| > |u_n - w_n|$ for large n, hence $u = v = w$. It is enough to consider the case when $u = 0$. We may also assume that $\delta_n = (u_n - v_n)/(u_n - w_n) = (1 - \overline{v}_n)/(1 - \overline{w}_n) \to \delta$ with $|\delta| \leqslant 1$.

Assume that $\overline{v}_n \to \overline{v}$ and $\overline{w}_n \to \overline{w}$, where $\overline{v}$ and $\overline{w}$ are complex numbers or ∞. Observe that

$$\Delta_n = \frac{1 - \overline{v}_n^k}{1 - \overline{w}_n^k} = \frac{\overline{w}_n^{-k} - (\overline{v}_n/\overline{w}_n)^k}{\overline{w}_n^{-k} - 1} = \delta_n \frac{1 + \overline{v}_n + \cdots + \overline{v}_n^{k-1}}{1 + \overline{w}_n + \cdots + \overline{w}_n^{k-1}} \to \infty$$

If $\overline{v} = \overline{w} = 1$, then $\Delta_n \to \delta$, a contradiction. If $\overline{w} = \infty$, then $\overline{v}_n/\overline{w}_n \to \delta$ and $\Delta_n \to \delta^k$, a contradiction. Since $\overline{v} = \infty$ implies $\overline{w} = \infty$ and $\overline{w} = 1$

implies $\overline{v} = 1$ (because $|1-\overline{v}| \leqslant |1-\overline{w}|$), both $\overline{v}$, $\overline{w}$ are finite, and $\overline{w} \neq 1$. Now $\delta_n \to \infty$ implies that $\overline{w} \neq 1$ is a k-th root of unity. By Proposition 2.10 and since $w_n/u_n \to \overline{w}$, it is impossible that w_n, u_n are both in R or both in L for infinitely many values of n. Thus we may assume that $w_n \in R$ and $u_n \in L$. However, in this case R' and L' are not transverse, a contradiction. □

Consider two simple arcs R', L' with common endpoint at 0 and disjoint otherwise. Suppose that λ is a complex number with $|\lambda| > 1$. Furthermore, suppose that $\lambda R' \supset R'$ and $\lambda L' \supset L'$.

Theorem 2.12. *If R', L' are as above, and $R' \cup L'$ is smooth except possibly at 0, then $R' \cup L'$ is a quasi-arc.*

Proof. Assume the contrary: there are three sequences x_n, y_n, $z_n \in R' \cup L'$ such that

- the point y_n is always between x_n and z_n in the arc $R' \cup L'$ (in particular, the three points x_n, y_n, z_n are always different);
- we have $\delta_n = |x_n - z_n|/|x_n - y_n| \to 0$ as $n \to \infty$.

It follows from the second assumption that $|x_n - z_n| \to 0$ since the denominator is bounded. We can now make a number of additional assumptions on x_n, y_n, z_n by passing to subsequences. Assume that x_n and z_n converge to the same limit. If this limit is different from 0, then straightforward geometric arguments yield a contradiction (it is obvious that every closed subarc of $R' \cup L'$ not containing 0 is a quasi-arc). Thus we may assume that x_n, $z_n \to 0$. Since y_n is between x_n and z_n, we also have $y_n \to 0$. From now on, we rely on the assumption that all three sequences x_n, y_n, z_n converge to 0.

Take $r > 0$ sufficiently small, and let A be the annulus $\{z \in \mathbb{C} \,|\, r < |z| < |\lambda| r\}$. Assume that $x_n \neq 0$ for all n (otherwise, for a suitable subsequence, $z_n \neq 0$ for all n, and we may interchange x_n and z_n). For every n, there exists a positive integer m_n such that $\lambda^{m_n} x_n \in \overline{A}$. Set x'_n, y'_n, z'_n to be $\lambda^{m_n} x_n$, $\lambda^{m_n} y_n$, $\lambda^{m_n} z_n$, respectively. We may assume that $x'_n \in R'$ rather than L'. (By the invariance property of $R' \cup L'$, we must have $x'_n \in R' \cup L'$.) Passing to a subsequence, arrange that $x'_n \to x \in \overline{A}$ as $n \to \infty$. Since $|x'_n - z'_n|/|x'_n - y'_n| = \delta_n \to 0$, then $z'_n \to x$. Since the intersections of R' and L' with an open neighborhood of $\overline{A}$ are smooth open arcs, it follows that $z'_n \in R'$ for large n, hence $y'_n \to x$ and $\delta_n \not\to 0$. A contradiction. □

3. Reducing the admissible collection

Let P and $\mathcal{Z}$ be as in the Main Theorem. Let c be a critical point of P. A cut $\Gamma = R \cup L \cup \{c\}$ formed by c and two rays R, L landing at

c such that $P(R) = P(L)$, is a *critical cut*. A pullback of a critical cut is a *precritical cut*.

3.1. **Outline of the proof of the Main Theorem.** The proof of the Main Theorem, under the assumption that $P|_{A_P(\mathcal{Z})}$ is not injective, will go as follows.

