

IMMEDIATE RENORMALIZATION OF COMPLEX POLYNOMIALS

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ABSTRACT. A cubic polynomial P with a non-repelling fixed point b is said to be *immediately renormalizable* if there exists a (connected) quadratic-like invariant filled Julia set K^* such that $b \in K^*$. In that case exactly one critical point of P does not belong to K^* . We show that if, in addition, the Julia set of P has no (pre)periodic cutpoints then this critical point is recurrent.

1. INTRODUCTION

In the introduction we assume knowledge of basics of complex dynamics.

We study polynomials P with connected Julia sets $J(P)$. An (external) ray with a rational argument always lands at a point that is eventually mapped to a repelling or parabolic periodic point. If two external rays like that land at a point $x \in J(P)$, then such rays are said to form a *rational cut (at x)*. The family of all rational cuts of a polynomial P may be empty (then one says that the *rational lamination* of P is empty); if it is non-empty it provides a combinatorial tool allowing one to study properties of P even in the presence of such complicated irrational phenomena as Cremer or Siegel periodic points.

Consider quadratic polynomials with connected Julia set. It is known that any quadratic polynomials $Q \notin \overline{\text{PHD}_2}$ has rational cuts (PHD_2 is the *Principal Hyperbolic Domain* of the Mandelbrot set). Thus, any arc from $\overline{\text{PHD}_2}$ towards the rest of the Mandelbrot set consists of polynomials with rational cuts.

The purpose of this paper is to investigate a similar phenomenon in the cubic case. Then there is a “gray area” \mathcal{G} in-between the set of cubic polynomials with rational cuts, and the set $\overline{\text{PHD}_3}$, the *Principal Hyperbolic Domain* of the cubic connectedness locus. We conjecture

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that \mathcal{G} is empty. Thus, the set \mathcal{G} is the true object of our study even though, paradoxically, in the end we want to establish that it is empty. As a step in this direction we prove that a polynomial from \mathcal{G} must have specific properties. Our approach is not unusual: according to the nature of the contrapositive argument one studies a phenomenon in great detail only to discover that an elaborate list of its properties leads to contradictions thus disproving its existence.

d:q11

Definition 1.1 ([DH85]). A *polynomial-like* map is a proper holomorphic map $f : U \rightarrow f(U)$ of degree $k > 1$, where $U, f(U) \subset \mathbb{C}$ are open Jordan disks and $\bar{U} \subset f(U)$. The *filled Julia set* $K(f)$ of f is the set of points in U that never leave U under iteration of f . The *Julia set* $J(f)$ of f is the boundary of $K(f)$. Call U a *basic neighborhood* of $K(f)$ and assume that if f is given, then its basic neighborhood is fixed. If $k = 2$, then the corresponding polynomial-like maps are said to be *quadratic-like*.

We can now state our main result.

t:recur

Theorem 1.2. *Let f be a cubic polynomial with empty rational lamination that has a quadratic-like restriction with a connected quadratic-like filled Julia set $K^*(f) = K^*$. Then the critical point of f that does not belong to K^* is recurrent.*

In the situation of Theorem 1.2 we will always denote a connected quadratic-like filled Julia set by K^* ; also, we will fix its neighborhood U^* on which f is quadratic-like and denote $f|_{U^*}$ by f^* .

2. PRELIMINARIES: POLYNOMIAL-LIKE MAPS AND CUBIC POLYNOMIALS WITH EMPTY RATIONAL LAMINATION

s:prelim

By *classes* of polynomials, we mean affine conjugacy classes. For a polynomial f , let $[f]$ be its class, let $K(f)$ be its filled Julia set, and let $J(f)$ be its Julia set. The *connectedness locus* \mathcal{M}_d of degree d is the set of classes of degree d polynomials whose critical points *do not escape* (i.e., have bounded orbits). Equivalently, \mathcal{M}_d is the set of classes of degree d polynomials f whose Julia set $J(f)$ is connected. The connectedness locus \mathcal{M}_2 of degree 2 is called the *Mandelbrot set*; the connectedness locus \mathcal{M}_3 of degree 3 is called the *cubic connectedness locus*. The *principal hyperbolic domain* PHD_3 of \mathcal{M}_3 is defined as the set of classes of hyperbolic cubic polynomials whose Julia sets are Jordan curves. Equivalently, $[f] \in \text{PHD}_3$ if both critical points of f are in the immediate basin attraction of the same (super-)attracting fixed point. A polynomial is *hyperbolic* if the orbits of all critical points converge to (super-)attracting cycles.

ss:pl

2.1. Polynomial-like maps.

d:ql2

Definition 2.1 ([DH85]). Two polynomial-like maps $f : U \rightarrow f(U)$ and $g : V \rightarrow g(V)$ of degree k are said to be *hybrid equivalent* if there is a quasi-conformal map φ from a neighborhood of $K(f)$ to a neighborhood of $K(g)$ conjugating f to g in the sense that $g \circ \varphi = \varphi \circ f$ wherever both sides are defined and such that $\bar{\partial}\varphi = 0$ almost everywhere on $K(f)$.

The terminology is explained by the following fundamental result.

Straightening Theorem ([DH85]). *Let $f : U \rightarrow f(U)$ be a polynomial-like map. Then f is hybrid equivalent to a polynomial P of the same degree. Moreover, if $K(f)$ is connected, then P is unique up to (a global) conjugation by an affine map.*

We will need the next definition.

d:plrays

Definition 2.2. Let f be a polynomial, and for some Jordan disk U^* the map $f^* = f|_{U^*}$ be polynomial-like. Let g be a monic polynomial hybrid equivalent to f^* . Then the corresponding filled Julia set $K(f^*) = K^*$ of f^* is called the *polynomial-like filled invariant Julia set*. The curves in $\mathbb{C} \setminus K(f^*) = K^*$ corresponding (through the hybrid equivalence) to dynamic rays of g are called *polynomial-like rays* of f^* . If the degree of f^* is two, then we will talk about *quadratic-like rays*. Denote polynomial-like rays $R^*(\beta)$, where β is the argument of the external ray of g corresponding to $R^*(\beta)$. We will also call them *K^* -rays* to distinguish them from rays external to $K(f) = K$ called *K -rays*.

Here K^* -rays are defined in a bounded neighborhood of K^* while K -rays are defined on the entire plane. By Straightening Theorem combined with well known results from complex dynamics the composition $\psi^* : \mathbb{C} \setminus K^* \rightarrow \mathbb{C} \setminus \bar{\mathbb{D}}$ of the hybrid conjugacy for K^* (see Straightening Theorem) and the inverse Riemann map for $\mathbb{C} \setminus K$ with real derivative at infinity transfers the dynamics of f to the plane on which, loosely, the role of K^* is played by the unit circle while the rest of the plane (i.e., the set $\mathbb{C} \setminus K^*$) remains “the same”. Thus, the rest of K (i.e., the set $K \setminus K^*$) is transferred to $\mathbb{C} \setminus \bar{\mathbb{D}}$ and looks like a collection of pieces “growing” out of $\bar{\mathbb{D}}$. In terms of dynamics the map P is transferred by the map ψ^* to a Jordan annulus $\widehat{U}^* = \mathbb{S}^1 \cup \psi^*(U^*)$ to produce the map $z^s : \widehat{U}^* \rightarrow \mathbb{S}^1 \cup \psi^*(f(K^*))$ ($\mathbb{S}^1 \subset \mathbb{C}$ is the unit circle centered at the origin) with the appropriate choice of s .

Evidently, if f is a polynomial of degree d and $T \subsetneq J(f)$ is a proper polynomial-like invariant Julia set then the degree of $f|_T$ is less than d .

In particular, if f is a *cubic* polynomial and $K^* \subset K(f)$ is a polynomial-like filled invariant Julia set, then the polynomial-like map $f|_{K^*}$ is quadratic-like. The following lemma is proven in [BOT16] (it is based upon Theorem 5.11 from McMullen’s book [McM94]).

`1:7.2` **Lemma 2.3** (Lemma 6.1 [BOT16]). *Let f be a complex cubic polynomial with a non-repelling fixed point a . Then the quadratic-like filled invariant Julia set K^* with $a \in K^*$ (if any) is unique.*

`ss:emp-lam`

2.2. Polynomials with empty rational lamination. As was said in the Introduction, we want to study cubic polynomials $f \in \mathcal{M}_3$ without rational cuts (equivalently, with empty rational lamination). This reduces the family of polynomials of interest to us.

`1:gray-1` **Lemma 2.4.** *Suppose that a cubic polynomial f has empty rational lamination. Then f must have exactly one fixed non-repelling point and all other periodic points of f are repelling.*

Proof. By Theorem 7.5.2 [BFMOT12] if all fixed points of f are repelling then at one of them the combinatorial rotation number is not 0 and hence several rays must land, a contradiction. Also, if f has a fixed non-repelling point and a distinct periodic non-repelling point, by Kiwi [Kiw00] the rational lamination of f is non-empty, a contradiction. \square

We will need the following corollary.

`c:7.2` **Corollary 2.5.** *A cubic polynomial f with empty rational lamination contains, in its filled Julia set $K(f)$, at most one set K^* ; this set must contain a unique non-repelling fixed point of f .*

Proof. By Lemma 2.4, the map f has a unique non-repelling fixed point, say, a , and all other periodic points of f are repelling. Thus, if K^* does not contain a , then all its periodic points are repelling. In particular, by Theorem 7.5.2 [BFMOT12] the map f^* has a fixed point b such that K -rays landing at b rotate under f^m , a contradiction. Thus, K^* contains a ; by Lemma 2.3 it is unique. \square

`s:fucode`

3. PRELIMINARIES: FULL CONTINUA AND THEIR DECORATIONS

In this section we consider certain ordered by inclusion pairs of full continua on the plane (a compact set $X \subset \mathbb{C}$ is *full* if $\mathbb{C} \setminus X$ is connected). This is a natural situation occurring in complex dynamics, both when studying polynomials and their parameter spaces. Indeed, let a cubic polynomial f have a connected filled Julia set $K(f) = K$. Suppose that K^* and U^* exist; Then the situation is exactly like the one described above because $K^* \subset K$. Another example is when one

takes the filled Main Cardioid of the Mandelbrot set \mathcal{M}_2 . It is easy to give other dynamical or parametric examples.

Let $X \subset Y$ be two full planar continua. We would like to represent Y as the union of X and *decorations*.

d:dec

Definition 3.1. Components of $Y \setminus X$ are called *decorations* (of Y relative to X), or just *decorations* (if X and Y are fixed).

Decorations are connected but not closed; thus, decorations may behave differently from what common intuition suggests. In Lemma 3.2 we discuss topological properties of decorations. Given sets A and B , say that A *accumulates in* B if $\overline{A} \setminus A \subset B$.

1:triv

Lemma 3.2. Any decoration D of Y rel. X accumulates in X . The set $\overline{D} \setminus D = \overline{D} \cap X$ is a continuum. The sets \overline{D} and $D \cup X = \overline{D} \cup X$ are full continua.

Proof. Suppose, by way of contradiction, that there exists $x \in \overline{D} \setminus (D \cup X)$. Then we have $D \subset A = D \cup \{x\} \subset \overline{D}$ while $A \cap X = \emptyset$. Since D is connected, and since $D \subset A \subset \overline{D}$, then A is connected too. Hence D is not a component of $K \setminus X$, a contradiction.

The continuum \overline{D} is full as otherwise its complementary domains would be complementary to Y .

Now, by the first paragraph, $\overline{D} \setminus D = \overline{D} \cap X$ is compact. Suppose that $\overline{D} \cap X$ is disconnected. Then there exists a bounded component U of $\mathbb{C} \setminus (X \cup D)$ that at least partially accumulates to X and partially to D . Since Y is full, then $U \subset Y$; hence U is a subset of a decoration that accumulates (partially) to points of D . By the first paragraph this implies that $U \subset D$, a contradiction. Thus, $\overline{D} \cap X$ is connected; then $\overline{D} \cap X$ is a full continuum as both X and \overline{D} are full. \square

We will use the inverse Riemann map $\psi : \mathbb{C} \setminus X \rightarrow \mathbb{C} \setminus \overline{\mathbb{D}}$ with real derivative at infinity. Loosely, one can say that under the map ψ the continuum X is replaced by the closed unit disk $\overline{\mathbb{D}}$ while the rest of the plane is conformally deformed. Thus, under ψ the decorations become subsets of $\mathbb{C} \setminus \overline{\mathbb{D}}$.

1:triv2

Corollary 3.3. Let D be a decoration of Y rel. X . Then $\overline{\psi(D)} \setminus D$ is a (perhaps, degenerate) continuum (arc or unit circle) $I_D \subset \mathbb{S}^1$.

