Commun. Math. Phys. 191, 61 – 70 (1998)

Communications in
Mathematical Physics

Collet-Eckmann Maps are Unstable

Alexander Blokh[1,⋆], **Michał Misiurewicz**[2]

[1] Department of Mathematics, University of Alabama in Birmingham, University Station, Birmingham, AL 35294-2060, USA. E-mail: ablokh@math.uab.edu
[2] Department of Mathematical Sciences, IUPUI, 402 N. Blackford Street, Indianapolis, IN 46202-3216, USA. E-mail: mmisiure@math.iupui.edu

Received: 17 October 1996 / Accepted: 16 April 1997

Abstract: We show that a smooth interval map satisfying the Collet-Eckmann condition at some critical point is not structurally stable in C^r topology for any r.

1. Introduction

We want to address an old problem of structural stability of smooth interval maps. The reader can find a good account of this problem in [MS]. Basically, one expects that the only way an interval map can be structurally stable is that the trajectories of all critical points are attracted to attracting periodic points. However, this is known only for C^1-stability ([J]).

It is widely believed that this problem resembles the Closing Lemma, that is, a perturbation necessary to change the topological type of a map has to be global (or at least a proof is possible only for global perturbations). Here we show that some maps admit the proof of instability in any C^r topology with local perturbations. These maps are the ones satisfying the Collet-Eckmann condition at some critical point. Positive Lyapunov exponent at a critical point causes "sensitivity" at this point, so a local perturbation has global effects.

There is a slight difficulty when we speak of structural stability of interval maps. Let us illustrate it with an example. Let $f : [0, 1] \to [0, 1]$ be a convex C^∞ unimodal map with $f(0) = 0$ and $a = \sup_{x\in[0,1]} f(x) < 1$. Set $g_\varepsilon(x) = f(x) + \varepsilon$ for $0 < \varepsilon \le 1 - a$. The maps g_ε converge to f in C^∞ topology as $\varepsilon \to 0$. None of g_ε is conjugate to f, since for each of them 0 is not a fixed point while for f it is. Yet this is not the kind of instability of f we want. Basically, the change of behavior of the map was achieved by the change of the interval on which it was defined (and rescaling back). In our example the map may have really changed its topological type, but to establish that is much more difficult than to prove the "instability" as above.

⋆ He was partially supported by NSF grant DMS 9626303.

To avoid problems like this, let us define the *essential interval* for a continuous interval map f as the smallest interval containing trajectories of all interior local extrema of f. Clearly, this interval is invariant for f. Now, continuous interval maps are said to be *essentially conjugate* if they are conjugate when restricted to their essential intervals. Finally, we say that an interval map f of class C^r is *C^r-structurally stable* if there exists a neighborhood of f in C^r topology such that every map from this neighborhood is essentially conjugate with f.

Note that we can also speak of C^r-structural stability of f even if f is not of class C^r. Namely, we require that there is a C^r-neighborhood of the 0 function such that for every h from this neighborhood $f + h$ is essentially conjugate with f. This may be not too interesting for all maps f, but it makes sense for instance if f is piecewise C^r.

Sometimes we are interested in changing the topological type of a map by introducing a small local perturbation. To describe this phenomenon, we introduce the notion of structural sensitivity at a point. Namely, we say that a map f is *C^r-structural sensitive at x* if for every C^r-neighborhood $\mathcal{V}$ of the function 0 and for every neighborhood U of x there exists $h \in \mathcal{V}$ which is 0 outside U and such that $f + h$ is not essentially conjugate to f.

We will use the following terminology. Let f be a continuous interval map. A point x is called a *periodic sink* (from one side) if there exists $n > 0$ and a (one-sided) neighborhood U of x such that $f^n(x) = x$, $f^n(U) \subset U$ and the diameter of $f^k(U)$ tends to 0 as $k \to \infty$. The *basin of attraction* of x is then the set $\bigcup_{k=0}^{\infty} f^{-k}(U)$. Note that if f^n has a (one-sided) derivative at x, its absolute value is less than or equal to 1. It is well known that any point from the boundary of the basin of attraction of a periodic sink is either periodic or preperiodic.