Step 1: reduction. Take only the nondegenerate cuts from $\mathcal{Z}$ with periodic root points that are outward repelling. These cuts define a polynomial-like Julia set by Theorem 1.2. Replacing P with the straightening of the corresponding polynomial-like map and using the conditions of the Main Theorem, we may assume that every periodic cut in $\mathcal{Z}$ is degenerate and all cuts from $\mathcal{Z}$ are either precritical cuts that eventually map to degenerate cuts with repelling periodic root points, and their images. Step 1 is made in this section.

Step 2: carrot modification. After the reduction step, we assume that every periodic cut in $\mathcal{Z}$ is degenerate and has repelling root point. Terminal segments of these degenerate cuts can be fattened to so called *carrots*. Carrots are quasi-disks; they are almost the same as "sectors" used in [BD88]. Moreover, if C is a carrot corresponding to a periodic cut, then $C \cap K_P$ is only the root point of the cut. Carrots corresponding to preperiodic cuts are defined differently and contain $K_P \setminus A_P(\mathcal{Z})$. Finally, we modify P in carrots and near infinity and obtain a quasi-regular polynomial f. Application of Theorem 2.8 to f concludes the proof of the Main Theorem. Step 2 is made in the next section.

If $P|_{A_P(\mathcal{Z})}$ is injective, then $A_P(\mathcal{Z})$ is a point, which is proved in Theorem 4.11 by a different (but simpler) method.

3.2. **Polynomial-like maps.** Let U and V be Jordan disks such that $U \Subset V$. Recall the following classical definitions of Douady and Hubbard [DH85].

Definition 3.1 (Polynomial-like maps)**.** Let $f : U \to V$ be a proper holomorphic map. Then f is said to be *polynomial-like* (PL). The *filled Julia set* $K(f)$ of f is defined as the set of points in U, whose forward f-orbits stay in U.

Similarly to polynomials, the set $K(f)$ is connected if and only if all critical points of f are in $K(f)$.

Definition 3.2 (Hybrid equivalence)**.** Let $f_1 : U_1 \to V_1$ and $f_2 : U_2 \to V_2$ be two PL maps. Consider Jordan neighborhoods W_1 of $K(f_1)$ and W_2 of $K(f_2)$. A quasiconformal homeomorphism $\phi : W_1 \to W_2$ is called a *hybrid equivalence* between f_1 and f_2 if $f_2 \circ \phi = \phi \circ f_1$ whenever both parts are defined, and $\overline{\partial}\phi = 0$ on $K(f_1)$.

Recall the following classical theorem of Douady and Hubbard [DH85].

Theorem 3.3 (PL Straightening Theorem). *A polynomial-like map $f : U \to V$ is hybrid equivalent to a polynomial of the same degree restricted on a Jordan neighborhood of its filled Julia set.*

3.3. **Eliminating some periodic cuts.** Let $\mathcal{Z}$ be an admissible collection of cuts. It is a union of a finite family of forward orbits of cuts. Among them there might exist degenerate cuts whose backward orbit in $\mathcal{Z}$ does not include nondegenerate cuts. These cuts make no impact upon the avoiding set $A_P(\mathcal{Z})$ (recall that it was defined in Definition 1.1) and will be called *fictitious*.

Definition 3.4. An admissible family of cuts is *legal* if it is a union of finite orbits of (pre)critical cuts each of which eventually maps to a degenerate cut with repelling periodic root point.

Define $\mathcal{Z}_{pc} \subset \mathcal{Z}$ as the subset consisting of all periodic nondegenerate cuts $\Gamma \in \mathcal{Z}$. (Here "pc" is from "**p**eriodic **c**uts".) An admissible family of cuts is legal if has no fictitious cuts, $\mathcal{Z}_{pc} = \varnothing$, and all critical root points eventually map to repelling periodic points.

Let V_E be a Jordan neighborhood of K_P bounded by an equipotential curve. A standard thickening of $V_E \setminus \bigcup_{\Gamma \in \mathcal{Z}_{pc}} \overline{W}_\Gamma$ yields a Jordan disk V and a polynomial-like map $P : U \to V$, where U is the component of $P^{-1}(V)$ containing $A_P(\mathcal{Z})$. By Theorem 3.3, the PL-map $P : U \to V$ is hybrid equivalent (by a map ψ) to $\tilde{P} : \tilde{U} \to \tilde{V}$, where $\tilde{P}$ is a polynomial and $\tilde{U}$ is a neighborhood of its filled Julia set.

Lemma 3.5. *There is a legal family $\tilde{\mathcal{Z}}$ of cuts in the dynamical plane of $\tilde{P}$ such that $A_{\tilde{P}}(\tilde{\mathcal{Z}}) = \psi(A_P(\mathcal{Z}))$.*

Lemma 3.5 is left to the reader. If the Main Theorem is proved for $\tilde{P}$ and $A_{\tilde{P}}(\tilde{\mathcal{Z}})$, then it would follow for $A_P(\mathcal{Z})$. This reduces the Main Theorem to the case of legal families of cuts.

4. Carrots

From now on we assume that $\mathcal{Z}$ is a legal family of cuts.

