OVER-ROTATION INTERVALS OF BIMODAL INTERVAL MAPS

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Abstract. We describe all bimodal over-twist patterns and give an algorithm allowing one to determine what the left endpoint of the over-rotation interval of a given bimodal map is. We then define a new class of polymodal interval maps called *well behaved*, and generalize onto them the above results.

1. INTRODUCTION

The paper is devoted to studying rotation theory for interval maps. We divide Introduction in several subsections.

1.1. **Motivation for rotation theory.** The celebrated Sharkovsky Theorem [Sha64, Sha-tr] elucidates the rich combinatorial behavior of periodic orbits of continuous interval maps. To state it let us first introduce the *Sharkovsky ordering* for the set $\mathbb{N}$ of positive integers:

$$3 \succ_s 5 \succ_s 7 \succ_s \dots \succ_s 2\cdot 3 \succ_s 2\cdot 5 \succ_s 2\cdot 7 \succ_s \dots$$
$$\succ_s \dots 2^2\cdot 3 \succ_s 2^2\cdot 5 \succ_s 2^2\cdot 7 \succ_s \dots \succ_s 8 \succ_s 4 \succ_s 2 \succ_s 1$$

If $m \succ_s n$, say that m is ***shar**per* than n. Denote by $Sh(k)$ the set of all positive integers m such that $k \succ_s m$, together with k, and by $Sh(2^\infty)$ the set $\{1, 2, 4, 8, \dots\}$ which includes all powers of 2. Denote also by $P(f)$ the set of the periods of cycles of a map f (by the *period* we mean the *least* period). Theorem 1.1 was proven by A. N. Sharkovsky.

Theorem 1.1 ([Sha64, Sha-tr])**.** *If $f : [0,1] \to [0,1]$ is a continuous map, $m \succ_s n$ and $m \in P(f)$, then $n \in P(f)$. Therefore there exists $k \in \mathbb{N} \cup \{2^\infty\}$ such that $P(f) = Sh(k)$. Conversely, if $k \in \mathbb{N} \cup \{2^\infty\}$ then there exists a continuous map $f : [0,1] \to [0,1]$ such that $P(f) = Sh(k)$.*

The above theorem provides a full description of possible sets of periods of cycles of continuous interval maps. Moreover, it shows that various periods *force* one another in the sense that if $m \succ_s n$ then, for a continuous interval map f, the existence of a cycle of period m *forces* the existence of a cycle of period n.

However, the period is a rough characteristic of a cycle as there are a lot of cycles of the same period. A much finer way of describing cycles is by considering the permutations induced by cycles. Moreover, one can define *forcing* relation among permutations in a natural way,

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and, by [Bal87], this relation is a partial order. Still, a drawback here is that while the forcing relation among cyclic permutations is very fine, the results are much more complicated than for periods and a transparent description of possible sets of permutations exhibited by the cycles of a continuous interval map is apparently impossible (see, e.g., [ALM00]). This motivates one to look for another, middle-of-the-road way of describing cycles, a way not as crude as periods but not as fine as permutations, which would still allow for a transparent description.

1.2. **Functional rotation numbers and rotation theory on the circle.** Looking for a new way of describing cycles of interval maps, it is natural to use as prototypes the results for circle maps of degree one due to Misiurewicz [Mis82] who used the notion of the *rotation number*. This notion was first introduced by Poincarè [Poi] for circle homeomorphisms, then extended to circle maps of degree one by Newhouse, Palis and Takens [NPT83], and then studied, e.g., in [BGMY80], [Ito81], [CGT84], [Mis82], [Mis89], [ALMM88] (see [ALM00] with an extensive list of references). One can define rotation numbers in a variety of cases using the following approach [MZ89], [Zie95]. Let X be a compact metric space, $\phi : X \to \mathbb{R}$ be a bounded measurable function (often called an *observable*), $f : X \to X$ be a continuous map. Then for any x the set $I_{f,\phi}(x)$ of all limits of the sequence $\frac{1}{n}\sum_{i=0}^{n-1}\phi(f^i x)$ is called the *ϕ-rotation set* of x. It is easy to see that $I_{f,\phi}(x)$ is a closed interval. If $I_{f,\phi}(x) = \{\rho_\phi(x)\}$ is a singleton, then the number $\rho_\phi(x)$ is called the *ϕ-rotation number* of x. The union of all ϕ-rotation sets of points of X is called the *ϕ-rotation set* of the map f and is denoted by $I_f(\phi)$.

If x is an f-periodic point of period n then its rotation number $\rho_\phi(x)$ is well-defined, and a useful related concept of the *ϕ-rotation pair* of x can be introduced; namely, the pair $(\frac{1}{n}\sum_{i=0}^{n-1}\phi(f^i x), n)$ is said to be the *ϕ-rotation pair* of x. Evidently, for all points from the same cycle their ϕ-rotation pairs (t, n) are the same, and their ϕ-rotation numbers are $\frac{t}{n}$.

For functions ϕ related to the dynamics of f one might get additional results about ϕ-rotation sets; e.g., this happens for rotation numbers in the circle degree one case [Mis82]. Let $f : S^1 \to S^1$ be a continuous map of degree 1, $\pi : \mathbb{R} \to S^1$ be the natural projection which maps an interval $[0, 1)$ onto the whole circle. Fix a lifting F of f. Define $\phi_f : S^1 \to \mathbb{R}$ so that $\phi_f(x) = F(X)-X$ for any point $X \in \pi^{-1}x$; then ϕ_f is well-defined, the classical rotation set of a point z is $I_{f,\phi_f}(z) = I_f(z)$ and the classical rotation number of z is $\rho_{f,\phi_f}(z) = \rho(z)$ whenever exists.

The *rotation set* of the map f is $I_f = \cup I_f(x)$; it follows from [NPT83], [Ito81] that I_f is a closed interval (cf [Blo94]). The sum $\sum_{i=0}^{n-1}\phi_f(f^i x) = m$ taken along the orbit of an n-periodic point x is an integer which defines a pair $(m, n) \equiv rp(x)$ called the *rotation pair* of x; denote the set of all rotation pairs of periodic points of f by $RP(f)$. For real $a \le b$ let $N(a, b) = \{(p, q) \in \mathbb{Z}^2_+ : p/q \in (a, b)\}$ (in particular $N(a, a) = \emptyset$). For $a \in \mathbb{R}$ and $l \in \mathbb{Z}_+ \cup \{2^\infty\}$ let $Q(a, l)$ be empty if a is irrational;

otherwise let it be $\{(ks, ns) : s \in Sh(l)\}$ where $a = k/n$ with k, n coprime.

Theorem 1.2 ([Mis82]). *For a continuous circle map f of degree 1 such that $I_f = [a, b]$ there exist $l, r \in \mathbb{Z}_+ \cup \{2^\infty\}$ such that $RP(f) = N(a, b) \cup Q(a, l) \cup Q(b, r)$.*

Observe that, equivalently, one can talk about continuous degree one maps F of the real line to itself. Each such map F is a lifting of (is locally monotonically semiconjugate to) a continuous degree one map f of the unit circle. For brevity in the future we will talk about *(lifted) periodic points of F* meaning points $x \in \mathbb{R}$ that project to f-periodic points of the unit circle $\mathbb{S}$ under a canonical projection $\pi : \mathbb{R} \to \mathbb{S}$. Then Theorem 1.2 can be viewed as a result describing possible rotation pairs and numbers of periodic points of continuous degree one maps of the real line to itself.

1.3. **Rotation theory on the interval.** The choice of ϕ_f is crucial for Theorem 1.2 and is dynamically motivated. It turns out that, with an appropriate choice of the observable, results similar to Theorem 1.2 can be obtained for interval maps too. First it was done when the rotation numbers for interval maps were introduced in [Blo94 - Blo95b] (see also [BK98]).

Namely, one defines the *rotation pair* of a non-degenerate cycle as (p, q), where q is the period of the cycle and p is the number of its elements which are mapped to the left of themselves. Let us denote the rotation pair of a cycle P by $rp(P)$ and the set of the rotation pairs of all cycles of a map f by $RP(f)$. The number p/q is called the *rotation number* of the cycle P. We introduce the following partial ordering among all pairs of integers (p, q) with $0 < p < q$. We will write $(p, q) \succ (r, s)$ if either $1/2 \le r/s < p/q$, or $p/q < r/s \le 1/2$, or $p/q = r/s = m/n$ with m and n coprime and $p/m \succ_s r/m$ (notice that $p/m, r/m \in \mathbb{N}$).

Theorem 1.3. *If $f : [0, 1] \to [0, 1]$ is continuous, $(p, q) \succ (r, s)$ and $(p, q) \in RP(f)$ then $(r, s) \in RP(f)$.*

This theorem makes it possible to give a full description of the sets of rotation pairs for continuous interval maps (as in Theorem 1.1, all theoretically possible sets really occur), see [Blo95b]. This description is similar to the one for circle maps of degree one (see [Mis82]).

A further development came with another choice of the observable made in [BM97]. The results of [BM97] imply those of [Blo94 - Blo95b], hence from now on we will study the new invariants introduced in [BM97]. Let $f : [0, 1] \to [0, 1]$ be continuous, $Per(f)$ be its set of periodic points, and $Fix(f)$ be its set of fixed points. It is easy to see that if $Per(f) = Fix(f)$ then $\omega(y)$ is a fixed point for any y. Assume from now on that $Per(f) \ne Fix(f)$ and define a function $\chi_f = \chi$ as follows:

$$\chi(x) = \begin{cases} 1/2 & \text{if } (f(x) - x)(f^2(x) - f(x)) \le 0, \\ 0 & \text{if } (f(x) - x)(f^2(x) - f(x)) > 0. \end{cases}$$

For any non-fixed periodic point y of period $p(y)$ the integer $l(y) = \sum_{i=0}^{n-1} \chi(f^i y)$ is at most $p(y)/2$ and is the same for all points from the orbit of y. The pair $orp(y) = (l(y), p(y))$ is called the *over-rotation pair* of y, and *coprime over-rotation pair* if p, q are coprime. Notice that in an over-rotation pair (p, q) both p and q are integers and $0 < p/q \leq 1/2$.

The set of all over-rotation pairs of periodic non-fixed points of f is denoted by $ORP(f)$ and the χ-rotation number $\rho_\chi(y) = \rho(y) = l(y)/p(y)$ is called the *over-rotation number* of y. Observe that by Theorem 1.1 and by the assumption that $Per(f) \neq Fix(f)$ it follows that f has a point of period 2 and that the over-rotation number of this point is 1/2; in other words, the set of all over-rotation numbers of periodic points of f includes 1/2 and, therefore, 1/2 belongs to the union of all χ-rotation sets $I_{f,\chi}(x)$ defined above.

Theorem 1.4 ([BM97]). *If $(p, q) \gg (k, l)$ and $(p, q) \in ORP(f)$ then $(k, l) \in ORP(f)$.*

Theorem 1.4 implies Theorem 1.1. Indeed, let f be an interval map and consider odd periods. For any $2n + 1$ the closest to 1/2 over-rotation number of a periodic point of period $2n + 1$ is $\frac{n}{2n+1}$. Clearly $\frac{n}{2n+1} < \frac{n+1}{2n+3} < \frac{1}{2}$. Hence for any periodic point x of period $2n + 1$ its over-rotation pair $orp(x)$ is $\gg$-stronger than the pair $(n + 1, 2n + 3)$, and by Theorem 1.4 the map f has a point of period $2n + 3$. Also, for any m we have $(n, 2n + 1) \gg (m, 2m)$, so by Theorem 1.4 the map f has a point of any even period $2m$. Applying this to the maps $f, f^2, f^4, \dots$ one can prove Theorem 1.1 for all periods but the powers of 2; additional arguments covering the case of powers of 2 are quite easy.

Theorem 1.4 implies a full description of sets $ORP(f)$ for interval maps, close to that from Theorem 1.2; in fact, the same description applies to sets $RP(f)$ (see Theorem 1.3) except that a set $ORP(f)$ is always located to the left of 1/2 because over-rotation numbers are less than or equal to $\frac{1}{2}$). To state the corresponding result we introduce new notation. Let $\mathcal{M}$ be the set consisting of 0, 1/2, all irrational numbers between 0 and 1/2, and all pairs (α, n), where α is a rational number from $(0, 1/2]$ and $n \in \mathbb{N} \cup \{2^\infty\}$. Then for $\eta \in \mathcal{M}$ the set $Ovr(\eta)$ is equal to the following. If η is an irrational number, 0, or 1/2, then $Ovr(\eta)$ is the set of all over-rotation pairs (p, q) with $\eta < p/q \leq 1/2$. If $\eta = (r/s, n)$ with r, s coprime, then $Ovr(\eta)$ is the union of the set of all over-rotation pairs (p, q) with $r/s < p/q \leq 1/2$ and the set of all over-rotation pairs (mr, ms) with $m \in Sh(n)$. Notice that in the latter case, if $n \neq 2^\infty$, then $Ovr(\eta)$ is equal to the set of all over-rotation pairs (p, q) with $(nr, ns) \gg (p, q)$, plus (nr, ns) itself.

Theorem 1.5. *Given a continuous interval map f, there exists $\eta \in \mathbb{N}$ such that $ORP(f) = Ovr(\eta)$. Conversely, if $\eta \in \mathcal{M}$ then there exists a continuous map $f : [0, 1] \to [0, 1]$such that $ORP(f) = Ovr(\eta)$.*

The closure of the set of over-rotation numbers of periodic points of f is an interval $I_f = [\rho_f, 1/2], 0 \leq \rho_f \leq 1/2$, called the *over-rotation interval* of f.

