RECURRENT CRITICAL POINTS AND TYPICAL LIMIT SETS FOR CONFORMAL MEASURES

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Abstract. For a rational $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ with a conformal measure μ we show that if there is a subset of the Julia set $J(f)$ of positive μ-measure whose points are not eventual preimages of critical or parabolic points and have limit sets not contained in the union of the limit sets of *recurrent* critical points, then μ is non-atomic, $\mu(J(f)) = 1$, $\omega(x) = J(f)$ for μ-a.e. point $x \in J(f)$ and f is conservative, ergodic and exact. The proof uses a version of the Lebesgue Density Theorem valid for Borel measures and conformal balls.

1. Introduction

Everywhere below $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a rational map, all measures considered are probability Borel and ω-limit sets of points are called *limit* sets. A measure μ is *conformal* if for an *exponent* $\alpha > 0$ we have

$$\mu(f(A)) = \int_A |f'(z)|^\alpha d\mu, \tag{1}$$

whenever $f|A$ is 1-to-1 (e.g., Lebesgue measure m on $\widehat{\mathbb{C}}$ is conformal with exponent 2). It is shown in [Sul83] that f has at least one conformal measure supported on $J(f)$ (see also [DU91]).

We study limit sets of points. By [Sul85] the limit set of a point in the Fatou set is a non-repelling cycle or the union of simple closed curves on which the appropriate power of the map is conjugate to an irrational rotation. If $m(J(f)) = 0$, then this is the full list of limit sets of m-a.e. point in $\widehat{\mathbb{C}}$, which justifies studying the behavior of points in $J(f)$ with respect to a conformal measure μ living on $J(f)$.

Theorem 1.1. *Suppose that f is a rational map and μ is a conformal measure with exponent α. Then the following possibilities hold; more precisely, if case 1 does not hold then case 2 holds.*

(1) *The set of eventual preimages of critical and parabolic points in $J(f)$ together with the set of points $z \in J(f)$ such that $\omega(z)$ is a subset of the union of limit sets of recurrent critical points has full μ-measure in $J(f)$.*

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(2) *Otherwise f is exact, ergodic, conservative, μ is unique, non-atomic,* $\operatorname{supp}(\mu) = J(f)$, $\omega(z) = J(f)$ *for μ-almost every point z and α is the minimal exponent for which a conformal measure with support on $J(f)$ exists.*

We were inspired by [Lyu83] where it is proven that for rational maps f with $J(f) \neq \widehat{\mathbb{C}}$ the limit set of m-a.e. point in $J(f)$ is contained in the union of limit sets of *all* critical points. Let us briefly overview some other results in this direction. In case $m(J(f)) > 0$ it is shown in [McM94] that if the limit set of m-a.e. point in $J(f)$ is not contained in the union of limit sets of all critical points then $J(f) = \widehat{\mathbb{C}}$ and f is ergodic with respect to m. In [Bar99] it is shown that in the second case the map is also exact and conservative. Similar results were obtained in [GPS90]. It was pointed out to us that the results in the latter paper can be generalized to conformal measures using standard arguments. Related stronger results with assumptions on measure are known: if $J(f)$ is expanding [Bow75, Sul83, Wal78] or if f is from a certain class of unimodal polynomials [Pra96] then f is ergodic with respect to μ. In these papers, as in many others, the specifics are quite important (see [Prz96] and [DMNUrb] for recent results and references).

We would like to emphasize that in the cited papers (e.g., in [GPS90]) the results about the limit behavior concern the union of limit sets of *all* critical points while we replace it by the union of limit sets of only *recurrent* critical points. Namely, making *no assumptions* on maps we show that either the limit set of μ-a.e. point in $J(f)$ which is not a preimage of a critical or parabolic point is contained in the union of the limit sets of the recurrent critical points, or μ is a so-called *lim sup full* measure which implies a lot of properties. So for the limit sets the following possibilities hold: 1) μ-a.e. point z has limit set in the union of limit sets of *recurrent* critical points or is an eventual preimage of a critical or parabolic point; 2) $\omega(z) = J(f)$ for μ-a.e. z.

Our approach involves a version of the Lebesgue Density Theorem for a Borel probability measure and a class of conformal images of round balls. The proof mostly uses topological and geometrical methods on the plane and relies upon some results of [Mor47]; we also use analytical results from [Pom92]. Note that by different methods the Main Theorem is proven in [BMO] for Lebesgue measure.

