MONOTONE IMAGES OF CREMER JULIA SETS

ALEXANDER BLOKH AND LEX OVERSTEEGEN

Abstract. We show that if P is quadratic polynomial with a fixed Cremer point and Julia set J, then for any monotone map $\varphi : J \to A$ from J onto a locally connected continuum A, A is a single point.

1. Introduction

Let $P : \mathbb{C} \to \mathbb{C}$ be a complex polynomial of degree d and let J_P be its Julia set. The topological structure of connected Julia sets $J = J_P$ and the dynamics of $P|_J$ has been studied in a number of papers. The best case, from the topological point of view, is the case when J is locally connected. Then J is homeomorphic to the quotient space of the unit circle $S^1/\sim = J_\sim$ with respect to a specific equivalence relation $\sim$, called an *invariant lamination*. In this case the map $\sigma : S^1 \to S^1$, defined by $\sigma(z) = z^d$ on the unit circle in the complex plane $\mathbb{C}$, induces a map $f_\sim : J_\sim \to J_\sim$ which is conjugate to the restriction $P|_J$. Spaces like $J_\sim$ are called below *topological Julia sets* while the induced maps $f_\sim$ on them are called *topological polynomials*. Thus, in the locally connected case, topological polynomials acting on topological (locally

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connected) Julia sets are good (one-to-one) models for true complex polynomials acting on their Julia sets.

Even if J is not locally connected this approach works in many cases. Given a polynomial P, call its irrational neutral periodic points *CS-points*; a CS-point p is said to be a *Cremer point* if the power of the map which fixes p is not linearizable in a small neighborhood of p. Suppose that P is a polynomial with connected Julia set and no CS-points. In his fundamental paper [Kiw04] Jan Kiwi obtained for such P an invariant lamination $\sim_P$ on S^1 such that $P|_{J_P}$ is semi-conjugate to the induced map $f_{\sim_P} : J_{\sim_P} \to J_{\sim_P}$ by a monotone map $m : J_P \to J_{\sim_P}$ (by *monotone* we mean a continuous map whose point preimages are connected). In addition Kiwi proved in [Kiw04] that for any P-periodic point $p \in J_P$ the set J_P is locally connected at p and $m^{-1} \circ m(p) = \{p\}$.

Thus, Kiwi's approach allows one to describe the dynamics of these polynomials restricted to their Julia sets by means of a certain monotone map onto a locally connected continuum; this dynamically motivated monotone map is a semiconjugacy between the polynomial and the corresponding induced map (in this case the induced map is a topological polynomial). The aim of this paper is to show that in some cases the entire approach which uses modeling of the Julia set by means of a monotone map onto a locally connected continuum breaks down for topological reasons. By a *basic Cremer polynomial* we mean a quadratic polynomial P with a *fixed* Cremer point. Our main result is Theorem 2.2.

Theorem 2.2. *If P is a basic Cremer polynomial and $\varphi : J_P \to A$ is a monotone map onto a locally connected continuum A, then A is a single point.*

Let K be a continuum such that if $\varphi : K \to A$ is a monotone map onto a locally connected continuum A then A is a single point. We call such continua *incompressible* and show in Theorem 2.2 that if P is a basic Cremer polynomial then its Julia set J_P is an incompressible continuum. Thus, in the case of a basic Cremer polynomial, studying the Julia set by means of a monotone map onto a locally connected continuum is impossible, and one needs a different approach (see, e.g., [BO06]).

2. Main Theorem

An *unshielded* continuum $K \subset \mathbb{C}$ is a continuum which coincides with the boundary of the infinite complementary component to K. Given an unshielded continuum K we denote by R_α the external (conformal) ray corresponding to the external angle α and by $\Pi_\alpha = \overline{R_\alpha} \setminus R_\alpha$ the corresponding principal set (usually, the continuum is fixed in the beginning of the argument, so we can omit K from the notation; if we do not want to specify the angle we will omit α too). A crosscut C of K is an open arc in $\mathbb{C} \setminus K$ whose closure meets K in two distinct points. Given an external ray R, a crosscut C is said to be *R-transversal* if $\overline{C}$ intersects $\overline{R}$ (topologically transversely) only once; if $t \in R$ then by C_t we always denote an R-transversal crosscut such that $\overline{C_t} \cap \overline{R} = \{t\}$. The *shadow of C*, denoted by $\mathrm{Sh}(C)$, is the bounded component of $\mathbb{C} \setminus C \cup K$. Given an external ray R we define the *(induced) order* on R so that $x <_R y \, (x, y \in R)$ if and only if the point x is "closer to K **on the ray** R than y".