An interval J will be called a *wanderval* if $f^n|_J$ is a homeomorphism for every n, the images of J are pairwise disjoint, and the orbit of J does not converge to the orbit of a periodic (even one-sided) sink (we can talk about convergence of the orbit of J since by the second condition all points of J have the same ω-limit set). It is easy to see that this is equivalent to $f^n|_J$ being homeomorphism for every n and the existence of a point $x \in J$ whose ω-limit set is not a periodic orbit. Indeed, if the first set of conditions is satisfied then the absence of *flat spots* (i.e. intervals on which f is a constant) implies that the orbit of no point $x \in J$ converges to a periodic orbit. Suppose now that the second set of conditions holds. In that case if some images of J intersect each other then the union of all images of J is an invariant union of finitely many intervals such that the restriction of f on this union is monotone on every component; this implies that all points of J have periodic orbits as their ω-limit sets.

The non-existence of wandervals for smooth interval maps was proven in a series of papers with the most general result obtained in [MMS]. The question of their existence in a piecewise-smooth setting remains unsolved in general, however under the assumption of the exponential growth of the derivative at a point we prove that this point is not contained in a wanderval.

2. Main Theorem

We look at the class of all continuous maps $f : [0, 1] \to [0, 1]$ for which there exist points $0 = a_0 < a_1 < \ldots < a_s = 1$ such that f is of class C^1 on each of the intervals $[a_i, a_{i+1}]$, with non-zero derivative on each (a_i, a_{i+1}). We will refer to these maps as *piecewise smooth maps*. The points $a_0, a_1, \ldots, a_s$ will be called *singular points*. Points at which f has local extrema (except 0 and 1) will be called *turning points*.

We will say that a piecewise smooth map f *satisfies the Collet-Eckmann condition* at a turning point c of f if

$$|(f^n)'(f(c))| \geq \alpha\lambda^n \tag{2.1}$$

for all n, where $\alpha > 0$ and $\lambda > 1$ (in other words, if the lower Lyapunov exponent at $f(c)$ is positive). If only one-sided derivatives of f^n at $f(c)$ exist, we mean (2.1) for both of them.

We introduce a function $r : [0,1]^2 \to \mathbb{R} \cup \{\infty\}$ as follows:

$$r(x,y) = \frac{|f(x) - f(y)|}{|x-y|\,|f'(x)|}$$

if $x \neq y$, and $r(x,x)$=1 (if only one-sided derivatives at x exist, we take as $f'(x)$ the derivative from the side where y is). We will call this function *relative stretching*, since it measures how the interval is being stretched relative to the derivative at one of its endpoints. In fact, this makes sense only if x and y belong to the same lap of f (by a *lap* we mean a maximal interval on which f is monotone; there may be singular points in the interior of a lap). We will call the infimum of $r(x,y)$ over the pairs of points x, y from the same lap the *shrinkability* of f. Note that if a point x is *critical* (i.e. the derivative of f at x vanishes) then $r(x,y) = \infty$ for all $y \neq x$.

If a, b are two consecutive singular points, let us look at r restricted to $[a,b]^2$. Clearly, r is continuous off the diagonal. If $x \neq y$ and $f'(x) \neq 0$ then $r(x,y) = f'(z)/f'(x)$ for some $z \in (x;y)$ (by $(x;y)$ we mean (x,y) if $x < y$ and (y,x) if $y < x$). Hence, since f is of class C^1 on $[a,b]$ and $f'(x) \neq 0$ for all $x \in (a,b)$ we conclude that r is also continuous on the diagonal, at all points (x,x) such that $f'(x) \neq 0$.

Let f be a piecewise smooth map and let c be a turning point of f. For a given $\varepsilon > 0$ and a neighborhood U of c we denote by $B(f,\varepsilon,U,c)$ the set of maps $g : [0,1] \to [0,1]$ such that $|g(x) - f(x)| \leq \varepsilon$ for every $x \in U$, $g(x) = f(x)$ for every $x \notin U$, $|g(c) - f(c)| = \varepsilon$, and c is a local extremum of g.