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2. Preliminaries

In Section 2, T is an endomorphism of a measure space (X, μ). We call T *non-singular* provided for any μ-measurable set A, $\mu(A) = 0$ implies $\mu(T(A)) = 0$. A useful property of non-singular maps is that if $A_n \subset B$ and $\mu(A_n) \to \mu(B)$ then $\lim \mu(T(A_n)) = \mu(T(B))$. A map T is called *conservative* if for all sets A with $\mu(A) > 0$, there exists a $k > 0$ such that $\mu(T^k(A) \cap A) > 0$; T is called *ergodic* if $\mu(A) \in \{0, 1\}$ whenever $T^{-1}(A) = A$; T is called *exact* if $\mu(A) \in \{0, 1\}$ whenever $T^{-n} \circ T^n(A) = A$ for all $n > 0$ (it follows from [Roh64] that for rational maps and non-singular measures this definition of exact is equivalent to the traditional one). Every exact map is ergodic. If X is compact then $\operatorname{supp}(\mu)$ is the set of all points whose all neighborhoods have positive μ-measure; then $\mu(\operatorname{supp}(\mu)) = 1$.

A set A is *(μ-)lim sup full* [Bar99] if $\limsup \mu(T^n(A)) = 1$; T is *(μ-)lim sup full* if every set of positive measure is lim sup full.

Theorem 2.1. [Bar99] *Let T be a non-singular d-to-1 lim sup full endomorphism. Then T is conservative and exact.*

Theorem 2.1 applies to rational maps and their conformal measures. We rely upon it in order to simplify some of the proofs. Other properties of lim sup full maps are obtained in Lemma 2.2.

Lemma 2.2. *Let T be a non-singular endomorphism.*

1. *If $L \subset X$ with $\mu(L) > 0$ is invariant and a positive measure subset of L is lim sup full then $\mu(L) = 1$ and f is lim sup full; in particular, a lim sup full map is exact.*
2. *If T is lim sup full and atomic then μ is concentrated on a single fixed point (in particular, if X is a topological space and $\text{supp}(\mu)$ is not a single point, then μ is non-atomic).*
3. *If T is a lim sup full continuous map of a metric compactum X, then $\text{supp}(\mu)$ is T-invariant and $\omega(z) = \text{supp}(\mu)$ for μ-a.e. z.*

Proof. We prove only statement 3 and do this by way of contradiction. Notice that if $\mu(G) > 0$ then $\mu(T(G)) > 0$ for any set G. Indeed, otherwise by non-singularity $\mu(T^k(G)) = 0$ for all $k \geq 1$, a contradiction to T being lim sup full. This and continuity of T immediately implies that $\text{supp}(\mu)$ is invariant.

We may assume that μ is non-atomic and consider the restriction $T|\text{supp}(\mu)$ (WLOG from now on we assume $\text{supp}(\mu) = X$ and so $\mu(U) > 0$ for all open subsets U of X). Suppose that the set B of points whose limit set is not X has positive μ-measure. Choose a countable basis for X and for any element U of the basis let $A_U = \{x : \omega(x) \cap U = \emptyset\}$. Then the union of all such sets A_U is B, so there exists U such that $\mu(A_U) > 0$. By ergodicity this implies that $\mu(A_U) = 1$, a contradiction with $\mu(U) > 0$ (the latter follows from $\text{supp}(\mu) = X$). ☐

In Lemma 2.3 we first state some well known properties of rational maps and then apply Lemma 2.2 to conformal measures. Observe, that some maps f have an *exceptional* set $E(f)$ which is a fully invariant set of one (two) attractive fixed points or an attracting two-periodic orbit [Bea91], so $\mu(E(f)) = 0$ for a conformal measure μ.

Lemma 2.3. *Let f be a rational map non-singular with respect to a measure μ such that $\mu(\widehat{\mathbb{C}} \setminus E(f)) > 0$.*

1. *If V is open and $V \cap J(f) \neq \emptyset$, then for any compact $K \subset \widehat{\mathbb{C}} \setminus E(f)$ there exists n with $K \subset f^n(V)$ (so, $\widehat{\mathbb{C}} \setminus E(f) \subset \cup f^n(V)$).*
2. *$J(f) \subset \text{supp}(\mu)$.*
3. *Let $\mu(E(f)) = 0$. If V is open, $V \cap J(f) \neq \emptyset$ and $\limsup \mu(f^n(R) \cap V) = \mu(V)$ for a set R, then R is μ-lim sup full.*
4. *Let $\mu(E(f)) = 0$. If f is μ-lim sup full, then $\text{supp}(\mu) = J(f), \mu$ is non-atomic and $\omega(x) = J(f)$ for μ-a.e. x.*

Proof. 1. See [Bea91].

2. If V is open, $V \cap J(f) \neq \emptyset$ and $\mu(V) = 0$, then by non-singularity and statement 1 we have $\mu(\widehat{\mathbb{C}} \setminus E(f)) = 0$, a contradiction.

