A MODEL OF THE CUBIC CONNECTEDNESS LOCUS

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ABSTRACT. We construct a model of the cubic connectedness locus.

1. INTRODUCTION

The concepts of *renormalization* and *tuning* appear in the context of the Mandelbrot set $\mathcal{M}_2 = \{c \in \mathbb{C} \mid Q_c^n(0) \not\to \infty\}$, where $Q_c(z) = z^2 + c$, and serve to explain a self-similar structure of $\mathcal{M}_2$. *Self-similarity* of $\mathcal{M}_2$ means, in particular, that there are infinitely many homeomorphic copies of $\mathcal{M}_2$ in $\mathcal{M}_2$, the so-called *baby Mandelbrot sets*. Baby Mandelbrot sets accumulate to any boundary point of $\mathcal{M}_2$. If c is in a baby Mandelbrot set, then Q_c is obtained from another quadratic polynomial by "tuning", i.e. consistently pinching closures of periodic Fatou components and their pull-backs. A baby Mandelbrot set consists of all tunings of a given polynomial different from $z \mapsto z^2$ and is contained in a unique maximal one.

If we collapse the closure of the main cardioid and all maximal baby Mandelbrot sets, we will obtain a dendrite $D(\mathcal{M}_2)$ that reveals the *macro*-structure of $\mathcal{M}_2$. A self-similar description of $\mathcal{M}_2$ involves knowing $D(\mathcal{M}_2)$ together with a subset of marked points in $D(\mathcal{M}_2)$, where each marked point is a collapsed maximal baby Mandelbrot set.

In this paper, we use these ideas to give a model of the cubic connectedness locus $\mathcal{M}_3$. Basically, we consider, for every polynomial f, its *chief* g, that is, a polynomial such that f is a tuning of g, and g is not a non-trivial tuning of another polynomial. Our model for $\mathcal{M}_3$ relies upon studying chiefs of cubic polynomials. For $d = 2$, the chiefs define maximal baby Mandelbrot sets. Thus, our approach gives a macro-view of $\mathcal{M}_3$, similar to $D(\mathcal{M}_2)$.

We are not aware of any other models of $\mathcal{M}_3$.

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2. Statement of the main results

The *parameter space* of complex degree d polynomials P is the space of affine conjugacy classes $[P]$ of these polynomials. The *connectedness locus* $\mathcal{M}_d$ consists of classes of all degree d polynomials P, whose Julia sets J_P are connected. The *Mandelbrot set* $\mathcal{M}_2$ has a complicated self-similar structure understood through the "pinched disk" model [13, 14, 20].

In this paper, we find a combinatorially defined upper semi-continuous (USC) partition of $\mathcal{M}_3$. A property of a polynomial is *combinatorial* if it can be stated based only upon knowing which pairs of rational external rays land at the same point and which pairs do not. A combinatorial USC partition of $\mathcal{M}_3$ yields a continuous map of $\mathcal{M}_3$ to a quotient space of CrP, the space of unordered cubic critical portraits. Let us describe our approach.

Let P be such that $[P] \in \mathcal{M}_3$. A point x is *(P-)legal* if it eventually maps to a repelling periodic point. An unordered pair of rational angles $\{\alpha, \beta\}$ is *(P-)legal* if the external rays with arguments α and β land at the same legal point of P. Write $\mathcal{Z}_P$ for the set of all P-legal pairs of angles; call $\mathcal{Z}_P$ the *l-set* of P. Such sets are closely related to *rational* and *real laminations* of polynomials introduced by J. Kiwi [16, 18].

A cubic polynomial $P \in \mathcal{M}_3$ is *visible* if $\mathcal{Z}_P \neq \varnothing$ and *invisible* otherwise. If P is visible, denote by $\mathcal{C}_P$ the set of all critical portraits *compatible* with $\mathcal{Z}_P$ (i.e., no critical chord from a critical portrait in $\mathcal{C}_P$ separates a P-legal pair of angles). Clearly, $\mathcal{C}_P$ is closed. The set $\mathcal{C}_P$ is *the combinatorial counterpart* of P.