Our main aim is to prove Theorem 2.2. However in order to do so we first prove a geometric Lemma 2.1 which could be of independent interest. Given a ray R we call a family of R-transversal crosscuts C_t,

$t \in R$ an *R-defining* family of crosscuts if for each $t \in R$ there exists a R-transversal crosscut C_t such that $\operatorname{diam}(C_t) \to 0$ as $t \to K$ and $\operatorname{Sh}(C_t) \subset \operatorname{Sh}(C_s)$ if $t <_R s$.

Lemma 2.1. *Let K be an unshielded continuum and R be an external ray to K. Then there exists an R-defining family of R-transversal crosscuts C_t, $t \in R$.*

Proof. Given a point $t \in R$, any R-transversal crosscut C_t consists of two semi-open arcs connecting t to K. On the uniformization plane one of them will "grow" from the point corresponding to t in the positive (counterclockwise) direction with respect to the ray; such semi-open arcs will be called *positive arcs at* t. Similarly we define *negative arcs at* t. The infimum of the diameters of all positive arcs at t is denoted by $p(t)$; similarly we define $n(t)$ for negative arcs at t.

By way of contradiction and without loss of generality we may assume that there exists $\gamma > 0$ and a sequence $t_i \to K$ in R such that $n(t_i) > \gamma$, $i = 1, 2, \dots$. By [Mil00] we can choose a sequence of pairwise disjoint transversal crosscuts C_{h_i}, $h_i \to K$ so that the area of their shadows $\operatorname{Sh}(C_{h_i})$ and the diameters $\operatorname{diam}(C_{h_i})$ converge to 0. Hence we can find a crosscut $C_{h_j} = C_j$ so that the area of $\operatorname{Sh}(C_j)$ is less than $\gamma^2/99$ and $\operatorname{diam}(C_j) \le \gamma/99$. Then the negative "half" of C_j, the part of the ray R contained in $\operatorname{Sh}(C_j)$, and the set K enclose an open simply connected domain U on the plane, the "negative half" of $\operatorname{Sh}(C_j)$.

Choose $t = t_i <_R h_j$. Then the arc length of the subarc $[t, h_j]$ in R is more than $2\gamma/3$. Choose a point $x \in U$ so that there is a straight segment from t to x inside U of length less than $\gamma/9$ (since R is a smooth curve such segment exists). Consider all closed balls $\overline{B}$ contained in $\overline{U}$ such that $x \in \overline{B}$. By compactness this family contains a ball $\overline{B} = \overline{B}(y, \varepsilon)$ of maximal radius. Set $\partial\overline{B} = S$ and show that the set $A = S \cap \partial U$ has more than one point. Clearly, A is non-empty (otherwise a ball with the same center and slightly bigger radius will contain x and will be contained in $\overline{U}$, a contradiction). Suppose that $A = \{z\}$ is a single point. A tiny shift of y away from z along the line zy creates a new point y'. We are about to construct a ball centered at y' of bigger than ε radius contained in $\overline{U}$ and containing x which will contradict the assumptions about $\overline{B}$. Consider two cases.

(1) The angle $\angle xyz$ is obtuse. Consider the ball $\overline{B}' = \overline{B}(y', \varepsilon)$. If y' is sufficiently close to y, then $x \in \overline{B}'$. Moreover, the boundary S' of $\overline{B}'$ consists of two arcs, L' and L'', where L' is outside B and $L'' \subset B$. Then L' is disjoint from ∂U because it is very close to the half-circle of S which is cut off S by the diameter of B perpendicular to yz and hence positively distant from ∂U. On the other hand, L'' is disjoint from ∂U because $L'' \subset B$. Hence $\overline{B}' \subset U$ and a slightly bigger ball with the same center will contain x and will be contained in U, a contradiction.