For every point $x \in [0,1]$ its *itinerary* (for f) is the sequence $(i_n(x))_{n=0}^{\infty}$, where $i_n(x)$ is the point $f^n(x)$ if this is a turning point and the lap of f to which $f^n(x)$ belongs otherwise.

Main Theorem. *Let f be a piecewise smooth map with non-zero shrinkability, satisfying the Collet-Eckmann condition at a turning point c. Then there is a neighborhood U of c such that for every $\varepsilon > 0$ if $g \in B(f,\varepsilon,U,c)$ then either g has more local extrema than f, or the g-trajectory of c is attracted to the orbit of a periodic (at least one-sided) sink, or the itineraries of c for f and g are different.*

Proof. We may assume from the very beginning that we are considering only those $g \in B(f,\varepsilon,U,c)$ which have c as their only local extremum in U. Indeed, we may choose U such that f has only one local extremum in it. Then, if g has more than one local extremum in U, it has more local extrema than f in the whole interval, and we are done.

Since we will often consider behavior of f separately at each side of some point x, we will speak about right and left halves of x. The point 0 has only right half, and the point 1 only left one. Now every half-point has the derivative of f well defined at it. Moreover, we can tell to which half of $f(x)$ a given half of x is mapped by f.

Let λ and α be constants from (2.1). We choose η and μ such that

$$\eta < 1 < \mu < \eta\lambda. \tag{2.2}$$

Let p be the number of singular half-points that are not periodic. We choose an integer $N > p$ such that

$$\alpha\mu^N \geq 1 \tag{2.3}$$

and

$$\left(\frac{\mu}{\eta\lambda}\right)^N < \left(\frac{K}{\eta}\right)^p, \tag{2.4}$$

where K is the shrinkability of f.

For every non-periodic singular half-point we choose its neighborhood (one-sided, of course) such that every piece of trajectory of length N passes through it at most once. Let V be the union of these neighborhoods. Thus, every piece of a trajectory of length N visits V at most p times. For every periodic singular half-point that is a sink we choose its neighborhood (again one-sided) contained in the basin of attraction. Let W be the union of these neighborhoods. Since f satisfies the Collet-Eckmann condition at c, the trajectory of $f(c)$ does not visit W at all. Then we choose $\delta > 0$ such that if $x \notin V \cup W$, $|x-y| < \delta$ and x, y do not lie on opposite sides of a periodic singular point then $r(x,y) \geq \eta$. To see that this is possible, notice that for a small δ, if x, y are as above then they belong to some closed interval J on which f' is continuous and non-zero. The function r is continuous on J^2 and it is equal to 1 on the diagonal, hence it is larger than η in some neighborhood of the diagonal.

We are now ready to choose U required in the theorem. The choice depends on the trajectory of c, so we will consider several cases. In what follows we denote $f^i(c)$ by c_i.

Case 1. The point c is periodic. Then clearly for any $g \in B(f, \varepsilon, U, c)$ with sufficiently small ε and U the point c will not be periodic of the same period. Therefore the itineraries of c for f and g will be different. Remember that this case is possible, since we allow non-zero one-sided derivatives at the turning points.

Case 2. The point c is preperiodic (but not periodic). Then c_k is a periodic repelling point for some $k > 0$. Let us work with the halves of c_k and their neighborhoods (i.e. one-sided neighborhoods in the usual sense). Then there are the following two possibilities for the behavior of f.

Case 2a. There are no inverse images of turning points in some closed one-sided neighborhood of c_k. Then there is an interval $(c_k; a)$ invariant for some iterate of f, and a is a periodic sink from the appropriate side. For sufficiently small ε and U and for any $g \in B(f, \varepsilon, U, c)$ with $g(c)$ on the appropriate side of c_1 the g-trajectory of c will be attracted by the orbit of a which remains a periodic sink from the appropriate side for g.