Proof. Follows from Lemma 3.2. \square

Observe that the set I_D can, indeed, coincide with the entire unit circle (e.g., D can spiral onto $\overline{\mathbb{D}}$). The arcs $I_D \neq \mathbb{S}^1$ are also possible as $\psi(D)$ may approach an arc I_D by imitating the behavior of the function $\sin(1/x)$ as $x \rightarrow 0^+$. Moreover, two distinct decorations D and T may

well have equal arcs I_D and I_T , or it may be so that, say, $I_D \subsetneq I_T$, or I_D and I_T can have a non-trivial intersection not coinciding with either arc (all these examples can be constructed by varying the behavior of components similar to the behavior function $\sin(1/x)$ as $x \rightarrow 0^+$). However there are some cases in which one can guarantee that each decoration has a degenerate arc I_D .

Ray Assumption on X and Y . *Suppose that there is a dense set $\mathcal{A} \subset \mathbb{S}^1$ and a family of curves R_x landing in \mathcal{A} and disjoint from $\psi(Y)$.*

Suppose that Ray Assumption holds for X and Y . Moreover, suppose that there is a neighborhood U of Y and a homeomorphism $\varphi : U \rightarrow W \subset \mathbb{C}$. Then Ray Assumption holds for $\varphi(X) \subset \varphi(Y)$ too.

1:triv3

Lemma 3.4. *Suppose that Ray Assumption holds for $X \subset Y$. Then for every decoration D the arc I_D is degenerate.*

Proof. Suppose that I_D is a non-degenerate arc. Choose three points $x, y, z \in \mathcal{A} \cap I_D$. It follows that $\psi(D)$ is contained in one of the two disjoint open strips formed by the curves R_x, R_y and R_z . However then $\psi(D)$ can accumulate to only one of the circle arcs formed by the points x, y and z , a contradiction. \square

Definition 3.5. Under Ray Assumption and in the above notation, the *argument* of a decoration D is the angle α such that $I_D = \{\alpha\}$.

Since we study decorations in the complex dynamical setting, making the ray assumption is not overly restrictive because, as we will now see, it holds in important for us dynamical cases.

1:raya1

Lemma 3.6. *Suppose that K^* is a connected filled invariant polynomial-like Julia set contained in a connected filled Julia set K of a polynomial P . Then $K^* \subset K$ satisfy the ray assumption.*

Proof. Choose a periodic repelling point $x \in K^*$ and a K -ray R landing at x . Under ψ^* this ray becomes a curve $\psi^*(R)$ landing at the appropriate point of \mathbb{S}^1 , and these points are dense in \mathbb{S}^1 . The remark after we define Ray Assumption now shows, that $K^* \subset K$ satisfy it. \square

s:para-sli

4. CUBIC PARAMETER SLICES

Let \mathcal{F} be the space of polynomials

$$f_{\lambda,b}(z) = \lambda z + bz^2 + z^3, \quad \lambda \in \mathbb{C}, \quad b \in \mathbb{C}.$$

An affine change of variables reduces any cubic polynomial to the form $f_{\lambda,b}$. Clearly, 0 is a fixed point for every polynomial in \mathcal{F} . Define the

λ -slice \mathcal{F}_λ of \mathcal{F} as the space of all polynomials $g \in \mathcal{F}$ with $g'(0) = \lambda$, i.e. polynomials $f(z) = \lambda z + bz^2 + z^3$ with fixed $\lambda \in \mathbb{C}$. We write \mathcal{P}_λ for the set of polynomials $f \in \mathcal{F}_\lambda$ for which there are polynomials $g \in \mathcal{F}$ arbitrarily close to f with $|g'(0)| < 1$ and the Julia set being a Jordan curve. Clearly, the class $[f]$ of f then belongs to $\overline{\text{PHD}_3}$. Also, denote by \mathcal{F}_{nr} the space of polynomials $f_{\lambda,b}$ with $|\lambda| \leq 1$ (“nr” from “non-repelling”). For a fixed λ with $|\lambda| \leq 1$ the λ -connectedness locus \mathcal{C}_λ , of the λ -slice of the cubic connectedness locus is defined as the set of all $f \in \mathcal{F}_\lambda$ such that $K(f)$ is connected. This is a full continuum [BrHu88, Z99]. We study sets $\mathcal{C}_\lambda \subset \mathcal{F}_\lambda$ as we want to see to what extent results concerning the quadratic Mandelbrot set hold for \mathcal{C}_λ .

ss:imre-vs-hyp

4.1. Immediately renormalizable polynomials vs the closure of the principal hyperbolic component. Let us describe what happens to quadratic-like invariant filled Julia sets of $f \in \mathcal{F}_{nr}$ that contain 0 when f is slightly perturbed (assuming such a set exists for a given f). In the rest of the paper by the “quadratic counterpart” of f^* we mean a quadratic polynomial hybrid conjugate to f^* by Straightening Theorem.

l:cnct

Lemma 4.1. *Let $f \in \mathcal{F}_{nr}$ be a polynomial, K^* be a quadratic-like filled invariant Julia set containing 0. Then K^* is connected. Every cubic polynomial $g \in \mathcal{F}_{nr}$ sufficiently close to f has a quadratic-like Julia set B^* containing 0; the set B^* here is also connected. Moreover, if 0 is an attracting fixed point for g then g has a quadratic-like Julia set which is a Jordan curve; in particular, $g \notin \text{PHD}_3$.*

Proof. Since 0 is non-repelling, then, by the Fatou-Shishikura inequality, the critical point of the quadratic counterpart of f^* cannot escape. Hence K^* is connected. Let $f^* : U^* \rightarrow V^*$, $f^* = f|_{U^*}$ be the associated quadratic-like map. If g is very close to f then $0 \in U^*$ and, moreover, we can arrange for a new Jordan disk W^* with $g(W^*) = V^*$. By the above, the associated quadratic-like Julia set B^* is connected. Finally, if $f_i \rightarrow f$ are polynomials with 0 as an attracting fixed point then, by the above, for large i the polynomial f_i has a quadratic-like filled Julia set coinciding with the closure of the basing of immediate attraction of 0. Therefore, $[f_i] \notin \text{PHD}_3$ for large i as desired. \square

Call a cubic polynomial $f \in \mathcal{F}_{nr}$ *immediately renormalizable* if there are Jordan domains U^* and V^* such that $0 \in U^*$, and $f^* = f|_{U^*} : U^* \rightarrow V^*$ is a quadratic-like map; denote by K^* the filled quadratic-like Julia set of f^* (in the future we *always* use the notation U^* , V^* , f^* and K^* when talking about immediately renormalizable maps). Denote the set of all **immediately renormalizable** polynomials by ImR , and let

$\text{ImR}_\lambda = \mathcal{F}_\lambda \cap \text{ImR}$. Let \mathcal{P} be the set of polynomials $f \in \mathcal{F}_{nr}$ with the following property: there are polynomials $g \in \mathcal{F}$ arbitrarily close to f with $|g'(0)| < 1$ such that $[g] \in \text{PHD}_3$. Then clearly $[f] \in \overline{\text{PHD}_3}$ (observe, that there may be polynomials outside of \mathcal{P} whose classes are also in $\overline{\text{PHD}_3}$). Also, set $\mathcal{P}_\lambda = \mathcal{P} \cap \mathcal{F}_\lambda$. Corollary 4.2 follows from Lemma 4.1.

`c:cnct`

Corollary 4.2. *If $f \in \text{ImR}$, then K^* is connected. The set ImR is open in \mathcal{F}_{nr} . The set ImR_λ is open in \mathcal{F}_λ for any $\lambda, |\lambda| \leq 1$. The sets ImR and \mathcal{P} are disjoint.*

We want to study the sets ImR and \mathcal{P} ; Corollary 4.2 shows that they are disjoint, so this investigation may be done in parallel. In fact a lot is known about the sets ImR_λ and \mathcal{P}_λ in the case when $|\lambda| < 1$. Namely, P. Roesch proved the following theorem.

`t:roesch`

Theorem 4.3 ([Roe06]). *\mathcal{P}_λ is a Jordan disk for any λ with $|\lambda| < 1$.*

We want to combine this result with [BOPT16a] where sufficient conditions on polynomials for being immediately renormalizable are given.

`t:when-imr`

Theorem 4.4 ([BOPT16a]). *If $f \in \mathcal{F}_\lambda, |\lambda| \leq 1$ belongs to the unbounded complementary domain of \mathcal{P}_λ in \mathcal{F}_λ , then f is immediately renormalizable.*

Combining these theorems and Lemma 4.1, we get Corollary 4.5.

`c:<1`

Corollary 4.5. *If $|\lambda| < 1$ then $\text{ImR}_\lambda = \mathbb{C} \setminus \mathcal{P}_\lambda$.*

We study the neutral case $|\lambda| = 1$; to obtain more general results, as much as possible we study neutral slices without using specifics of the number λ .

`l:lambda<0`

Lemma 4.6. *If $|\lambda| < 1$ then $[z^3 + \lambda z] \in \text{PHD}_3$.*

Proof. We claim that $J(f)$ is a Jordan curve. Let U be the basin of immediate attraction of 0 (which is an attracting fixed point of f). Since $f^n(-z) = -f^n(z)$ for every n then U is centrally symmetric with respect to 0. Since there exists a critical point $c \in U$ and $f'(-z) = f'(z)$ for any z , then $-c$ is critical. The central symmetry of U with respect to 0 now implies that $-c \in U$. Since both critical points of f belong to U , the claim follows. \square

Theorem 4.7 develops Lemma 4.6.

`l:0in`

Theorem 4.7. *For any $\lambda, |\lambda| < 1$, we have $0 \in \text{Int}(\mathcal{P}_\lambda)$. For any $\lambda, |\lambda| \leq 1$, we have $0 \in \mathcal{P}_\lambda$, and \mathcal{P}_λ is a continuum.*

Proof. The first claim is proven in Lemma 4.6. To prove the rest, observe that if $|\lambda| = 1$ then $\mathcal{P}_\lambda = \limsup \mathcal{P}_\tau$ where $\tau \rightarrow \lambda, |\tau| < 1$. Since $0 \in \text{Int}(\mathcal{P}_\tau)$ for all these numbers τ , then $0 \in \mathcal{P}_\lambda$, and \mathcal{P}_λ , being the lim sup of the continua \mathcal{P}_τ which all share a common point 0, is also a continuum as claimed. \square

ss:stru-fla

4.2. The structure of the slice \mathcal{F}_λ . Following [BOT16], define the set $\mathcal{CU}_\lambda, |\lambda| \leq 1$ as the set of all polynomials $f \in \mathcal{F}_\lambda$ with connected Julia sets and such that the following holds:

- (1) f has no repelling periodic cutpoints in $J(f)$;
- (2) f at most one non-repelling cycle not equal to 0, and all its points have multiplier 1.

The set \mathcal{CU}_λ is a centerpiece, literally and figuratively, of the λ -slice \mathcal{C}_λ of the cubic connectedness locus. A big role in studying polynomials from \mathcal{C}_λ is played by studying properties of the quadratic polynomial $z^2 + \lambda z$ whose fixed point 0 has multiplier λ . Aiming at most general results, we consider the general case of λ with irrational argument without any extra-conditions. To state some theorems proven earlier we need new notions. For a closed subset $A \subset \mathbb{S}^1$ of at least 3 points, call its convex hull $\text{CH}(A)$ a *gap*. Given a chord $\ell = \overline{ab}$ of the unit circle with endpoints a and b , set $\sigma_3(\ell) = \overline{\sigma_3(a)\sigma_3(b)}$ (we abuse the notation and identify the angle-tripling map $\sigma_3 : \mathbb{R}/\mathbb{Z}$ with the map $z^3 : \mathbb{S}^1 \rightarrow \mathbb{S}^1$; similarly we treat the map σ_2). For a closed set $A \subset \mathbb{S}^1$, call each complementary arc of A a *hole* of A . Given a compactum $A \subset \mathbb{C}$ let the *topological hull* $\text{Th}(A)$ be the complement to the unbounded complementary domain of A .

sss:family

4.2.1. Family of invariant quadratic gaps. Let us discuss properties of quadratic σ_3 -invariant gaps [BOPT16]. For our purposes it suffices to consider gaps G such that $G \cap \mathbb{S}^1$ has no isolated points. “Invariant” means that an edge of a gap G maps to an edge of G , or to a point in $G \cap \mathbb{S}^1$; “quadratic” means that after collapsing holes of G the map $\sigma_3|_{\text{Bd}(G)}$ induces a locally strictly monotone two-to-one map of the unit circle to itself that preserves orientation and has no critical points. For convenience, normalize the length of the circle so that it equals 1. Let \mathfrak{V} be a quadratic σ_3 -invariant gap with no isolated points. Then there is a unique arc $I_{\mathfrak{V}}$ (called the *major hole* of \mathfrak{V}) complementary to $\mathfrak{V} \cap \mathbb{S}^1$ whose length is greater than or equal to $1/3$; the length of this arc is at most $1/2$. The edge $M_{\mathfrak{V}}$ of \mathfrak{V} connecting the endpoints of $I_{\mathfrak{V}}$ is called the *major* of \mathfrak{V} . If $M_{\mathfrak{V}}$ is critical then itself and \mathfrak{V} are said to be of *regular critical type*; if $M_{\mathfrak{V}}$ is periodic then itself and \mathfrak{V} are said to be of *periodic type*. Collapsing edges of \mathfrak{V} to points, we

construct a monotone map $\tau : \mathfrak{V} \rightarrow \mathbb{S}^1$ that semiconjugates $\sigma_3|_{\text{Bd}(\mathfrak{V})}$ and $\sigma_2 : \mathbb{S}^1 \rightarrow \mathbb{S}^1$.