In this paper we will define, for every $[P] \in \mathcal{M}_3$, a *closed* subset $\mathcal{A}_P$ of CrP called an *alliance*. The main properties of alliances are:

(1) if P is visible, then $\mathcal{C}_P \subset \mathcal{A}_P$;
(2) distinct alliances are disjoint;
(3) the alliances form an USC partition of CrP.

One special alliance is said to be *prime*. It contains $\mathcal{C}_P$ for all visible polynomials P with a non-repelling fixed point, some other combinatorial counterparts, and is associated with all invisible polynomials P. The other alliances are called *regular*; they are combinatorial counterparts of certain visible polynomials, and there are uncountably many of them.

Main Theorem. *All alliances form a USC partition $\{\mathcal{A}_P\}$ of CrP. The union of regular alliances is open and dense in CrP. The map $P \mapsto \mathcal{A}_P$ is continuous and maps $\mathcal{M}_3$ onto the quotient space $\mathrm{CrP}/\{\mathcal{A}_P\}$.*

Thurston [20] gave a detailed, conjecturally homeomorphic, model of the quadratic connected locus $\mathcal{M}_2$. The situation with the cubic connectedness locus $\mathcal{M}_3$ is different. Indeed, $\mathcal{M}_3$ is complex 2-dimensional. Cubic polynomials are richer dynamically than quadratic ones (critical points are essential for the dynamics of polynomials, and cubic polynomials generically have two critical points) which makes the cubic case highly intricate combinatorially [7, 10]. This results into a breakdown of some crucial steps of [20] (e.g., cubic invariant laminations admit wandering triangles [4, 5]). Also, $\mathcal{M}_3$ is known to be non-locally connected [19] and to contain copies of various non-locally connected quadratic Julia sets [12]. All this makes the cubic case much harder and significantly complicates a complete description of $\mathcal{M}_3$.

3. Critical portraits and laminations

We assume familiarity with complex polynomial dynamics, including Julia sets, external rays, etc. All cubic polynomials in this paper are assumed to be *monic*, i.e., of the form z^3+ a quadratic polynomial, and to have connected Julia sets. We can parameterize the external rays of a cubic polynomial P by *angles*, i.e., elements of $\mathbb{R}/\mathbb{Z}$. The external ray of argument $\theta \in \mathbb{R}/\mathbb{Z}$ is denoted by $R_P(\theta)$. Clearly, P maps $R_P(\theta)$ to $R_P(3\theta)$.

For sets A, B, let $A \vee B$ be the set of all unordered pairs $\{a, b\}$ with $a \in A, b \in B$; thus, the *l-set* $\mathcal{Z}_P$ of P consists of all pairs $\{\alpha, \beta\} \in (\mathbb{Q}/\mathbb{Z}) \vee (\mathbb{Q}/\mathbb{Z})$ such that $R_P(\alpha)$ and $R_P(\beta)$ land at the same legal point of P.

A *chord* $\overline{ab}$ is a closed segment connecting points a, b of the unit circle $\mathbb{S} = \{z \in \mathbb{C} \,|\, |z| = 1\}$. If $a = b$, the chord is *degenerate*. Two *distinct* chords of $\mathbb{D}$ *cross* if they intersect in $\mathbb{D}$ (alternatively, they are called *linked*). Sets of chords are *compatible* if chords from distinct sets do not cross.

Write σ_d for the self-map of $\mathbb{S}$ that takes z to z^d. A chord $\overline{ab}$ is (σ_d-) *critical* if $\sigma_d(a) = \sigma_d(b)$. Let CCh be the set of all σ_3-critical chords with the natural topology; CCh is homeomorphic to $\mathbb{S}$. A *critical portrait* is a pair $\{\overline{c}, \overline{y}\} \in \mathrm{CCh} \vee \mathrm{CCh}$ such that $\overline{c}$ and $\overline{y}$ do not cross. Let CrP be the space of all critical portraits. It is homeomorphic to the Möbius band [21].