Case 2b. There are inverse images of at least one turning point of f in an arbitrary small one-sided neighborhood of c_k. We can choose one of these inverse images sufficiently close to c_k, and then we can choose next inverse images by following the periodic orbit of c_k backwards. In this way we get inverse images of a turning point arbitrarily close to c with an additional property that their trajectories before hitting a turning point miss some fixed (small) neighborhood U of c. Let us take a sufficiently small $\varepsilon > 0$ and $g \in B(f, \varepsilon, U, c)$ with $g(c)$ on the appropriate side of c_1. Then there is a point $a \in (g^k(c); c_k)$ such that $f^m(a)$ is a turning point and $f^i(a) \notin U$ for $i < m$. Suppose that the itineraries of c for f and g are the same. Then by induction we see that $f^i(a) \in (g^{k+i}(c); c_{k+i})$ for $i \leq m$ (even if $g^{k+i}(c) \in U$, the induction step works). Since c_{k+m} and $g^{k+m}(c)$ are on the opposite sides of the turning point $f^k(a)$, we get a contradiction.

We have shown that if c is a periodic or preperiodic point then choosing a neighborhood U of c according to Cases 1, 2a or 2b we can guarantee that for every $\varepsilon > 0$ if $g \in B(f, \varepsilon, U, c)$ and c is the only local extremum of g in U then the itineraries of c under the maps g and f are different, or c is attracted to an orbit of a periodic sink.

Case 3. The point c is neither periodic nor preperiodic. Let us show that there are inverse images of turning points arbitrary close to c_1 on both sides of c_1. Indeed, suppose that this is not the case. Then there is an interval J containing c_1 such that all iterates of f are monotone on J. The discussion following the definition of a wanderval implies now that if $\omega(c)$ is not a periodic orbit then J is a wanderval. However, because of (2.1), c cannot be attracted to a periodic (even one-sided) sink. It cannot be also on the boundary of a basin of attraction of a periodic sink (because then it would be periodic or preperiodic). Piecewise-smoothness of f implies that these are the only two ways the set $\omega(c)$ can be a periodic orbit. Hence, $\omega(c)$ is not a periodic orbit and thus there exists a point a_1 in a small neighborhood of c_1 such that $(c_1; a_1)$ is a wanderval. We denote $f^{i-1}(a_1)$ by a_i. We have

$$\frac{|c_{n+1} - a_{n+1}|}{|c_1 - a_1|\,|(f^n)'(c_1)|} = \prod_{k=1}^{n} r(c_k, a_k).$$

The numbers $r(c_k, a_k)$ are usually greater than or equal to η. There are several cases when they may be smaller than η, and then they are larger than or equal to K. The first case is when $c_k \in V \cup W$. This can happen at most p times during each of N consecutive steps. The second case is when $|c_k - a_k| \geq \delta$. However, the images of a wanderval are pairwise disjoint, so this can happen at most $1/\delta$ times no matter how big n is. The third case would be when c_k and a_k lie on the opposite sides of a periodic singular point. Then this periodic point would belong to a wanderval, so this case is impossible. Therefore (we use (2.4))

$$\liminf_{n\to\infty} \left(\frac{|c_{n+1} - a_{n+1}|}{|c_1 - a_1|\,|(f^n)'(c_1)|} \right)^{1/n} \geq (\eta^{N-p} K^p)^{1/N} > \frac{\mu}{\lambda}\,.$$

Together with (2.1) this proves that $|c_{n+1} - a_{n+1}|$ grows exponentially (at least as a constant times μ^n), a contradiction.

Therefore there are inverse images of turning points of f arbitrary close to c_1 on both sides of c_1. Clearly the same holds for the point c. If the orbit of such an inverse image x comes closer to c (if coming from the other side, we look at the first image to decide whether it is closer), we replace x by this point of the orbit of x. In such a way we see that there are arbitrarily small neighborhoods (a, b) of c such that $f(a) = f(b)$, some image of a is a turning point d of f, and the orbit of a before getting to d does not pass through (a, b). We will refer to such neighborhoods as *very nice*.

By (2.4), we have

$$\frac{\eta\lambda}{\mu} \left(\frac{K}{\eta} \right)^{p/N} > 1.$$

Therefore we can choose an integer $M \geq N$ so large that

$$\left(\frac{\eta\lambda}{\mu} \left(\frac{K}{\eta} \right)^{p/N} \right)^{M} \geq 2 \left(\frac{\eta}{K} \right)^{2+1/\delta}\,. \tag{2.5}$$

Next we choose a very nice neighborhood U of c, such that $f^k(U) \cap U = \emptyset$ for $k = 1, 2, \dots, M - 1$. Moreover we choose U so that orbits of periodic singular points are disjoint from U.