Definition 4.1 (Prototype carrot). The *prototype carrot*, or simply *proto-carrot* $\mathsf{C}(\rho_0, \theta_0)$, with polar parameters (ρ_0, θ_0) is the "triangular" region in $\overline{\mathbb{D}}$ given by the inequalities $\rho_0 \leqslant \rho \leqslant e^{-|\theta - \theta_0|}$ in the polar coordinates (θ, ρ). Here θ is an *angular coordinate* so that $\theta - \theta_0$ can be either positive or negative, and ρ is the *radial coordinate*, i.e., the distance to the origin. We assume that the parameter $\rho_0 < 1$ is close to 1. A proto-carrot is bounded by a circle arc and two symmetric

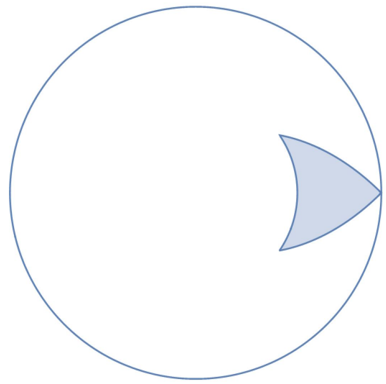

FIGURE 2. A proto-carrot.

segments of logarithmic spirals, see Figure 2. All proto-carrots are homeomorphic.

A part of the boundary of $\mathsf{C}(\rho_0, \theta_0)$ near point $e^{2\pi i\theta_0}$ is given by $\rho = e^{-|\theta-\theta_0|}$ and consists of two analytic curves meeting at $e^{2\pi i\theta_0}$ and invariant under the map $z \mapsto z^d$ (regardless of d). Since rotation by θ_0 composed with z^d equals z^d composed with the rotation by $d \cdot \theta_0$, the next proposition follows.

Proposition 4.2. *Take any $\theta_0 \in \mathbb{R}/\mathbb{Z}$. The map $z \mapsto z^d$ takes $\mathsf{C}(\rho_0, \theta_0)$ to $\mathsf{C}(\rho_0^d, d\theta_0)$, provided that ρ_0 is close to 1.*

Let us recall the following concept.

Definition 4.3 (Stolz angle)**.** A *Stolz angle* in $\mathbb{D}$ at a point $u \in \mathbb{S}^1$ is by definition a convex cone with apex at u bisected by the radius and with aperture strictly less than π.

The proto-carrot $\mathsf{C}(\rho, \theta)$ approaches the unit circle within some Stolz angle at $e^{2\pi i\theta}$ (the Implicit Function Theorem shows that the aperture of such Stolz angles can be made arbitrarily close to $\pi/2$). The following theorem proved in [CG92, Theorem 2.2] describes an important property of Stolz angles.

Theorem 4.4. *Consider a simply connected domain $D \subset \overline{\mathbb{C}}$ that is not the sphere minus a singleton, and let $\psi : \mathbb{D} \to D$ be a conformal isomorphism. Suppose that a point $z_0 \in \mathrm{Bd}(D)$ is accessible from D. Then there is a point $u_0 \in \mathbb{S}^1$ with the following property: $\psi(u) \to z_0$ as $u \to u_0$ inside any Stolz angle with apex at u_0.*

Let $E(\rho)$ be the equipotential $\psi_P(\{z \in \mathbb{D} \mid |z| = \rho\})$. Write $U(\rho)$ for the open Jordan domain bounded by $E(\rho)$.

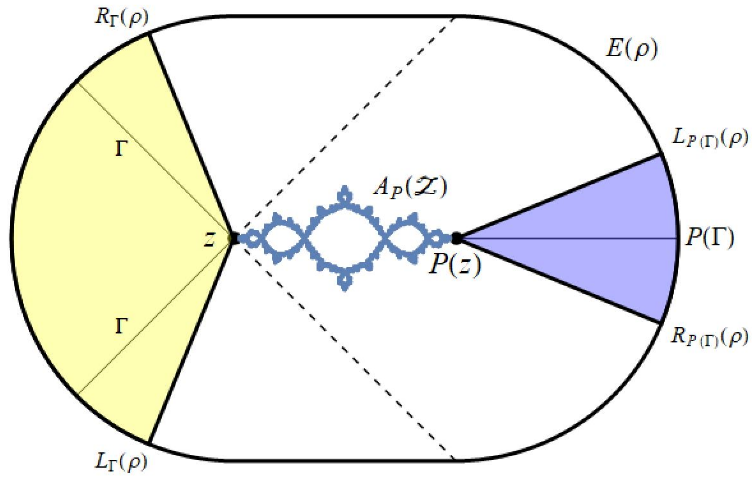

FIGURE 3. Definition of a carrot $C_\Gamma(\rho)$, where $z = z_\Gamma$ is critical. The carrot $C_{P(\Gamma)}(\rho)$ is the (dark) shaded sector on the right and $C_\Gamma(\rho)$ is the (light) shaded sector on the left. Here $P(\Gamma)$ is degenerate, the local degree of P at z is 4, and the set $P^{-1}(P(\Gamma)) \setminus \Gamma$ consists of two dashed lines.

Definition 4.5 (Carrots)**.** For a periodic (necessarily degenerate) cut $\Gamma \in \mathcal{Z}$ with a repelling root point, define the *carrot* $C_\Gamma(\rho)$ as $\psi_P(\mathsf{C}(\rho, \theta_\Gamma))$. This is the closed triangular region bounded by three arcs $R_\Gamma(\rho)$, $L_\Gamma(\rho)$, and $E_\Gamma(\rho)$. Here $R_\Gamma(\rho)$ and $L_\Gamma(\rho)$ are simple topological arcs landing at z_Γ; the latter follows from Theorem 4.4. The arc $E_\Gamma(\rho)$ is a part of the equipotential curve $E(\rho)$.