3. Follows from 1. and properties of non-singular maps.

4. By Lemma 2.2(3), $\omega(z) = \text{supp}(\mu)$ for μ-a.e. z. By 2., $J(f) \subset \text{supp}(\mu)$. Since there are no points z with $J(f) \subsetneqq \omega(z)$ we conclude that $\text{supp}(\mu) = J(f)$. Non-atomicity follows from statement 2 of Lemma 2.2. □

3. Density Theorems

In this section by a *cover* we always mean a cover of the plane by Borel sets and by a *bounded set* a bounded Borel subset of the plane. Our aim is to prove a version of the Lebesgue Density Theorem for a Borel measure which is finite on bounded sets (the family of all such measures is denoted by $\mathcal{FOB}$) and a suitable family of *conformal balls.* We begin with definitions and results not related to conformal maps.

Definition 3.1. *We say that a cover $\mathcal{Q}$ is a* Vitali *cover with multiplier $M > 1$ provided that for every element $Q \in \mathcal{Q}$ there exists a special point x_Q (called a* center *of Q) and a number $r(Q) > 0$ (called an* inner radius*) such that:*

(1) *For two closed Euclidean balls centered at x_Q, $B(x_Q, r(Q))$ and $B(x_Q, Mr(Q))$, we have*
$$B(x_Q, r(Q)) \subset Q \subset B(x_Q, Mr(Q)),$$

(2) *for every $z \in Q$, Q contains the convex hull of the set*
$$\{z\} \cup B(x_Q, r(Q)).$$

A cover is closed *if all its elements are closed.*

It follows that any closed element of a Vitali cover is the closure of its interior. Indeed, suppose that Q is a closed element of a Vitali cover and $z \in \partial Q$. Then since the entire convex hull of the set $\{z\} \cup B(x_Q, r(Q))$ is contained in Q we see that $z \in \overline{\text{Int}(Q)}$. Hence indeed $Q = \overline{\text{Int}(Q)}$. We choose closed covers for the sake of definiteness. By $Q(x)$ we denote an element of a Vitali cover centered at $x = x_Q$. Notice, that a cover with multiplier M is also a cover with multiplier $M' > M$ and the choice of center and inner radius in the definition is not fixed. In particular if $Q \in \mathcal{Q}$ is an element of a Vitali cover with multiplier M, inner radius $r(Q)$ and center x_Q, then there exists a (sufficiently small) open set O containing x_Q such that for each $y \in O$, the set Q can be considered as an element of a Vitali cover $\mathcal{Q}'$ with multiplier $M' = 3M$, center $x'_Q = y$ and inner radius $r'(Q) = r(Q)/2$. We need this observation later.

Lemma 3.2. *If $\mathcal{Q} = \{Q_\alpha\}$ is a Vitali cover then $\overline{\mathcal{Q}} = \{\overline{Q}_\alpha\}$ is also a Vitali cover.*

Proof. We only need to verify condition 2 for points $z \in \overline{Q} \setminus Q$. Since $z \in \overline{Q}$, there exists a sequence $z_i \in Q$ such that $\lim z_i = z$. For each $w \in B(x_Q, r(Q))$ the straight line segment $[w, z_i]$ is contained in Q. Hence, $[w, z] = \lim[w, z_i] \subset \overline{Q}$ as desired. □

The following theorem proved by Morse [Mor47] shows that one can choose subcovers of Vitali covers with nice properties on bounded sets.

Theorem 3.3. [Mor47] *There exists a constant C (depending on M) such that for every closed Vitali cover $\mathcal{Q}$ with multiplier M, every bounded set A and a family of elements $\{Q(x) \in \mathcal{Q}\}_{x \in A}$ one can select a countable subfamily Q_k such that:*

(1) $\cup_k Q_k$ *covers* A,
(2) *no point of the plane is contained in more than* C *elements of the sequence* Q_k,
(3) *the sequence* Q_k *can be partitioned into at most* C *families of pairwise disjoint sets.*

Theorem 3.3 can be used to obtain versions of the Lebesgue Density Theorem (see [dG75]).

Definition 3.4. *If* $\{Q_k(x)\}$ *is a sequence of elements of a Vitali cover with* $\operatorname{diam}(Q_k(x)) \to 0$ *as* $k \to \infty$ *then we say that it is* centrally nested *at* x *and denote it by* $Q_k(x) \to x$.

Definition 3.5. *We call a Vitali cover* $\mathcal{Q}$ infinitesimal at x *if there exists sequence* $\{Q_k(x)\}$ *of elements of* $\mathcal{Q}$ *centrally nested at* x. *We call* $\mathcal{Q}$ infinitesimal on a set A *if it is infinitesimal at every point of* A. *Finally, we call* $\mathcal{Q}$ infinitesimal *if it is infinitesimal on the plane.*

Theorem 3.6. *Let* $\mu \in \mathcal{FOB}$, $\mathcal{Q}$ *be a closed Vitali cover,* W *be an open set and* $B \subset W$ *be a bounded set on which* $\mathcal{Q}$ *is infinitesimal. Then there exists a sequence* $\{T_k\}$ *of pairwise disjoint sets from* $\mathcal{Q}$ *contained in* W *such that*

$$\mu(B \setminus \cup T_k) = 0. \tag{2}$$

Proof. Let us first prove that for some number $b < 1$ depending only on M, any open set U and any bounded set $A \subset U$ such that $\mathcal{Q}$ is infinitesimal on A there exists a finite collection $\{S_k\}$ of pairwise disjoint sets from $\mathcal{Q}$ which are subsets of U with $\mu(A \setminus \cup S_k) \le b\mu(A)$.