Take some $\varepsilon > 0$ and $g \in B(f, \varepsilon, U, c)$. Recall that due to the arguments from the beginning of the proof we may assume that c is the only local extremum of g in U. Set

$$b_n = g_n(c),\ \alpha_n = |(f^n)'(c_1)| \text{ and } \gamma_n = \frac{|c_{n+1} - b_{n+1}|}{\varepsilon \alpha_n}.$$

Suppose that the itineraries of c for f and g are the same. Then c_{n+1} and b_{n+1} are in the same lap of f (the laps of g are the same as for f) and if a is an endpoint of U and $x \in U$ then $f(x)$ and $g(x)$ lie on the same side of $f(a)$ (otherwise g would have extra turning points).

We claim that for a given $n > 0$ either both b_n and c_n are in U or both are outside U. First observe that the g-orbit of a and the f-orbit of a are the same. Let us now show by induction that if an endpoint a of U belongs to $[c_n; b_n]$, then $f^i(a) = g^i(a) \in [c_{n+i}; b_{n+i}]$ for $i = 0, 1, \dots, m$, where m is the smallest integer such that $f^m(a)$ is a turning point. Indeed, if $f^i(a) = g^i(a) \in [c_{n+i}; b_{n+i}]$ then applying f and g to this we get $f^{i+1}(a) = g^{i+1}(a) \in f[c_{n+i}; b_{n+i}] \cap g[c_{n+i}; b_{n+i}]$. Since f and g are either both increasing or both decreasing on $[c_{n+i}; b_{n+i}]$, we get $f^{i+1}(a) = g^{i+1}(a) \in [c_{n+i+1}; b_{n+i+1}]$, which completes the induction step. Thus, $f^m(a) = g^m(a) \in [c_{n+m}; b_{n+m}]$. This is a contradiction, since by our assumptions there is no turning point in $[c_{n+m}; b_{n+m}]$. This proves the claim.

Consider again two cases depending on the behavior of the trajectory of c.

Case 3a. The point c is non-recurrent. Choose U so small that no c_n is in U for $n > 0$. Let us show that this U satisfies the requirements of the theorem. Indeed, $c_n \notin U$ for all n, thus $b_n \notin U$ for all n. Hence, the f-orbit and the g-orbit of b_1 are the same. Therefore if the itineraries of c_1 and b_1 for f and g respectively are the same then there are no inverse images of turning points of f in a one-sided neighborhood $[c_1; b_1)$ of c_1, a contradiction.

Case 3b. The point c is recurrent. Let U be the set chosen after formula (2.5). We are going to look at how γ_n behaves when n grows. More precisely, we call n *special* if $c_n \in U$ (this includes $n = 0$) and prove by induction that if n is special then $\gamma_n \geq (\mu/\lambda)^n$. This is clearly true for $n = 0$. Now we show how to make an induction step. Note that the situation is very similar to what we encountered when we were proving that c_1 is not an endpoint of a wanderval.

Assume that n is special and $n + m$ is the next special number. By the definition of U we have $m \geq M$. Our goal is to estimate γ_{n+m} from below; to this end we estimate from below quotients $\gamma_{n+1+k}/\gamma_{n+k}$ for $k = 0, 1, 2, \dots, m - 2$. We have

$$\gamma_{n+1+k}/\gamma_{n+k} = \frac{|c_{n+2+k} - b_{n+2+k}|}{\varepsilon \alpha_{n+1+k}} \cdot \frac{\varepsilon \alpha_{n+k}}{|c_{n+1+k} - b_{n+1+k}|},$$

on the other hand we have $b_{n+1+k} = f^k(b_{n+1})$, and thus $b_{n+2+j} = f(b_{n+1+j})$, so

$$\gamma_{n+1+k}/\gamma_{n+k} = \frac{|f(c_{n+1+k}) - f(b_{n+1+k})|}{|c_{n+1+k} - b_{n+1+k}||f'(c_{n+1+k})|} = r(c_{n+1+k}, b_{n+1+k}).$$