It remains to define carrots for strictly preperiodic cuts in $\mathcal{Z}$. Take such a cut $\Gamma \in \mathcal{Z}$; we may assume by induction that the carrot $C_{P(\Gamma)}(\rho^d)$ is already defined. Let W be the wedge corresponding to Γ. There is a pullback $R_\Gamma(\rho)$ of $R_{P(\Gamma)}(\rho^d)$ and a pullback $L_\Gamma(\rho)$ of $L_{P(\Gamma)}(\rho^d)$ with the following properties:

(1) both $R_\Gamma(\rho)$ and $L_\Gamma(\rho)$ land at z_Γ, the root point of Γ;
(2) the arc $E_\Gamma(\rho)$ of $E(\rho)$ with endpoints $R_\Gamma(\rho) \cap E(\rho)$ and $L_\Gamma(\rho) \cap E(\rho)$ is the component of $E(\rho) \setminus P^{-1}(R_{P(\Gamma)}(\rho^d) \cup L_{P(\Gamma)}(\rho^d))$ containing $W \cap E(\rho)$.

The set $C_\Gamma(\rho)$ is then defined as a triangular region bounded by three arcs $R_\Gamma(\rho)$, $L_\Gamma(\rho)$ and $E_\Gamma(\rho)$.

Schematic Figure 3 illustrates the definition of a carrot in the case when z is critical.

4.1. **Carrots are quasi-disks.** We will need the following geometric property of carrots.

Proposition 4.6. *Let $\Gamma \in \mathcal{Z}$ have periodic repelling root point z_Γ. Then $C_\Gamma(\rho)$ is a quasi-disk, for every ρ sufficiently close to* 1.

Proof. By our construction, it is enough to prove that $R_\Gamma(\rho) \cup L_\Gamma(\rho)$ is a quasi-arc locally near z_Γ. Observe that this property is independent of ρ. Consider a local holomorphic coordinate u near z_Γ such that $u = 0$ at z_Γ, and P^m takes the form $u \mapsto \lambda u$. Here λ is the derivative of P^m at z_Γ, hence $|\lambda| > 1$. A local coordinate u with the properties stated above exists by the classical Königs linearization theorem. Set R', L' to be the images of $R_\Gamma(\rho)$, $L_\Gamma(\rho)$ in the u-plane and apply Theorem 2.12 to R', L'. Since a holomorphic local coordinate change takes quasi-arcs to quasi-arcs, we obtain the desired. □

Lemma 4.7. *All carrots $C_\Gamma(\rho)$ with $\Gamma \in \mathcal{Z}$ and $\rho \in (0, 1)$ sufficiently close to* 1 *are quasi-disks. Moreover, the map $P : R_\Gamma(\rho) \cup L_\Gamma(\rho) \to R_{P(\Gamma)}(\rho^d) \cup L_{P(\Gamma)}(\rho^d)$ is quasi-symmetric.*

Proof. Take $\Gamma \in \mathcal{Z}$ with root point $z = z_\Gamma$. If z is periodic, then it is repelling, and the map $P : \mathrm{Bd}(C_\Gamma(\rho)) \to \mathrm{Bd}(C_{P(\Gamma)}(\rho^d))$ is quasi-symmetric (note that P is conformal on a neighborhood of the boundary of $C_\Gamma(\rho)$). In this case, $C_\Gamma(\rho)$ is a quasi-disk by Proposition 4.6. Suppose now that z is strictly preperiodic. We may assume by induction that $C_{P(\Gamma)}(\rho^d)$ is a quasi-disk. If z is not critical, then it follows immediately that $C_\Gamma(\rho)$ is also a quasi-disk, as a pullback of $C_{P(\Gamma)}(\rho^d)$ under a map that is one-to-one and conformal in a neighborhood of $\mathrm{Bd}(C_\Gamma(\rho))$. It also follows that the map $P : \mathrm{Bd}(C_\Gamma(\rho)) \to \mathrm{Bd}(C_{P(\Gamma)}(\rho^d))$ is quasi-symmetric.

Finally, assume that z is critical. Let n be the smallest positive integer with $P^n(z)$ periodic, and let m be the minimal period of $P^n(z)$. It suffices to prove that

$$P^n : R_\Gamma(\rho) \cup L_\Gamma(\rho) \to R_{P^n(\Gamma)}(\rho^{d^n}) \cup L_{P^n(\Gamma)}(\rho^{d^n})$$

is QS. Indeed, we may choose local coordinates x near z and y near $P^n(z)$ so that the map P^n takes the form $y = x^k$ for some integer $k > 1$ (this integer is the local degree of P^n at z). Moreover, the y coordinate can be chosen so that P^m takes the form $y \mapsto \lambda y$. In these coordinates, Lemma 2.11 applies and yields the desired. □

4.2. **The carrot modification of P.** Fix $\rho \in (0, 1)$ close to 1. Then the carrots $C_\Gamma(\rho)$ with $\Gamma \in \mathcal{Z}$ as defined above are disjoint. Recall that $U(\rho)$ is the bounded component of $\mathbb{C} \setminus E(\rho)$. Then $K_P \subset U(\rho)$. In this section, we modify P to form a new map $P^c : \overline{U}(\rho) \to \overline{U}(\rho^d)$. First, let us define P^c so that $P^c = P$ outside of $\bigcup_{\Gamma\in\mathcal{Z}_{cr}} C_\Gamma(\rho)$, where $\mathcal{Z}_{cr}$ is the set of all *critical* cuts in $\mathcal{Z}$, i.e., cuts with critical root points.