Indeed, since $A \subset U$ and $\mathcal{Q}$ is infinitesimal on A then for any $x \in A$ there exists an element $Q(x)$ of $\mathcal{Q}$ such that $Q(x) \subset U$. Choose such sets $Q(x)$ for all points of A and apply Theorem 3.3 to this cover of A. It follows that there are at most C families of pairwise disjoint elements from this cover of A such that the union of the elements of all these families covers A. Choose the sequence $Q_1, Q_2, \dots$ the union of whose elements intersected with A has the greatest μ-measure. Then $\mu(A \cap (\cup Q_k)) \ge \frac{1}{C}\mu(A)$. Therefore we can choose a finite collection $S_1, \dots, S_m$ of sets from this family so that $\mu(A \cap (\cup_{i=1}^m S_i)) \ge \frac{1}{2C}\mu(A)$. Set $b = 1 - \frac{1}{2C}$ and notice that

$$\mu(A \setminus \cup_{i=1}^m S_k) \le (1 - \frac{1}{2C})\mu(A) = b\mu(A).$$

Now set $W_0 = W$ and $B_0 = B$, and apply the above claim to $B_0 \subset W_0$. This gives a finite collection $\{T_i\}_{i=1}^{h_1}$ of pairwise disjoint elements of $\mathcal{Q}$ such that, denoting $B \setminus \cup_1^{h_1} T_k$ by B_1, we have $\mu(B_1) \le b\mu(B)$. Set $W_1 = W_0 \setminus \cup_1^{h_1} T_k$ and apply the same claim, but now to $B_1 \subset W_1$. This leads to a new finite collection $\{T_i\}_{i=h_1+1}^{h_2}$ of $\mathcal{Q}$, the set B_2 with $\mu(B_2) \le b\mu(B_1) \le b^2\mu(B)$ and the corresponding open set W_2. Induction completes the proof. □

Theorem 3.8 is similar to the classical Lebesgue Density Theorem. To state it we need the following definition.

Definition 3.7. *Let* $\mathcal{Q}$ *be a closed infinitesimal Vitali cover. A point* x *is said to be a point of* $\mathcal{Q}$*-density of a set* R *with respect to a measure* μ *if, for any centrally nested sequence* $\{Q_k(x)\}$ *of elements of* $\mathcal{Q}$*, the following holds:*

$$\lim_{Q_k(x) \to x} \frac{\mu(Q_k(x) \cap R)}{\mu(Q_k(x))} = 1 \tag{3}$$

Theorem 3.8. *Let $\mu \in \mathcal{FOB}$, R be a bounded set with $\mu(R) > 0$ and $\mathcal{Q}$ be a closed infinitesimal Vitali cover. Then μ-a.e. $x \in R$ is a point of $\mathcal{Q}$-density of R with respect to μ.*

Proof. Consider the set

$$A = \{x \in R | \exists Q_k(x) \to x, \lim_{Q_k(x)\to x} \frac{\mu(Q_k(x) \cap R)}{\mu(Q_k(x))} \neq 1\}$$

The statement of the theorem is that $\mu(A) = 0$. To show this we represent the set A as the union of a countable family of subsets and show that they all are of zero μ-measure. We adapt standard arguments from the proof of the classical Lebesgue Density Theorem.

Suppose that $a < 1$ and define the set A_a as follows:

$$A_a = \{x \in R | \exists Q_k(x) \to x, \frac{\mu(Q_k(x) \cap R)}{\mu(Q_k(x))} < a\}$$

Since $A = \cup_{n=1}^{\infty} A_{1-1/n}$ it suffices to show that $\mu(A_a) = 0$ for $a < 1$.

Fix $\delta > 0$ and consider the set A_a^δ of all points $x \in R$ such that there exist $Q(x) \in \mathcal{Q}$ of diameter less than δ with $\frac{\mu(Q_k(x)\cap R)}{\mu(Q_k(x))} < a$. Then $A_a = \cap_\delta A_a^\delta$. Now we use the observation made after the definition of a Vitali cover and work with elements of $\mathcal{Q}$ with a new inner radius $r'(Q) = r(Q)/2$ and a new multiplier $M' = 3M$; considered like that they form a Vitali cover $\mathcal{Q}'$. Every center $x \in A_a^\delta$ of the appropriate $Q(x)$ has a small neighborhood O_x of points that are centers of the same set as an element of $\mathcal{Q}'$. For every $x \in A_a^\delta$ we pick such a neighborhood and denote the union of all these neighborhoods by $\tilde{A}_a^\delta$. The set $\tilde{A}_a^\delta$ is open, contains the set A_a^δ and every point of this set is a center of some $Q \in \mathcal{Q}'$ with $\frac{\mu(Q\cap R)}{\mu(Q)} < a$.