Thus we can estimate $\gamma_{n+k+1}/\gamma_{n+k}$ from below by η or K. As before, we normally use η, but we have to use K in some cases. The first case is when $c_{n+1+k} \in V \cup W$. This can happen at most p times during each of N consecutive steps. The second case is when $|c_{n+1+k} - b_{n+1+k}| \geq \delta$. This can happen at most $1/\delta$ times. The reason is that the intervals $(c_j; b_j)$ for $j = n+1, \dots, n+m-1$ are pairwise disjoint. Indeed, suppose that they are not. Then one of them intersects the interval $(c_{n+m}; b_{n+m})$ (because the images of non-disjoint intervals are non-disjoint). The latter interval is contained in U, while the one intersecting it is disjoint from U, a contradiction.

The third case would be when c_{n+1+k} and b_{n+1+k} lie on the opposite sides of a periodic singular point. We claim that this is impossible. Call this periodic point x. By the choice of U the trajectory of x is disjoint from U. On the other hand, we assumed that the itineraries of c_1 and b_1 for f and g respectively coincide. This together with the fact that f and g coincide outside U implies that $f^{m-k-1}(x) \in [c_{n+m}; b_{n+m}] \subset U$, a contradiction.

Hence, the estimate of $\gamma_{n+1+k}/\gamma_{n+k}$ from below by K will be used at most $1+mp/N$ times when the first case occurs, and at most $1/\delta$ times when the second case occurs.

For $\gamma_{n+m}/\gamma_{n+m-1}$ we have to modify slightly the estimate, since perhaps b_{n+m+1} is different from $f(b_{n+m})$. Since $|b_{n+m+1} - f(b_{n+m})| \leq \varepsilon$, we get

$$\gamma_{n+m} \geq \frac{|f(c_{n+m}) - f(b_{n+m})| - \varepsilon}{\varepsilon \alpha_{n+m-1} |f'(c_{n+m})|} \geq \gamma_n \eta^{m-j} K^j - \frac{1}{\alpha_{n+m}},$$

where $j = mp/N + 1/\delta + 2$. Using (2.5) and (2.1), and since $m \geq M$, we get

$$\gamma_{n+m} \geq \gamma_n \cdot 2 \left(\frac{\mu}{\lambda}\right)^m - \frac{1}{\alpha \lambda^{n+m}}.$$

Since

$$\left(\frac{\mu}{\lambda}\right)^n \cdot 2 \left(\frac{\mu}{\lambda}\right)^m - \frac{1}{\alpha \lambda^{n+m}} \geq \left(\frac{\mu}{\lambda}\right)^{n+m}$$

(because of (2.3) and since $n+m \geq M \geq N$), we get $\gamma_{n+m} \geq (\mu/\lambda)^{n+m}$. This completes the induction step.

For every special n we get

$$|c_{n+1} - b_{n+1}| \geq \gamma_n \varepsilon \alpha_n \geq \varepsilon \alpha \mu^n,$$

a contradiction since $\mu > 1$. This completes the proof. $\square$

Let us make an observation that leads to a result which seems to be of some interest by itself. Namely, in the beginning of the whole proof and in the beginning of Case 3 we did not use the assumption that the point c is a turning point. We used only the fact that the lower Lyapunov exponent at $f(c)$ was positive. We concluded that this point did not belong to a wanderval. Hence, we get the following result.

Proposition 2.1. *Let f be a piecewise smooth map with non-zero shrinkability. Then at every point contained in a wanderval the lower Lyapunov exponent is non-positive.*

Some time ago one of the authors (AB) got interested in the question of existence of wandervals under the assumptions less restrictive than those from [MMS]. In particular it would be nice to find out whether a piecewise smooth map with some singular points at which the order of degeneracy is different to the left and to the right may have wandervals (in the unimodal case this question is quoted in [MS], where it is noted that the proof from [MMS] breaks down under new milder assumptions). We would like to point out that Proposition 2.1 answers the question in a rather specific situation. It also shows that it may make sense to consider a question of whether points with specific properties may be contained in a wanderval (one could call this a "pointwise" approach to the problem of existence of wandervals). Note also that we only need C^1-smoothness while a higher smoothness is necessary for all previous results (see, e.g., [MMS]).