Motivated by studying l-sets of visible polynomials, Thurston [20] defined *invariant laminations* as families of chords with certain dynamical properties. We use a slightly different approach (see [3]).

Definition 3.1 (Laminations)**.** A *prelamination* is a family $\mathcal{L}$ of chords called *leaves* such that distinct leaves are unlinked and all points of $\mathbb{S}$ are leaves. If, in addition, the set $\mathcal{L}^+ = \bigcup_{\ell \in \mathcal{L}} \ell$ is compact, then $\mathcal{L}$ is called a *lamination*.

From now on $\mathcal{L}$ denotes a lamination (unless we specify that it is a prelamination). *Gaps* of $\mathcal{L}$ are the closures of components of $\mathbb{D} \setminus \mathcal{L}^+$. A gap G is *countable* (*finite, uncountable*) if $G \cap \mathbb{S}$ is countable and infinite (finite, uncountable). Uncountable gaps are called *Fatou* gaps. For a closed convex set $H \subset \mathbb{C}$, maximal straight segments in $\mathrm{Bd}(H)$ are called *edges* of H.

Convergence of prelaminations $\mathcal{L}_i$ to a set of chords $\mathcal{E}$ is understood as Hausdorff convergence of leaves of $\mathcal{L}_i$ to chords from $\mathcal{E}$. Evidently, $\mathcal{E}$ is a lamination. A lamination $\mathcal{L}$ is *nonempty* if it contains nondegenerate leaves, otherwise it is *empty* (denoted $\mathcal{L}_\varnothing$). Say that $\mathcal{L}$ is *countable* if it has countably many nondegenerate leaves and *uncountable* otherwise; $\mathcal{L}$ is *perfect* if it has no isolated leaves (thus, Fatou gaps of perfect laminations have no critical edges).

If $G \subset \overline{\mathbb{D}}$ is the convex hull of $G \cap \mathbb{S}$, define $\sigma_d(G)$ as the convex hull of $\sigma_d(G \cap \mathbb{S})$. *Sibling ($\sigma_d$)-invariant laminations* modify Thurston's [20] invariant geodesic laminations. A *sibling of a leaf* ℓ is a leaf $\ell' \neq \ell$ with $\sigma_d(\ell') = \sigma_d(\ell)$. Call a leaf ℓ^* such that $\sigma_d(\ell^*) = \ell$ a *pullback* of ℓ. Note that the map σ_d can be extended continuously over $\mathcal{L}^+$ by extending linearly over all leaves of $\mathcal{L}$. We also denote this extended map by σ_d.

Definition 3.2 ([3])**.** A (pre)lamination $\mathcal{L}$ is *sibling (σ_d)-invariant* if

(1) for each $\ell \in \mathcal{L}$, we have $\sigma_d(\ell) \in \mathcal{L}$,
(2) for each $\ell \in \mathcal{L}$ there exists $\ell^* \in \mathcal{L}$ with $\sigma_d(\ell^*) = \ell$,
(3) for each $\ell \in \mathcal{L}$ such that $\sigma_d(\ell)$ is a nondegenerate leaf, there exist d **pairwise disjoint** leaves $\ell_1, \dots, \ell_d$ in $\mathcal{L}$ such that $\ell_1 = \ell$ and $\sigma_d(\ell_1) = \dots = \sigma_d(\ell_d)$.

Collections of leaves from (3) above are *full sibling collections*. Their leaves cannot intersect even on $\mathbb{S}$. By cubic (resp., quadratic) laminations, we always mean sibling σ_3-(resp., σ_2-) invariant laminations. When dealing with cubic laminations, we write σ instead of σ_3. From now on $\mathcal{L}$ (possibly with sub- and superscripts) denotes a cubic sibling invariant (pre)lamination.