Let $\tilde{A}_a = R \cap \left(\cap_n \tilde{A}_a^{1/n}\right)$; $\tilde{A}_a$ is a measurable set, containing A_a, and for any $x \in \tilde{A}_a$ there is a sequence of sets $\{Q_k(x)\}, Q_k(x) \in \mathcal{Q}', Q_k(x) \to x$ with $\frac{\mu(Q_k(x)\cap R)}{\mu(Q_k(x))} < a$ for any k. These sets form the cover $\mathcal{Q}''$ of $\tilde{A}_a$.

Now, the fact that μ is a Borel measure implies that there are open sets containing $\tilde{A}_a$ whose μ-measure is arbitrarily close to $\mu(\tilde{A}_a)$. Choose an open set $U \supset \tilde{A}_a$. Consider the cover $\mathcal{Q}''$ of $\tilde{A}_a$ and apply Theorem 3.6. By this theorem there is a sequence $\{S_k\}$ of pairwise disjoint sets from $\mathcal{Q}''$ such that $\mu(\tilde{A}_a \setminus \cup S_k) = 0$ and $\cup S_k \subset U$. On the other hand, $\frac{\mu(S_k\cap R)}{\mu(S_k)} < a < 1$ by the choice of $\mathcal{Q}''$. This implies that

$$\mu(\tilde{A}_a) \le \mu(\cup(S_k \cap R)) \le a\mu(\cup S_k) \le a\mu(U)$$

and pushing $\mu(U)$ to $\mu(\tilde{A}_a)$ we have $\mu(\tilde{A}_a) \le a\mu(\tilde{A}_a)$ and thus $\mu(\tilde{A}_a) = 0$. Since $A_a \subset \tilde{A}_a$ this implies that $\mu(A_a) = 0$. □

Definition 3.9. *Let U, V be open sets in $\mathbb{C}$ and $\varphi : U \to V$ be a conformal isomorphism. Then $d = \sup_{x,y\in U} |\frac{\varphi'(x)}{\varphi'(y)}|$ is called the* distortion *of φ and φ is called a K*-conformal isomorphism *if $d \le K$.*

Let us now define conformal balls.

Definition 3.10. *By a* conformal ball*, or* cball*, we mean a conformal image of the round ball $B(\mathbf{0}, 1)$ in the complex plane, where $\mathbf{0} = (0, 0)$. A cball Q is called a* KN-ball centered *at x provided there exist $r > 0$, $N > 1$ and a K-conformal isomorphism $\varphi : B(\mathbf{0}, r) \to \mathbb{C}$*

such that $\varphi(B(\mathbf{0}, r/N)) = Q$, $\varphi(\mathbf{0}) = x$ *and* $\varphi'(\mathbf{0}) = 1$. *We say that* φ *is a* generating map *for* Q *and that* r/N *is the* conformal radius *of* Q.

We apply Theorem 3.8 to conformal balls. More precisely, we show that the collection of KN-balls , for any $N > 1$ and K sufficiently close to 1, forms a Vitali cover. It will follow that the Lebesgue Density Theorem holds for this collection with respect to any measure from $\mathcal{FOB}$. Recall that a set T is said to be *convex* if, for every $w, b \in T$, every straight line segment $[b, w]$ is contained in T. Now we can state the following:

Theorem 3.11. [Pom92, page 65] *A* KN*-ball* Q *generated by* φ *with conformal radius* r/N *is convex iff*

$$Re[1 + z\varphi''(z)/\varphi'(z)] > 0 \textit{ for all } z \in B(\mathbf{0}, r/N).$$

Lemma 3.12 follows from Theorem 3.11.

Lemma 3.12. *For each* $N > 1$ *there is* $\delta > 0$ *such that if* $\varphi : B(\mathbf{0}, r) \to \mathbb{C}$ *is a conformal isomorphism with* $\varphi'(0) = 1$ *and*

$$1/(1+\delta) < \left|\frac{\varphi'(x)}{\varphi'(y)}\right| < 1 + \delta \textit{ for all } x, y \in B(\mathbf{0}, r), \tag{4}$$

then $\varphi(B(\mathbf{0}, r/N))$ *is convex.*

Proof. We will first establish the following:

Claim. For each $N > 1$ and $0 < \epsilon < 1/4$ there exists $\delta > 0$ such that for each conformal isomorphism $\psi : B(\mathbf{0}, 1) \to \mathbb{C}$ with $\psi'(\mathbf{0}) = 1$ and

$$1/(1+\delta) < \left|\frac{\psi'(x)}{\psi'(y)}\right| < 1 + \delta \text{ for all } x, y \in B(\mathbf{0}, 1), \tag{5}$$

we have for all $z \in B(\mathbf{0}, 1/N)$

$$|\psi'(z) - 1| < \epsilon, \ |\psi''(z)| < \epsilon \text{ and } \mathrm{Re}[1 + z\psi''(z)/\psi'(z)] > 0. \tag{6}$$