The map τ is uniquely defined by the fact that it is monotone and semiconjugates $\sigma_3|_{\text{Bd}(\mathfrak{V})}$ and $\sigma_2 : \mathbb{S}^1 \rightarrow \mathbb{S}^1$. Indeed, if there had been another map τ' like that then there would have existed a non-trivial orientation preserving homeomorphism of the circle to itself conjugating σ_2 with itself. However it is easy to see that the only such map is the identity (recall, that σ_2 has a unique fixed point).

The family of all invariant quadratic gaps can be parameterized (see [BOPT16]). Namely, by Lemmas 3.22 and 3.23 of [BOPT16], there exists a Cantor set $Q \subset \mathbb{S}^1$ such that if we collapse every hole of Q to a point, we obtain a topological circle whose points are in one-to-one correspondence with *all* quadratic invariant gaps \mathfrak{U} such that $\mathfrak{U} \cap \mathbb{S}^1$ is a Cantor set. Moreover, the following holds:

- (1) for each point $\bar{\theta} \in \mathbb{S}^1$ of Q that is not an endpoint of a hole of Q , the critical chord $\overline{(\theta + 1/3)(\theta + 2/3)}$ is the major of a quadratic invariant gap \mathfrak{U} such that $\mathfrak{U} \cap \mathbb{S}^1$ is a Cantor set;
- (2) for each hole $(\bar{\theta}_1, \bar{\theta}_2)$ of Q the chord $\overline{(\theta_1 + 1/3)(\theta_2 + 2/3)}$ is the periodic major of a quadratic invariant gap \mathfrak{U} such that $\mathfrak{U} \cap \mathbb{S}^1$ is a Cantor set.

Thus, to choose the *tag* of a quadratic invariant gap \mathfrak{V} we first take its major $M_{\mathfrak{V}}$ and then choose the edge or vertex ℓ of \mathfrak{V} distinct from $M_{\mathfrak{V}}$ but with the same image as $M_{\mathfrak{V}}$. Evidently, (1) ℓ is an edge of \mathfrak{V} if $M_{\mathfrak{V}}$ is not critical (and is, therefore, of periodic type), (2) ℓ is a vertex of \mathfrak{V} if $M_{\mathfrak{V}}$ is critical (and is, therefore, of regular critical type).

The convex hull \mathfrak{Q} of Q in the plane is called the *Principal Quadratic Parameter Gap* (see Figure 1). The set Q plays a somewhat similar role to that of the following set appearing in quadratic dynamics: the set of arguments of all parameter rays (rays to the Mandelbrot set) landing at points of the main cardioid. Holes of \mathfrak{Q} will play an important role. The *period of a hole* (θ_1, θ_2) of \mathfrak{Q} is defined as the period of $\theta_1 + 1/3$ under the angle tripling map. This period also equals to the period of $\theta_2 + 2/3$ under the angle tripling map. The only period 1 (invariant) holes of \mathfrak{Q} are $(1/6, 1/3)$ and $(2/3, 5/6)$; these will play a special role.

sss:la-slice

4.2.2. *Properties of the λ -slice.* For every polynomial $f \in \mathcal{F}_\lambda$ and every angle $\alpha \in \mathbb{R}/\mathbb{Z}$, we will define the *dynamic ray* $R_f(\alpha)$. Also, for every angle θ , in the parameter plane of \mathcal{F}_λ we define the *parameter ray* $\mathcal{R}_\lambda(\theta)$. We use rays to show that the picture in \mathcal{F}_λ resembles the picture in the parameter plane of quadratic polynomials.

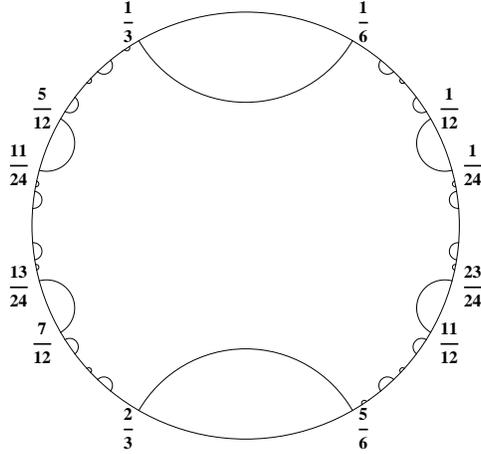


FIGURE 1. The Principal Quadratic Parameter Gap

fig:Qf

t:main-bot16

Theorem 4.8 (Main Theorem of [BOT16]). *Fix λ with $|\lambda| \leq 1$. The set \mathcal{CU}_λ is a full continuum. The set \mathcal{C}_λ is the union of \mathcal{CU}_λ and a countable family of limbs \mathcal{LI}_H of \mathcal{C}_λ parameterized by holes H of \mathfrak{Q} . The union is disjoint. For a hole $H = (\theta_1, \theta_2)$ of \mathfrak{Q} , the following holds.*

- (1) *The parameter rays $\mathcal{R}_\lambda(\theta_1)$ and $\mathcal{R}_\lambda(\theta_2)$ land at the same point $f_{root(H)}$.*
- (2) *Let $\mathcal{W}_\lambda(H)$ be the component of $\mathbb{C} \setminus \overline{\mathcal{R}_\lambda(\theta_1) \cup \mathcal{R}_\lambda(\theta_2)}$ containing the parameter rays with arguments from H . Then, for every $f \in \mathcal{W}_\lambda(H)$, the dynamic rays $R_f(\theta_1 + 1/3)$, $R_f(\theta_2 + 2/3)$ land at the same point, either a periodic and repelling point for all $f \in \mathcal{W}_\lambda(H)$, or the point 0 for all $f \in \mathcal{W}_\lambda(H)$. Moreover, $\mathcal{LI}_H = \mathcal{W}_\lambda(H) \cap \mathcal{C}_\lambda$.*
- (3) *The dynamic rays $R_{f_{root(H)}}(\theta_1 + 1/3)$, $R_{f_{root(H)}}(\theta_2 + 2/3)$ land at the same parabolic periodic point, and $f_{root(H)}$ belongs to \mathcal{CU}_λ .*

Figure 2 shows the parameter slice $\mathcal{F}_{e^{2\pi i/3}}$ in which several parameter rays and several wakes are shown.

Given a compact set $A \subset \mathbb{C}$, let the *topological hull* of A be the unbounded complementary domain of A .

t:cubio

Theorem 4.9 ([BOT16]). *We have that $\text{Th}(\mathcal{P}_\lambda) \subset \mathcal{CU}_\lambda$. The set \mathcal{CU}_λ is a full continuum.*

In this paper we study the set $\mathcal{CU}_\lambda \setminus \text{Th}(\mathcal{P}_\lambda)$. By Theorems 4.8 and 4.9, except for vertices $f_{root(H)}$ of parameter wakes $\mathcal{W}_\lambda(H)$, no points of \mathcal{CU}_λ belong to those parameter wakes. The only places at which points of $\mathcal{CU}_\lambda \setminus \text{Th}(\mathcal{P}_\lambda)$ may be located can be associated with parameter rays

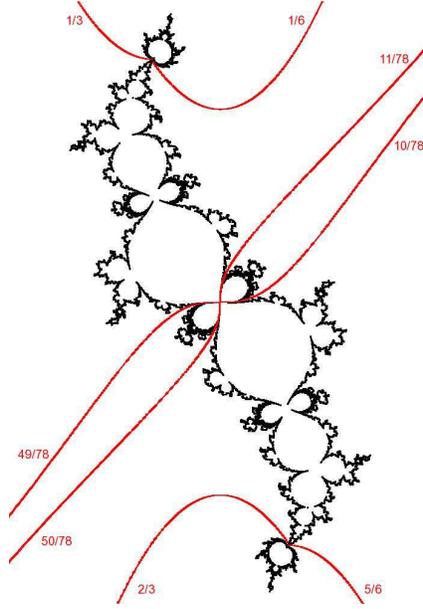
FIGURE 2. Parameter slice $\mathcal{F}_{e^{2\pi i/3}}$ with some parameter rays

fig:Fc1d3

$\mathcal{R}_\lambda(\theta)$ where $\theta \in \Omega$ is a parameter, associated with a regular critical quadratic invariant gap \mathfrak{U} with regular critical major $M_{\mathfrak{U}}$.

5. DECORATIONS

The notation below will be used in what follows. Namely, we assume that f is an immediately renormalizable cubic polynomial; recall that by Lemma 2.3 its filled Julia set K^* is unique. Let $f : U^* \rightarrow V^*$ be a quadratic-like map where V^* is very tight around K^* . Let ω_1 be the critical point of f belonging to K^* ; let ω_2 be the other critical point of f (this notation will be used in what follows). In order to indicate the dependence on f , we may write $K^*(f)$, $\omega_2(f)$, etc. Observe that in this section we put *no* restrictions on the rational lamination of f . A *pullback* of a connected set D under f is defined as a connected component of $f^{-1}(D)$.

1:pull

Lemma 5.1. *A pullback of a connected set A under f maps onto A . In particular, there exists a pullback \tilde{K}^* of K^* disjoint from K^* and mapped by f onto K^* in the one-to-one fashion.*

Proof. The first claim of the lemma is proven in Lemma 4.1 of [LT90]. Moreover, since K^* is a quadratic-like Julia set, it has a pullback \tilde{K}^* disjoint from itself. By the first claim, \tilde{K}^* is a continuum that maps onto K^* . Moreover, $f|_{\tilde{K}^*}$ is one-to-one. Indeed, otherwise there are

points $x \neq y \in \tilde{K}^*$ with $f(x) = f(y) = z \in K^*$. It follows that z has overall four preimages (two in K^* and two in \tilde{K}^*), a contradiction. \square

The notation \tilde{K}^* will be used from now on. Also, from now on by decorations we mean those of K rel. K^* .

Definition 5.2. A decoration is said to be *critical* if it contains \tilde{K}^* . Thus there is only one critical decoration denoted D_c . All other decorations are said to be *non-critical*.

Let $v_2 = f(\omega_2)$ be the critical value associated with the point ω_2 .

l:v-nin-k

Lemma 5.3. *Neither ω_2 nor v_2 belong to K^* .*

Proof. We have $\omega_2 \notin K^*$ since $\omega_1 \in K^*$ and K^* contains at most one critical point. If $v_2 \in K^*$, then there are two preimages of v_2 in K^* and two preimages of v_2 outside of K^* (both numbers take multiplicities into account). This contradicts f being three-to-one. \square

For $x \notin K^*$ let $D(x)$ be the decoration containing x ; set $D_v = D(v_2)$ and call it *critical value decoration*. Initial dynamical properties of decorations are listed in Theorem 5.4. Set L to be $\{\omega_2\}$ (if $\omega_2 \in J(f)$), or the closure of the Fatou domain of f containing ω_2 (if any). In Theorem 5.4 we use the following *E-construction*, and we use the same notation whenever we implement it.

The E-construction. Draw a K -ray E landing at a periodic repelling point $x \neq f(\omega_1), x \in K^*$. Construct the two pullback K -rays E' and E'' of E landing at distinct points $x', x'' \in K^*$ where $x' \neq x''$ by the choice of x . The set $\mathbb{C} \setminus (K^* \cup E' \cup E'')$ consists of components Z_1 and Z_2 . Assume that Z_1 contains all K -rays with arguments from an open arc I_1 of length $1/3$ while Z_2 contains all K -rays with arguments from the open arc I_2 of length $2/3$ disjoint from I_1 . Notice that since $x' \neq x''$ then some periodic repelling points of K^* belong to $\overline{Z_1}$ and are accessible by K -rays from within Z_1 (the same claim holds for Z_2).

t:decor

Theorem 5.4. *The critical decoration D_c maps onto the entire $K(f)$ while any other decoration maps onto a decoration in the one-to-one fashion. Any decoration $D \neq D_v$ has three homeomorphic pullbacks: D_1 such that $\overline{D_1} \setminus D_1 \subset \tilde{K}^*$ and D_2, D_3 that are decorations. D_i itself is a decoration for $i = 2, 3$. Decoration D_v has a homeomorphic pullbacks D'_v which is itself a decoration, and a pullback T that maps onto D_v in the two-to-one fashion, contains ω_2 , is contained in D_c , and accumulates to both K^* and \tilde{K}^* .*

Proof. We prove Theorem 5.4 step by step.

Step 1. *If D is a decoration of f , then every pullback of D is a subset of some decoration of f . Moreover, if D' and D'' are decorations and $f(D') \cap D'' \neq \emptyset$, then $f(D') \supset D''$.*

Proof of Step 1. Let Γ be a pullback of D . Clearly, $\Gamma \subset K \setminus K^*$. Since Γ is connected, it must lie in some decoration. Now, if D' and D'' are decorations and $f(D') \cap D'' \neq \emptyset$ then we can choose a pullback D''' of D'' which is non-disjoint from D' . By the above $D''' \subset D'$; by Lemma 5.1, $f(D''') = D''$. Hence $f(D') \supset D''$ as desired.