1.4. **The role of periodic points.** As the reader may have noticed by now, we introduced the functional rotation numbers considering all points x; on the other hand, our main focus has been on periodic points and their over-rotation numbers and pairs. It is then natural to consider the role of functional rotation numbers of periodic points in functional rotation sets of maps, including their density in those sets. On the other hand, evidently, Theorem 1.5 is modeled after Theorem 1.2. It is then also natural to consider other parallels between classical rotation numbers defined for the circle maps of degree one, and over-rotation numbers for interval maps. In fact, this is one of the main ideas of the present paper as applies to N-bimodal and other similar classes of interval maps. However first we study a different (but related) analogy between circle maps of degree one and interval maps, namely we study the role of periodic points and their (over-)rotation numbers in both circle and interval cases.

Indeed, even without the complete description of possible rotation pairs of periodic points of circle maps of degree one given in Theorem 1.2, one can show that for any degree one continuous circle map f either f is monotonically conjugate to an irrational rotation of the circle, or the rotation numbers of its periodic points are dense in its rotation interval. This fact is related to a more general problem of establishing the connection between the ϕ-rotation numbers of *periodic points* of a map f, and the *ϕ-rotation set* $I_f(\phi)$ of the map f for any function ϕ. We describe this connection in the case of interval maps and circle maps of degree one (see, e.g., [Blo95c]); our explanation is based upon the so-called "spectral decomposition" for one-dimensional maps [Blo86, Blo87a, Blo87b], [Blo95a] (we will only use it for interval and circle maps).

To state the appropriate results we need a few basic definitions as well as a couple of less standard ones (see, e.g., [DGS76]). Given a cycle A of period n, a unique invariant probability measure ν_A concentrated on A is the measure assigning to each point of A the weight $\frac{1}{n}$; let us call ν_f a *CO-measure* [DGS76] (comes from "**c**losed **o**rbit"). Recall that for Borel measures on compact spaces one normally considers their *weak* topology defined by the continuous functions [DGS76].

Theorem 1.6 ([Blo86, Blo87a, Blo87b, Blo95a]). *Suppose that $f : I \to I$ is a continuous interval map or a circle map with non-empty set of periodic points. Then any invariant probability measure μ for whom there exists a point x with $\mu(\omega_f(x)) = 1$ can be approximated by CO-measures arbitrary well. In particular, CO-measures are dense in all ergodic invariant measures of f.*

For the sake of completeness let us also state the result which describes maps of compact one-dimensional branched manifolds (abusing the language we will call them *graphs* from now on) which do *not* have periodic points (see [AK79] for the circle and [Blo84, Blo86, Blo87a, Blo87b] for maps of any graph). Observe that we do not assume our graphs to be connected; also, to avoid trivialities let us assume that our maps are *onto* (otherwise we can simply consider the nested sequence of images of the space and take their intersection which will still be a graph and on which the map will be onto). A natural one-dimensional map without periodic points is an irrational circle rotation. An extension of that is a map which permutes (not necessarily cyclically) a finite collection of circles so that this collections falls into several cycles of circles and in each cycle the appropriate power of the map fixes the circles and acts on each of them as an irrational rotation (it is easy to see that then in each cycle of circles it is the rotation by the same irrational angle, but these angles may change from a cycle to a cycle). Let us call such maps *multiple irrational circle rotations.*

It turns out that multiple irrational circle rotations are prototypes of all graph maps without periodic points. By a *monotone* map of a topological space we mean a map such that point-preimages (sometimes called *fibers*) are connected.

Theorem 1.7 ([Blo84])**.** *Suppose that $f : X \to X$ is a continuous map of a graph to itself with no periodic points. Then there exists a monotone map from X to a union Y of several circles which semiconjugates f and a multiple irrational circle rotation.*

In Subsection 1.2 we explained that given a degree one circle map $f : \mathbb{S} \to \mathbb{S}$, one can define the function ϕ_f by choosing a lifting $F : \mathbb{R} \to \mathbb{R}$, then for any $x \in \mathbb{S}$ a lifting X of x, and then setting $\phi_f(x) = F(X) - X$ so that ϕ_f which is well-defined and continuous (because f is of degree one and continuous). Classical rotation numbers and sets of points of $\mathbb{S}$ under f are in fact ϕ_f-rotation numbers and sets. Evidently, Theorem 1.6, the definition of weak topology on probability invariant measures of f, and the definition of the classical rotation numbers and sets imply that rotation numbers of periodic points are dense in the rotation set of a circle map $f : \mathbb{S} \to \mathbb{S}$ of degree one provided f has some periodic points. Actually, the fact that the classical rotation set of a degree one circle map is a closed interval can also be deduced from the "spectral decomposition", however this goes way beyond the scope of the present paper.

The situation with over-rotation numbers is similar but slightly more complicated. The issue here is that for over-rotation numbers, the dynamics in small neighborhoods of fixed points can play a misleading role. To explain this, let us draw analogy with the case of the topological entropy (see [AKM65] where the concept was introduced and [ALM00, DGS76] for a detailed description of its properties). It is known that for continuous interval maps it can happen so that the entropy of such maps is large (even infinite) while it is assumed on smaller and smaller invariant sets converging to fixed points of the map. Similarly, it can happen that the dynamics in a small neighborhood of, say, an attracting fixed point a is chaotic in the sense that points "switch sides", i.e. map from the left of a to the right of a, in a chaotic fashion while still being attracted to a. That may lead to a rich set of sequences $\chi(f^i(x))$ and large χ-rotational sets of such points while having no bearing upon the set of periodic points of the map at all. To avoid this "artificial" richness we consider only *admissible* points.

Namely, by a *limit measure* of a point x we mean a limit of ergodic averages of the δ-measure concentrated at x; clearly, any limit measure is invariant [DGS76]. If μ is a unique limit measure of x, then x is said to be *generic* for μ. Call a point x *admissible* if any limit measure μ of x is such that $\mu(Fix(f)) = 0$ where $Fix(f)$ is the set of all fixed points of f; since μ is invariant, this implies that in fact the set of all points x which are eventual preimages of fixed points of f is of zero μ-measure too. Since the set of discontinuities of χ is contained in the union of the set of fixed points $Fix(f)$ of f and their preimages, we see that for an admissible point x the set of discontinuities of χ is of zero limit measure for any limit measure of x. Now, let x be an admissible point. Take a number $u \in I_{f,\chi}(x)$; the definitions and properties of measures imply that $u = \int \chi(x)\, d\mu$ where μ is a limit measure of x. By Theorem 1.6 and by definitions μ can be approximated arbitrarily well by a CO-measure concentrated on a non-fixed periodic orbit. Hence u can be approximated arbitrarily well by the over-rotation number of a non-fixed periodic orbit, and so $u \in I_f$. Thus, $I_{f,\chi}(x) \subset I_f$ as long as x is admissible (see Theorem 1.8).

Additional arguments allow us to prove Theorem 1.8, describing the connection between I_f and the pointwise χ-rotation sets $I_{f,\chi}(x)$.

Theorem 1.8 ([Blo94, Blo95c])**.** *The following statements are true.*

(1) *If f is continuous and $\rho_f < 1/2$ then for any $a \in (\rho_f, 1/2]$ there is an admissible point x, generic for a measure μ, such that $I_f(x) = \{a\}$.*
(2) *If x is an admissible point, then $I_{f,\chi}(x) \subset I_f = [\rho_f, 1/2]$.*
(3) *If f is piecewise-monotone and $\rho_f \neq 0$ then there exists an invariant measure μ such that f is minimal on the support of μ and there exists a point x, generic for μ and such that $I_{f,\chi}(x) = \{\rho_f\}$.*

Theorem 1.8(3) cannot be extended for all continuous interval maps as one can design a map f which has a sequence of invariant intervals with their "own" maps that have increasing to $[u, \frac{1}{2}]$ over-rotation intervals; evidently, for such a map f the conclusions of Theorem 1.8 do not hold.

In a recent paper by Jozef Bobok [Bo] the case covered in Theorem 1.8(3) is studied in great detail and depth resulting into a much more precise claim. Recall that a dynamical system is said to be *strictly ergodic* if it has a unique invariant measure. To state Theorem 1.9 in full generality we need a couple of notions on which we will elaborate later in Subsection 2.1. Namely, a cyclic permutation π *forces* a cyclic permutation θ if a continuous interval map f which has a cycle inducing π always has a cycle inducing θ. By [Bal87] forcing is a partial ordering. One can talk about the *over-rotation pair* $orp(\pi)$ and the *over-rotation number* $\rho(\pi)$ of a cyclic permutation π. We call a cyclic permutation π an *over-twist* if it does not force other cyclic permutations of the same over-rotation number.

Theorem 1.9 ([Bo]). *Let a point x and a measure μ be as defined in Theorem 1.8(3). Then the map $f|_{\omega(x)}$ is strictly ergodic with μ being the unique invariant measure of $f|_{\omega(x)}$. Moreover, if ρ_f is rational then x is periodic and can be chosen so that the permutation induced by the orbit of x is an over-twist of over-rotation number ρ_f.*

The above results allow one to make conclusions about the dynamics of an interval map based upon little information: if one knows ρ_f then one can, e.g., describe all possible over-rotation numbers of f-periodic points (except, in the non-piecewise monotone case, the number ρ_f itself if it is rational). In other words, numerical information about a map, compressed to I_f, implies various types of the limit behavior of periodic points reflected by their rotation numbers. Can one say more? In particular, can we explicitly describe at least some permutations induced by periodic orbits of f? By definition the affirmative answer can be given if one can explicitly describe the over-twists of given over-rotation numbers. In addition, it is important to design a practical approach (an algorithm) to figuring out what the over-rotation interval of a map f is.

In this paper we address these issues for bimodal interval maps of the type "increasing-decreasing-increasing" (so-called *N-bimodal maps* or bimodal maps *of type N*). The paper develops ideas from [BS13] (the results of [BS13] are described in Section 2). In particular, one of the tools used in [BS13] was a special disconnected conjugacy of a unimodal map to a discontinuous map of the interval which can be lifted to the degree one discontinuous map of the real line; in the present paper we show that this tool apply to a wider class of maps, including N-bimodal ones.

Our paper is divided into sections as follows:

(1) Section 2 contains preliminaries.
(2) In Section 3 we will show that, given an N-bimodal map f we can construct its lifting to a degree one map of the real line which admits a continuous monotonically increasing lower bound function G whose classical rotation number gives us the left endpoint of the over-rotation interval of f.
(3) In Section 4, as an application we will describe the bimodal permutations which are forcing-minimal among all permutations with the same over-rotation number (i.e., N-bimodal *over-twist permutations*).
(4) In Section 5, we describe a general class of continuous maps, called *well behaved maps*, for which a construction similar to the one from Section 3 goes through allowing for finding the orbit on which the left endpoint of the over-rotation interval is assumed. If f is a *map* like that and the over-rotation interval of f is $I_f = [\frac{p}{q}, \frac{1}{2}]$ where $\frac{p}{q} \in \mathbb{Q}, p, q \in \mathbb{Z}, g.c.d(p, q) = 1, q \neq 0$ then our construction gives a transparent prescription as to where a periodic orbit x of f with over-rotation number $\frac{p}{q}$ must be located.

2. PRELIMINARIES

This section is divided into short subsection devoted to certain topics in one-dimensional dynamical systems.

2.1. **Combinatorial dynamics in one-dimension.** We need definitions from *one-dimensional combinatorial dynamics* ([ALM00]). A map f has a *horseshoe* if there are two closed intervals I, J with disjoint interiors whose images cover their union. In particular, f has a horseshoe if there exist points a, b, c such that either $f(c) \le a = f(a) < b < c \le f(b)$ (set $I = [a, b], J = [b, c]$) or $f(c) \ge a = f(a) > b > c \ge f(b)$ (set $I = [b, a], J = [c, b]$). It is easy to see [BM97] that if a map has a horseshoe then it has periodic points of all possible over-rotation numbers. A *(cyclic) pattern* is the family of all cycles on the real line that induce the same cyclic permutation of the set $T_n = \{1, 2, \dots, n\}$ or its flip; a map (not even necessarily one-to-one) of the set T_n into itself is called a *non-cyclic pattern*. If one considers the family of all cycles on the real line that induce the same cyclic permutation (i.e., one does not allow for a flip), this family is called a *cyclic* ***oriented*** *pattern*. If an interval map f has a cycle P from a pattern Π associated with permutation π, we say that P is a *representative* of π in f and f *exhibits* π (on P); if f is *monotone (linear)* on each complementary to P interval, we say that f is *P-monotone (P-linear)* [MN90]. In what follows the same terminology will apply to permutations, patterns and cycles, so for brevity we will be introducing new concepts for, say, permutations. Observe also, that permutations are understood up to orientation. Finally, notice that in what follows we will interchangeably talk about permutations and patterns.

A permutation π is said to have a *block structure* if there is a collection of pairwise disjoint segments $I_0, \dots, I_k$ with $\pi(T_n \cap I_j) = T_n \cap I_{j+1}, \pi(T_n \cap I_k) = T_n \cap I_0$; the intersections of T_n with intervals I_j are called *blocks* of π. A permutation without a block structure is said to be with *bo block structure*, or, equivalently, *irreducible*. If we collapse blocks to points, we get a new permutation π', and then π is said to have a block structure *over* π'. A permutation π *forces* a permutation θ if any continuous interval map f which exhibits π also exhibits θ. By [Bal87] forcing is a partial ordering. If π has a block structure over a pattern θ, then π forces θ. By [MN90] for each permutation π there exists a unique irreducible pattern π' over which π has block structure (thus, π' is forced by π).