Suppose the claim is not true. Then there exists a sequence $\psi_n : B(\mathbf{0}, 1) \to \mathbb{C}$ such that inequality (5) holds for each ψ_n with $\delta = 1/n$, $\psi_n'(0) = 1$, $\psi_n(0) = 0$ and for each n there is a point $x_n \in B(\mathbf{0}, 1/N)$ with either $|\psi_n'(x_n) - 1| \geq \epsilon$, or $|\psi_n''(x_n)| \geq \epsilon$. Since the family $\{\psi_n\}$ is normal, we may assume that $\psi_n \to \psi_\infty$. Then $\psi_\infty'(0) = 1$ and so, $|\psi_\infty'(z)| = 1$. Hence, $\psi_\infty(z) = z$, $\psi_\infty'(z) = 1$ and $\psi_\infty''(z) = 0$ for all $z \in B(\mathbf{0}, 1)$. Since $\psi_n(z)$, $\psi_n'(z)$ and $\psi_n''(z)$ converge uniformly on $B(\mathbf{0}, 1/N)$ to $z, 1$ and 0 respectively we get a contradiction. This completes the proof of the claim. By Theorem 3.11, $\psi(B(\mathbf{0}, 1/N))$ is convex.

Let $\delta > 0$ be given by the above Claim and let $\varphi : B(\mathbf{0}, r) \to \mathbb{C}$ be a conformal isomorphism such that $\varphi'(0) = 1$ and satisfying (4). Let $g(z) = rz$; then $\psi = g^{-1} \circ \varphi \circ g : B(\mathbf{0}, 1) \to \mathbb{C}$ is a conformal isomorphism with $\psi'(z) = \varphi'(g(z))$ for all z. Hence ψ satisfies the conditions of the Claim and $\psi(B(\mathbf{0}, 1/N))$ is convex. Since $\varphi(B(\mathbf{0}, r/N)) = g \circ \psi(B(\mathbf{0}, 1/N))$ the required result is established. □

The following theorem allows us to apply our version of the Density Theorem to KN-balls .

Theorem 3.13. *For any* $N > 1$, *there is* $\delta > 0$ *such that for any* $1 < K < 1 + \delta$ *the closures of* KN*-balls form a Vitali cover.*

Proof. Fix $N > 1$. By Lemma 3.12 we can choose $0 < \delta < 1/4$ such that for each $(1+\delta)$-conformal isomorphism $\varphi : B(\mathbf{0}, r) \to \mathbb{C}$, the set $Q = \varphi(B(\mathbf{0}, r/N))$ is convex. Let $1+\delta = K_0$. By Lemma 3.2, it suffices to show that a K_0N-ball Q centered at $\mathbf{0}$ satisfies the conditions of Definition 3.1. Let $\varphi : B(\mathbf{0}, r) \to \mathbb{C}$ be the generating map for Q. Hence, $\varphi(\mathbf{0}) = \mathbf{0}$ and $\varphi'(\mathbf{0}) = 1$.

Since Q is convex, condition 2 of Definition 3.1 of a Vitali cover is automatically satisfied. Condition 1 follows immediately from the Koebe Theorem and the inequality $|\varphi'| < 5/4$ which together imply that $B(\mathbf{0}, r/(4N)) \subset Q \subset B(\mathbf{0}, 5r/(4N))$. Hence the collection of K_0N-balls is an infinitesimal Vitali cover with multiplier 5. □

Putting Theorems 3.8 and 3.13 together yields the following:

Corollary 3.14. *There exists K such that the collection $\mathcal{Q}_{K2}$ of closures of $K2$-balls is a Vitali cover and thus for any $\mu \in \mathcal{FOB}$ and any bounded set R with $\mu(R) > 0$ we have that μ-a.e. $x \in R$ is a point of $\mathcal{Q}_{K2}$-density of R with respect to μ.*

4. Main Theorem

For $x \in \widehat{\mathbb{C}}$ and $n > 0$, consider $r_n(x) = \sup\{r : B(f^n(x), r)$ can be pulled back to x univalently$\}$. Then $r_n(x) > 0$ if and only if all points $x, \dots, f^{n-1}(x)$ are not critical; otherwise we set $r_n(x) = 0$ which fully defines $r_n(x)$. Let $L(f)$ be the set of all points $x \in J(f)$ with $r_n(x) \not\to 0$. Then $L(f)$ is forward invariant. Sometimes $L(f)$ is called the *conical set* of f [DMNUrb]. Below K is the constant from Corollary 3.14.