Step 2. *If D is a non-critical decoration then $f(D)$ is a decoration.*

Proof of Step 2. The set $f(D)$ is connected and disjoint from K^* (by definition of a non-critical decoration). Hence it is contained in one decoration. By Step 1, the set $f(D)$ coincides with this decoration.

Step 3. $\omega_2 \in D_c$.

Proof of Step 3. Let $\omega_2 \notin D_c$. Since by Lemma 3.2, $A = D_c \cup K^*$ is compact, a neighborhood of ω_2 is disjoint from A . It follows that the set L is contained in a decoration $D \neq D_c$. Choose a neighborhood U of $f(L)$ so that a pullback W of U with $\omega_2 \in W$ is disjoint from A . Then W is a neighborhood of L that maps two-to-one onto U .

Choose a K -ray R that enters U and denote the first (coming from infinity) point of intersection of R and $\text{Bd}(U)$ by x . Denote the union of the segment of R from infinity to x and a curve $I' \subset U$ from x to v_2 by R' . The pullback R'' of R' containing ω_2 consists of two segments of two K -rays each of which maps to the segment of R from infinity to x , and an arc I'' that double-covers I' . Clearly, R'' partitions the plane in two half-planes on one of which f acts in the one-to-one fashion while on the other one it acts in the two-to-one fashion. However by the construction R'' is disjoint from A and points of $K^* \subset A$ have three preimages in A , a contradiction. Thus, $\omega_2 \in D_c$.

Step 4. *We have $\omega_2 \in Z_2$, and, hence, $D_c \subset Z_2$. The map f is a homeomorphism on Z_1 . If D is a decoration then Z_1 contains a unique homeomorphic pullback D' of D which is itself a non-critical decoration. No decoration has a unique pullback.*

Proof of Step 4. Both Z_1 and Z_2 contain points from $K \setminus K^*$; in fact, their union contains the entire $K \setminus K^*$. Hence, all the decorations are contained in $Z_1 \cup Z_2$. Notice though, that the K -rays approaching points of $K \setminus K^*$ have arguments from the disjoint open arcs of angles at infinity, namely, from I_1 and I_2 , respectively. Since all K -rays in Z_1 have arguments from the arc I_1 , then they all have distinct images. We claim that $\omega_2 \notin Z_1$. Indeed, suppose that $\omega_2 \in Z_1$. By Step 3 then $D_c \subset Z_1$ and so $K^* \subset Z_1$. Choose a repelling periodic point $y \in K^*$ accessible by a K -ray R to K^* from within Z_1 . Then choose the first

preimage $y' \in \tilde{K}^*$ of $f(y)$. Clearly, y' is also accessible by a K -ray R' which itself is a pullback of $f(R)$. However both rays must have arguments from I_1 , a contradiction (recall that I_1 is an arc of length $1/3$). Thus, $\omega_2 \in Z_2$ and, hence, $D_c \subset Z_2$.

Let us show that $f|_{Z_1}$ is a homeomorphism. It is a homeomorphism on the union of all K -rays with arguments from I_1 . Suppose that points $x, y \in Z_1$ are such that $f(x) = f(y) = z$. We may assume that $z \notin E \cup \{v_2\}$. Since $\tilde{K}^* \subset D_c \subset Z_2$, then $z \notin K^*$. Hence we can construct a ray H from z to infinity bypassing $K^* \cup v_2 \cup E$. Consider pullbacks H_x and H_y of H such that $x \in H_x$ and $y \in H_y$. By the choices we made, $H_x \subset Z_1, H_y \subset Z_1$, and $H_x \cap H_y = \emptyset$. It follows that there exist distinct points $x' \in H_x$ and $y' \in H_y$ with the same image and such that both x' and y' belong to $\mathbb{C} \setminus K$. However then x' and y' must belong to K -rays with arguments from I_1 , a contradiction. Hence $f|_{Z_1}$ is one-to-one. By Brouwer's Invariance of Domain Theorem, $f|_{Z_1}$ is a homeomorphism onto $f(Z_1)$.

Let D be a decoration. Choose $y \in D \cap J(f)$ and a sequence $y_i \in \mathbb{C} \setminus K$ of points that converge to y and belong to K -rays $R_i \neq E$. Choose K -rays $R'_i \in Z_1$ with $f(R'_i) = R_i$ and then points $y'_i \in R'_i$ such that $f(y'_i) = y_i$. Then $y'_i \rightarrow y'$ where $f(y') = y$, and hence $y' \in D'$ where $D' \subset Z_1$ is a decoration. The rest of the claim follows.

Step 5. *If $D \neq D_v$ is a decoration then it has three pullbacks each of which maps onto D homeomorphically. Two of the pullbacks accumulate into K^* ; one pullback accumulates into \tilde{K}^* .*

Proof of Step 5. By Lemma 3.2 the set $K^* \cup D$ is a full continuum; clearly, $v_2 \notin K^* \cup D$. Apply the E-construction to D . Then draw a ray R from v_2 to infinity so that $R \cap (K^* \cup D \cup E) = \emptyset$. Construct the pullback C of R passing through ω_2 ; the set C cuts the plane in two half-planes, X and Y , such that X maps onto $\mathbb{C} \setminus R$ in the one-to-one fashion while Y maps onto $\mathbb{C} \setminus R$ in the two-to-one fashion. The cut C is disjoint from $K^* \cup \tilde{K}^*$ as well as from the pullbacks of D . Hence C separates K^* and \tilde{K}^* ; it is easy to see that $\tilde{K}^* \subset X$ and $K^* \subset Y$. It follows that there exists a unique pullback D_1 of D contained in X . Since it has to accumulate to points mapped to K^* , the set D_1 accumulates into \tilde{K}^* . Hence $D_1 \subset D_c$. Since $D_c \subset Z_2$, then $D_1 \subset Z_2$.

In addition to D_1 , by Lemma 5.1 there may be either one pullback of D mapped onto D in the two-to-one fashion, or two pullbacks of D mapped onto D in the one-to-one fashion. By Step 4, D has a homeomorphic pullback $D_2 \subset Z_1$ which is a non-critical decoration. Clearly, $D_2 \neq D_1$. Hence D has three homeomorphic pullbacks of which D_1 accumulates to \tilde{K}^* and D_2 accumulates to K^* . Let D_3 be

the remaining pullback of D . If it accumulated to \tilde{K}^* it would follow that $f|_{\tilde{K}^*}$ is not one-to-one, a contradiction. Hence S accumulates to K^* .

Step 6. *The decoration D_v has two pullbacks. One of them, say, T , maps onto D_v in the two-to-one fashion, contains ω_2 , is contained in D_c , and accumulates to both K^* and \tilde{K}^* ; the other one is the homeomorphic pullback D'_v defined in Step 4.*

Proof of Step 6. Clearly, D_v cannot have three pullbacks as otherwise the point v_2 will have four preimages (counted with multiplicity), a contradiction. By Step 4, D_v has a homeomorphic pullback $D'_v \subset Z_1$. Thus, we only need to study the remaining two-to-one pullback T of D_v . Clearly, $\omega_2 \in T$ which implies that $T \subset D_c$. To prove that T accumulates in both K^* and \tilde{K}^* , notice that there are points of T close to \tilde{K}^* . Indeed, a neighborhood of \tilde{K}^* maps homeomorphically onto a neighborhood of K^* and therefore contains points of the preimage of D_v . These points cannot belong to D'_v because $\overline{D'_v} \setminus D'_v \subset K^*$ by Lemma 3.2. Hence they belong to T as claimed. Thus, T accumulates into \tilde{K}^* . To show that T accumulates into K^* too, choose a sequence of points $y_i \in D_v$ such that $y_i \rightarrow y \in K^*$. For each i , let $y'_i, y''_i \in T$ be two distinct preimages of y_i in T . We may assume that they converge to y', y'' respectively. If $y', y'' \notin K^*$ then $y', y'' \in \tilde{K}^*$ which implies that in any neighborhood of \tilde{K}^* there are pairs of points with the same image, a contradiction. Hence T accumulates into both K^* and \tilde{K}^* . \square

6. QUADRATIC ARGUMENTS

s:quadarg

Consider K^* -rays $R^*(\alpha)$. Clearly, $f(R^*(\alpha)) \supset R^*(2\alpha)$ (the curve $f(R^*(\alpha))$ extends the ray $R^*(2\alpha)$ into the annulus between the basic neighborhood of K^* and its image). A *crosscut* (of K^*) is a closed arc I with endpoints $x, y \in K^*$ such that $[I \setminus \{a, b\}] \subset \mathbb{C} \setminus K^*$. If a_n is a crosscut then the *shadow* $\text{Shad}(a_n)$ of a crosscut a_n is the bounded complementary component of $a_n \cup K^*$. A sequence of crosscuts $a_n, n = 1, 2, \dots$ is *fundamental* if $a_{n+1} \subset \text{Shad}(a_n)$ for every n and the diameter of a_n converges to 0 as $n \rightarrow \infty$. Two fundamental sequences of crosscuts are *equivalent* if crosscuts of one sequence are eventually contained in the shadows of crosscuts of the other one, and vice versa. This is an equivalence relation whose classes are called *prime ends*. In what follows the set of endpoints of a closed arc I is denoted by $\text{end}(I)$.

By the Carathéodory theory, every quadratic-like ray $R^*(\alpha)$ to K^* corresponds to a certain prime end $E^*(\alpha)$ represented by a fundamental sequence of crosscuts $\{a_n\}$. Moreover, for every a_n a *tail* of $R^*(\alpha)$ is contained in $\text{Shad}(a_n)$ (a *tail* of $R^*(\alpha)$ is defined by a point $x \in R^*(\alpha)$

and coincides with the component of $R^*(\alpha) \setminus \{x\}$ that accumulates into K^*). It is convenient to consider also the associated picture in $\mathbb{C} \setminus \mathbb{D}$; the picture on the K^* -plane is transferred to that in $\mathbb{C} \setminus \mathbb{D}$ by means of the map ψ^* introduced earlier.

Namely, for every n , the set $\psi^*(a_n \setminus \text{end}(a_n))$ is an arc $I_n \subset \mathbb{C} \setminus \overline{\mathbb{D}}$ without endpoints such that \bar{I} is a closed arc with endpoints $x_n, y_n \in \mathbb{S}^1$. One can choose the circle arc I'_n positively oriented from x'_n to y'_n such that $\alpha \in I'_n$ and consider the Jordan curve $Q_n = I_n \cup I'_n$; then the radial ray R_α with the initial point at $z_\alpha \in \mathbb{S}^1$ where $z_\alpha \in \mathbb{S}^1$ is the point of the circle with argument α , intersected with the simply connected domain U_{a_n} with boundary Q_n , contains a small subsegment of R_α with an endpoint z_α . Observe that $(\psi^*)^{-1}(U_{a_n})$ is the shadow of the crosscut a_n . The *impression* of $E^*(\alpha)$ is the intersection of the closures of $\text{Shad}(a_n)$. We say that a prime end $E^*(\alpha)$ is *disjoint* from a set $S \subset \mathbb{C} \setminus K^*$ if $\text{Shad}(a_n) \cap S = \emptyset$ for all sufficiently large n . Riemann map of $\mathbb{C} \setminus K^*$. In what follows, when talking about crosscuts, we will use this notation.

1:cross-primends

Lemma 6.1. *Suppose that $X \subset \mathbb{C} \setminus K^*$ is a connected set, and $\psi^*(X)$ accumulates on exactly one point $z_\alpha \in \mathbb{S}^1$ with argument α . Then X is non-disjoint from $E^*(\alpha)$ and disjoint from any other prime end.*

Proof. Let a be a crosscut associated with $E^*(\alpha)$. Consider the set U_a . Since $\psi^*(X)$ accumulates on z_α , then $\psi^*(X)$ is non-disjoint from U , and hence X is non-disjoint from $\text{Shad}(a)$. By definition, X is non-disjoint from $E^*(\alpha)$. Also, for any point $t = e^{2\pi\beta i} \neq z_\alpha$ we can find a crosscut b associated with β and so small that \bar{U}_b is disjoint from $\psi^*(X)$. Then X is disjoint from $\text{Shad}(b)$ and hence X is disjoint from $E^*(\beta)$. \square

Recall that a set A *accumulates in* B if $\bar{A} \setminus A \subset B$.