Suppose now that $\Gamma \in \mathcal{Z}_{cr}$. Set $P^c = P$ on $R_\Gamma(\rho) \cup L_\Gamma(\rho)$. Note that $E_\Gamma(\rho)$ wraps around the entire $E(\rho^d)$ under P. Define P^c on $E_\Gamma(\rho)$ as a QS isomorphism between $E_\Gamma(\rho)$ and $E_{P(\Gamma)}(\rho^d)$. Thus, by the remark made above, P^c is necessarily different from P on $E_\Gamma(\rho)$. Finally, let $P^c : C_\Gamma(\rho) \to C_{P(\Gamma)}(\rho^d)$ be a QS map that extends the already defined map $P^c : \mathrm{Bd}(C_\Gamma(\rho)) \to \mathrm{Bd}(C_{P(\Gamma)}(\rho^d))$. The existence of such extension is guaranteed by Theorem 2.7. The map $P^c : \overline{U}(\rho) \to \overline{U}(\rho^d)$ is a *carrot modification* of P. Clearly, $P^c : U(\rho) \to U(\rho^d)$ is a proper map; let d_c be its topological degree. Observe that $d_c < d$ provided that $\mathcal{Z}_{cr} \neq \varnothing$.

Lemma 4.8. *There is a quasi-regular degree d_c polynomial $f : \mathbb{C} \to \mathbb{C}$ such that $f = P^c$ on $\overline{U}(\rho)$.*

Proof. The map $P^c : U(\rho) \to U(\rho^d)$ is glued of several (finitely many) quasi-regular maps along quasi-arcs. Such map is itself quasi-regular, as follows from the "QC removability" of quasi-arcs, cf. Proposition 4.9.9 of [Hub06].

Set $\mathbb{D}(\rho) = \{z \in \mathbb{D} \mid |z| < \rho\}$; this is the disk of radius ρ around 0. Clearly, there is a quasi-regular map $g : \overline{\mathbb{D}}(\rho) \to \overline{\mathbb{D}}(\rho^d)$ such that $g(z) = z^{d_c}$ in a neighborhood of 0, and $g = \psi_P^{-1} \circ P^c \circ \psi_P$ on the boundary of $\mathbb{D}(\rho)$. It suffices to define f as P^c on $\overline{U}(\rho)$ and as $\psi_P \circ g \circ \psi_P^{-1}$ on $\mathbb{C} \setminus \overline{U}(\rho)$. □

Let A_0 be a topological annulus such that the bounded complementary component of A_0 lies in $\overline{U}(\rho)$, and $\overline{\partial} f = 0$ in the unbounded complementary component. We may assume that A_0 is bounded by $E(\rho)$ from the inner side and by $E(\rho^{d^{T_0}})$ from the outer side. Here $T_0 > 1$ is some positive integer. Set $A = A_0 \cup A_{cr}$, where $A_{cr} = \bigcup_{\Gamma\in\mathcal{Z}_{cr}} C_\Gamma(\rho)$. Then by definition $\overline{\partial} f = 0$ outside of A. In order to verify the assumptions of Theorem 2.8, it remains to prove the following lemma.

Lemma 4.9. *Define T_{cr} as the cardinality of $\mathcal{Z}_{cr}$; set $T = T_{cr} + T_0$. The forward f-orbit of any point $x \in \mathbb{C}$ can visit A_{cr} at most T_{cr} times. Therefore, it can visit A at most T times.*

Proof. It suffices to prove the first statement. Define the subset $\mathsf{X} \subset \mathbb{D}$ consisting of all points, whose polar coordinates (ρ, θ) satisfy

$$\rho \leqslant e^{-|\theta - \theta_\Gamma|}$$

for at least one periodic $\Gamma \in \mathcal{Z}$ (then Γ is necessarily degenerate by our assumption on $\mathcal{Z}$). Clearly, X is forward invariant under the map

$(\rho,\theta)\mapsto(\rho^d,d\theta)$. Moreover, $\psi_P(\mathsf{X})$ includes all $C_\Gamma(\rho')$ for all periodic $\Gamma\in\mathcal{Z}$ and all $\rho'\in(0,1)$.

Suppose now that $\rho_0\in(0,1)$ is sufficiently close to 1. A forward P-orbit of a point $x\in A_{cr}$ may visit A_{cr} at most T_{cr} times before it first enters $\psi_P(\mathsf{X})$ (that is, it may visit each $C_\Gamma(\rho_0)$ with $\Gamma\in\mathcal{Z}_{cr}$ at most once). Thus it suffices to prove that no point of $\psi_P(\mathsf{X})$ can map to $C_\Gamma(\rho_0)$ with $\Gamma\in\mathcal{Z}_{cr}$ under an iterate of f. Since $\psi_P(\mathsf{X})$ is forward invariant, it suffices to choose ρ_0 so that $C_\Gamma(\rho_0)\cap\psi_P(\mathsf{X})=\varnothing$ for all $\Gamma\in\mathcal{Z}_{cr}$.