The following construction is a key ingredient of one-dimensional combinatorial dynamics. Let π be a (non-cyclic) permutation, Π be its pattern, P be a finite set with a map $f : P \to P$ of pattern Π, and f be a P-linear map; assume also that the convex hull of P is $[0, 1]$. Say that the closure $\overline{I}$ of a component I of $[0, 1] \setminus P$ *π-covers* the closure $\overline{J}$ of another such component J if $\overline{J} \subset f(\overline{I})$. Construct the oriented graph G_π whose vertices are closures of the components of $[0, 1] \setminus P$ and whose edges (arrows) go from $\overline{I}$ to $\overline{J}$ if and only if $\overline{I}$ π-covers $\overline{J}$. Clearly, G_π does not depend on the actual choice of P and the definition is consistent.

A cycle is *divergent* if it has points $x < y$ such that $f(x) < x$ and $f(y) > y$. A cycle that is not divergent will be called *convergent*. It is well-known that if a pattern is divergent then for any cycle P of this pattern the P-linear map has a horseshoe; on the other hand, it is easy to see that if a pattern is convergent then a cycle P of this pattern cannot give rise to the P-linear map with a horseshoe.

2.2. **Rotation theory on the interval.** One can talk about the *over-rotation pair* $orp(\pi)$ and the *over-rotation number* $\rho(\pi)$ of a permutation π. We call a permutation π an *over-twist permutation* (or just an *over-twist*) if it does not force other permutations of the same over-rotation number; the pattern of an over-twist permutation is said to be an *over-twist* pattern. Theorem 1.4 and the properties of forcing imply the existence of over-twist patterns of any given rational over-rotation number between 0 and 1; in fact, it implies that a map which has a periodic point of rational over-rotation number ρ exhibits an over-twist pattern of rotation number ρ. By Theorem 1.4 an over-twist pattern has a coprime over-rotation pair; in particular, over-twists of over-rotation number $1/2$ are of period 2, so from now on we consider over-twists of over-rotation numbers distinct from $1/2$.

Suppose that π is a convergent pattern and that P is a periodic orbit of pattern π. Let f be a P-linear map. Then f has a unique fixed point a. Consider the set $Q = P \cup \{a\}$ and denote its pattern by π'. We will work with the oriented graph $G_{\pi'}$. Suppose that there is a real-valued function ψ defined on arrows of $G_{\pi'}$. It is well-known [ALM00] that the maximal and the minimal averages of ψ along all possible paths (with growing lengths) in $G_{\pi'}$ are assumed, in particular, on periodic sequences. If the values of ψ on arrows are all rational, then the maximum and the minimum of those averages are rational too.

We choose a specific function ψ as follows. Associate to each arrow in $G_{\pi'}$ the number 1 if it corresponds to the movement of points from the right of a to the left of a. Otherwise associate 0 to the arrow. As explained above, this yields rational maximum and rational minimum of limits of averages of ψ taken along all possible paths (with growing lengths) in $G_{\pi'}$, and these extrema are assumed on periodic sequences. Given a cycle of f, one can consider its cycles and compute out for them their over-rotation numbers; simultaneously, ψ-rotation numbers can be computed out for the associated paths (loops) in the oriented graph $G_{\pi'}$. Evidently, the over-rotation numbers of f-cycles and the ψ-rotation numbers of the associated loops in $G_{\pi'}$ are the same.

We also need to introduce some classical concepts.

Definition 2.1. A map $F : \mathbb{R} \to \mathbb{R}$ is said to be of degree 1 if $F(x+1) - F(x) = 1$ for any $x \in \mathbb{R}$.

Classical results of Poincaré [Poi] apply to all monotonically increasing maps of the real line of degree one [RT86] for whom every point $y \in R$ has the same classical rotation number defined as the limit of the sequence $F^n(x)/n$.

3. N-BIMODAL MAPS

Given a rational number $\rho, 0 < \rho \leq 1/2$, we want to describe all over-twist patterns of N-bimodal type of over-rotation number ρ. In the beginning of this section we outline our approach to the problem.

Our arguments are based upon an extension of a construction from [BS13] onto the N-bimodal case. This gives rise to a special lifting of a given N-bimodal interval map f to a degree one *discontinuous* map F_f of the real line. The construction is designed to guarantee that over-rotation numbers of f-periodic points of the interval coincide with the classical Poincaré rotation numbers of the corresponding points under F_f. Even though the classical tools of [Mis82] do not apply to F_f (after all, F_f is discontinuous), our construction implies the existence of a *continuous non-strictly monotonically increasing* function $G_f \leq F_f$ with important properties.

Namely, the monotonicity of G_f implies that G_f is semiconjugate to a circle map $\tau_f : \mathbb{S} \to \mathbb{S}$ and τ_f is monotone (i.e., point preimages under τ_f are connected) circle map which is either locally constant or locally increasing (we consider counterclockwise direction on $\mathbb{S}$ as positive). One can define the G_f-rotation number for every point $y \in \mathbb{R}$, and for all y's this number will be the same; denote it by ρ'_f. The set A_f of points y such that $G_f(y) < F_f(y)$ is, evidently, open; it follows from the construction, that G_f is a constant on each component of A_f. It is well-known that then there exist points x whose G_f-trajectories avoid A_f, and if ρ'_f is rational then there exists a point x on which G_f acts so that the associated to x points $x' \in [0, 1]$ and $x'' \in \mathbb{S}$ are both periodic. Evidently, the classical rotation pair of x'' and the over-rotation pair of x' coincide. This implies that the over-rotation pair of x' is coprime.

Moreover, we have that (1) $G_f \leq F_f$, (2) the rotation numbers of points of F_f equal the over-rotation numbers of the corresponding points of $[0, 1]$, (3) on x and, inductively, on all its images, we have $F_f = G_f$, and (4) the classical rotation number ρ'_f of G_f can be computed on the orbit of x. This implies that the left endpoint ρ_f of the over-rotation interval of f equals ρ'_f and that it is assumed on the f-orbit of x'. We use results of [BB19] to deduce then that the pattern of x' is an over-twist. The classical rotation numbers of x in the sense of F_f and in the sense of G_f are the same because the maps are the same on the trajectory of x. Any f-periodic point $y \in [0, 1]$ has its over-rotation number equal to the rotation number of y in the sense of F_f; since $G_f \leq F_f$, it is greater than or equal to that of x (on whose trajectory $G_f = F_f$). Hence on the trajectory of x the over-rotation number is minimal among all cycles of f.

Definition 3.1. By an N-bimodal map we shall mean a continuous map $f : [0, 1] \to [0, 1]$ which satisfies the following properties:

(1) f has a unique fixed point a_f, a point of local maxima M_f, and a point of local minima m_f such that $M_f < a_f < m_f$;
(2) $f(M_f) = 1$ and $f(m_f) = 0$.

If $f(0) > a_f > f(1)$ then it is easy to see that the interval $[0, a_f]$ maps into the interval $[a_f, 1]$ and vice versa. Clearly, the over-rotation interval of f is degenerate and coincides with $\{1/2\}$. We consider this case as trivial and do not deal with it in the rest of the paper. Thus, from now we assume that the point a_f has preimages from at least one side; we may assume that $f(1) \geq a_f$ and, therefore, there is always a preimage a''_f of a_f with $m_f > a''_f > a_f$. We will also use the following notation. Let $d_1(f)$ be the unique point in the interval (M_f, a_f) such that $f(d_1(f)) = f(1)$. Clearly, $d_1(f)$ exists by the intermediate value theorem. Similarly, let $d_2(f)$ be the unique point in the interval (a_f, m_f) such that $f(d_2(f)) = f(0)$ ($d_2(f)$ is actually defined only if $f(0) \leq a_f$, otherwise we assume $d_2(f)$ to be undefined).The N-bimodal map in Case1 and Case2 are shown in Figure 1 and 2 respectively.

FIGURE 1. ***An N-bimodal map*** f ***in case when*** $d_2(f)$ ***is defined***

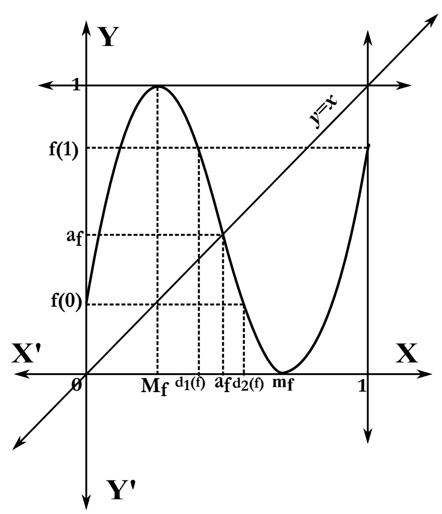
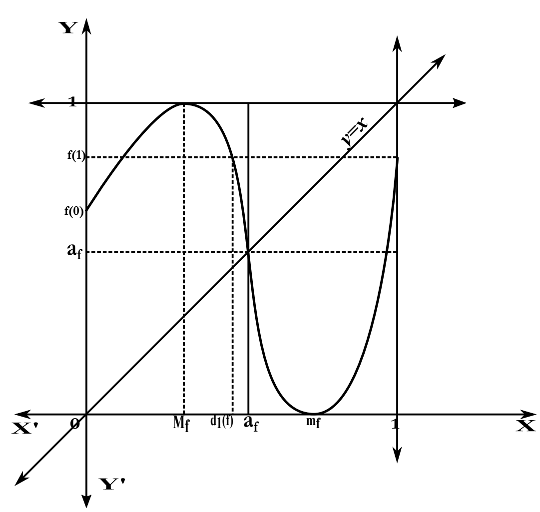


FIGURE 2. ***An N-bimodal map*** f ***in case*** $d_2(f)$ ***is not defined***



3.1. **Disconnected lifting of an N-bimodal map f.** In this subsection we construct a few maps all of which are based upon a given N-bimodal map f. For the sake of simplicity of notation we often are not using subscript f; this is justified because at this point there are no other maps and, therefore, omitting the subscript will not lead to ambiguity. However in the future we may occasionally use subscripts to emphasize dependence of our construction upon a given map.

Consider the map $g_f = \sigma_f \circ f \circ \sigma_f^{-1}$ where the map σ_f is defined below. However, before defining σ we need to make an observation. Our aim is to study periodic *non-fixed* points of f and their over-rotation numbers; we do it by working with periodic *non-fixed* points of g_f. Therefore the behavior of f at a_f is not important for us. This allows us to ignore the fact that with the adopted below definitions the map σ_f and, as a result, the function g_f is multivalued at a_f and its preimages (indeed, this does not have any bearing upon the over-rotation numbers of *non-fixed* f-periodic points as they never map to a_f). Let us now denote the over-rotation interval I_f of f by $I_f = [\mu, \frac{1}{2}]$, and consider a discontinuous conjugacy $\sigma_f : [0,1] \to [0,1]$ defined by

$$\sigma_f(x) = \begin{cases} x & \text{if } 0 \le x \le a_f \\ a_f + 1 - x & \text{if } a_f \le x \le 1 \end{cases} \tag{3.1}$$

The map σ_f flips the interval $[a_f, 1]$ symmetrically with respect to the midpoint $\frac{1+a_f}{2}$ of $[a_f, 1]$ so that σ_f^2 is the identity on the entire $[0,1]$. We now define the map $g_f : [0,1] \to [0,1]$, $g = \sigma_f \circ f \circ \sigma_f^{-1}$ to which σ_f conjugates the map f. In what follows by σ'_f we mean the map σ_f restricted upon $[a_f, 1]$; moreover, if we flip points of the plane in the vertical direction with respect to the line $y = \frac{1+a_f}{2}$ we shall say that we apply *vertical* σ'_f, and if we flip points of the plane in the horizontal direction with respect to the line $x = \frac{1+a_f}{2}$ we shall say that we apply *horizontal* σ'_f.

Case 1: $f(0) \le a_f$. Then a_f has two preimages, a'_f and a''_f, and we have $0 < a'_f < M_f < a_f < m_f < a''_f < 1$. Let us now describe the graph of the function g_f by giving the expression for $g_f(x)$ depending on the location of x.

(a) On the interval $[0, a'_f]$, $g_f(x) = f(x)$, that is, the graph of g_f is the same as the graph of f in the interval $[0, a'_f]$.

(b) On the interval $[a'_f, a_f]$, $\sigma_f(x) = x$ and $f(\sigma_f(x)) \ge a_f$ so that $g_f(x) = a_f + 1 - f(x)$; the graph of g_f is obtained from that of f by applying the vertical σ'_f to it.

(c) On the interval $[a_f, a_f+1-a''_f]$, $g_f(x) = a_f+1-f(a_f+1-x)$. In other words, this part of the graph of g_f can be obtained from the part of the graph of f located above the interval $[a''_f, 1]$ by first applying the horizontal σ'_f to it, and then applying the vertical σ'_f to it.

(d) On the interval $[a_f + 1 - a''_f, 1]$, $g_f(x) = f(a_f + 1 - x)$. So, the graph of g in the interval, $[a_f, 1]$ can be obtained from the graph of f located above $[a_f, a'_f]$ by applying the horizontal σ'_f to it.