1:accum

Lemma 6.2. *Suppose that a connected subset $\hat{D} \subset \mathbb{C} \setminus \overline{\mathbb{D}}$ accumulates on a non-degenerate arc $A \subsetneq \mathbb{S}^1$. Let \hat{R} be an arc in $\mathbb{C} \setminus \overline{\mathbb{D}}$ landing at an interior point y of A . Then \hat{R} crosses \hat{D} .*

Proof. By way of contradiction assume that $\hat{R}_1 \cap \hat{D} = \emptyset$. Extend \hat{R}_1 to infinity still avoiding \hat{D} . Since $A \neq \mathbb{S}^1$, then there exists a ray \hat{R}_2 from a point $x \in \mathbb{S}^1 \setminus A$ to infinity that is disjoint from \hat{D} . It follows that \hat{D} is contained in a component of $\mathbb{C} \setminus (\hat{R}_1 \cup \hat{R}_2 \cup I)$ where I is one of the two circle arcs with endpoints x and y . However then \hat{D} can only accumulate on the part of A that is contained in I , a contradiction. \square

Proposition 6.3 uses Lemma 6.2.

p:pends

Proposition 6.3. *Every decoration D is disjoint from all prime ends of K^* except exactly one.*

Proof. The (connected) set $D' = \psi^*(D)$ accumulates to a closed arc $A \subset \mathbb{S}^1$. Suppose that A is non-degenerate, and bring this to a contradiction. First, for every repelling periodic point $x \in K$ there exists a K -ray landing at x . Also, K -rays are disjoint from D . Choose two distinct repelling periodic points in K^* , draw K -rays landing at them, and then map all this by ψ^* to $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$. This will result in two curves landing at two distinct points of \mathbb{S}^1 and disjoint from D' . Thus, D' does not accumulate onto the entire \mathbb{S}^1 .

Now, choose a periodic K -ray R that lands at a repelling periodic point $w \in K^*$. Under the map ψ^* it is associated to a curve $\psi^*(R)$ that lands at a certain point $x \in \mathbb{S}^1$ and is otherwise disjoint from $\overline{\mathbb{D}}$. Pulling back R under f following backward orbit of w in K^* corresponds to pulling back $\psi^*(R)$ under z^2 . Since A is non-degenerate, there exists a number N such that some N -th pullback of R lands at a point $w' \in K^*$ while the corresponding N -th pullback of $\psi^*(R)$ is a curve Q in $\mathbb{C} \setminus \overline{\mathbb{D}}$ that lands at an interior point of A . Since Q is disjoint from D' , we see by Lemma 6.2 that $A = \{\alpha\}$ is degenerate. By Lemma 6.1, D is disjoint from all prime ends of K^* except for $E^*(\alpha)$. \square

Suppose that $E^*(\alpha)$ is the only prime end non-disjoint from D . Then α is called the *quadratic argument* of D and is denoted by $\alpha(D)$. By Proposition 6.3 each decoration has only one quadratic argument (and so quadratic arguments are well defined) while different decorations may a priori have the same quadratic argument. A useful interpretation of these concepts is as follows. Using the map ψ^* for $\mathbb{C} \setminus K^*$ we can transfer all decorations to the set $\mathbb{C} \setminus \overline{\mathbb{D}}$; then for any decoration D the set $\psi^*(D)$ accumulates to the point $e^{2\pi i \alpha(D)}$ of the unit circle with argument $\alpha(D)$ (i.e., $\overline{\psi^*(D)} \setminus \psi^*(D)$ consists of one point from the unit circle with argument $\alpha(D)$). Moreover, if U^* is a basic neighborhood of K^* then ψ^* conjugates z^2 restricted onto $\psi^*(U^* \setminus K^*)$ and f restricted onto $U^* \setminus K^*$.

The dynamics on the uniformization plane immediately implies the next lemma stated here without proof.

1:z2

Lemma 6.4. *If $D \neq D_c$ is a decoration then $\alpha(f(D)) = \sigma_2(\alpha(D))$. On the other hand, in the notation of Theorem 5.4, we have that $\alpha(D_v) = \sigma_2(\alpha(D_c))$.*

In what follows we use Riemann maps and their inverses for either the entire filled Julia set K of f , or a quadratic-like Julia set K^* . In the former case we talk about K -plane and z^3 -plane (which are associated

to each other under the appropriate Riemann map), in the latter case, similarly, we talk about K^* -plane and z^2 -plane.

Corollary 6.5 follows from Proposition 6.3.

c:2pbdec

Corollary 6.5. *Consider a decoration D with quadratic argument α . Then both $\alpha/2$ and $(\alpha + 1)/2$ are quadratic arguments of decorations containing pullbacks of D . These two decorations are different.*

s:cubarg

7. CUBIC ARGUMENTS

So far we have been working on establishing general facts concerning the situation when a cubic polynomial f has a connected quadratic-like filled Julia set K^* (clearly, $K^* \subset K(f)$). In this section we begin looking into more specific cases; as the first step we describe some results and concepts, mostly taken from [BOT16].

Consider an immediately renormalizable polynomial $f \in \mathcal{F}_\lambda$ with $|\lambda| \leq 1$, and define an invariant quadratic gap $\mathfrak{U}(f)$ associated with 0; as before, by $f^* : U^* \rightarrow V^*$, we denote the corresponding quadratic-like map. When $J(f)$ is disconnected, gaps analogous to $\mathfrak{U}(f)$ were studied in [BCLOS16] where tools developed in [LP96] were used; however this approach is based upon the fact that $J(f)$ is disconnected in an essential way and, hence, is very different from that used in [BOT16] and here. Once we introduce $\mathfrak{U}(f)$, we shall see that it coincides with the similar gap in the disconnected case. Recall that by Lemma 2.3 if f is a cubic polynomial with a non-repelling fixed point a , then there exists at most one quadratic-like filled invariant Julia set K^* containing a ; by Corollary 2.5 if f has empty rational lamination then it has a unique non-repelling fixed point a and at most one quadratic-like filled invariant Julia set that, if it exists, must contain a .

To define the gap $\mathfrak{U}(f)$ associated with K^* , we use (pre)periodic points of f . Since K^* is a quadratic-like filled Julia set, then K^* is a component of $f^{-1}(K^*)$.

d:uf

Definition 7.1. Let $\widehat{X}(f) = \widehat{X}$ be the set of all σ_3 -(pre)periodic points $\bar{\alpha} \in \mathbb{S}^1$ such that $R_f(\alpha)$ lands in K^* . Let $X(f) = X$ be the closure of \widehat{X} . Let $\mathfrak{U}(f)$ be the convex hull of X . Let \widetilde{K}^* be the component of $f^{-1}(K^*)$ different from K^* (such a component of $f^{-1}(K^*)$ exists because $f|_{K^*}$ is two-to-one). Let $Y(f) = Y$ be the closure of the set of all preperiodic points $\bar{\alpha} \in \mathbb{S}^1$ with $R_f(\alpha)$ landing in \widetilde{K}^* . Observe, that \widetilde{K}^* is disjoint from U^* (otherwise points of $f(\widetilde{K}^* \cap U^*)$ must belong to K^* , a contradiction with dynamics of points of $\widetilde{K}^* \cap U^*$).

From now on we fix an immediately renormalizable polynomial $f \in \mathcal{F}_{nr}$ and do not refer to f in our notation (we write \mathfrak{U} instead of $\mathfrak{U}(f)$ etc). Lemma 7.2 summarizes some results of Section 7 of [BOT16].

1:inv-quad

Lemma 7.2. *The set \tilde{K}^* is disjoint from U^* . The gap \mathfrak{U} is an invariant quadratic gap of regular critical or periodic type. The map $\sigma_3|_{\widehat{X}}$ is two-to-one, and Y lies in the closure of the major hole of X .*

This shows that the results of [BOPT16] and [BOT16] apply to \mathfrak{U} (recall that these results are described in Subsection 4.2). E.g., consider the map τ defined there for any quadratic invariant gap of σ_3 (τ collapses edges of \mathfrak{U} and semiconjugates $\sigma_3|_{\text{Bd}(\mathfrak{U})}$ and σ_2). The map ψ^* maps K -rays to their counterparts on the z^2 -plane. On the other hand, the Riemann map defined by K sends the radial rays with rational arguments in $\mathbb{C} \setminus \overline{\mathbb{D}}$, to K -rays, including K -rays landing in K^* . Composing these two maps we obtain a map η that associates radial rays from the z^3 -plane with arguments from \widehat{X} to ψ^* -images of K -rays on the z^2 -plane landing in \mathbb{S}^1 . Thus, if $R(\beta)$ is a radial ray on z^3 -plane and $\beta \in \widehat{X}$, then the ray $\eta(R(\beta))$ is contained in z^2 -plane and lands at a point of \mathbb{S}^1 ; this defines a map $\tau' : \widehat{X} \rightarrow \mathbb{S}^1$. By construction we see that τ' semiconjugates $\sigma_3|_{\widehat{X}}$ and σ_2 ; the uniqueness of the map τ shows that τ' is the restriction of τ onto \widehat{X} .

Observe that K -rays with arguments from $\text{Bd}(\mathfrak{U})$ do not necessarily have principal sets contained in K^* . Nevertheless the map τ allows us to relate decorations of K^* and their quadratic arguments with edges and vertices of the gap \mathfrak{U} .

1:cridi3

Lemma 7.3. *The quadratic argument of D_c is $\tau(M_{\mathfrak{U}})$.*

Proof. If the quadratic argument of D_c is not $\tau(M_{\mathfrak{U}})$, then there is an edge/vertex v of \mathfrak{U} such that $\tau(v)$ is the quadratic argument of D_c and we can find angles $\alpha, \beta \in \widehat{X}$ such that the arc $I = (\alpha, \beta)$ contains v but does not contain the endpoints of $M_{\mathfrak{U}}$. For K -rays $R(\alpha), R(\beta)$ with arguments α, β , consider the component W of $\mathbb{C} \setminus [R(\alpha) \cup R(\beta) \cup K^*]$ containing K -rays with arguments from I . By Lemma 7.2 Y lies in the closure of the major hole of X . Hence $\tilde{K}(f^*)$ is disjoint from W despite the fact that $\tilde{K}(f^*) \subset D_c \subset W$. \square

In the end of this section we study the issue of landing at points of K^* for periodic and preperiodic angles from \mathfrak{U} .

1:who-land

Lemma 7.4. *Let $\alpha \in \mathfrak{U}$ be a (pre)periodic angle that never maps to an endpoint of the major $M_{\mathfrak{U}}$ of \mathfrak{U} . Suppose that the K -ray $R(\alpha)$ with argument α lands at a point x . Then $x \in K^*$.*

In our notation the claim of the lemma simply means that $\alpha \in \widehat{X}$.

Proof. Let $\alpha = 0 \in \mathfrak{U}$, yet $x \notin K^*$. Let $y \in K^*$ be the fixed point at which $R^*(0)$ lands. Then the only K -ray that can land at y is $R(\frac{1}{2})$ which implies that $M_{\mathfrak{U}} = \overline{0\frac{1}{2}}$, a contradiction with the assumptions of the lemma. Similarly, if $\alpha = \frac{1}{2}$, then $x \in K^*$.

We claim that the lemma holds for a (pre)periodic angle $\alpha \in \mathfrak{U}$ if it holds for $\beta = \sigma_3(\alpha)$. By way of contradiction assume that $x \notin K^*$; then $x \in \widetilde{K}^*$, $\alpha \in Y$ and, by Lemma 7.2, Y is contained in the closure of the major hole of \mathfrak{U} . It follows that α is an endpoint of $M(\mathfrak{U})$, a contradiction with the assumptions of the lemma. By the first paragraph we conclude that the lemma holds if α eventually maps to a σ_3 -fixed angle.

We claim that if $\beta \in \mathfrak{U} \cap \widehat{X}$ and $\sigma_3(\alpha) = \beta$ then $\alpha \in \widehat{X}$. Indeed, by Lemma 7.2 $\sigma_3|_{\widehat{X}}$ is two-to-one. If $\alpha \notin \widehat{X}$ then all three σ_3 -preimages of β belong to \mathfrak{U} . However this is only possible if two of them are endpoints of $M_{\mathfrak{U}}$. It follows that α is an endpoint of $M_{\mathfrak{U}}$, again a contradiction. This and the first paragraph of the proof imply the claim of the lemma if α eventually maps to a σ_3 -fixed angle.