Take any $\Gamma\in\mathcal{Z}_{cr}$. The set of all angles θ such that $R_P(\theta)$ is separated from $A_P(\mathcal{Z})$ by Γ is an arc I_Γ of $\mathbb{R}/\mathbb{Z}$ whose length is an integer multiple of $1/d$ (indeed, the endpoints of this arc are mapped to the same point under the d-tupling map). Define

$$\mathsf{C}(\rho_0,I_\Gamma)=\{(\rho,\theta)\mid \exists\theta_0\in I_\Gamma\quad \rho_0\leqslant\rho\leqslant e^{-|\theta-\theta_0|}\}.$$

Then all points in $C_\Gamma(\rho_0)\setminus K_P$ are necessarily in $\psi_P(\mathsf{C}(\rho_0,I_\Gamma))$. It is clear that no θ_Γ with periodic Γ can belong to I_Γ. Therefore, $\mathsf{C}(\rho_0,I_\Gamma)$ is disjoint from X for ρ_0 sufficiently close to 1. It follows that $C_\Gamma(\rho_0)\cap\psi_P(\mathsf{X})=\varnothing$, as desired. □

The set $A_P(\mathcal{Z})$ is a fully invariant set for f. We have $d_c\geqslant 2$ since the map $P:A_P(\mathcal{Z})\to A_P(\mathcal{Z})$ is not injective by the assumptions of the Main Theorem. Thus all assumptions of Theorem 2.8 are fulfilled. Then there is a QC map $\Psi:\overline{\mathbb{C}}\to\overline{\mathbb{C}}$ and a rational map $Q:\mathbb{C}\to\mathbb{C}$ of degree d_f such that $Q\circ\Psi=\Psi\circ f$. It can be arranged that $\Psi(\infty)=\infty$. With this normalization, $Q^{-1}(\infty)=\{\infty\}$, therefore, Q is a degree d_c polynomial.

Theorem 4.10. *The set $\Psi(A_P(\mathcal{Z}))$ coincides with K_Q.*

Proof. Since $A_P(\mathcal{Z})$ is P-stable, $\Psi(A_P(\mathcal{Z}))\subset K_Q$. It remains to prove that any point $y=\Psi(x)$ with $x\notin A_P(\mathcal{Z})$ escapes to infinity under the iterations of Q. Equivalently, x escapes to infinity under the iterations of f. Indeed, if the forward f-orbit of x is outside of K_P and outside of all carrots, then $f^n(x)=P^n(x)\to\infty$. If x is in K_P but not in $A_P(\mathcal{Z})$, then $f^k(x)\in C_\Gamma(\rho)$ for some $k\geqslant 0$ and some $\Gamma\in\mathcal{Z}$. Possibly replacing ρ with ρ^{d^l} with a suitable l and x with $f^l(x)$, we may assume that Γ is periodic. However, in this case $C_\Gamma(\rho)$ is in the P-basin of infinity, hence $f^n(x)=P^{n-k}\circ f^k(x)\to\infty$. □

Since $\deg(Q)=\deg(f)=d_c<\deg(P)$, the proof of the Main Theorem is now complete.

4.3. **The case when P is injective on $A_P(\mathcal{Z})$.** Theorem 4.11 completes the proof of the Main Theorem.

Theorem 4.11. *Suppose that all assumptions of the Main Theorem are fulfilled and $P : A_P(\mathcal{Z}) \to A_P(\mathcal{Z})$ is one-to-one. Then $A_P(\mathcal{Z})$ is a single repelling point.*

Proof. Replace P with a suitable iterate to arrange that all periodic cuts in $\mathcal{Z}$ are fixed. Let Γ be such a fixed cut, and z_Γ its root point. Then $P(z_\Gamma) = z_\Gamma$. We claim that there are no critical points of P in $A_P(\mathcal{Z}) \setminus \mathrm{Rt}_\mathcal{Z}$. Indeed, consider a critical point $c \in A_P(\mathcal{Z})$. A point $w \in A_P(\mathcal{Z})$ near $P(c)$ has at least two preimages z, z' near c. If both are in the principal component of $\mathbb{C} \setminus \bigcup \mathcal{W}_\mathcal{Z}$, then both are in $A_P(\mathcal{Z})$, a contradiction. Thus, say, z' is not in the principal component; then it must be separated from $A_P(\mathcal{Z})$ by a cut from $\mathcal{Z}$. Since w can be chosen arbitrarily close to $P(c)$, the point c itself must belong to $\mathrm{Rt}_\mathcal{Z}$.