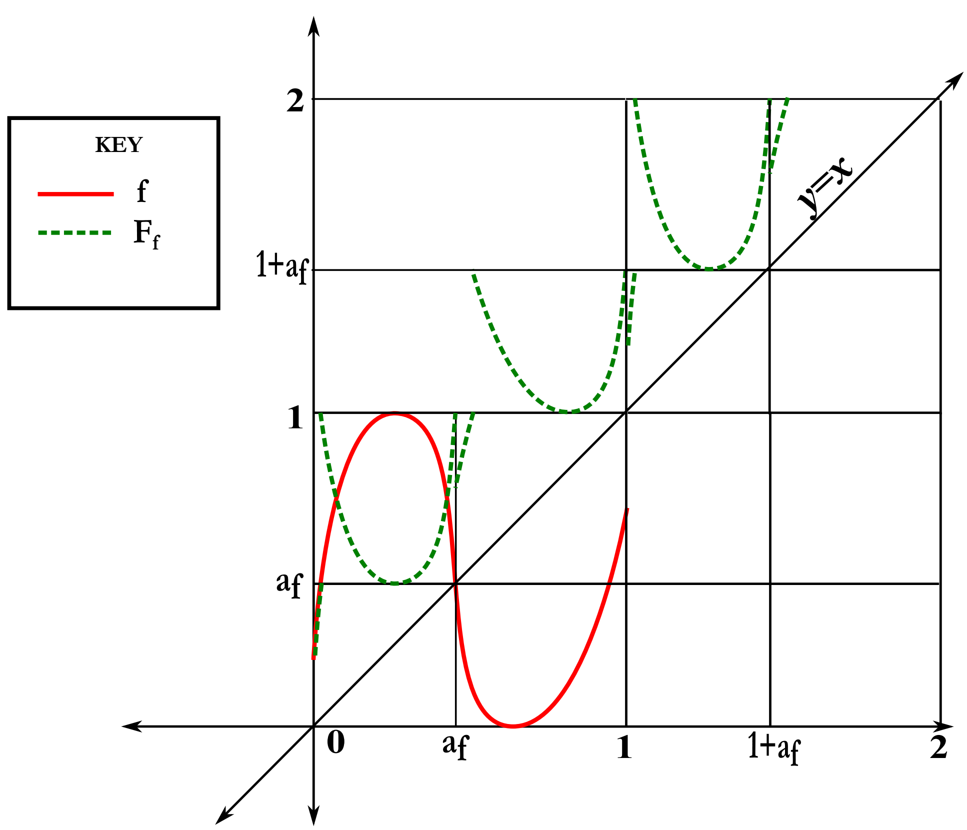
It immediately follows from the definitions, that $I_f = I_g$. Hence in studying over-rotation numbers of periodic points we can concentrate upon the map g_f. We do so by defining a *degree one* lifting F_f of g_f to the real line. The lifting is designed so that the classic rotation numbers of points of F_f corresponding to periodic points of g_f in fact equal over-rotation numbers of these periodic points of g_f.

Here is how we define a degree one lifting $F_f : \mathbb{R} \to \mathbb{R}$;the idea is to keep $F_f = g_f$ everywhere except for the interval $[a_f + 1 - a''_f, 1]$ located to the right of a_f on which points are mapped to the left of a_f so that when we compute out the corresponding over-rotation number the number 1 should be added:

$$F_f(x) = \begin{cases} g_f(x) = f(x) & \text{if } 0 \le x \le a'_f \\ g_f(x) = a_f + 1 - f(x) & \text{if } a'_f \le x \le a_f \\ g_f(x) = a_f + 1 - f(a_f + 1 - x) & \text{if } a_f \le x < a_f + 1 - a''_f \\ g_f(x) + 1 = 1 + f(a_f + 1 - x) & \text{if } a_f + 1 - a''_f \le x \le 1 \end{cases} \tag{3.2}$$

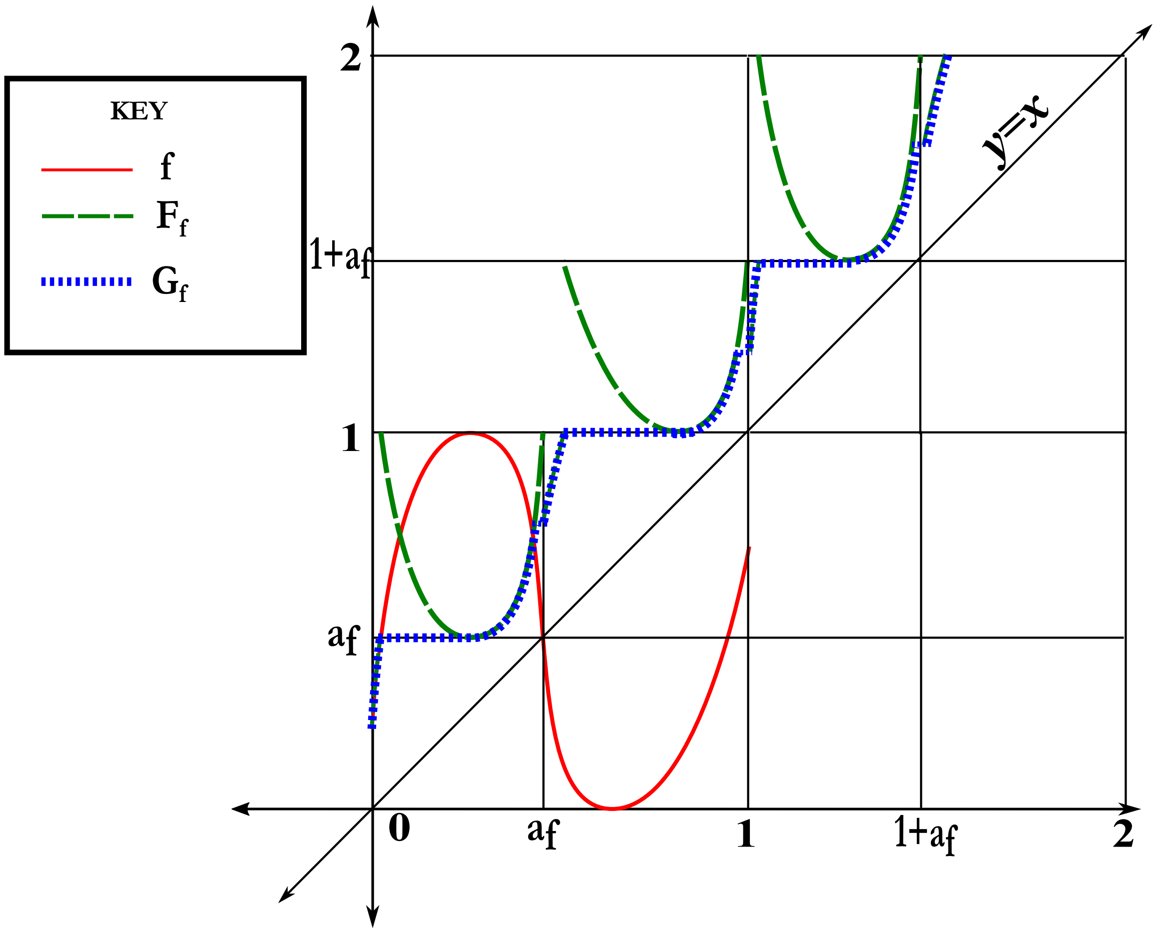
Then we extend F_f onto the real line as a degree one map; this means that on each $[n, n+1]$ where $n \in \mathbb{Z}$ we set $F_f(x+n) = F_f(x) + n$ for all $n \in \mathbb{Z}$ and $x \in [0,1]$. The graph of the F_f is shown in Figure 3.

FIGURE 3. ***Construction of the map F_f for an N-bimodal map f in Case 1***



The last step in this series of maps is a continuous map $G_f : \mathbb{R} \to \mathbb{R}$.

FIGURE 4. ***Construction of the map G_f for an N-bimodal map f in Case 1***



The map G_f is defined on each $[n, n+1], n \in \mathbb{Z}$ in the following fashion:

$$G_f(x) = \begin{cases} F_f(x) & \text{if } n \leq x \leq n + a'_f \\ n + a_f & \text{if } n + a'_f \leq x \leq n + M_f \\ F_f(x) & \text{if } n + M_f \leq x \leq n + d_1(f) \\ F_f(d_1) & \text{if } n + d_1(f) \leq x \leq n + a_f \\ F_f(x) & \text{if } n + a_f \leq x \leq n + a_f + 1 - a''_f \\ n + 1 & \text{if } n + a_f + 1 - a''_f \leq x \leq n + a_f + 1 - m_f \\ F_f(x) & \text{if } n + a_f + 1 - m_f \leq x \leq n + a_f + 1 - d_2(f) \\ F_f(n) + 1 & \text{if } n + a_f + 1 - d_2(f) \leq x \leq n + 1 \end{cases} \tag{3.3}$$

The graph of G_f is shown in the Figure 4.

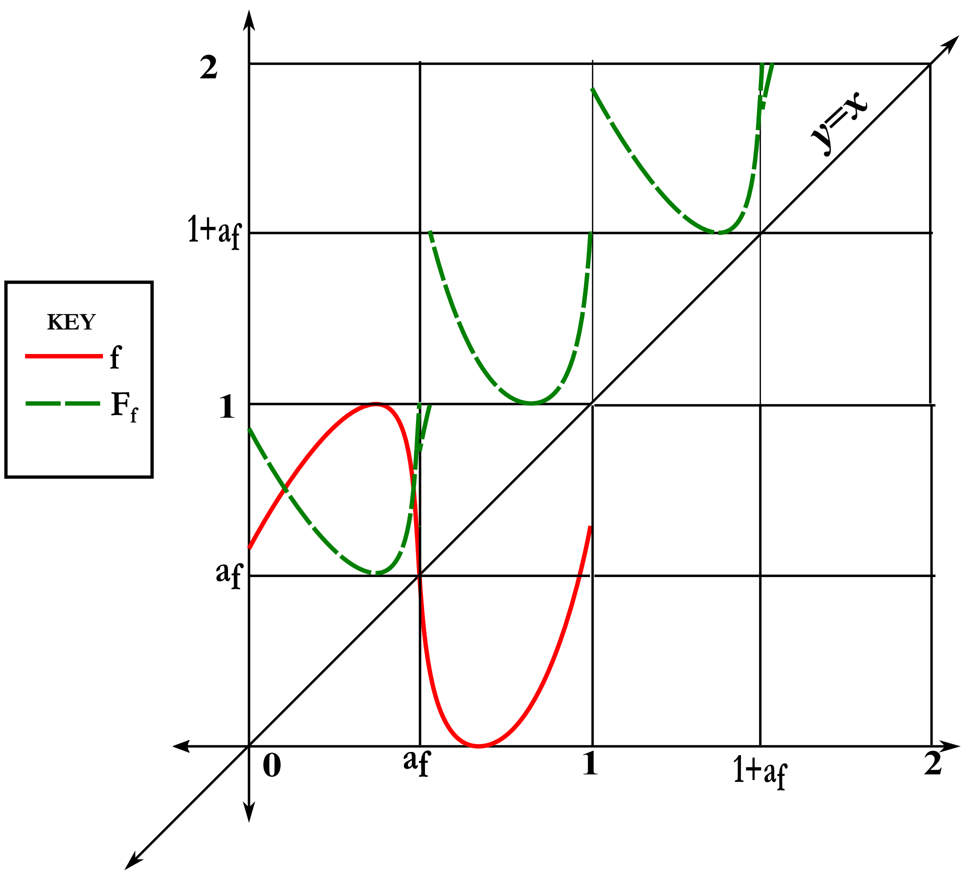
Case 2: $f(0) > a_f$. In this case a_f has only one preimage, namely a''_f, and we have $0 < M_f < a_f < m_f < a''_f < 1$. The points a'_f and $d_2(f)$ are undefined. The functions g_f, F_f and G_f will be slightly different. On the interval $[0, a_f]$, $g_f(x) = a_f + 1 - f(x)$, i.e. the graph of g_f can be obtained from the graph of f by applying vertical σ_f. The function g_f

is same as in the earlier case on the interval $[a_f, 1]$.The map $F_f : \mathbb{R} \to \mathbb{R}$ is now defined as follows:

$$F_f(x) = \begin{cases} g_f(x) = a_f + 1 - f(x) & \text{if } 0 \le x \le a_f \\ g_f(x) = a_f + 1 - f(a_f + 1 - x) & \text{if } a_f \le x < a_f + 1 - a''_f \\ g_f(x) + 1 = 1 + f(a_f + 1 - x) & \text{if } a_f + 1 - a''_f \le x \le 1 \end{cases} \tag{3.4}$$

Then, as before, we define F_f on each $[n, n+1]$, $n \in \mathbb{Z}$ as $F_f(x+n) = F_f(x) + n$ for all $n \in \mathbb{Z}$ and $x \in [0, 1]$. The graph of F_f is shown in Figure 5.