Assume now that α never maps to a σ_3 -fixed angle. The angle $\tau(\alpha)$ is (pre)periodic under σ_2 and never maps to 0 under iterations of σ_2 . The K^* -ray $R^*(\tau(\alpha))$ with quadratic argument $\tau(\alpha)$ lands at a point $x' \in K^*$. Let a K -ray $R(\gamma)$ land at x' . We claim that $\alpha = \gamma$. Indeed, $\alpha \in \mathfrak{U}$ by the assumptions, and $\gamma \in \mathfrak{U}$ since $R(\gamma)$ lands at $x' \in K^*$. Moreover, neither α nor γ ever map to 0 or $\frac{1}{2}$. It now follows from the construction that both angles (recall that α and γ belong to \mathfrak{U}) behave in the same fashion with respect to the partition of \mathfrak{U} in two arcs by the σ_3 point that belongs to \mathfrak{U} , and its preimage in \mathfrak{U} (or, in the appropriate exceptional cases, by the major $M(\mathfrak{U}) = \overline{0\frac{1}{2}}$ and its preimage-edge in \mathfrak{U}). Together with the assumptions of the lemma that α never maps to an endpoint of the major $M_{\mathfrak{U}}$ this implies that $\alpha = \gamma$ and hence the landing point x of $R(\alpha)$ belongs to K^* as desired. \square

8. SECTORS

Suppose that f is an immediately renormalizable cubic polynomial. Define a few objects depending on f and denote them with f as the subscript; yet, if f is fixed, we may omit it from notation. Consider a pair of external rays $R(\alpha)$, $R(\beta)$ landing in K^* . The set $\Sigma_f = K^* \cup R(\alpha) \cup R(\beta)$ divides the plane into two components, one of which contains all external rays with arguments in (α, β) and the other contains all external rays with arguments in (β, α) . To formally justify this claim, collapse

K^* to a point (i.e., consider the equivalence relation \sim on $\overline{\mathbb{C}}$, whose classes are K^* and single points in $\overline{\mathbb{C}} \setminus K^*$). By Moore's theorem, the quotient space $\overline{\mathbb{C}}/\sim$ is homeomorphic to the sphere. The image of $\Sigma_f(\alpha, \beta)$ under the quotient map, together with the image of the point at infinity, form a Jordan curve. The statement now follows from the Jordan curve theorem. Let $S^\circ(\alpha, \beta)_f$ be the component of $\overline{\mathbb{C}} \setminus \Sigma_f(\alpha, \beta)$ containing all external rays with arguments in (α, β) . Observe that $S^\circ(\alpha, \beta)$ is defined only if the rays $R(\alpha)$, $R(\beta)$ both land in K^* . The sets $S^\circ(\alpha, \beta)$ will be called *open sectors*, and the sets $\Sigma(\alpha, \beta)$ will be called *cuts*. Images of sectors contain K^* iff sectors contain \tilde{K}^* .

An open sector $S^\circ(\alpha, \beta)$ is associated with its *argument arc* $(\alpha, \beta) \subset \mathbb{R}/\mathbb{Z}$ that consists of arguments of all rays included in $S^\circ(\alpha, \beta)$. Note, that this sector does not have to coincide with the union of those rays as open sectors may contain decorations. More generally, consider a subset $T \subset \mathbb{C}$. We call the set T_f (f)-*radial* if any ray intersecting T lies in T . For a radial set T we can define the *argument set* $\arg(T)$ of T as the set of all $\gamma \in \mathbb{R}/\mathbb{Z}$ with $R(\gamma) \subset T$. Every open sector is a radial set, whose argument set is an open arc. It is clear that, for any radial set T , we have

$$\arg(f(T)) = \sigma_3(\arg(T)), \quad \arg(f^{-1}(T)) = \sigma_3^{-1}(\arg(T)).$$

The following properties of open sectors are almost immediate.

1:pb-opsec

Lemma 8.1. *Let S° be an open sector and T° an f -pullback of S° . Then $\arg(T^\circ)$ is the union of one or several components of $\sigma_3^{-1}(\arg(S^\circ))$. The number of critical points in T° equals the number of components minus one. If $\omega_2 \notin T^\circ$ and the closure of T° intersects K^* , then T° is an open sector mapping 1-1 onto S° . Any pullback of S° is disjoint from K^* .*

Proof. The first claim ($\arg(T^\circ)$ is a union components of $\sigma_3^{-1}(\arg(S^\circ))$) is immediate. Note that $f : T^\circ \rightarrow S^\circ$ is proper. Therefore, this map has a well-defined degree. The degree is clearly equal to the number of components in $\arg(T^\circ)$. On the other hand, by the Riemann–Hurwitz formula, the degree is equal to the number of critical points in T° plus one. Thus the second claim follows.

Let us prove the third claim of the lemma. The only critical point that can lie in T° is ω_2 . Since we assume that $\omega_2 \notin T^\circ$, then $\arg(T^\circ)$ has only one component. Let $\arg(T^\circ) = (\alpha, \beta)$. Then T° is bounded by $R(\alpha) \cup R(\beta)$ and a part of $K^* \cup \tilde{K}^*$. If both $R(\alpha)$ and $R(\beta)$ land in K^* , then, by definition, T° coincides with the open sector $S^\circ(\alpha, \beta)$. If both $R(\alpha)$ and $R(\beta)$ land in \tilde{K}^* , then T° is disjoint from K^* as its image S° does not contain K^* ; this implies that $\overline{T^\circ}$ is disjoint with

K^* , a contradiction with our assumptions. Finally, if one of the two rays $R(\alpha)$, $R(\beta)$ lands in K^* and the other lands in \tilde{K}^* , then we get a contradiction too. Suppose, say, that $R(\alpha)$ lands in K^* and $R(\beta)$ lands in \tilde{K}^* . The sets $K^* \cup R(\alpha)$ and $\tilde{K}^* \cup R(\beta)$ are closed disjoint non-separating sets, whose union cannot separate the plane. Thus the only possibility is that $T^\circ = S^\circ(\alpha, \beta)$ and $f : T^\circ \rightarrow S^\circ$ is a homeomorphism. The last claim of the lemma now easily follows. \square

1:img-sec

Lemma 8.2. *Consider an open sector $S^\circ(\alpha, \beta)$, whose argument arc is mapped one-to-one under σ_3 . Then $f(S^\circ(\alpha, \beta)) = S^\circ(3\alpha, 3\beta)$. Moreover, $S^\circ(\alpha, \beta)$ maps one-to-one onto $S^\circ(3\alpha, 3\beta)$.*

Proof. Let T° be the f -pullback of $S^\circ(3\alpha, 3\beta)$ that includes rays with arguments in (α, β) . Clearly, the rays $R(\alpha)$, $R(\beta)$ are on the boundary of T° . Since these rays land in K^* and $T^\circ \cap K^* = \emptyset$, we must have $T^\circ \subset S^\circ(\alpha, \beta)$. On the other hand, if $T^\circ \neq S^\circ(\alpha, \beta)$, then, by Lemma 8.1, the arguments of rays in T° form two intervals of \mathbb{R}/\mathbb{Z} rather than one. A contradiction. Therefore, $T^\circ = S^\circ(\alpha, \beta)$ as desired. \square

Definition 8.3 (minimal sectors). Let x be a point outside of K^* . The *minimal sector* $S(x)$ of x is defined as the intersection of all $\overline{S^\circ}(\alpha, \beta)$ such that $x \in S^\circ(\alpha, \beta)$.

Note that, by definition, a minimal sector is always a closed set. It is clear from the definition that a minimal sector is bounded by at most two external rays and a piece of K^* . It may coincide with the union of a single ray and its impression. The next lemma shows that minimal sectors are related to decorations and immediately follows from the definitions. Recall that if $x \notin K^*$ then $D(x)$ is the decoration containing x . Recall also that the map $\tau : \mathfrak{U} \rightarrow \mathbb{S}^1$ collapses to points all edges of the gap \mathfrak{U} and semiconjugates $\sigma_3|_{\text{Bd}(\mathfrak{U})}$ and σ_2 .

1:dec-sec

Lemma 8.4. *Let $x \notin K^*$. Consider the edge (possibly degenerate) $\overline{ab} = \tau^{-1}(\alpha(D(x)))$ of \mathfrak{U} . Then $S(x) \setminus K^*$ is the union of all decorations with quadratic argument $\alpha(D(x))$ and all K -rays with arguments from the complementary arc of $\text{Bd}(\mathfrak{U})$ in \mathbb{S}^1 with endpoints a and b .*

Define the *critical sector* as the minimal sector $S(\omega_2)$.

1:noncr-sec

Lemma 8.5. *Suppose that $\omega_2 \notin S(x)$. Then $f(S(x)) = S(f(x))$.*

Proof. We first prove that $f(S(x)) \subset S(f(x))$. Indeed, $S(x)$ is the intersection of all $\overline{S^\circ}(\alpha, \beta)$ with $x \in S^\circ(\alpha, \beta)$. The f -image of the intersection lies in the intersection of images. Taking only those $S^\circ(\alpha, \beta)$, for which $|\beta - \alpha| < 1/3$, we see by Lemma 8.2 that $f(S(x))$ lies in the

intersection of $\overline{S^\circ}(3\alpha, 3\beta) \ni f(x)$. The latter set obviously coincides with $S(f(x))$.

We now prove that $S(f(x)) \subset f(S(x))$, i.e., every point z in $S(f(x))$ has the form $f(y)$ for some $y \in S(x)$. Take an open sector $S^\circ(\alpha, \beta) \ni x$ that contains only one f -preimage of z ; call this preimage y . We may also assume that $\omega_2 \notin S^\circ(\alpha, \beta)$ and that (α, β) maps one-to-one under σ_3 . Then, by Lemma 8.2, we have $f(S^\circ(\alpha, \beta)) = S^\circ(3\alpha, 3\beta)$. Let S° be any open sector in $S^\circ(3\alpha, 3\beta)$ containing z . Then the pullback T° of S° in $S^\circ(\alpha, \beta)$ must be an open sector, and it must contain a preimage of z . The only option is that $y \in T^\circ$. Since y is contained in all such T° , we have $y \in S(x)$. \square

l:bdminsec

Lemma 8.6. *For any $x \in \mathbb{C} \setminus K^*$, the rays on the boundary of $S(x)$ map onto the rays on the boundary of $S(f(x))$ under f .*

Proof. Consider an open sector S° around x . Its argument arc covers one or several components of $\sigma_3^{-1}(\arg(S(f(x))))$. Moreover, S° can be chosen so that the endpoints of $\arg(S^\circ)$ are arbitrarily close to $\sigma_3^{-1}(\arg(S(f(x))))$. The lemma follows. \square

9. BACKWARD STABILITY

In this section we study backward stability of decorations. The aim, to begin with, is to show that under certain circumstances decorations shrink as we pull them back. Our arguments are based upon the following theorem of Mañé.

t:mane

Theorem 9.1 ([Man93]). *If $f : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$ is a rational map and $z \in \overline{\mathbb{C}}$ a point that does not belong to the limit set of any recurrent critical point, then for some Jordan disk W around z , some $C > 0$ and some $0 < q < 1$ the spherical diameter of any component of $f^{-n}(W)$ is less than Cq^n .*

Neighborhoods satisfying Theorem 9.1 are called *Mañé neighborhoods*.

l:sn

Lemma 9.2. *Fix $q \in (0, 1)$ and $b > 0$. Consider a sequence of positive numbers s_n such that either $s_{n+1} = qs_n$ or $s_{n+1} \leq 2qs_n + b$. In the former case call n the good index, and in the latter case call n the bad index. Suppose that the distance between adjacent bad indices tends to infinity. Then $s_n \rightarrow 0$.*

Proof. It suffices to show that $s_n \rightarrow 0$ as n runs through all bad indices $n_1 < n_2 < \dots$; fix $\varepsilon > 0$ and N such that $q^N < 1/8$ and $q^{N-1}b < \varepsilon$. Then, for i large, we have $n_{i+1} - n_i \geq N$, and

$$s_{n_{i+1}} = q^{n_{i+1}-n_i-1}(2qs_{n_i} + b) = q^{n_{i+1}-n_i}(2s_{n_i} + q^{-1}b) \leq \frac{s_{n_i}}{4} + \varepsilon.$$

Since the map $h(x) = x/4 + \varepsilon$ has a unique attracting point $4\varepsilon/3$ which attracts all points of \mathbb{R} , then s_{n_i} becomes eventually less than 4ε . Since $\varepsilon > 0$ is arbitrary, it follows that $s_n \rightarrow 0$ as $i \rightarrow \infty$, as desired. \square

Recall that we consider immediately renormalizable polynomials f and use for them the notation introduced above. Consider two rays $R = R(\alpha)$ and $L = R(\beta)$ landing in K^* . Also, take any equipotential \widehat{E} of P . Let $\Delta = \Delta(R, L, \widehat{E})$ be the bounded complementary component of $K^* \cup R \cup L \cup \widehat{E}$ such that the external rays that penetrate into Δ have arguments that belong to the positively oriented arc from α to β . Evidently, Δ is the intersection of $S^\circ(\alpha, \beta)$ with the Jordan disk enclosed by \widehat{E} . Hence results of the previous section dealing with sectors apply to Δ and similar sets. Let Δ' be a pullback of Δ such that $\overline{\Delta'} \cap K^* \neq \emptyset$; then say that Δ' is a pullback of Δ adjacent to K^* . If Δ' is a P^n -pullback of Δ such that $\overline{\Delta'} \cap K^* \neq \emptyset$, we call Δ' an iterated pullback of Δ adjacent to K^* . Let $\Delta = \Delta_0$ and for every n the set Δ_n be a pullback of Δ_{n-1} adjacent to K^* . Then the sequence of sets $\Delta_n, n = 0, 1, \dots$ is called a backward pullback orbit of Δ adjacent to K^* . Set \widetilde{U}^* to be the pullback of U^* containing \widetilde{K}^* .