3. Discussion of Assumptions

We made three assumptions in Main Theorem: piecewise smoothness, non-zero shrinkability and the Collet-Eckmann condition at a turning point c. Now we will discuss them.

Our definition of piecewise smoothness is as unrestrictive as possible. In particular, it does not prevent existence of wandervals, even ones that do not come close to the turning points. An example of such a map can be easily derived from Denjoy's example of a circle diffeomorphism with wandervals (see [CN]).

We compensate by assuming that a turning point c satisfies the Collet-Eckmann condition. Then the trajectory of this point has nothing to do with wandervals (see Proposition 2.1).

The assumption on non-zero shrinkability is in fact a condition on the behavior of the map close to singular points. Let a be a singular half-point. We will say that f is *non-flat* at a if either $f'(a) \neq 0$ or there exists a neighborhood W (one-sided) of a and a constant $L > 0$ such that for every $t \in W \setminus \{a\}$,

$$L \leq \frac{|f(t) - f(a)|}{|t - a|\,|f'(t)|} \leq 1. \tag{3.1}$$

Lemma 3.1. *If a is the only singular point in $W = [a; b)$ and (3.1) holds for all $t \in (a; b)$, then $r(x, y) \geq L$ for every $x, y \in W$.*

Proof. The sign of f' on W is constant; we may assume it is positive. We may also assume that a is the left endpoint of W. The proof in the remaining three cases is similar. We have by (3.1),

$$\left(\frac{f(t) - f(a)}{t - a}\right)' = \frac{f'(t)(t - a) - (f(t) - f(a))}{(t - a)^2} \geq 0,$$

so the function $(f(t) - f(a))/(t - a)$ is non-decreasing. Hence, if $a < x < y < b$, then $(f(y) - f(a))/(y - a) \geq (f(x) - f(a))/(x - a)$. Therefore

$$\frac{f(y) - f(x)}{y - x} - \frac{f(y) - f(a)}{y - a} = \left(\frac{f(y) - f(a)}{y - a} - \frac{f(x) - f(a)}{x - a}\right)\frac{x - a}{y - x} \geq 0,$$

so we get

$$\frac{f(y) - f(x)}{y - x} \geq \frac{f(y) - f(a)}{y - a} \geq \frac{f(x) - f(a)}{x - a}.$$

Hence, by (3.1) we get $r(x, y) \geq L$ and $r(y, x) \geq L$. This completes the proof. □

Corollary 3.2. *If f is smooth on $[a, b]$, non-flat at a and b, and f' is non-zero on (a, b) then f has non-zero shrinkability on $[a, b]$.*

To justify our use of the term "non-flat", we prove the following lemma.

Lemma 3.3. *Let $l > 1$ be an integer and let f be a function of class C^{l-1} defined in a (one-sided) neighborhood of a point a. Assume that $f^{(i)}(a) = 0$ for $i = 1, \dots, l - 1$, and that $f^{(l)}(a)$ exists and is non-zero. Then f is non-flat at a.*

In particular, if a is a non-degenerate critical point of f (that is, $f'(a) = 0$ but $f''(a) \neq 0$) then f is non-flat at a.

Proof. By the definition of the derivative, we have

$$\lim_{t\to a} \frac{f^{(l-1)}(t)}{t-a} = f^{(l)}(a).$$

Thus, by l'Hôpital's rule,

$$\lim_{t\to a} \frac{f'(t)}{(t-a)^{l-1}} = \frac{f^{(l)}(a)}{(l-1)!} \quad \text{and} \quad \lim_{t\to a} \frac{f(t)}{(t-a)^{l}} = \frac{f^{(l)}(a)}{l\,!} .$$

Therefore

$$\lim_{t\to a} \frac{f(t)}{(t-a)f'(t)} = \frac{1}{l} ,$$

so (3.1) holds in a neighborhood of a with $L = 1/(2l)$. Thus, f is non-flat at a. □

On the other hand, assume that (3.1) holds. If $t > a$ and $f'(t) > 0$ (the other three cases are similar) we get from the first inequality of (3.1),

$$[\ln(f(t) - f(a))]' \leq \frac{1}{L(t-a)} .$$

Integrating from t (close to a) to some $b > a$ we get

$$\ln(f(b) - f(a)) - \ln(f(t) - f(a)) \leq \frac{1}{L}[\ln(b-a) - \ln(t-a)].$$

Therefore

$$f(t) - f(a) \geq \text{const.}(t-a)^{1/L}.$$

This means that f really cannot be flat (in the common sense) at a.