FIGURE 5. ***Construction of the map F_f for an N-bimodal map f in Case 2***

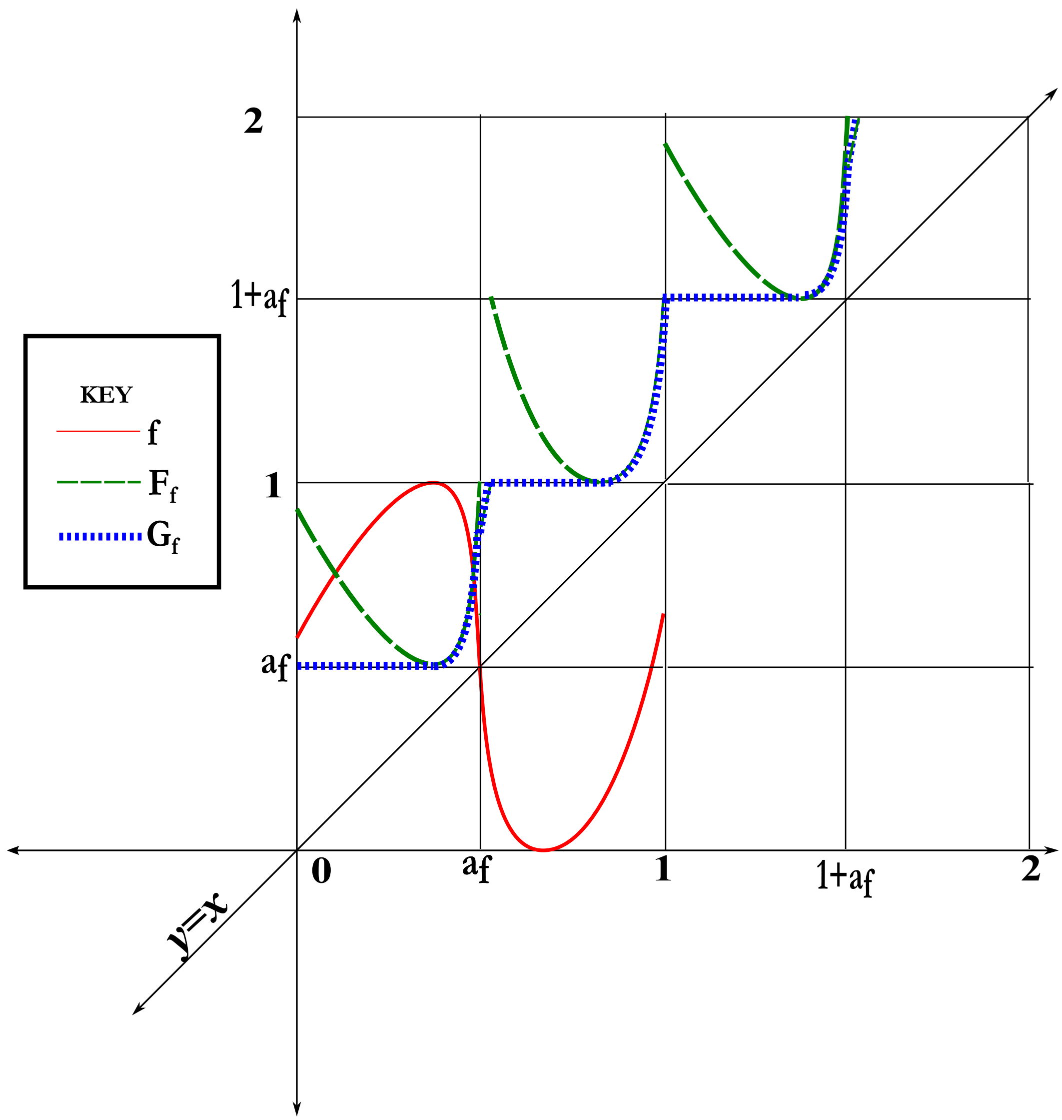


On each $[n, n+1]$, $n \in \mathbb{Z}$ the map $G_f : \mathbb{R} \to \mathbb{R}$ will now be defined as follows:

$$G_f(x) = \begin{cases} n + a_f & \text{if } n \le x \le n + M_f \\ F_f(x) & \text{if } n + M_f \le x \le n + d_1(f) \\ F_f(d_1) & \text{if } n + d_1(f) \le x \le n + a_f \\ F_f(x) & \text{if } n + a_f \le x \le n + a_f + 1 - a''_f \\ n + 1 & \text{if } n + a_f + 1 - a''_f \le x \le n + a_f + 1 - m_f \\ F_f(x) & \text{if } n + a_f + 1 - m_f \le x \le n + 1 \end{cases} \tag{3.5}$$

The graph of G_f is shown in Figure 6.

FIGURE 6. ***Construction of the map G_f for an N-bimodal map f in Case 2***



In what follows we will consider the relation of the classical Poincaré rotation numbers of points of the real line under F_f and over-rotation numbers of points of $[0, 1]$ in the sense of the map f (equivalently, the map g_f). We reserve the just introduced notation for the maps g_f, F_f and G_f (assuming that a map f is given). Notice, that, as one can see from the above, we will often deal with (continuous) functions on the real line that have open intervals on which the functions are constants. Let us call maximal such interval *flat spots* (of the corresponding function). Moreover, the same terminology trivially applies to circle maps too.

3.2. **Rotation numbers for f and F.** We are ready to prove the next theorem that relates the rotation numbers of the above constructed maps. By a *minimal* set of a map h of a compact space to itself we mean an invariant compact set Z all of whose points have dense trajectories in Z (thus, a compact invariant subset of Z coincides with Z). For the map f considered in Theorem 3.2 we assume that all agreements and notation discussed in the beginning of Section 3 hold; this time, however, we will use the subscripts to emphasize the dependence of the construction upon a given map. Moreover, it is useful to compare this theorem with Theorem 1.9.

Theorem 3.2. *Let $f : [0,1] \to [0,1]$ be an N-bimodal map. Then the continuous non-strictly monotonically increasing function $G_f : \mathbb{R} \to \mathbb{R}$ has the rotation number ρ'_f coinciding with the left endpoint ρ_f of the over-rotation interval $[\rho_f, \frac{1}{2}]$ of the map f. Furthermore, there exists a minimal f-invariant set Z_f such that for every point $y \in Z_f$ we have $I_{f,\chi} = \rho_f$, and there are two possibilities:*

1. *$\rho'_f = \rho_f$ is rational, Z_f is a periodic orbit, and $f|_{Z_f}$ is canonically conjugate to the circle rotation by ρ_f restricted on one of its cycles so that the over-rotation pair of Z_f coincides with the classical rotation pair of the circle rotation by ρ_f (in particular, the over-rotation pair of Z_f is coprime) and the over-rotation interval of the Z_f-linear map is $[\rho_f, 1/2]$;*
2. *$\rho'_f = \rho_f$ is irrational, Z_f is a Cantor set, and $f|_{Z_f}$ is canonically at most two-to-one semi-conjugate to the circle rotation by ρ_f.*

Moreover, define the set Y_f as

$$Y_f = [0, a'_f] \cup [M_f, d_1(f)] \cup [d_2(f), m_f] \cup [a''_f, 1]$$

if $f(0) \le a_f$ and, therefore, a'_f exists and is well-defined, or as

$$Y_f = [M_f, d_1(f)] \cup [a_f, m_f] \cup [a''_f, 1]$$

if $f(0) > a_f$. Then $Z_f \subset Y_f$ and for every point y whose trajectory is contained in Y_f we have that $I_{f,\chi}(y) = \{\rho_f\}$.

Proof. In Subsection 3.1 we introduced the maps F_f and G_f; in doing so, we considered Case and Case 2 depending upon whether $f(0) < a_f$ (Case 1) or $f(0) \ge a_f$ (Case 2). In the proof of this theorem we will consider only Case 1 as Case 2 is completely analogous to Case 2.

Evidently, the classical rotation numbers and rotation pairs of the map F_f coincide with the over-rotation numbers and over-rotation pairs of the map g_f, and hence with the over-rotation numbers and over-rotation pairs of the map f. Thus, the classical rotation set I_{F_f} of the map F_f coincides with $I_f = [\rho_f, \frac{1}{2}]$. Moreover, comparing the maps F_f and G_f we see that:

1. $F_f = G_f$ except for the collection $\mathcal{C}$ of intervals $(n + a'_f, n + M_f)$, $(n + d_1(f), n + a_f)$, $(n + a_f + 1 - a''_f, n + a_f + 1 - m_f)$, $(n + a_f + 1 - d_2(f), n + 1)$, $n \in \mathbb{Z}$, on each of which G_f is a constant;
2. $G_f(x) \le F_f(x)\ \forall x \in \mathbb{R}$;
3. G_f is continuous and non-strictly monotonically increasing.

By [RT86], for every $z \in \mathbb{R}$, $\lim_{n\to\infty} \frac{G_f^n(z)}{n} = \rho'_f$ exists and is independent of z; ρ'_f is the *classical rotation number* of the map G_f. Since G_f is of degree one, it induces a degree one map τ_f of the unit circle $\mathbb{S}$. Since the map G_f is non-strictly monotonically increasing, then τ_f preserves the cyclic orientation in the non-strict sense; equivalently, one can say that τ_f is monotone and locally non-strictly increasing (we consider counterclockwise direction on the circle as positive). By [ALM00], there exists a point $z \in [0, 1)$ whose orbit is disjoint from the union of open intervals from $\mathcal{C}$ (thus, the G_f-orbit of z is the same as the F-orbit of z) and has the following properties:

(1) if ρ'_f is rational then $\pi(z)$ is periodic and in terms of circular order τ_f acts on the orbit of z as the rotation by the angle ρ'_f;
(2) if ρ'_f is irrational, then $\pi(z) \in \omega_{\tau_f}(z)$ where $\omega_{\tau_f}(z)$ is a *minimal* set (i.e. τ_f-orbits of all points of $\omega_{\tau_f}(z)$ are dense in $\omega_{\tau_f}(z)$) such that collapsing arcs of $\mathbb{S}$ complementary to $\omega_{\tau_f}(z)$ we can semi-conjugate τ_f to the (irrational) rotation of $\mathbb{S}$ by the angle τ_f.

In the end this construction yields a (semi-)conjugation of the original map f on the limit set $\omega_f(z)$ and the rotation by τ_f on a special set, say, A_f so that (a) if ρ'_f is rational then z is f-periodic, A_f is a periodic orbit, and we deal with conjugation, while (b) if ρ'_f is irrational then $\omega_f(z)$ is a Cantor set, $A_f = \mathbb{S}$, and we deal with semi-conjugacy which is at most two-to-one. In either case the (semi-)conjugacy acts as follows (in our explanation we assume that the circle is normalized so that its length is 1): the points of $\omega_f(z)$ that belong to $[0, a_f]$ are put on the arc $[0, a_f]$ of the circle maintaining the same order while the points of $\omega_f(z)$ that belong to $[a_f, 1]$ are put on the circle arc $[a_f, 1]$ in the reverse order.

It follows that in either case the map $f|_{\omega_f(z)}$ has a unique invariant measure μ_f (i.e., it is *strictly ergodic*), every point $x \in \omega_f(z)$ is admissible, and for every point $x \in \omega_f(z)$ we have $I_{f,\chi}(x) = \{\rho'_f\}$. Moreover, in both cases the measure μ_f can be transformed, in a canonical fashion, to a specific invariant measure related to the circle rotation by the angle ρ'_f: in the rational case the corresponding measure is just a CO-measure concentrated on the f-periodic orbit of z whereas in the irrational case it corresponds, in a canonical fashion, to the Lebesgue measure on the unit circle invariant under the irrational rotation by the angle ρ'_f.

Let us now relate ρ'_f and the left endpoint ρ_f of the over-rotation interval $[\rho_f, \frac{1}{2}]$ of f. By the above and by Theorem 1.8 we see that $\rho'_f \in I_f = [\rho_f, \frac{1}{2}]$, and hence $\rho_f \leq \rho'_f$. On the other hand, G_f is monotonically increasing and $G_f \leq F_f$ which by induction implies that $G_f^n(X) \leq F_f^n(X) \forall n$. Indeed, the base of induction is the fact that, by construction, $G_f \leq F_f$. Assume that the desired inequality is proven for n; then, for every $X \in \mathbb{R}$, we have

$$G_f^n(G_f(X)) \leq G_f^n(F_f(X)) \leq F_f^n(F_f(X)) \, \forall n \in \mathbb{N}$$

$$\implies G_f^{n+1}(X) \leq F_f^{n+1}(X) \,\forall n \in \mathbb{N}, X \in \mathbb{R}$$

which proves the desired inequality. This implies that

$$\lim_{n\to\infty} \frac{G_f^{n+1}(X)}{n} = \rho'_f \leq \lim_{n\to\infty} \frac{F_f^{n+1}(X)}{n}$$

which implies that $\rho'_f \leq \rho_f$. Thus, $\rho'_f = \rho_f$.

Since G_f is non-decreasing and continuous, the map $G_f : \mathbb{R} \to \mathbb{R}$ can be monotonically semi-conjugate to a monotone circle map h_f of degree one. It is well-known (see, e.g., [ALM00]) that there exists a closed invariant set A_f of h_f on which the rotation number ρ_f is realized; moreover, A_f can be chosen to avoid flat spots of h_f. Then there are two possibilities depending on whether ρ_f is rational or irrational. If $\rho_f = p/q$ is rational with p, q given in lowest terms (i.e., p, q are coprime) then A_f can be chosen to be a *periodic orbit* of period q. It lifts to the G_f-orbit A'_f of a point $x \in \mathbb{R}$ such $G_f^q(x) = x+p$ so that the classical rotation pair of x under the degree one map G_f (equivalently, F_f) is (p, q). If ρ_f is irrational then A_f can be chosen to be a Cantor set and h_f is monotonically semi-conjugate to an irrational rotation of the circle by the map collapsing flat spots of A_f.

We need to find the appropriate f-invariant set Z_f associate to the set A_f whose existence is claimed in the theorem. In general the situation is complicated here because of the fact that the map F_f associated with f does not have to coincide with G_f at points of the lifting of the set A_f to the real line; in other words, in general the set A_f is not easily transformed to a closed invariant set of f on which the over-rotation ρ is realized. However the specifics of the construction allow us to circumvent these complications.

Indeed, observe that the correspondence between the maps involved in our construction implies the existence of a continuous conjugacy ψ_f between f and h_f applicable outside of the closures of flat spots of h_f; the map ψ_f sends orbits of h_f to orbits of f while keeping the same (over-)rotation numbers in both cases. Now, if ρ_f is irrational, we can choose a point $y \in A_f$ that avoids closures of flat spots of h_f altogether. It follows that the point $\psi_f(y)$ gives rise to its f-limit set Z_f, and since the lifting of the set A_f to the real line stays away from a small neighborhood of a_f and its integer shifts (this follows from the fact that $\rho_f < 1/2$), then there is a continuous conjugacy between $f|_{Z_f}$ and $h_f|_{A_f}$. It is then easy to see that Z_f has all the desired properties.