These are properties of cubic sibling invariant laminations [3]:

gap invariance: if G is a gap of $\mathcal{L}$, then $H = \sigma(G)$ is a leaf of $\mathcal{L}$ (possibly degenerate), or a gap of $\mathcal{L}$, and in the latter case, the map $\sigma|_{\mathrm{Bd}(G)} : \mathrm{Bd}(G) \to \mathrm{Bd}(H)$ is an orientation preserving composition of a monotone map and a covering map (gap invariance is a part of Thurston's original definition [20]);

compactness: if a sequence of sibling invariant prelaminations converges to a set of chords $\mathcal{A}$, then $\mathcal{A}$ is a sibling invariant lamination.

A chord ℓ is *inside* a gap G if ℓ is, except for the endpoints, in the interior of G. A gap G of $\mathcal{L}$ is *critical* if either all edges of G are critical, or there is a critical chord inside G. A *critical set* of $\mathcal{L}$ is a critical leaf or a critical gap. We also define a *lap* of $\mathcal{L}$ as either a finite gap of $\mathcal{L}$ or a nondegenerate leaf of $\mathcal{L}$ not on the boundary of a finite gap.

The following facts are well-known (see, e.g., [2] or [11]). Fatou gaps of σ_d-invariant laminations are (pre)periodic. If U is a σ_d-periodic Fatou gap of period n and the map $\sigma_d^n|_{\mathrm{Bd}(U)}$ is of degree $k > 1$, then there is a monotone map from $\mathrm{Bd}(U)$ to $\mathbb{S}$ collapsing edges of U and semiconjugating $\sigma_d^n|_{\mathrm{Bd}(U)}$ with σ_k. Call Fatou gaps from the cycle of U *periodic Fatou gaps of degree* k. If now $\sigma_d^n|_{\mathrm{Bd}(U)}$ is of degree 1, then the monotone map collapsing edges of U can be chosen to semiconjugate $\sigma_d^n|_{\mathrm{Bd}(U)}$ and an irrational rotation of $\mathbb{S}$. In this case, U is a *Siegel gap*. In any cycle of Siegel gaps some will have critical edges. In general, if ℓ is a critical edge of a Fatou gap, then ℓ is isolated.

Lemma 3.3. *Suppose that $\mathcal{L}_i \to \mathcal{L}$ are σ_d-invariant laminations, and let G be a periodic lap of $\mathcal{L}$. Then G is also a lap of $\mathcal{L}_i$ for all sufficiently large i.*

Proof. Let G be a lap and ℓ an edge of G; we write k for the minimal period of ℓ. Then $\mathcal{L}_i$, for large i, must have a lap G_i with $G_i \to G$. Choose an edge ℓ_i of G_i so that $\ell_i \to \ell$. Then ℓ_i does not cross ℓ for large i as otherwise the leaves $\sigma_d^k(\ell_i)$ and ℓ_i would cross. Moreover, ℓ_i is disjoint from the interior of G for large i as otherwise $\sigma_d^k(\ell_i)$ would intersect the interior of G_i (note that ℓ_i maps farther away from ℓ under σ_d^k). By way of contradiction assume that $\mathcal{L}_i$ do not contain G. Then $G_i \supsetneqq G$ and $\ell_i \neq \ell$ for at least one edge ℓ of G. It follows that $\sigma_d^k(G_i) \supsetneqq G_i$, a contradiction. □

Let us define laminational analogs of the sets $\mathcal{C}_P$.

Definition 3.4. For a given σ_d-invariant $\mathcal{L}$, let $\mathcal{C}(\mathcal{L})$ be the family of all critical portraits, compatible with $\mathcal{L}$; if $\mathcal{K} \in \mathcal{C}(\mathcal{L})$ we say that $\mathcal{K}$ is a critical portrait *of* $\mathcal{L}$.