The assumption on non-zero shrinkability restricts severely possible behaviors near singular points that are not turning points. It is easy to check that if the singularities from both sides of such a point have the same order, shrinkability stays positive. However, if they are of different orders, shrinkability is zero.

Thus, our assumptions on the behavior on both sides of a singular point are quite different than the assumptions in [MS]. We don't care if the orders of the singularities are different if the point is a turning point, but it is important that they are the same if the point is not a turning point. In [MS] this is just the opposite.

The third assumption we are making is the Collet-Eckmann condition at one of the turning points. One can ask whether we can replace it by a subexponential expansion. We cannot do it when using our techniques. The trajectory of the turning point may be coming back to U with some fixed frequency, and we have to guarantee sufficient expansion between each two consecutive returns.

4. Corollaries

Main Theorem is stated in a rather technical way. However, it has important consequences, that can be stated in more general terms. In order to do it, we need a couple of lemmas. They are very simple, so we state them without proof.

Lemma 4.1. *If g satisfies one of the conditions from the conclusion of Main Theorem then it is not essentially conjugate to f.*

Lemma 4.2. *Let f be a piecewise smooth map and let U be a neighborhood of a turning point c of f. Let $h : [0,1] \to [0,1]$ be a function of class C^∞ that is 0 outside U and 1 in some neighborhood of c. Then the map εh tends to 0 in the C^∞ topology as $\varepsilon \to 0$. Moreover, $f + h \in B(f, \varepsilon, U, c)$ if $(f + h)([0,1]) \subset [0,1]$.*

Thus, we can always make perturbations of the type described in the Main Theorem. Now, our corollaries are the following.

Corollary 4.3. *Let f be a piecewise smooth map with non-zero shrinkability, satisfying Collet-Eckmann condition at some turning point. Then f is not C^r-structurally stable for any $r \le \infty$.*

Corollary 4.4. *Let f be a piecewise smooth map with non-zero shrinkability, satisfying the Collet-Eckmann condition at a turning point c. Then f is C^r-structurally sensitive at c for every $r \le \infty$.*

Readers for which the term "non-zero shrinkability" is too special, can use the results of the previous section and change the assumptions in the above corollaries from "a piecewise smooth map with non-zero shrinkability" to "a smooth map with non-degenerate critical points".

For unimodal maps the situation is somehow simpler. If a critical point c has different itineraries for f and g, then there is $t \in (0,1)$ such that c is periodic for $tf + (1-t)g$. Therefore, in view of Lemma 3.3, we can state another corollary as follows.

Corollary 4.5. *For any $1 \le r \le \infty$, in the space of C^r unimodal interval maps with nondegenerate critical points, the set of Collet-Eckmann maps is nowhere dense and every such map can be approximated by maps with the critical point belonging to a superattracting periodic orbit.*

References

[J] Jakobson, M.V.: Smooth mappings of the circle into itself. Mat. Sb. (N.S.) **85** (127), 163–188 (1971) (Russian; English translation: Math. USSR - Sbornik **14**, 161–185 (1971)

[MMS] Martens, M., de Melo, W. and van Strien, S.: Julia-Fatou-Sullivan theory for real one-dimensional dynamics. Acta Math. **168**, 273–318 (1992)

[MS] de Melo, W. and van Strien, S.: *One-Dimensional Dynamics* Berlin: Springer Verlag, 1983

[CN] Coven, E. and Nitecki, Z.: Nonwandering sets of the powers of maps of the interval. Ergod. Th. & Dynam. Sys. **1**, 9–31 (1981)

Communicated by Ya. G. Sinai