Suppose now that ρ_f is rational. If the corresponding periodic orbit A_f of h_f avoids closures of flat spots of h_f we are done by the arguments similar to the ones from the previous paragraph. Suppose now that A_f passes through an endpoint b of a flat spot of h_f. Choose a point y very close to b avoiding flat spots of h_f. Then the finite segment of the h_f-orbit of y consisting of points y, $h_f(y)$, $\dots$, $h_f^q(y) \approx y$ is transformed by ψ_f into a finite segment $\psi_f(y)$, $f(\psi_f(y))$, $\dots$, $f^q(\psi_f(y)) \approx \psi_f(y)$ which converges to an f-periodic orbit as $y \to b$; since in this case the lifting of A_f also stays away from small neighborhoods of a_f and its integer shifts, the limiting transition is legitimate and the limit periodic orbit Z_f of $\psi_f(y)$, $f(\psi_f(y))$, $\dots$, $f^q(\psi_f(y)) \approx \psi_f(y)$ has all the desired properties. The remaining claims of the theorem easily follow from the above analysis and are left to the reader.

Observe that, in case of a rational ρ_f, since ultimately Z_f is associated with a cycle on the circle with a map that acts as a rotation, and since the over-rotation pair of Z_f coincides with the classical rotation pair on that cycle, it follows that the over-rotation pair of Z_f is coprime. Moreover, if we apply the proven above results to the Z_f-linear map ψ, it follows that Z_f can play the role of the Z_ψ; in particular this implies that the over-rotation interval I_ψ of ψ equals $[\rho_f, 1/2]$ as desired. □

Theorem 3.2 yields a strategy in finding the over-rotation interval $I_f = [\rho_f, \frac{1}{2}]$ of f: it suffices to take any point y whose trajectory is contained in Y_f and compute out its over-rotational set $I_{f,\chi}$ which, by Theorem 3.2, must be a singleton $\{\rho_f\}$. Moreover, this theorem also allows one to describe all N-bimodal over-twist patterns which is done in the next section of the present paper.

4. N-BIMODAL OVER-TWIST PATTERNS

Let us apply our results to finding the N-bimodal over-twist pattern. Let $f : [0, 1] \to [0, 1]$ be an N-bimodal map for whom the notation and agreements introduced in Definition 3.1 hold. Moreover, we will also rely upon Theorem 3.2 and use the notation from that theorem. Also, to emphasize the dependence on f, like earlier, we will continue to use subscripts while writing $M_f, m_f, d_1(f)$ and $d_2(f)$ to avoid any sort of ambiguity.

First of all, we need to show that the patterns of the periodic orbits discovered in the previous section, are over-twist patterns. What we know is that, according to Theorem 3.2, if Π is such pattern then (1) the over-rotation pair $orp(\Pi) = (p, q)$ of Π is coprime, and (2) if P is a cycle of pattern Π and f is a P-linear map, then the over-rotation interval I_f of f is $[\rho(P), 1/2]$ where $\rho(P) = p/q$ is the over-rotation number of P. This shows that the following theorem [BB19] applies to the above situation.

Theorem 4.1 ([BB19]). *Let P be a cycle of covergent pattern π such that the P-linear map f has the over-rotation interval $[\rho(P), 1/2]$ where $\rho(P)$ is the over-rotation number of P. Moreover, suppose that the over-rotation pair of P is coprime. Then the pattern π is over-twist.*

Indeed, consider an N-bimodal interval map f with rational ρ_f. Consider the set Z_f from Theorem 3.2. Then the set Z_f from that theorem must be a periodic orbit of over-rotation number ρ_f. Moreover, by Theorem 3.2 the over-rotation pair of Z_f is coprime and the over-twist interval of the Z_f-linear map is $[\rho_f, 1/2]$. By Theorem 4.1 it follows that the pattern of Z_f is an over-twist pattern. This completes the proof of the following corollary.

Corollary 4.2. *Let f be an N-bimodal map such that ρ_f is rational. Let Z_f be a set defined in Theorem 3.2. Then Z_f is a cycle of over-twist pattern.*

We want to remark here, that without the assumption that the over-rotation pair of P is coprime Theorem 4.1 is not true. Indeed, there exist non-coprime patterns whose over-rotation number equals the left endpoint of the forced over-rotation interval, and, by Theorem 1.4 they are not over-twist patterns (by Theorem 1.4, over-twist patterns must have coprime over-rotation pairs). In the trivial cases these are patterns that have a block structure over over-twist patterns. However there are similar patterns that do not have block structure over over-twists. Such patterns are called *badly ordered* [BB19]; they present a surprising departure from the previously observed phenomenon according to which the results about over-rotation numbers on the interval and those about classical rotation numbers for circle maps of degree one are analogous.

Indeed, take a circle map f of degree one. Suppose that f has a cycle P of classical rotation pair (mp, mq) which does not a block structure over a rotation by p/q. Then by [ALM98] the rotation interval of f contains p/q in its interior (in fact, the results of [ALM98] are stronger and more quantitative but for our purposes the above quote is sufficient).

Let us show that *all* N-bimodal over-twist patterns can be described based upon Theorem 3.2.

Lemma 4.3. *Let f be a P-linear N-bimodal map where P is a periodic orbit of over-twist pattern π with over-rotation pair (p, q). Then $P \subset Y_f$ can be viewed as the set Z_f from Theorem 3.2.*

Recall that the set Y_f for a given N-bimodal map f is defined in Theorem 3.2. The set Y_f in Case 1 and Case 2 is shown in Figure 7 and Figure 8 respectively.

FIGURE 7. ***The set Y_f, shown in dotted line, for the N-bimodal map f in Case 1***

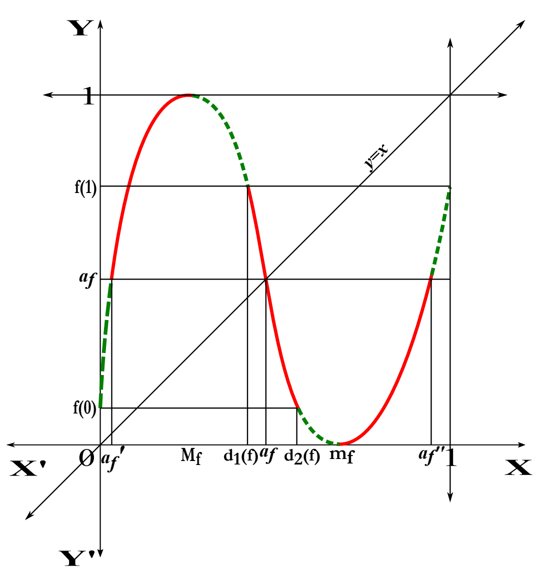
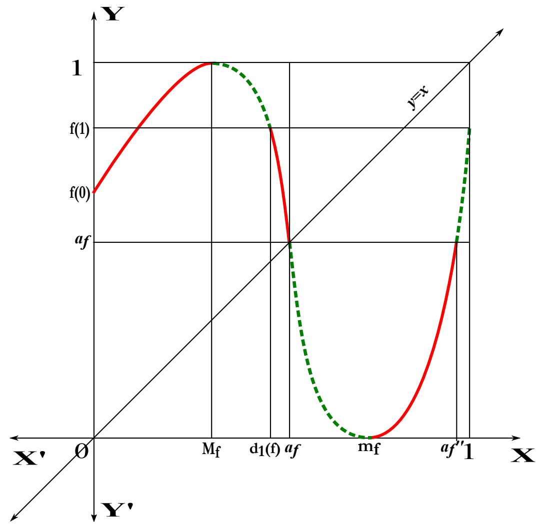


FIGURE 8. ***The set Y_f, shown in dotted line, for the N-bimodal map f in Case 2***



Proof. By Theorem 1.4, p and q must be coprime. Moreover, by Theorem 1.4 and by definition $I_f = [\frac{p}{q}, \frac{1}{2}]$. Let us show that then the set P is contained in $Y_f = [0, a'_f] \cup [M_f, d_1(f)] \cup [d_2(f), m_f)] \cup [a''_f, 1]$ (if $f(0) < a_f$) or in $[M_f, d_1(f)] \cup [a_f, m_f] \cup [a''_f, 1]$ (if $f(0) > a_f$). It is sufficient to consider only the case $f(0) < a_f$ as the other one is similar. Suppose that the containment claimed above fails. By Theorem 3.2 it follows that the set Z_f whose existence and properties are described in Theorem 3.2 is contained in Y_f and is disjoint from P. Thus, π forces a pattern γ of Z_f. However both π and γ have the same over-rotation pair (p, q), which is impossible because by the assumption π is an over-twist pattern. □

To describe all N-bimodal over-twist patterns we consider two cases.

First, assume that $f(0) \leq a_f$. Set $K_1(f) = [0, a'_f]$, $K_2(f) = [M_f, d_1(f)]$, $K_3(f) = [d_2(f), m_f]$, $K_4(f) = [a''_f, 1]$. By Theorem 3.2, $Z_f \subset Y_f = K_1(f) \cup K_2(f) \cup K_2(f) \cup K_3(f) \cup K_4(f)$ which is what we will rely upon giving an explicit description of N-bimodal over-twists of over-rotation number $\frac{p}{q}$. By definition, there must be p points of P in the interval $K_3(f) = [d_2(f), m_f]$ and p points of P in the interval $K_2(f) = [M_f, d_1(f)]$. Indeed, $[d_2(f), m_f]$ is the only component of Y_f to the right of a_f whose points map to the left of a_f and hence contribute to the over-rotation number. Since the number of points mapped from the left of a_f to the right of a_f has to be the same, there must be p points of P in the interval $K_2(f) = [M_f, d_1(f)]$. The remaining $q - 2p$ points are contained in the intervals $K_1(f)$ and $K_4(f)$. If there are r points of P in the interval $[0, a'_f]$, then there would be $s = q - 2p - r$ points in the interval $[a''_f, 1]$. This defines the number of points in the intervals $K_1(f)$, $K_2(f)$, $K_3(f)$, and $K_4(f)$. Clearly, $r \geq 0$ and $s \geq 0$, i.e. $0 \leq r \leq q - 2p$.

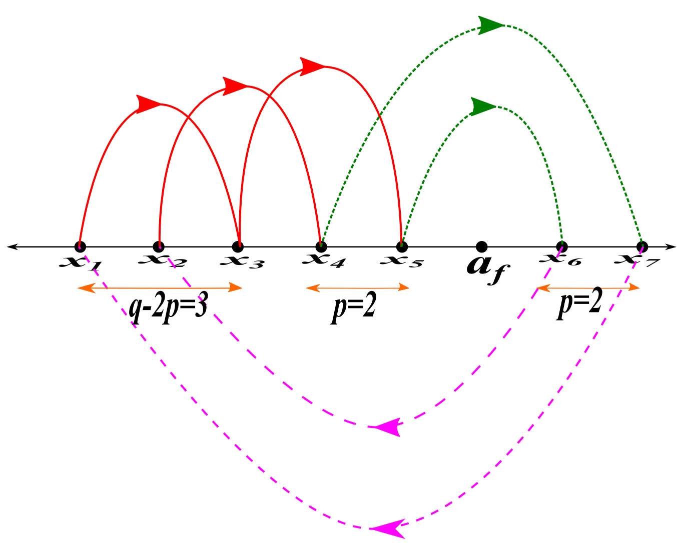
If $r = 0$ or $s = q - 2p - r = 0$ (i.e., $r = q - 2p$) , our over-twist pattern reduces to a unimodal over-twist pattern described in [BS13].

First recall that the unique unimodal over-twist pattern of over-rotation number $\frac{p}{q}$ is denoted by $\gamma_{\frac{p}{q}}$ and its action on the q points $x_1, x_2, \dots, x_q$ of a periodic orbit P which exhibits this pattern is as follows: the first $q - 2p$ points of the orbit from the left are shifted to the right by p points, the next p points are flipped (that is, the orientation is reversed, but the points which are adjacent remains adjacent) all the way to the right. Finally, the last p points of the orbit on the right are flipped all the way to the left. Thus $\gamma_{\frac{p}{q}}$ can be described by the permutation $\pi_{\frac{p}{q}}$ defined as follows:

$$\pi_{\frac{p}{q}}(j) = \begin{cases} j+p & \text{if } 1 \leq j \leq q-2p \\ 2q-2p+1-j & \text{if } q-2p+1 \leq j \leq q-p \\ q+1-j & \text{if } q-p+1 \leq j \leq q \end{cases} \tag{4.1}$$

The unimodal over-twist pattern $\gamma_{\frac{2}{7}}$ is shown in Figure 9.