Lemma 4.1. *Let $x \in L(f)$ and μ be an α-conformal measure. Then there exists a point ζ, a number $\gamma > 0$, a sequence of integers $n_i \to \infty$ and a sequence of $K2$-balls $W_i' \to x$, centered at x, such that $f^{n_i} : W_i' \to B(\zeta, \gamma) = B$ is univalent, onto, has distortion at most $\sqrt{K}$ and $\min\{|(f^{n_i})'(z)| : z \in W_i'\} \to \infty$. Moreover, there exists $c = c(\alpha)$ such that $\frac{\mu(B)}{c}|(f^{n_i})'(x)|^{-\alpha} \le \mu(W_i') \le c\mu(B)|(f^{n_i})'(x)|^{-\alpha}$.*

Proof. Let $x \in L(f)$. Then there exists $\eta' > 0$, a sequence $n_i \to \infty$ and a sequence of compact disks $W_i \ni x$ such that $f^{n_i} : W_i \to B(f^{n_i}(x), \eta')$ is univalent. We may assume that $f^{n_i}(x) \to \zeta$ for some $\zeta \in J(f)$. By the Koebe Theorem, we can choose $\eta < \eta'$ such that for all i and any $y, z \in W_i$ with $f^{n_i}(y), f^{n_i}(z) \in B(f^{n_i}(x), \eta)$ we have $|(f^{n_i})'(y)/(f^{n_i})'(z)| < \sqrt{K}$. Let us fix s which is slightly bigger than 1. For sufficiently large i $f^{n_i}(x)$ is so close to ζ that the Moebius transformation φ_i which keeps $B(\zeta, \eta/2s)$ invariant and maps ζ into $f^{n_i}(x)$ has distortion at most $\sqrt{K}$ on $B(\zeta, \eta/s)$ and maps $B(\zeta, \eta/s)$ into $B(\zeta, \eta)$.

Let W_i' be the connected component of the set $f^{-n_i}(B(\zeta, \eta/2s))$ containing x. We show that W_i' are elements of $\mathcal{Q}_{K2}$ (i.e. $K2$-balls) centered at x. Indeed, consider the map $f^{-n_i}\varphi_i : B(\zeta, \eta/s) \to \mathbb{C}$ with the appropriately chosen branch of inverse function f^{-n_i}. Under this map ζ is mapped onto x, $W_i' = f^{-n_i}\varphi_i(B(\zeta, \eta/2s))$ and the distortion of the map is at most K. Hence $W_i'(x) \in \mathcal{Q}_{K2}$. Moreover, $\operatorname{diam}(W_i') \to 0$ and $\min\{|(f^{n_i})'(x)| : x \in W_i'\} \to \infty$ as $i \to \infty$ (see, e.g., [McM94, Theorem 3.6, p. 39]). Hence, $W_i'(x) \to x$ is a sequence of elements of $\mathcal{Q}_{K2}$ centrally nested at x, and we can set $\gamma = \eta/2s$. The last statement of the lemma follows from the definition of a conformal measure. □

Lemma 4.2. *Let μ be a conformal measure of exponent α with $\mu(L(f)) > 0$. Then μ is a unique measure with this property and the following holds:*

(1) *f is lim sup full, exact, ergodic, conservative, μ is non-atomic,* $\text{supp}(\mu) = J(f)$ *and $\omega(z) = J(f)$ for μ-a.e. $z \in \widehat{\mathbb{C}}$;*

(2) *α is the minimal exponent for which a conformal measure with support on J exists.*

Proof. The uniqueness of the measure μ follows from [DMNUrb]. Let us prove the rest of the lemma.

(1) Assume that $x \in L(f) \cap R$ is a point of $\mathcal{Q}_{K2}$-density of a set R with respect to μ and prove that then R is lim sup full. Choose a sequence of sets W_i' with all properties from Lemma 4.1. Then $\lim \frac{\mu(W_k'(x)\cap R)}{\mu(W_k'(x))} = 1$. On the other hand the distortion of the map $f^{n_i} : W_i' \to B(\zeta, \gamma)$ is at most K. This and the properties of conformal measures imply that $\mu(f^{n_i}(R) \cap B(\zeta, \gamma)) \to \mu(B(\zeta, \gamma)) > 0$. By statement 3 of Lemma 2.3 R is lim sup full.

Assume now that $R \subset L(f)$ is such that $\mu(R) > 0$. By Theorem 3.8 μ-a.e. point of R is a point of $\mathcal{Q}_{K2}$-density of R with respect to μ. Hence R is lim sup full. By statement 1 of Lemma 2.2, $\mu(L(f)) = 1$ (and so $\mu(J(f)) = 1$) and f is lim sup full. By statement 4 of Lemma 2.3, $\text{supp}(\mu) = J(f)$, μ is non-atomic and $\omega(x) = J(f)$ for μ-a.e. x. The rest follows from Theorem 2.1.