(2) The angle $\angle xyz$ is not obtuse. Let H be the line segment through x and perpendicular to the segment yz. Then the component L of $S \setminus H$ *not* containing z is positively distant from ∂U. Let $p \in S \cap H$. Since the angle of the triangle $\triangle y'zp$ at p is greater than the angle of this triangle at z, we see that $d(y', z) > d(y', p) \geq d(y', x)$. On the other hand, since $\angle xyz$ is not obtuse then $\angle pyy'$ is not acute, and so $d(p, y') = \varepsilon' > d(p, y) = \varepsilon \geq d(x, y)$. Set $B' = B(y', \varepsilon')$. As before, the boundary S' of $\overline{B}'$ consists of two arcs, L' and L'', where L' is outside B and $L'' \subset B$. Then L' is disjoint from ∂U because it is very close to L and L'' is disjoint from ∂U because $L'' \subset B$, a contradiction since $\varepsilon' > \varepsilon$.

Thus, $\overline{B}$ must intersect ∂U at at least two points. Since the area of $\mathrm{Sh}(C_j)$ is less than $\gamma^2/99$ then $\varepsilon < \gamma/17$. If there is a point $a \in C_j \cap S$ then there is a negative arc at t - the concatenation of the straight segment from t to x, the segment inside $\overline{B}$ from x to a, and the appropriate part of C_j - of diameter less than $\gamma/9 + 2\gamma/17 + \gamma/99 < \gamma$, a contradiction. If there is a point $b \in K \cap S$ then there is a negative arc at t - the concatenation of the straight segment from t to x and the segment inside $\overline{B}$ from x to b - of diameter less than $\gamma/9+2\gamma/17 < \gamma$, a contradiction. Hence $M = \overline{B} \cap \partial U = S \cap \partial U \subset R$. On the other hand, by a theorem of Jørgensen (see [Jør56] and [Pom92]) M is connected. Hence M is a non-degenerate subarc of S. Since $d(S, K) > 0$, we can construct another ball $\overline{B}'$ which intersects M only at its endpoints such that $\overline{B}' \cap K = \emptyset$. Then $\overline{B}' \cap R$ cannot be connected since $\overline{B}' \cap R$ misses the entire arc M (except for its endpoints) which contradicts the theorem of Jørgensen. Hence $n(t) \to 0, p(t) \to 0$ as $t \to K$ which shows that there is a family of R-transversal crosscuts C_t, $t \in R$, such that $\mathrm{diam}(C_t) \to 0$ as $t \to K$.

This family can be modified to satisfy the second condition of the lemma so that $\mathrm{Sh}(C_t) \subset \mathrm{Sh}(C_s)$ if $t <_R s$. Observe that C_t is the union of a negative and a positive arc at t. We modify negative arcs and positive arcs separately to satisfy the second condition of the lemma, and since it does not matter which side we consider we denote the one-sided arcs we deal with by S_t. Choose C_t so that for all $s \le_R t$ we have $\mathrm{diam}(C_s) \le \varepsilon$ for a small ε, follow the ray beyond t towards K, and denote the segment of the ray from t to a point $s \in R$ with $s <_R t$ by $Q(t, s)$. Let Π be the principal set of R and consider two cases.

(1) Suppose that for every $s, s <_R t$ we have $d(s, t) \le 3\varepsilon$. By definition S_t is positively distant from Π; let $\delta = \mathrm{dist}(S_t, \Pi) > 0$ and choose

$u <_R t$ so that for all $s \leq_R u$ we have $\mathrm{diam}(S_s) < \min(\varepsilon/9, \delta/99)$ and $\mathrm{dist}(u, \Pi) < \min(\varepsilon/9, \delta/99)$. Then S_u is disjoint from S_t by the choice of δ. Since the ray is smooth, it is easy to see that we can create a family of short pairwise disjoint arcs A_v from points $v \in Q(t, u)$ to S_t of diameter less than 4ε where each connector ends at a point $e_v \in S_t$; moreover, these arcs can be chosen disjoint from S_u and each other and such that $A_v \cap R = \{v\}$. Denote the union of A_v and the piece of S_t from e_v to K by S'_v. Then the family $S'_v, v \in Q(t, u)$ together with $S_t = S'_t$ and $S_u = S'_u$ satisfies the second condition of the lemma and $\mathrm{diam}(S'_v) < 5\varepsilon$.