FIGURE 9. ***The Unimodal over-twist pattern*** $\gamma_{\frac{2}{7}}$

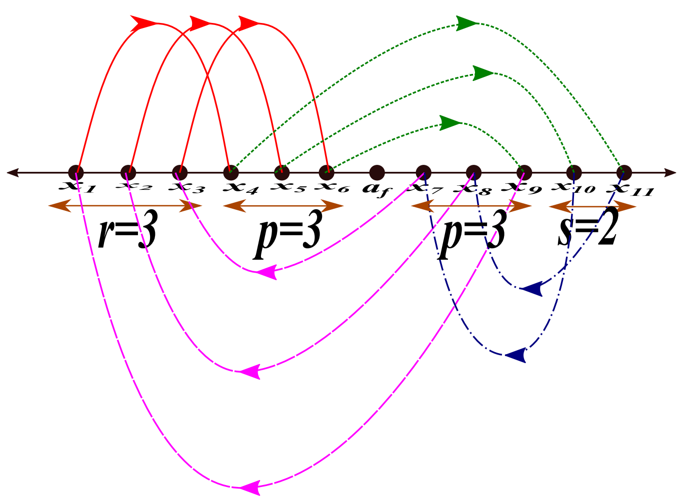


To study over-twist patterns which are strictly bimodal, we set the restriction $r \geq 1$ and $s \geq 1$. Then, $r \in \{1, 2, \dots, q-2p-1\}$. Clearly, for each fixed value of r from the set $\{1, 2, \dots, q-2p-1\}$ we get a distinct bimodal over-twist pattern of over-rotation number $\frac{p}{q}$. Thus, for the over-rotation number $\frac{p}{q}$, there are $q-2p-1$ possible distinct bimodal over-twist patterns each of which can be characterized by three parameters r, p, q. We will denote each such patterns by $\Gamma_{r,\frac{p}{q}}$. Let the permutation corresponding to the over-twist pattern be denoted by $\Pi_{r,p,q}$. It follows that $\Pi_{r,p,q}$ should be described as follows:

$$\Pi_{r,p,q} = \begin{cases} j+p & \text{if } 1 \leq j \leq r \\ q-j+r+1 & \text{if } r+1 \leq j \leq r+p \\ 2p-j+r+1 & \text{if } r+p+1 \leq j \leq r+2p \\ j-p & \text{if } r+2p+1 \leq j \leq q \end{cases} \tag{4.2}$$

In other words, this is what the pattern $\Gamma_{r,\frac{p}{q}}$ does with the q points $x_1, x_2, \ldots, x_q$ of the periodic orbit. The first r points $x_1, x_2, \ldots, x_r$ from the left of the orbit are shifted to the right by p points. The next p points $x_{r+1}, x_{r+2}, \ldots, x_{r+p}$ map forward onto the last (the rightmost) p points of the orbit with a flip (i.e., with orientation reversed) but without any expansion so that $f(x_{r+1})) = x_q, \ldots f(x_{r+p}) = x_{q-p-1}$. The images of the next p points $x_{r+p+1}, \ldots x_{r+2p}$ are just the first (the leftmost) p points of the orbit with a flip, so that $f(x_{r+p+1}) = x_p, \ldots, f(x_{r+2p}) = x_1$. Finally, the images of the last (the rightmost) $s = q - 2p - r$ points $x_{r+2p+1}, x_{r+2p+2}, \ldots, x_q$,are exactly the points $x_{r+p+1}, \ldots, x_{q-p}$ respectively. Observe that the unimodal case $\pi_{\frac{p}{q}}$ from [BS13] described above is a particular case of $\Pi_{r,p,q}$ with $r = 0$. As an example of a bimodal permutation $\Pi_{r,p,q}$, taking $r = 3$, $p = 3$ and $q = 11$, we get a bimodal over-twist pattern of period 11 given by the permutation $\Pi_{3,\frac{3}{11}} = (1, 4, 11, 8, 2, 5, 10, 7, 3, 6, 9)$ depicted in Figure 10.

FIGURE 10. ***The Bimodal over-twist pattern*** $\Gamma_{3,\frac{3}{11}}$



Finally, consider the case when $f(0) \geq a_f$. Then, by Theorem 3.2, the set $Z_f = P$ is a periodic orbit contained in $Y_f = [M_f, d_1(f)] \cup [a_f, m_f] \cup [a''_f, 1]$. In such a case we see that there must be p points in each of the intervals $[M_f, d_1(f)]$ and $[a_f, m_f]$ and $q - 2p$ points in the interval $[a''_f, 1]$. In such a case, the corresponding pattern is the unimodal over-twist pattern of over-rotation number $\frac{p}{q}$; in fact, the corresponding permutation is the flip of the permutation $\pi_{\frac{p}{q}}$. Observe, that according to our analysis overall there are $q - 2p - 1$ N-bimodal oriented over-twist patterns.

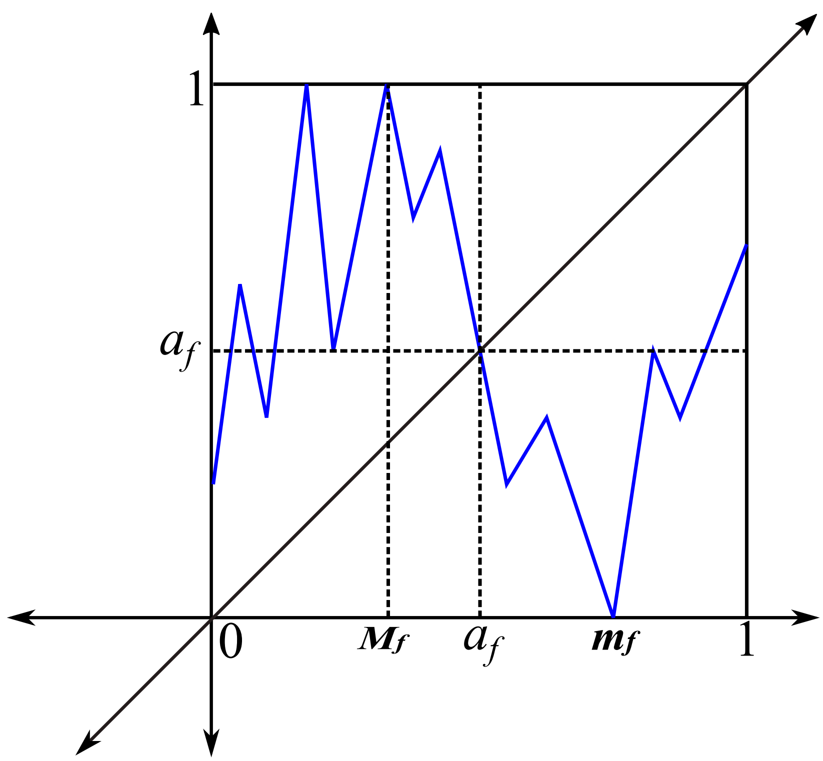
5. WELL-BEHAVED CONTINUOUS MAPS

In this section we extend the above results onto a wider class of continuous interval maps which we call *well behaved*. To avoid unnecessary complications, for the sake of brevity, and to focus upon the most interesting and broadly studied class of maps we will assume that the maps in question are (strictly) piecewise-monotone; however, this is not crucial and similar arguments can be applied in the general continuous case.

Definition 5.1. Let $f : [0, 1] \to [0, 1]$ be a continuous map with a unique fixed point $a_f \in (0, 1)$ (clearly, then $x < f(x)$ for any $x \in [0, a_f)$ and $f(x) < x$ for any $x \in (a_f, 1]$; in particular, $\min_{x\in[0,a_f]} f(x) > 0$ and $\max_{x\in[a_f,1]} f(x) < 1$). Without loss of generality we may assume that $\min_{x\in[0,1]} f(x) = 0$ and $\max_{x\in[0,1]} f(x) = 1$. Let $M_f = \max\{x : f(x) = 1\}$ and $m_f = \min\{x : f(x) = 0\}$ (evidently, $0 < M_f < a_f$ and $a_f < m_f < 1$). If for all $x \in [M_f, a_f], f(x) > a_f$ and for all $x \in [a_f, m_f], f(x) < a_f$ we will call f *well behaved*. Let $\mathcal{W}$ be the family of all well behaved maps.

Figure 11 gives an example of a well behaved map.

FIGURE 11. ***A well-behaved continous map*** f



In the preceding sections of the paper we constructed, for a given N-bimodal map f, the canonical discontinuous lifting of f. It turns out that this construction can be extended onto well behaved maps. Indeed, let $f \in \mathcal{W}$. Like in the bimodal case, let us consider the discontinuous conjugacy $\sigma_f : [0, 1] \to [0, 1]$ defined by

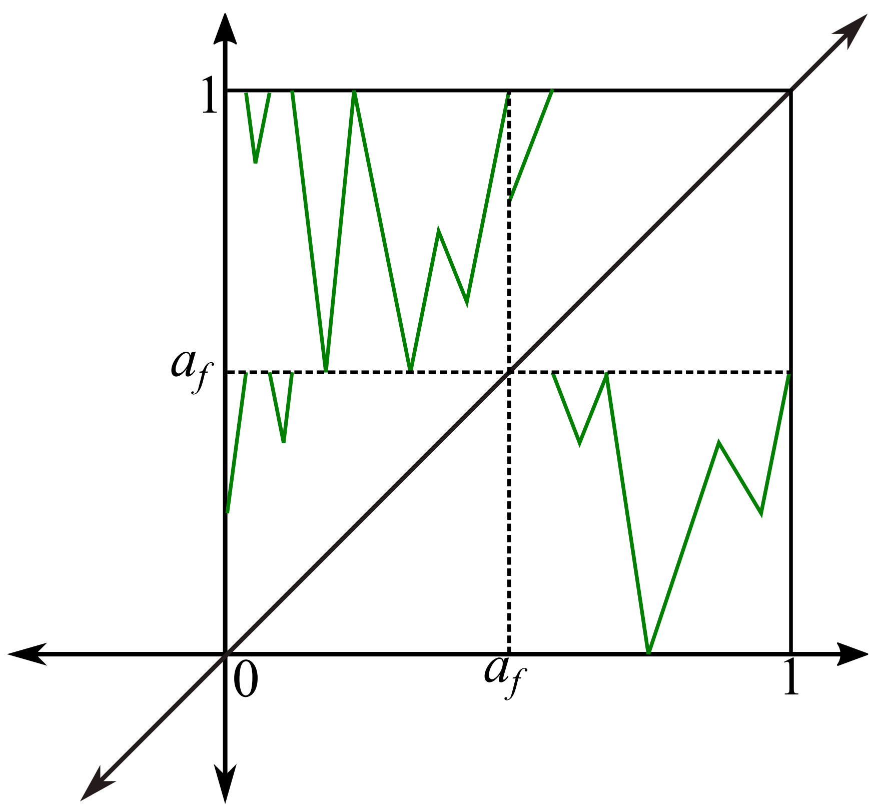
$$\sigma_f(x) = \begin{cases} x & \text{if } 0 \le x \le a_f \\ a_f + 1 - x & \text{if } a_f \le x \le 1 \end{cases} \tag{5.1}$$

which conjugates f to a map $g_f : [0, 1] \to [0, 1]$ so that $g_f = \sigma_f \circ g_f \circ \sigma_f^{-1}$. As before, we ignore the fact that our maps are going to be multivalued at a_f and its preimages as it does not impact the over-rotation interval of the map that depends only upon the over-rotation numbers of periodic *non-fixed* points. As before, by σ'_f we mean the map σ_f restricted upon $[a_f, 1]$; moreover, if we flip points of the plane in the vertical direction with respect to the line $y = \frac{1+a_f}{2}$ we shall say that we apply *vertical* σ'_f, and if we flip points of the plane in the horizontal direction with respect to the line $x = \frac{1+a_f}{2}$ we shall say that we apply *horizontal* σ'_f.

Let the preimages of a_f in the interval $[0, a_f]$ be denoted sequentially by $a^i_f, i = 1, 2, \dots, k$ for some $k \in \mathbb{N}$ such that $a^1_f < a^2_f < \cdots < a^k_f = a_f$. On each interval (a^i_f, a^{i+1}_f) either $f(x) > a_f$ or $f(x) < a_f$, and by our assumption $f(x) > a_f$ on (a^{k-1}_f, a_f).

Each interval $[a^i_f, a^{i+1}_f], i = 1, 2, \dots, k-1$ is of one of two types. If $f(x) \leq a_f$ on $[a^i_f, a^{i+1}_f]$, then $g_f(x) = f(x)$ and the graph of g_f is same as the graph of f; if $f(x) \geq a_f$ on $[a^i_f, a^{i+1}_f]$, then $g_f(x) = a_f + 1 - f(x)$ and the graph of g_f can be obtained from the graph of f by applying the vertical σ'_f. In this way, we can construct the graph of the map g_f in the interval $[0, a_f]$ (see Figure 12).

FIGURE 12. ***Construction of the map*** g_f ***for the well behaved continuous map*** f



Now we construct the graph of the map g_f on the interval $[a_f, 1]$.

(1) Apply the horizontal σ'_f to the entire graph of f on the interval $[a_f, 1]$. Define $h_f : [a_f, 1] \to [0, 1]$ by $h_f(x) = f(a_f + 1 - x)$.

(2) Let the preimages of a_f under the function h_f in the interval $[a_f, 1)$ be denoted sequentially by $b_f^1, b_f^2, b_f^3, ...b_f^l$ where $b_f^1 = a$ $b_f^i < b_f^{i+1} \forall i$. In each of the intervals, $[b_f^i, b_f^{i+1}]$, the function $h_f(x) - a_f$ will have the same sign. By our assumption in the interval $(a_f, b_f^2) = (b_f^1, b_f^2)$, $h_f(x) < a_f$.

(3) If for some $i \in \{1, 2...k\}$ in the interval $[b_f^i, b_f^{i+1}]$ we have $h_f(x) \leq a_f$, then in that interval, $g_f(x) = h_f(x) = f(a_f + 1 - x)$. On the other hand, if in the interval $[b_f^i, b_f^{i+1}]$ we have $h_f(x) \geq a_f$, then in that interval $g(x) = a + 1 - h(x) = a + 1 - f(a_f + 1 - x)$, that is, in that case the graph of g_f can be obtained by applying the vertical σ'_f to the graph of h_f.