A lamination $\mathcal{L}$ is *clean* if any pair of distinct non-disjoint leaves of $\mathcal{L}$ is on the boundary of a finite gap. Clean laminations give rise to equivalence relations: $a \sim_{\mathcal{L}} b$ if either $a = b$ or a, b are in the same lap of $\mathcal{L}$. In that case the quotient $\mathbb{S}/\sim_{\mathcal{L}} = J_{\mathcal{L}}$ is called a *topological Julia set* and the map $f_{\mathcal{L}} : J_{\mathcal{L}} \to J_{\mathcal{L}}$, induced by σ_d, is called a *topological polynomial*. By Lemma 3.16 of [8], any clean lamination has the following property: if one endpoint of a leaf is periodic, then the other endpoint is also periodic with the same minimal period. Limits of clean σ_d-invariant laminations are called *limit laminations*, cf. [9] (e.g., clean laminations are limit laminations).

Definition 3.5 (Perfect laminations [11])**.** The maximal perfect subset $\mathcal{L}^p$ of $\mathcal{L}$ is called the *perfect part* of $\mathcal{L}$; a lamination $\mathcal{L}$ is *perfect* if $\mathcal{L} = \mathcal{L}^p$.

Equivalently, one can define $\mathcal{L}^p$ as the set of all leaves $\ell \in \mathcal{L}$ such that arbitrarily close to ℓ there are uncountably many leaves of $\mathcal{L}$. Evidently, perfect laminations are clean and, hence, limit laminations.

Definition 3.6 (Chiefs)**.** If $\mathcal{L}$ is nonempty, a *chief* of $\mathcal{L}$ is defined as a minimal, by inclusion, nonempty sublamination of $\mathcal{L}$.

The next lemma follows from [11].

Lemma 3.7. *The set $\mathcal{L}^p$ is an invariant lamination. If $\mathcal{L}$ is uncountable, then $\mathcal{L}^p \subset \mathcal{L}$ is nonempty. A chief is perfect or countable.*

Proof. By Lemma 3.12 of [11], the set $\mathcal{L}^p$ is an invariant lamination. If $\mathcal{L}$ is uncountable, then by definition $\mathcal{L}^p \subset \mathcal{L}$ is nonempty. If a chief $\mathcal{L}$ is uncountable but not perfect, its perfect part $\mathcal{L}^p \subsetneqq \mathcal{L}$ is a nonempty sublamination, a contradiction with the assumption that $\mathcal{L}$ is a chief. □

Given a chord $\ell = \overline{ab}$, let $|\ell|$ denote the length of the smaller circle arc with endpoints a and b (computed with respect to the Lebesgue measure on $\mathbb{S}$ such that the total length of $\mathbb{S}$ is 1); call $|\ell|$ the *length* of ℓ.

Lemma 3.8. *Any nonempty lamination contains leaves of length $\geqslant \frac{1}{d+1}$.*

Proof. Indeed, for a nondegenerate leaf ℓ so that $|\ell| < \frac{1}{d+1}$, either $|\sigma_d(\ell)| = d|\ell|$ or $|\sigma_d(\ell)| \geqslant \frac{1}{d+1}$. □

Lemma 3.9. *If $\mathcal{L}$ is nonempty, then $\mathcal{L}$ contains a chief.*

Proof. Let $\mathcal{L}_\alpha$ be a nested family of laminations. Definition 3.2 implies that then $\bigcap \mathcal{L}_\alpha$ is a sibling invariant lamination too. If all $\mathcal{L}_\alpha$ are nonempty, then by Lemma 3.8 each of them has a leaf of length at least $\frac{1}{d+1}$ and so $\bigcap \mathcal{L}_\alpha$ is nonempty. Now the desired statement follows from the Zorn lemma. □

The next lemma follows from the definitions and the compactness property of invariant laminations.