(2) Let ν be a conformal measure with exponent $\beta < \alpha$, $\omega = \mu + \nu$. Note that, by uniqueness of μ on $L(f)$, $\nu(L(f)) = 0$ and, since $\nu(J(f)) > 0$, $\nu(B) > 0$ for any open set B meeting $J(f)$. For each $x \in L(f)$, let $W_i' \to x$ be a sequence of $K2$-balls, centrally nested at x, as in Lemma 4.1. Then $\omega(L(f) \cap W_i')/\omega(W_i') \to 1$ for ω-a.e. $x \in L(f)$.

Now, by Lemma 4.1, $\mu(W_i') \leq c(\alpha)\mu(B)|(f^{n_i})'(x)|^{-\alpha}$ while $\nu(W_i') \geq \frac{\nu(B)}{c(\beta)}|(f^{n_i})'(x)|^{-\beta}$. Since by Lemma 4.1 $\min\{|(f^{n_i})'(z)| : z \in W_i'\} \to \infty$, we get $\mu(W_i')/\nu(W_i') \to 0$. It is then easy to see that $\limsup \omega(L(f) \cap W_i')/\omega(W_i') = 0$, contradicting the fact this limit must be 1 for ω-a.e. $x \in L(f)$. ☐

Note that in the proof of statement 1 of Lemma 4.2 one can use tools from [GPS90]. Note also that if the set of points z such that $\omega(z)$ is not a subset of the union of the limit points of *all* critical points has positive measure, then $\mu(L(f)) > 0$.

For the sake of brevity we denote the union of the limit sets of recurrent critical points of f by $PR(f)$.

Lemma 4.3. *If $z \notin L(f)$ is neither an eventual preimage of a critical point nor an eventual preimage of a parabolic periodic then $\omega(z) \subset PR(f)$.*

Proof. Suppose that $z \notin L(f)$ is neither an eventual preimage of a critical point nor an eventual preimage of a parabolic periodic point and contrary to the conclusion of the lemma $\omega(z) \not\subset PR(f)$. Since z is not an eventual preimage of a parabolic periodic point then $\omega(z)$ does not coincide with a parabolic cycle P. Indeed, assuming the contrary and replacing f by its power we may suppose $P = \{a\}$ to be a fixed parabolic point. Suppose that z is not eventually mapped into a. Then the orbit of z is infinite. It is well-known (see, e.g., [CG93, pages 35–41]) that a is repelling for $f|J(f)$ in the following sense: there exists a metric d on $J(f)$ such that for some $\varepsilon > 0$ and any point $x \neq a$ which is at most ε-distant from a we have $d(f(x), a) > d(x, a)$. Since the orbit of z is infinite and accumulates on $\{a\}$, there are arbitrarily large n such that $0 < d(f^n(z), a) < \varepsilon$. For any such n there exists $m > n$ such that $d(f^m(z), a) \geq \varepsilon$ (otherwise by the definition of a repelling point z will always stay farther away from a than $f^n(z)$). Thus there exists a sequence of numbers m growing to infinity such that $d(f^m(z), a) \geq \varepsilon$, a contradiction with $\omega(z) = \{a\}$.

Since periodic points are not isolated in limit sets not coinciding with them we conclude that no parabolic point is isolated in $\omega(z)$. Since $\omega(z) \not\subset PR(f)$ we conclude that there exists a sequence $f^{n_k}(z) \to y \in \omega(z) \setminus PR(f)$ such that y is not a parabolic periodic point. By a result of Mañé [Mn93], for each $\epsilon > 0$ there exists a $\delta > 0$ such that components of $f^{-n}(B(y, \delta))$ have diameter less than ϵ and that $B(y, \delta)$ is disjoint from the union of parabolic cycles of f. On the other hand z is not an eventual preimage of any critical point of f. Since $z \notin L(f)$, the critical points that force $r_{n_k}(z) \to 0$ are contained in $\omega(z)$. Moreover, if c is such a critical point then $y \in \overline{orb(c)}$. Since $y \notin PR(f)$ we conclude that all critical points generating $r_{n_k}(z)$ belong to $\omega(z) \setminus PR(f)$. Choose one of these points and denote it c_1; we have $c_1 \in \omega(z) \setminus PR(f)$ and $y \in \overline{orb(f(c_1))}$. We can now apply the same argument to c_1 instead of y which yields a critical point $c_2 \in \omega(z) \setminus PR(f)$ such that $c_1 \in \overline{orb(f(c_2))}$. Repeating this argument and relying upon the finiteness of the number of critical points of f we will find a critical point $c \notin PR(f)$ with $c \in \overline{orb(f(c))}$, a contradiction which completes the proof. □

Proof of Theorem 1.1. Assume that the set of all points z which are not eventual preimages of critical or parabolic points such that $\omega(z) \not\subset PR(f)$ has positive μ-measure. Clearly, by Lemma 4.3 this set is a subset of $L(f)$. Now Lemma 4.2 completes the proof. □

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