Suppose that $A_P(\mathcal{Z})$ is not a singleton. Then Theorem 7.4.7 of [BFMOT13] is applicable to the P-invariant continuum $A_P(\mathcal{Z})$. This theorem states that there is a rotational fixed point in $A_P(\mathcal{Z})$. That is, either a non-repelling fixed point a or a repelling fixed point a such that the external rays of P landing at a undergo a nontrivial combinatorial rotation. If a is non-repelling, then there is a critical point c that is not preperiodic and not separated from a by $\mathcal{Z}$. In the attracting and parabolic cases this follows from classical results of Fatou [Fat20]. Suppose that a is a Cremer of Siegel fixed point. This case was considered in Theorem 4.3 [BCLOS16] (the proof is based upon [BM05] and classical results of Mañé [Man93]) that implies that then $A_P(\mathcal{Z})$ must contain a recurrent critical point. By the previous paragraph this leads to a contradiction. Thus a is repelling and rotational. However, since $P|_{A_P(\mathcal{Z})}$ is one-to-one, this implies that there are no other fixed points in $A_P(\mathcal{Z})$. Therefore, $a = z_\Gamma$; but the latter is non-rotational, a contradiction. We conclude that $A_P(\mathcal{Z})$ is a singleton. □

5. Proof of Theorem 1.5

Consider a continuum $Y \subset K_P$ such that $P : Y \to Y$ is a degree k branched covering. By definition, there are open neighborhoods U and V of Y and a degree k branched covering $\tilde{P} : U \to V$ such that $P = \tilde{P}$ on Y and $\tilde{P}^{-1}(Y) = Y$. For every $y \in Y$, the local multiplicity $\mu_Y(y)$ is defined as the multiplicity of y with respect to $\tilde{P}$. For all $z \neq P(y)$ very close to $P(y)$, exactly $\mu_Y(y)$ points of $P^{-1}(z) \cap Y$ are near y. If y is not critical then $\mu_Y(y) = 1$. On the other hand, some critical points of P in Y may also have multiplicity 1 with respect to Y (that is, with respect to $\tilde{P}$). Irregular points of Y are precisely the points $y \in Y$ with $\mu_{K_P}(y) > \mu_Y(y)$.

The proof of Theorem 1.5 splits into several steps.

5.1. **Reduction to the case when K_P is connected.** Let P and Y be as above. Suppose first that K_P is disconnected. Let $K_P(Y)$ be the component of K_P containing Y. Clearly, $K_P(Y)$ is a P-invariant continuum. Choose a tight equipotential E_{V^*} around $K_P(Y)$ so that the disk V^* bounded by E_{V^*} does not contain escaping critical points of P. Then $P : U^* \to V^*$ is a PL map with filled Julia set $K_P(Y)$, where U^* is the component of $P^{-1}(V^*)$ containing Y. By Theorem 3.3, the PL map $P : U^* \to V^*$ is *hybrid equivalent* to a PL restriction of a polynomial, say, P^*. Let Y^* be the subset of K_{P^*} corresponding to $Y \subset K_P(Y)$. Evidently, P^* and Y^* satisfy the assumptions of Theorem 1.5. Thus, we can consider only polynomials with connected Julia sets.

5.2. **Defining an admissible collection of cuts.** From now on, assume that the Julia set of P is connected. Start by defining a collection of cuts whose root points are irregular points of Y. Let a be an irregular point; it is necessarily a critical point of P. By the assumptions of Theorem 1.5, the point a is eventually mapped to a repelling periodic point. It follows from the Landing Theorem that there are preperiodic external rays landing at a. Recall that a cut $\Gamma = R \cup L \cup \{a\}$ formed by a and two rays R, L landing at a such that $P(R) = P(L)$ is a *critical cut*. The corresponding wedge W is called a *critical wedge* at a. A critical wedge W at a is Y-*empty* if $W \cap Y = \varnothing$.

Lemma 5.1. *Suppose that $\mu_Y(a) < \mu_{K_P}(a)$. Then there is at least one Y-empty critical wedge W at a.*

Proof. Consider all components of $Y \setminus \{P(a)\}$; let s be the number of them. By [BOT21] and Theorem 6.6 of [McM94], every component of $Y \setminus \{P(a)\}$ is separated from the next one in the cyclic order by an external ray landing at $P(a)$. Denoting components of $Y \setminus \{P(a)\}$ by $Y_1, \dots, Y_s$ and external rays landing at a and separating these components by $R_1, \dots, R_s$ we may assume that

$$Y_1 \prec R_1 \prec Y_2 \prec R_2 \prec \cdots \prec Y_s \prec R_s$$

where $\prec$ indicates positive (counterclockwise) circular direction.

If we pull this picture back to a we will see that there are $\mu_{K_P}(a)$ pullbacks of each ray R_i and $\mu_{K_P}(a)$ pullbacks of each set Y_j from the previous paragraph "growing" out of a. Since $\mu_Y(a) < \mu_{K_P}(a)$, not all pullbacks of sets Y_i are contained in Y, some of them are *not* contained in Y. However, it follows from the definitions, in particular, from the fact that $P|_Y$ coincides with $\tilde{P}|_Y$, that the circular order of the pullbacks of Y_i contained in Y must follow that of the sets Y_1, $\dots$, Y_s. Let us now choose a pullback of Y_1 that *is* contained in Y, and move from it in the positive direction. We will be encountering pullbacks of sets Y_i and pullbacks of rays R_j in the same order of increasing of their subscripts until we reach the next pullback of Y_1. However, $\mu_Y(a) < \mu_{K_P}(a)$. Hence at some moment in this process the pullback $Y_i' \subset Y$ of Y_i and the following it pullback $Y_{i+1}' \subset Y$ of Y_{i+1} are not located in the adjacent pullbacks of the wedges between the corresponding external rays. Rather, there will be a pullback R_i' of R_i and then the next (in the sense of positive circular order) pullback R_i'' of R_i such that there are no points of Y in between these rays. The wedge between R_i' and R_i'' is the desired Y-empty critical wedge W at a. $\square$

Define $\mathcal{Z}^{irr}$ ("irr" stands for "irregular") as the set of boundary cuts of all Y-empty critical wedges at all irregular points of Y. More precisely, for every irregular point $a \in Y$, mark specific s rays separating s components of $Y \setminus \{P(a)\}$. Then choose all Y-empty critical wedges at a bounded by pullbacks of the marked rays (cf. Lemma 5.1). The family of cuts $\mathcal{Z}^{irr}$ is clearly admissible.