1:Del

Lemma 9.3. *Suppose that $\{\Delta_n\}$ is a backward pullback orbit of Δ adjacent to K^* and $n_1 < n_2 < \dots$ are all positive integers n such that $\omega_2 \in \Delta_n$ and ω_2 is non-recurrent. If $n_{i+1} - n_i \rightarrow \infty$, then $\Delta_n \subset U^*$ for large n so that the distance between Δ_n and K^* tends to 0.*

Proof. Consider a finite covering \mathcal{U} of $\Delta \setminus U^*$ by Mañe neighborhoods. After adjustments we may assume that elements of \mathcal{U} are subsets of Δ . Similarly, fix a finite covering \mathcal{V} of \widetilde{U}^* by Mañe neighborhoods such that $\bigcup \mathcal{V} = \widetilde{U}^*$. Set $\mathcal{U}_1 = \mathcal{U}$ and define \mathcal{U}_n inductively as follows. Assuming by induction that $\Delta_n \setminus U^* \subset \bigcup \mathcal{U}_n \subset \Delta_n \cup U^*$, define \mathcal{U}_{n+1} as the set of all open sets U satisfying one of the following:

- (1) there is $U' \in \mathcal{U}_n$ such that $U \subset \Delta_{n+1}$ is a pullback of U' ;
- (2) the point ω_2 is in Δ_{n+1} , and $U \in \mathcal{V}$.

Neighborhoods in \mathcal{U}_{n+1} as in item (1) (resp., (2)) are called *type (1)* (resp., *type (2)*) neighborhoods.

Any $U \in \mathcal{U}_n$ is either obtained as a P^k -pullback of some type (1) neighborhood in \mathcal{U}_{n-k} with k being maximal with this property, or comes from \mathcal{V} but only at the moments when $\omega_2 \in \Delta_n$. In the former case set $s(U) = Cq^k$; by the Mañe theorem, $\text{diam}(U) \leq s(U)$. In the latter case set $s(U) = \text{diam}(U)$. Define $s_n = s_n(\Delta)$ as the sum of $s(U)$ over all $U \in \mathcal{U}_n$. By the triangle inequality $\text{diam}(\Delta_n \setminus U^*)$ is bounded

above by s_n . Thus it suffices to show that $s_n \rightarrow 0$ as $n \rightarrow \infty$. Now consider two cases of transition from n to $n+1$.

(1) Assume that $\omega_2 \notin \Delta_{n+1}$. Then Δ_{n+1} maps one-to-one to Δ_n ; moreover, no point of $\Delta_{n+1} \setminus U^*$ maps into U^* . It follows that all neighborhoods in \mathcal{U}_{n+1} are type (1). In this case we have $s_{n+1} = qs_n$.

(2) Assume that $\omega_2 \in \Delta_{n+1}$. Then there are at most twice as many type (1) neighborhoods in \mathcal{U}_{n+1} as neighborhoods in \mathcal{U}_n . Also, \mathcal{U}_{n+1} includes \mathcal{V} . Thus we have $s_{n+1} \leq 2qs_n + \text{diam}(\tilde{U}^*)$.

Thus, s_n satisfies Lemma 9.2 with $b = \text{diam}(\tilde{U}^*)$; in particular, $s_n \rightarrow 0$ as $n \rightarrow \infty$, and so the diameter of $\Delta_n \setminus U^*$ tends to 0. Replacing U^* with a smaller neighborhood of K^* and repeating the same argument yields $\Delta_n \subset U^*$ for large n . Evidently, this completes the proof. \square

If, in the setting of Lemma 9.3, the sequence $n_{i+1} - n_i$ does not tend to infinity, then it takes the same value infinitely many times. We now consider this case.

1:bnd

Lemma 9.4. *Suppose that $n_{i+1} - n_i$ takes the same value N infinitely many times. Then the quadratic argument of ω_2 is σ_2^N -fixed and the associated pullbacks $[\alpha_n, \beta_n]$ of $[\alpha, \beta]$ are shrinking segments around corresponding points of the orbit of the quadratic argument of ω_2 .*

Proof. Let R_n and L_n be the rays with quadratic arguments α_n and β_n landing in K^* and bounding Δ_n near K^* . Since Δ_{n+1} is a P -pullback of Δ_n , then $\alpha_n = 2\alpha_{n+1} \pmod{1}$ and $\beta_n = 2\beta_{n+1} \pmod{1}$, and the interval $[\alpha_{n+1}, \beta_{n+1}]$ is twice shorter than $[\alpha_n, \beta_n]$. By the assumption, $\omega_2 \in \Delta_n \cap \Delta_{n+N}$ for arbitrarily large n . Hence the intervals $[\alpha_n, \beta_n]$ contain the quadratic argument of ω_2 . Passing to the limit, we see that this quadratic argument is σ_2^N -fixed. The last claim now follows. \square

Standard arguments now yield the next lemma.

1:all-n

Lemma 9.5. *If ω_2 is non-recurrent and a decoration D has a quadratic argument $\alpha(D)$ which does not belong to the orbit of a periodic quadratic argument $\alpha(D_c)$, then there exists N such that any P^N -pullback of D adjacent to K^* is contained in U^* .*

Proof. If D_n is a P^n -pullback of D adjacent to K^* with $D_n \not\subset U^*$, then $f^i(D_n) \not\subset U^*$ for every $i, 0 \leq i \leq n$. By Lemma 9.4 any backward pullback orbit of D adjacent to K^* satisfies conditions of Lemma 9.3; thus, for any such orbit $D = D_0, D_1, \dots$ there exists the minimal n such that $D_n \subset U^*$. By way of contradiction suppose that for any N there exists a P^N -pullback of D adjacent to K^* and not contained in U^* . Then there are infinitely many such pullbacks. Now take the P -pullbacks of D adjacent to K^* (there are finitely many of them). Choose among

them one pullback that has P^N -pullbacks adjacent to K^* and not contained in U^* for any N , and then apply the same construction to it. Evidently, in the end we will construct a backward pullback orbit of D adjacent to K^* , a contradiction with Lemma 9.3. \square

All this implies Proposition 9.6.

p:alldec

Proposition 9.6. *If f is immediately renormalizable, and ω_2 is non-recurrent then every decoration is eventually mapped to D_c .*

Proof. Suppose that the quadratic argument $\alpha(D) = \gamma$ of D does not belong to a periodic orbit of $\alpha(D_c)$. Then by Lemma 9.5 for any U^* the P^n -pullbacks of D will be contained in U^* for any $n > N(U^*)$ where $N(U^*)$ depends on U^* . Consider now the union K_d of K^* and all decorations that are eventually mapped to D_c . We claim that the set K_d is backward invariant. Indeed, take any decoration $D \subset K_d$. Then any f -pullback of D is either a decoration in K_d or a subset of D_c . It follows that $f^{-1}(K_d) \subset K_d$, as desired. Also, the set K_d is compact as a union of K^* and a sequence of sets that are closed in $\mathbb{C} \setminus K^*$ and accumulate to K^* . Observe that if $\alpha(D_c)$ is periodic, the sets in K_d are decorations with periodic arguments from the σ_2 -orbit of $\alpha(D_c)$, or decorations with non-periodic arguments-preimages of $\alpha(D_c)$.

Now, $J(f)$ is a minimal by inclusion compact backward invariant subset of \mathbb{C} . (Equivalently, the backward orbit of any point from $J(f)$ is dense in $J(f)$.) Thus, $J(f) \subset K_d$ since K_d contains points of $J(f)$. On the other hand, consider a bounded Fatou component Ω . If $\Omega \not\subset K^*$ then the boundary of Ω is a subset of K_d . Therefore, it is a subset of some decoration $D \subset K_d$. It follows that Ω itself is included into D . We showed that $K = K_d$ which completes the proof of the proposition. \square

The next corollary follows from Proposition 9.6.

c:uni-dec

Corollary 9.7. *If f is immediately renormalizable and ω_2 is non-recurrent then for each quadratic angle γ there is at most one decoration with quadratic argument γ .*

Proof. Assume that $\alpha(D_c)$ is non-periodic. If there are two distinct decorations D' and D'' with the same quadratic argument γ , then there exists a unique number n such that $\sigma_2(\gamma) = \alpha(D_c)$. This is the only chance for D' and D'' to be mapped to D_c . It follows from Proposition 9.6 that $f(D') = f(D'') = D_c$. However by Lemma 8.2 this is impossible (for small open sectors with argument arc containing $\alpha(D_c)$ the map f restricted on them is one-to-one by Lemma 8.2).

Now suppose that $\alpha(D_c)$ is periodic of period n . Then, similar to the previous paragraph, we see that if there are two distinct decorations D' and D'' with the same quadratic argument γ then there must exist a decoration $D''' \neq D_c$ with quadratic argument $\alpha(D_c)$ such that $f^n(D''') = D_c$. Transferring this to the z^2 -plane we see that the map z^2 is not one-to-one in a small neighborhood of the point of the unit circle with argument $\alpha(D_c)$, a contradiction completing the proof. \square

The next lemma specifies properties of the gap \mathfrak{U} .

l:perio-type

Lemma 9.8. *If ω_2 is non-recurrent, then \mathfrak{U} is of periodic type.*

Proof. Consider the critical value decoration D_v . By Proposition 9.6, D_v eventually maps back to D_c . The rays on the boundary of the minimal sector $S(\omega_2)$ map to the rays on the boundary of $S(f(\omega_2))$ under f . This follows from Lemma 8.6. The sector $S(f(\omega_2))$ is eventually mapped to $S(\omega_2)$ by Lemma 8.5. Therefore, \mathfrak{U} is of periodic type. \square

10. MAIN THEOREM

s:mt

To prove the main result we rely upon recent powerful results obtained in [DL21]. To state them, we need to state a property of quadratic polynomials from the Main Cardioid of the Mandelbrot set established in [Chi08].

t:chi

Theorem 10.1 ([Chi08]). *Let Q be a quadratic polynomial with a fixed point a such that $P'(a) = e^{2\pi i\theta}$ with irrational θ . Then the limit set $\omega(c_Q)$ of the critical point c_Q of Q is a continuum.*

In the situation of Theorem 10.1 the set $\text{Th}(\omega(c_Q))$ is called the *mother hedgehog* of Q and is denoted by M_Q .

t:md

Theorem 10.2 ([DL21]). *Let Q be a quadratic polynomial with a fixed point a such that $Q'(a) = e^{2\pi i\theta}$ with irrational θ . Then $Q^{-1}(M_Q)$ is the union of M_Q and a continuum M'_Q such that $M_Q \cap M'_Q = \{c\}$.*

Theorem 10.2 allows one to view Q similar to how Siegel quadratic polynomials with locally connected Julia sets are viewed. By Theorem 10.2 there are infinite concatenations of pullbacks of M_Q analogous to external rays in that they partition $J(Q)$ in pieces and, thus, enable further study of topology $J(Q)$. Indeed, by Theorem 10.2 countably many pullbacks of M'_Q are attached to M_Q at preimages of c_Q that belong to M_Q . Each of them is eventually mapped onto M_Q . Hence each of them has countably many attached to it pullbacks of M_Q , etc. The entire grand orbit of M_Q can be viewed as a “spiderweb” of concatenated in various ways pullbacks of M_Q .

Each pullback T of M_Q can be characterized by a sequence of integers $\text{thr}(T)$ called the *thread* of T . It reflects the “journey” of T to M_Q , and, simultaneously, the concatenation (the *chain*) \mathcal{CH}_T (of pullbacks of M_Q) that connects M_Q and T . In what follows we denote chains by boldface capital letters (e.g., \mathbf{A} or \mathbf{X}) and threads by small boldface letters (e.g., \mathbf{a} and \mathbf{x}). To study chains of pullbacks of M_Q we need the next lemma.

1:chains

Lemma 10.3. *Distinct pullbacks S and T of M_Q can only intersect if they meet at a point that is a preimage of c_Q . Two chains share an initial finite string and are otherwise disjoint.*

Proof. Observe that M_Q is a full continuum. Therefore all pullbacks of M_Q (including M'_Q) are full continua too. It follows that if T is a pullback of M_Q and $c_Q \notin T$, then $P|_T$ is a homeomorphism to the image. Suppose that $S \neq T$ are pullbacks of M_Q and $x \in S \cap T$. Let us apply Q to S and T step by step. Then there are two cases.

(1) At some earliest moment n we have $Q^n(S) = M_Q$ and $Q^n(T) \neq M_Q$. Then $Q^n(T)$ is a Q^m -pullback of M'_Q for some $m \geq 0$. It follows that $Q^{m+n}(x) = c_Q$ as desired.

(2) At some earliest moment n we have $Q^n(S) = Q^n(T) = A$, for each $i < n$, $Q^i(S) \neq Q^i(T)$ and neither of these sets equals M_Q . Then $Q^n(S) \neq M_Q$ either because otherwise we must have $Q^{n-1}(S) = M_Q$, a contradiction. The set $Q^{n-1}(S \cup T) = B$ is a pullback of the set A . Together with the fact that $Q|_B$ is not a homeomorphism this implies that $c_Q \in Q^{n-1}(S) \cap Q^{n-1}(T)$. Since neither of these two sets equals M_Q and they are distinct, it follows that at most one of them equals M'_Q while the other one (denote it by Z) is a pullback of M'_Q . We have that $c_Q \in Z \cap M'_Q$ and, for some $k > 0$, $Q^k(Z) = M'_Q$ while $Q^k(M'_Q) = M_Q$. It follows that $Q^k(c_Q) = c_Q$, a contradiction.