(2) Suppose that there is the first point $u \in R$, $s <_R t$ such that $\mathrm{dist}(t, u) = 3\varepsilon$. Then S_u is disjoint from S_t, and we can proceed the same way as before. That is, we get a family of negative arcs $S'_v, v \in Q(t, u)$ which together with $S_t = S'_t$ and $S_u = S'_u$ satisfies the second condition of the lemma and $\mathrm{diam}(S'_v) < 5\varepsilon$.

Let us proceed with this construction. If case (1) takes place then on the next step we replace ε by $\varepsilon/9$. If the case (2) takes place we may need to make several steps until we finally get u such that for all $s \leq_R u$ we have $\mathrm{diam}(S_s) < \varepsilon/9$. From this time on we proceed with ε replaced by $\varepsilon/9$. Clearly, this way we complete the construction and thus the proof of the lemma. □

Given an external ray R_α and an R_α-defining family of crosscuts C_t one can define the impression by $\mathrm{Imp}(\alpha) = \cap_{t \in R_\alpha} \overline{Sh(C_t)}$. It can be easily shown that this definition is equivalent to the standard one and that $\mathrm{Imp}(\alpha)$ is independent of the choice of the R_α-defining family of crosscuts [Pom92].

Let us now state a few facts about basic Cremer polynomials P (see, e.g., [GMO99]). The notation introduced here will be used from now on. For convenience, parameterize quadratic polynomials P as $z^2 + v$. Denote the Cremer fixed point of P by p and the critical point of P by c ($c = 0$, however we will still denote the critical point of P by c). Also, denote by σ the angle doubling map of the circle. It is well-known that if $P'(p) = e^{2\pi i\rho}$ then there exists a special *rotational* Cantor set $F \subset S^1$ such that σ restricted on F is semiconjugate to the irrational rotation by the angle $2\pi\rho$ [BS94]; the semiconjugacy ψ is not one-to-one only on the endpoints of countably many intervals complementary to F in S^1 (ψ maps the endpoints of each such interval into one point). Of the complementary intervals the most important one is the *critical leaf (diameter)* with the endpoints denoted below by α and $\beta = \alpha + 1/2$ (for definiteness we assume that $0 < \alpha < 1/2$). The limit set $F = \omega(\alpha)$ is exactly the set of points whose entire orbits are contained in $[\alpha, \beta]$ where the arc is taken counterclockwise from α to β. By Theorem 4.3 of [GMO99] we have that $p \in \mathrm{Imp}(\gamma)$ for every $\gamma \in F$, and $\{p, c, -p\} \subset \mathrm{Imp}(\alpha) \cap \mathrm{Imp}(\beta) = K$.

Theorem 2.2. *If P is a basic Cremer polynomial and $\varphi : J_P \to A$ is a monotone map onto a locally connected continuum A, then A is a single point.*

Proof. Set $J = J_P$. By way of contradiction suppose that $\varphi : J \to A$ is a monotone map onto a locally connected non-degenerate continuum A. Since J (and hence all its subcontinua) is non-separating then by the Moore Theorem [Moo25] the map Φ, defined on the entire complex plane $\mathbb{C}$, and identifying precisely *fibers* (point-preimages) of φ has $\mathbb{C}$ as its range. This implies that $\Phi(J) = \varphi(J) = A$ is a *dendrite* (locally connected continuum containing no simple closed curve).

External (conformal) rays R_α in the J-plane are then mapped into continuous pairwise disjoint curves $\varphi(R_\alpha)$ in the A-plane; below we call the curves $\varphi(R_\alpha)$ *A-rays* even though the construction is purely topological. Clearly, if $R_\alpha = R$ lands then so does $\varphi(R)$ (i.e., $\varphi(R)$ converges to a point). Let us show that in fact $\varphi(R)$ lands even if R does not (and, hence, the principal set Π of R is not a singleton). By Lemma 2.1 there exists an R-defining family of crosscuts C_t. Since φ is continuous then $\mathrm{diam}(\varphi(C_t)) \to 0$ as $t \to K$. Suppose that there is a sequence $t_n \to K$ such that $\overline{\varphi(C_{t_n})}$ is an arc for all $t_n \in R$ (and hence a crosscut of A) and these crosscuts are all pairwise disjoint. Since A is locally connected then by Carathéodory theory $\varphi(C_{t_n})$ converges to a unique point $x \in A$ which implies that in fact $\varphi(C_t) \to x$ as $t \to K$ and $\varphi(R)$ lands. Otherwise denote by N_t the "negative half" of C_t. Without loss of generality we may assume that there exists $t \in R$ such that for all $s <_R t$ in R, all $\overline{\varphi(N_s)}$ have the same point, say, z, in common, which immediately implies that $\varphi(R)$ lands at z.