5.3. **Reducing to the case of no irregular points.** We keep the notation introduced above. By definition of $\mathcal{Z}^{irr}$, we have $Y \subset A_P(\mathcal{Z}^{irr})$. By the Main Theorem applied to P and $\mathcal{Z}^{irr}$, there is a polynomial P^* such that $P^* : K_{P^*} \to K_{P^*}$ is topologically conjugate to $P : A_P(\mathcal{Z}^{irr}) \to A_P(\mathcal{Z}^{irr})$. Let Y^* be the P^*-invariant continuum corresponding to Y under this conjugacy. We claim that Y^* contains no irregular points.

If $a \in Y^*$ is an irregular point, then $\mu_{Y^*}(a) < \mu_{K_{P^*}}(a)$. By Lemma 5.1, there is a Y^*-empty critical wedge at a. A corresponding Y-empty critical wedge at a must be included into $\mathcal{Z}^{irr}$; a contradiction. Thus all points of Y^* are regular.

Replacing P with P^* and Y with Y^*, we may now assume that $P : Y \to Y$ has no irregular points. However, then $P : Y \to Y$ satisfies the assumptions of Theorem 1.4. (Observe that the absence of irregular points is equivalent to the condition that Y is a component of $P^{-1}(Y)$.) The conclusion of Theorem 1.5 now follows from Theorem 1.4.

Proof of Corollary 1.6. Assume that Y is an invariant non-degenerate fiber of P. We claim that $P|_Y$ is a degree k branched covering for some $k > 1$. It is easy to see that planar fibers map onto (and locally onto) planar fibers (see, e.g., [Sch99] or [BCLOS16]); in particular, $P(Y) =$

Y. Observe that if $P|_Y$ is 1-to-1 then all the arguments of Theorem 4.11 apply to Y and imply that Y is a singleton, a contradiction. Hence there are points of Y with more than one preimage in Y.

Suppose that $z \in Y$ is irregular. Then z is critical, and there are pairs of points y', y'' arbitrarily close to z such that $P(y') = P(y'') = y$, where $y' \in Y$ and $y'' \notin Y$. We claim that then z is preperiodic, and there are several (rational) rays that land at z. Suppose otherwise. Choose a rational cut Γ'' that separates z and y''; set $\Gamma = P(\Gamma'')$. By the assumption, Γ'' does not contain z. But then there exists another cut $\Gamma' \subset P^{-1}(\Gamma)$ that separates y' from z, a contradiction. Hence z is preperiodic and there are rational rays landing at z. This implies that z maps to a repelling periodic point (recall that by the assumptions of Corollary 1.6 there are no parabolic points in Y) and fulfills one of the assumptions of Theorem 1.5.

By definition of a fiber, there is a wedge W_z at z such that $\mathrm{Bd}(W_z) = \Gamma_z$ is a critical cut, $Y \subset \overline{W}$, and $P^{-1}(P(\Gamma_z)) \cap W_z = \varnothing$. Consider the collection $\mathcal{Z}^{irr} = \{\Gamma_z\}$, where z runs through the set of all irregular points of Y. Clearly, $\mathcal{Z}^{irr}$ is admissible and satisfies the assumptions of the Main Theorem. It follows that the corresponding avoiding set $A = A_P(\mathcal{Z}^{irr}) \supset Y$ gives rise to a polynomial Q such that $Q : K_Q \to K_Q$ is topologically conjugate to $P : A \to A$. Moreover, the conjugacy extends as a positively oriented homeomorphism between neighborhoods of K_Q and A. (This extension is not a conjugacy, however.) Passing from P to Q, we may assume that Y has no irregular points at all. In this case, $P : U \to V$ is a degree k covering for some $k > 1$ and some neighborhoods U, V of Y. Moreover, $Y = P^{-1}(Y) \cap U$. Thus, Y satisfies all assumptions of Theorem 1.5, and we are done. □

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(Alexander Blokh and Lex Oversteegen) Department of Mathematics, University of Alabama at Birmingham, Birmingham, AL 35294-1170

(Peter Haïssinsky) Aix-Marseille Univ., CNRS, I2M, UMR 7373, 39, rue Frédéric Joliot Curie, 13453 Marseille Cedex 13, France

(Vladlen Timorin) Faculty of Mathematics, HSE University, Russian Federation, 6 Usacheva St., 119048 Moscow

(Vladlen Timorin) Independent University of Moscow, Bolshoy Vlasyevskiy Pereulok 11, 119002 Moscow, Russia

Email address, Alexander Blokh: ablokh@math.uab.edu

Email address, Peter Haïssinsky: peter.haissinsky@math.cnrs.fr

Email address, Lex Oversteegen: overstee@math.uab.edu

Email address, Vladlen Timorin: vtimorin@hse.ru