We leave the second claim of the lemma to the reader. □

Threads are one-sided sequences, infinite on the left. For each pullback S in \mathcal{CH}_T choose the least number $m(S)$ such that $Q^{m(S)}(S) = M_Q$ while M_Q itself is associated with infinite on the left string of zeros denoted by $\bar{0}$. Thus, if $\mathcal{CH}_T = (M_Q, S_1, \dots, S_{k-1}, S_k = T)$ then $\text{thr}(T) = (\bar{0}, m(S_1), \dots, m(T))$ so that $\text{thr}(M_Q) = (\bar{0})$, $\text{thr}(M'_Q) = (\bar{0}, 1)$ etc. Observe that given a pullback T of M_Q , its thread $\text{thr}(T)$ is a sequence with $\bar{0}$ followed by a finite string of strictly growing positive integers.

If $\text{thr}(T) = (\bar{0}, m_1, \dots, m_k)$ then S_1 is the Q^{m_1} -pullback of M_Q attached to M_Q . Under Q^{m_1} the set S_2 maps forward so that $Q^{m_1}(S_2)$ is attached to M_Q and is later, under the action of $Q^{m_2-m_1}$, mapped to M_Q . In general, each power Q^{m_j} moves all pullbacks $S_i \in \mathcal{CH}_T$, $j \leq$

$i \leq k$ forward so that the chain $Q^{m_j}(M_Q, S_1, \dots, S_{k-1}, S_k)$ equals

$$(M_Q, Q^{m_j}(S_{j+1}), Q^{m_j}(S_{j+2}), \dots, Q^{m_j}(S_k))$$

and the associated thread is $(\bar{0}, m_{j+1} - m_j, \dots, m_k - m_j)$. Thus, the action of Q on threads will be denoted by η and translates as $\eta : (\bar{0}, m_1, \dots, m_k) \mapsto (\bar{0}, m_1 - 1, \dots, m_k - 1)$ (notice that if $m_1 = 1$ the number $m_1 - 1 = 0$ should be viewed as a part of infinite strings of zeros).

Expanding this idea, consider infinite chains (of pullbacks of M_Q) and characterize them by their *threads*, i.e. infinite in both directions sequences of integers $(\bar{0}, m_1, m_2, \dots)$ where $0 < m_1 < \dots$ (we will still use the same notation for infinite chains/threads as for finite ones). The corresponding infinite chain of pullbacks of M_Q is (M_Q, S_1, \dots) that can be described as follows: the set S_1 is the Q^{m_1} -pullback of M_Q , attached to M_Q , the set S_2 is the Q^{m_2} -pullback of M_Q , attached to S_1 , etc. Clearly, the map η can be defined on infinite threads as follows: if $\mathbf{x} = (\bar{0}, m_1, m_2, \dots)$, then $\eta(\mathbf{x}) = (\bar{0}, m_1 - 1, m_2 - 2, \dots)$. This action corresponds to the action of Q on infinite chains so that if $\text{thr}(\mathbf{X}) = \mathbf{x}$, then $\text{thr}(Q(\mathbf{X})) = \eta(\mathbf{x})$ (i.e., $\text{thr} \circ Q = \eta \circ \text{thr}$). Clearly, the map η defined on infinite threads has periodic points. E.g., by definition $\mathbf{a} = (\bar{0}, m_1, m_2, \dots)$ is η -fixed if $m_1 - 1 = 0$, $m_2 - 1 = m_1$, etc, i.e. the only η -fixed thread is $(\bar{0}, 1, 2, 3, \dots)$. Of course, there are other periodic infinite chains.

Suppose that $(\bar{0}, m_1, \dots)$ is periodic of (minimal) period N (in what follows by “period” we always mean “minimal period”). It is easy to see that this means the following: there exists k such that $m_k = N$ and, moreover, for any $j = lk + r$, $0 \leq l, 0 \leq r < k$ we have $m_j = lN + m_r$. Recall that the *topological hull* $\text{Th}(A)$ of a compact set $A \subset \mathbb{C}$ is the complement of the unbounded complementary domain of A .

1:exponent

Lemma 10.4. *Let Q be a quadratic polynomial with a fixed point a such that $Q'(a) = e^{2\pi i\theta}$ with irrational θ . Then claims (1) – (4) hold.*

- (1) *Let T be a pullback of M_Q disjoint from M_Q . Then there exists $C_1 > 0$ and $0 < q < 1$ such that any Q^N -pullback of T has diameter less than $C_1 q^N$ for any N .*
- (2) *Distinct infinite chains converge to distinct points.*
- (3) *Suppose that $\text{thr}(\mathbf{X}) = \mathbf{x}$ is periodic of period N . Then \mathbf{X} converges to a periodic point of period N that does not belong to M_Q . Any preperiodic chain converges to a preperiodic point.*
- (4) *The only periodic point in M_Q is the fixed point a . All periodic points $y \neq a$ of Q are limits of the corresponding chains of pullbacks of M_Q .*

Proof. (1) The fact that $M_Q = \text{Th}(\omega(c_Q))$ implies that T is disjoint from $\omega(c_Q)$. By Mañé [Man93] (see Theorem 9.1) we can cover $\text{Th}(T)$ with small Mañé disks U_1, \dots, U_k so that their union is itself a Jordan disk V with $\bar{V} \cap M_Q = \emptyset$ so that any Q^n -pullback of each U_i is of diameter less than Cq^n for some $C > 0$ and $0 < q < 1$ (these depend solely on Q). Since $\bar{V} \cap \text{Th}(\omega(c_Q)) = \emptyset$ then any Q^N -pullback of V is a topological disk V' that homeomorphically maps onto V under Q^N (observe that since c_Q is recurrent, the orbit of c_Q is contained in $\omega(c_Q)$). Hence each U_i is represented in V' by exactly one pullback U'_i whose diameter, by Theorem 9.1, is less than Cq^N . Hence the diameter of V' is less than kCq^N and it remains to set $C_1 = kC$.

(2) Now, if two distinct chains of pullbacks of M_Q converge to the same point, it would follow that some points of these pullbacks are blocked from infinity by the union of these chains and, therefore, cannot belong to $J(Q)$, a contradiction. Observe that we are not claiming that all chains converge. However if two chains do converge, they cannot converge to the same as was just proven.

(3) If $\mathbf{x} = (\bar{0}, m_1, \dots)$, then there exists k such that $m_k = N$ and, moreover, for any $j = lk + r, 0 \leq l, 0 \leq r < k$ we have $m_j = lN + m_r$. Suppose that S is the j -th element of \mathbf{X} (associated with m_j). It is preceded in \mathbf{X} by another pullback S' of M_Q , associated with m_{j-1} , and until S' maps to M_Q under Q^{m_j} , the set S remains detached from M_Q . Thus, $Q^{m_{j-1}-1}(S)$ is a specific pullback of M_Q detached from M_Q . Evidently, there is a finite collection of such “last detached” pullbacks of M_Q that serve all elements of \mathbf{X} . By (1) we can choose numbers $C > 0$ and $0 < q < 1$ that serve all these “last detached” pullbacks, and, after straightforward adjustments, conclude that $\text{diam}(S) < Cq^{m_{j-1}}$. Evidently, this implies that \mathbf{X} converges to a Q^N -fixed point z . If the period of z were less than N , there would exist a chain distinct from \mathbf{X} but converging to z , a contradiction to (2). Moreover, z cannot belong to M_Q as otherwise some points of M_Q would be blocked from infinity (similar to the argument in (2)). The last part of claim (3) now follows.

(4) Claims (1) - (3) hold for some quadratic polynomial P_{sie} with Siegel fixed point b and locally connected Julia set. In case of such polynomials it is easy to check that all periodic points of P_{sie} except for b can be obtained as limits of periodic chains. However chains of Q and chains of P are in one-to-one correspondence with corresponding periodic threads. Hence the number of periodic points of any period n is the same for Q and for P_{sie} and coincides with the number of periodic threads of that period. Recall that except for b there are no Q -periodic points that belong to M_Q as M_Q is a Q -invariant Jordan

curve on which Q is conjugate with an irrational rotation. It follows from these counts of periodic points for Q and P_{sie} that no periodic point of Q , except for a , can belong to M_Q . \square

Let us now combine Lemma 10.4 and tools from [BFMOT12].

t:allc

Theorem 10.5. *Let P be a polynomial of any degree with a fixed non-repelling point $a \in B^*$ where B^* is an invariant filled quadratic-like Julia set of P . Assume that any periodic neutral point $x \neq a$ of P is parabolic. Then for a periodic point $z \in B^* \setminus \{a\}$ and a periodic external ray R of P the fact that z belongs to the impression I of R implies that $I = \{z\}$ (thus, $J(P)$ is locally connected at z).*

It is well-known that if the impression of a ray is degenerate then this ray lands at some point z and the continuum is ready.

Proof. First assume that a is either attracting or parabolic. Then P has no Cremer or Siegel points. Hence by Corollary 7.5.4 [BFMOT12] any periodic impression is degenerate.

The forthcoming argument which involves chains, thread etc applies to $P|_{B^*}$ as B^* is quadratic-like; we will, therefore, use the same notation as before despite the fact that P itself is not quadratic. Suppose now that $P'(a) = e^{2\pi i\theta}$ where θ is irrational. We will need the following construction. Let \mathbf{X} be the chain that converges to z . Let $(\bar{0}, m_1, \dots) = \text{thr}(\mathbf{X}) = \mathbf{x}$ be its thread. Choose a pullback S of the mother hedgehog M_P of P that belongs to \mathbf{X} assuming that $S \neq M_P$, $S|_{neM'_P}$. Evidently, on the plane there are pullbacks L, R of M_P attached to S from either side of the part of \mathbf{X} connecting S and z . The choice of L and R depends on θ , however, regardless of θ , such L and R exist. Then choose (pre)periodic chains \mathcal{CH}_l and \mathcal{CH}_r that extend L and R and converge to points y_l, y_r , and external rays Y_l, Y_r of P that land at y_l, y_r and have arguments θ_l, θ_r , resp. Set

$$Z = Y_l \cup \{y_l\} \cup L \cup S \cup R \cup \{y_r\} \cup Y_r$$

and use it in the proof. Repeat this construction for a different pullback $S' \in \mathbf{X}$ of M_P assuming that in \mathbf{X} the set S' is closer to M_P than S and construct a similar set Z' for S' so that Z separates Z' from z .

The set Z divides \mathbb{C} in two open subsets, $W_a \ni a$ and $W_z \ni z$. A ray whose impression contains z must be contained in W_z or coincide with Y_l or with Y_r . Now, the set Z' divides \mathbb{C} in two open subsets, $W'_a \ni a$ and $W'_z \ni z$, and $W'_z \supset Z \cup W_z$. It follows that the impression of any angle that contains z cannot contain a . By Corollary 7.5.4 from [BFMOT12] this impression is degenerate and, hence, coincides with $\{z\}$. This implies that $J(P)$ is locally connected at z as desired. \square

The next result is used in dealing with an easier case.

t:cutpts

Theorem 10.6 ([BOT21]). *Let P be a polynomial of any degree with connected filled Julia set $K(P)$. Let $x \in K(P)$ be a repelling or parabolic periodic point and all $K(P)$ -rays to x form m wedges W_i , where $1 \leq i \leq m$. Moreover, suppose that $x \in Q$ is a cutpoint of order n of an invariant continuum $Q \subset J$. Then $n \leq m$, each wedge W_i intersects Q over a connected (possibly empty) set, and every Q -ray to x is isotopic rel. Q to a $K(P)$ -ray that lands at x .*

We are finally ready to prove Theorem 1.2.

Proof of Theorem 1.2. Since ω_2 is non-recurrent, then by Lemma 9.8 \mathfrak{U} is of periodic type. Let $M = \overline{\alpha\beta}$ be its major and $I = (\alpha, \beta)$ be its major hole. Then α and β are σ_3 -periodic angles each of which is approached from the outside of I by periodic angles $\alpha_i \rightarrow \alpha$ and $\beta_i \rightarrow \beta$ whose rays land in J^* . These angles belong to \mathfrak{U} and correspond (through the map τ collapsing edges of \mathfrak{U}) to σ_2 -periodic angles α'_i and β'_i . Moreover, “quadratic” angles α'_i and β'_i converge to the same “quadratic” periodic angle $\gamma = \tau(M)$. Then by Theorem 10.5 the landing points of “quadratic” external rays with arguments α'_i and β'_i converge to the periodic landing point of the “quadratic” ray with argument γ . Evidently, the same will happen in K^* , i.e. the landing points of P -rays with arguments α_i and β_j converge to a periodic point in K^* . This point belongs to the impressions of α and of β . Hence, by Theorem 10.5, they are degenerate and coincide with this point, a contradiction with the assumption that the rational lamination of P is empty. \square

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