The graph of g_f thus constructed will be discontinuous but will have the same over-rotation interval as the map f, i.e. $I_{g_f} = I_f$. We now define a lifting F_f of degree one of the function f:

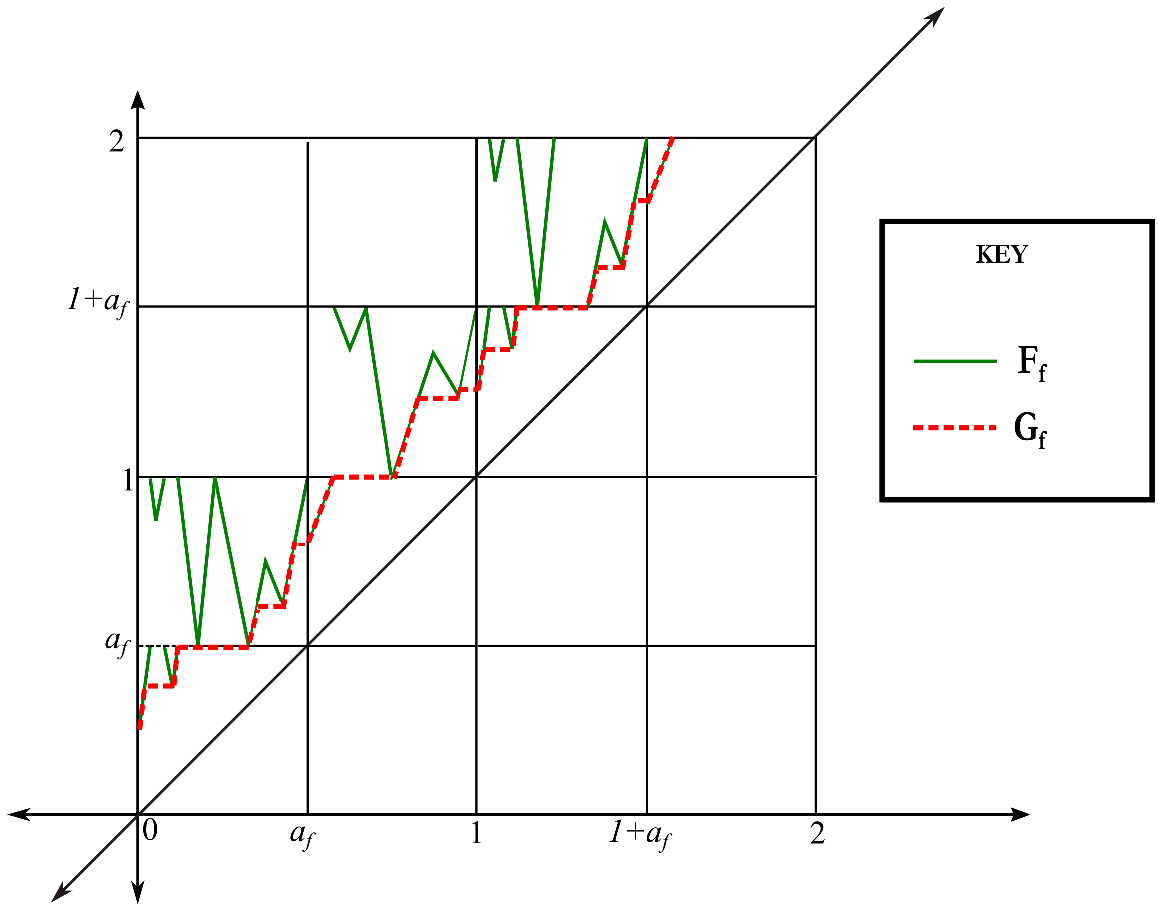
$$F_f(x) = \begin{cases} g_f(x) + 1 & \text{if } x \in [a_f, 1] \text{ and } g_f(x) < a_f \\ g_f(x) & \text{otherwise} \end{cases}$$

and then as usual if $x = k + y$ with $y \in [0, 1)$, then $F_f(x) = k + F_f(y)$. The map F_f so constructed will be a degree one map of the real line to itself, that is, an *old map* (we borrow our terminology here from [Mis82]). Obviously, by the construction the sets of classical rotation numbers and pairs of F_f coincide with the sets of over-rotation numbers and pairs of g and hence with the sets of over-rotation numbers and pairs of f. So, the classical rotation set I_{F_f} of the function F_f coincides with the over-rotation interval I_f of the function f.

Observe that by construction all discontinuities of F_f are at points that map to a_f and its integer shifts. Since the behavior of the map at these points is irrelevant to our studies that concentrate upon figuring out the left endpoint of the over-rotation interval as well as the dynamics of over-twist patterns of over-rotation number not equal to $1/2$, we see that a lot of arguments that apply in the continuous case apply to our functions too. Notice also, that by construction $F_f([0, 1]) \subset [0, 2]$.

Next we construct the *lower bound* function G_f similar to the corresponding function constructed previously for N-bimodal maps. However here we follow the classic approach from [ALM00]. The definition of the *lower bound function* G_f is as follows: $G_f = \inf\{F_f(y) : y \geq x\}$. Heuristically, one can get the graph of G_f from the graph of F_f in the following manner: take the graph of F_f and start to pour water onto it *from below* so long that it starts to pour out over the "edges". Then, the bottom level of water thus formed will give us the graph of the function G_f. This function is clearly non-decreasing (in fact, if the original function F_f is non-decreasing then $G_f = F_f$). We want to discover conditions on F_f that would imply that G_f is continuous because this would in turn imply the existence of a point x whose G_f-orbit of G_f avoids "flat spots" of G_f and, therefore, coincides with the F_f-orbit of x. This would imply that x has the lowest classic rotation number in the sense of F_F, and that the corresponding point $x' = \sigma_f(x)$ has the least possible over-rotation number in the sense of f.

FIGURE 13. ***Construction of the maps F_f and G_f for the well behaved continuous map f***



Lemma 5.2. *The range of any non-decreasing function $\psi : \mathbb{R} \to \mathbb{R}$ is closed; it coincides with $\mathbb{R}$ if and only if ψ is continuous. The number of points of discontinuity of ψ is at most countable.*

Proof. Clearly, $\lim_{x\to a^-} \psi(x) < \lim_{x\to a^+} \psi(x)$ for a point a of discontinuity of ψ; depending upon the value of ψ at a either one or two open intervals are going to be missing from the range of ψ. Since ψ is non-decreasing, such gaps in the range are disjoint for distinct points of discontinuity of ψ. Hence there are at most countably many points of discontinuity of ψ. The complement to the range is always the union of those gaps, hence the range is closed. The rest of the lemma is just as simple and is left to the reader. □

We now introduce a new class of maps $\mathcal{I}$ from $\mathbb{R}$ to $\mathbb{R}$.

Definition 5.3. A function $T : \mathbb{R} \to \mathbb{R}$ is called *eventually-increasing* if there exists a dense set $D_T \subset \mathbb{R}$ such that for any $z \in D_T$ $\exists y \in \mathbb{R}$ with $T(y) = z$ and $T(x) > z$ $\forall x > y$.

Thus, a map $T : \mathbb{R} \to \mathbb{R}$ is *eventually-increasing* if any horizontal line (level) from a dense family will intersect the graph of T so that there will exist a point of intersection after which (i.e., to the right of which) the graph of T will be strictly above that horizontal line.

Lemma 5.4. *Let $T : \mathbb{R} \to \mathbb{R}$ be eventually-increasing. Then, the lower bound function $S_T : \mathbb{R} \to \mathbb{R}$ defined by $S_T(x) = \inf\{T(y) : y \geq x\}$ is continuous.*

Proof. Given $z \in D_T$ choose y such that $T(y) = z$ and $T(x) > z$ for all $x > y$. Then by definition it follows that $S_T(y) = z$. Since D_T is dense, Lemma 5.2 implies the claim. □

We will now prove the main result of this section of the paper. In proving it we will keep intact all the notation and agreements introduced in the paper so far. However we also need to introduce some new concepts and notation. Suppose that $f \in \mathcal{W}$. Then a_f is a unique fixed point of f. We construct now a special branch of the inverse function of f, i.e. a function $h_f : [0,1] \to [0,1]$ such that $f \circ h_f(x) = x$ (observe that by definition f is onto). The function h_f, called the *canonical inverse (of f)* is constructed as follows. Suppose that $\alpha_f = \min\{f(x) : x \leq a_f\}$ and $\beta_f = \max\{f(x) : x \geq a_f\}$. If the entire segment $[0, a_f]$ maps to the right of a_f, then $\alpha_f = a_f$; similarly, if the segment $[a_f, 1]$ maps to the left of a_f, then $\beta_f = a_f$. Since the case when segments $[0, a_f]$ and $[a_f, 1]$ are flipped to the other side of a_f is trivial, we assume that at least one of them is not flipped to the other side of a_f. Thus, at least one of the numbers α_f, β_f is not equal to a_f. For the sake of definiteness from now on we assume that $\alpha_f < a_f$.

Now, let $z \in [\alpha_f, a_f]$. Then we define $h_f(z)$ as the greatest number $y \in [0, a_f]$ such that $f(y) = z$ (in particular, $h_f(a_f) = a_f$). Similarly, if $z \in [a_f, \beta_f]$, then $h_f(z)$ is the least number y such that $f(y) = z$. If now $z \notin [\alpha_f, \beta_f]$ then we define $h_f(z)$ as the closest to a_f number y such that $f(y) = z$. This completely defines the function h_f. A useful exercise for the reader here is to consider N-bimodal functions and describe their canonical inverses.

We can directly describe the set $\overline{h_f([0,1])}$ as follows. Set

$$L_1(f) = \{x \in [0, a_f] : f(x) \in [\alpha_f, a_f] \cup (\beta_f, 1], \quad x = \sup\{y \in [0, a_f] | f(y) = f(x)\}\}$$

$$L_2(f) = \{x \in [a_f, 1] : f(x) \in [a_f, \beta_f] \cup [0, \alpha_f), \quad x = \inf\{y \in [a_f, 1] | f(y) = f(x)\}\};$$

then it is easy to see that $\overline{h_f([0,1])} = L_1(f) \cup L_2(f) = Y_f$.

Using the introduced notation we now prove our main result.

Theorem 5.5. *Let $f : [0,1] \to [0,1], f \in \mathcal{W}$ be a well behaved map. Then the lower bound function $G_f : \mathbb{R} \to \mathbb{R}$ is continuous and increasing. Moreover, the classical rotation number of the map G_f equals the left endpoint ρ_f of the over-rotation interval $[\rho_f, \frac{1}{2}]$ of the map f. There exists a minimal f-invariant set Z_f such that for every point $y \in Z_f$ we have $I_{f,\chi}(y) = \rho_f$, and there are two possibilities:*

(1) ρ_f *is rational,* Z_f *is a periodic orbit, and* $f|_{Z_f}$ *is canonically conjugate to the rotation by* ρ_f *restricted on one of its cycles;*
(2) ρ_f *is irrational,* Z_f *is a Cantor set, and* $f|_{Z_f}$ *is canonically semi-conjugate by a map which is at most two-to-one to the circle rotation by* ρ_f*.*

Furthermore, $Z_f \subset Y_f$ *and for every point* y *whose trajectory is contained in* Y_f *we have* $I_{f,\chi}(y) = \{\rho\}$*.*

Proof. We first show that F_f is *eventually increasing*. Take a level $y = \lambda$ where $0 < \lambda < a_f$. By construction of F_f, all F_f-preimages of λ (associate to all points of intersection of the line $y = \lambda$ with the graph of F_f) lie strictly to the left of a_f. By construction, they are contained in $[a_f - 1, a_f]$, and there are two cases: (1) if $\alpha_f \le \lambda < a_f$ then some points like that belong to $[0, a_f)$, and (2) if $0 < \lambda < \alpha_f$ then all points like that belong to $(a_f - 1, 0)$. In either case though the continuity of f (and therefore the continuity of F_f outside the set of preimages of a_f and its integer shifts) implies that there is the greatest point y with $F_f(y) = \lambda$. Now, take $t > y$. If $t \ge a_f$ then by construction $F_f(t) \ge a_f > \lambda$. If $y < t < a_f$ then $F_f(t)$ cannot be less than λ as by construction at the right endpoint of the interval of continuity of F_f containing t the function F_f must reach out to $a_f > \lambda$, hence by the Intermediate Value Theorem there must exist preimages of λ to the right of y, a contradiction.

The level λ with $a_f < \lambda < 1$ is considered similarly; the difference with the previous case is only that now we have to rely upon the fact that on any interval of continuity of F_f between a_f and 1 the function F_f has to reach out to the level 1. This shows that the the necessary conditions for F_f to be eventually increasing are satisfied for all values except for countable families oTf integer shifts of a_f and integers themselves. Hence F_f is eventually increasing and $G_f \le F_f$ is continuous.The graph of F_f and G_f for the *well-behaved map* f is shown in Figure 13.

The remaining arguments literally repeat the arguments in the last part of the proof of Theorem 3.2 and are left to the reader. □

Call a pattern π *well behaved* if any cycle P of pattern π gives rise to a well behaved P-linear map $f_P = f$. Theorem 5.5, together with the arguments used in the proof of Corollary 4.2 and Lemma 4.3, implies the description of well-behaved over-twist patterns. Recall that since f is well behaved, for it there are several canonically defined sets, such as the set $L_1(f)$, the set $L_2(f)$, and their union Y_f.

Corollary 5.6. *Let* P *be a cycle of well behaved pattern* π *and let* f_P *be a* π*-linear map. Then* π *is an over-twist pattern if and only if* $P \subset Y_{f_P}$*.*

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