The union U of P-preimages of the points p and c is countable, and so is the set $\varphi(U)$. By Theorem 10.23 of [Nad92] A has countably many branch points. Hence A contains uncountably many *cutpoints* of order 2 which do not belong to $\varphi(U)$, i.e. points $x \notin \varphi(U)$ such that $A \setminus \{x\}$ consists of exactly 2 components. Choose such a cutpoint $x \in A$ and denote the two components of $A \setminus \{x\}$ by B and C. Let us show that there are at least two A-rays landing at x and cutting the entire plane into two half-planes each of which contains a component of $A \setminus \{x\}$. Indeed, consider A-rays $\varphi(R_{\alpha'})$ and $\varphi(R_{\beta'})$ landing in B. Then there are two arcs into which α', β' divide the circle, and exactly one of them contains only angles whose A-rays land in B. Hence the entire set of angles whose A-rays land in B is contained in an open arc, say, Q_B.

Similarly, the set of angles whose A-rays land in C is contained in an open arc Q_C. Clearly, $S^1 \setminus (Q_B \cup Q_C)$ is the union of two closed arcs or points, and two angles - one from each of the components - would give rise to the desired two rays. Denote these angles by α'' and β''.

It follows that the fiber $Z = \varphi^{-1}(x)$ contains both principal sets $\Pi_{\alpha''}$ and $\Pi_{\beta''}$. Also, Z cuts J into two connected sets (φ-preimages of B and C). Finally, no forward P-image of Z contains c or p. Let us now study the P-trajectory of Z. First we show that there exists no n such that $\sigma^n(\alpha'') = \sigma^n(\beta'') \pm 1/2$. Indeed, otherwise $P^n(Z)$ contains $\Pi_{\sigma^n(\alpha'')}$ and $\Pi_{\sigma^n(\beta'')} = -\Pi_{\sigma^n(\alpha'')}$. Since $c = 0 \notin P^n(Z)$ (by the choice of x) then there exists $y \in P^n(Z), y \neq 0$ such that $-y \in P^n(Z)$ too. Then $P|_{P^n(Z)}$ is not a homeomorphism. By a theorem of Heath (see [Hea96]) it follows that then $P^n(Z)$ *must* contain a critical point, a contradiction.

Now, given two angles θ, θ' we define $d(\theta, \theta')$ as the length of the shortest arc between θ and θ' (we normalize the circle so that its length is equal to 1). It is easy to see that $d(\sigma(\theta), \sigma(\theta')) = T(d(\theta, \theta'))$ where $T : [0, 1/2] \to [0, 1/2]$ is the appropriate scaling of the full tent map. The dynamics of T shows then that there exists m such that $d(\sigma^m(\alpha''), \sigma^m(\beta'')) \geq 1/3$ and by the previous paragraph we may also assume that $d(\sigma^m(\alpha''), \sigma^m(\beta'')) < 1/2$. Since the longest complementary arcs to the union of two Cantor sets $F \cup F + 1/2$ are of length $1/4$ we see that the shorter open arc complementary to $\sigma^m(\alpha''), \sigma^m(\beta'')$ contains points of the set F (or $F+1/2$) and then since its length is less than $1/2$ the other arc contains points of the same set too. However the closed connected set $P^m(R_{\alpha''} \cup Z \cup R_{\beta''})$ does not contain p (or, respectively, $-p$). Choose an angle of F (resp. $F+1/2$) which belongs to the arc of the circle at infinity corresponding to the part of the plane not containing p (resp. $-p$). Then its impression does not contain p (resp. $-p$), a contradiction. □

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(Alexander Blokh and Lex Oversteegen) Department of Mathematics, University of Alabama at Birmingham, Birmingham, AL 35294-1170

E-mail address, Alexander Blokh: ablokh@math.uab.edu

E-mail address, Lex Oversteegen: overstee@math.uab.edu