Lemma 3.10. *Let $\mathcal{L}$ be a chief. If $\ell \in \mathcal{L}$ is a nondegenerate leaf, then the iterated pullbacks of the nondegenerate iterated images of ℓ are dense in $\mathcal{L}$.*

4. Invariant gaps and prime portraits

An *invariant* gap is an invariant gap of a cubic lamination $\mathcal{L}$, not necessarily specifying $\mathcal{L}$. An infinite invariant gap is *quadratic* if it has degree 2. By Section 3 of [8], any quadratic invariant gap can be obtained as follows. A critical chord $\overline{c}$ gives rise to the complementary circle arc $L(\overline{c})$ of length $2/3$ with the same endpoints as $\overline{c}$. The set $\Pi(\overline{c})$ of all points with orbits in $\overline{L(\overline{c})}$ is nonempty, closed and forward invariant. Let $\Pi'(\overline{c})$ be the maximal perfect subset of $\Pi(\overline{c})$. The convex hulls $G(\overline{c})$ of $\Pi(\overline{c})$ and $G'(\overline{c})$ of $\Pi'(\overline{c})$ are invariant quadratic gaps, and any invariant quadratic gap is of one of these forms. For any invariant gap G, finite or infinite, a *major* of G is an edge $M = \overline{ab}$ of G, for which there is a critical chord $\overline{ax}$ or $\overline{by}$ disjoint from the interior of G. By Section 4.3 of [8], a degree 1 invariant gap has one or two majors; every edge of G eventually maps to a major and if G is infinite and of degree 1, at least one of its majors is critical. An invariant gap G is *rotational* if $\sigma = \sigma_3$ acts on its vertices (i.e., on $G \cap \mathbb{S}$) as a combinatorial rotation. For brevity say that a chord is *compatible* with a finite collection of gaps if it does not cross edges of these gaps.

For a critical chord ℓ, let $I(\ell)$ be the complement of $\overline{L(\ell)}$. Let $\mathcal{K} = \{\overline{c}, \overline{y}\}$ be a critical portrait. Call $\mathcal{K}$ *weak* if the forward orbit of $\overline{c}$ is disjoint from $I(\overline{y})$, or the forward orbit of $\overline{y}$ is disjoint from $I(\overline{c})$; otherwise call $\mathcal{K}$ *strong*.

Lemma 4.1. *The set of strong critical portraits is open and dense in* CrP *while the set of weak critical portraits is closed and nowhere dense in* CrP.

The proof of Lemma 4.1 is left to the reader.

Lemma 4.2. *If U is a degree one periodic infinite gap of σ_d for some $\mathrm{d} \geqslant 2$, then some image of U has critical edges.*

Proof. It is well-known that every edge of U eventually maps to a critical or a periodic leaf. If gaps from the orbit of U have no critical edges, it must have some periodic edges. Let $\ell_1, \dots, \ell_k$ be a maximal chain of concatenated edges of U such that $\sigma_d^n(\ell_i) = \ell_i, 1 \leqslant i \leqslant k$. Then σ_d^n restricted to a small arc $I \subset \mathrm{Bd}(U)$ adjacent to $L = \bigcup_{i=1}^k \ell_i$ repels points away from the appropriate endpoint of L. Since $\sigma_d|_{\mathrm{Bd})U)}$ is of degree 1, then points of I are attracted to a σ_3^n-fixed point $x \in \mathrm{Bd}(U)$. Since σ_3 is expanding on $\mathbb{S}$, this implies that a subarc of $\mathrm{Bd}(U)$ must collapse to a point under σ_d^n. Hence some image of U has critical edges. □

Together with results of [8], Lemma 4.2 implies Lemma 4.3.

Lemma 4.3. *A critical portrait $\{\overline{c}, \overline{y}\}$ is compatible with an invariant quadratic gap if and only if it is weak. Similarly, $\{\overline{c}, \overline{y}\}$ is compatible with an infinite invariant gap if and only if it is weak.*

Proof. Suppose that $\mathcal{K}$ is weak and forward orbit T of $\overline{y}$ is disjoint from $I(\overline{c})$; then $T \subset \Pi(\overline{c})$, and $\{\overline{c}, \overline{y}\}$ is compatible with $G(\overline{c})$. Assume now that $\{\overline{c}, \overline{y}\}$ is compatible with an infinite invariant gap U. If U is quadratic, the claim follows from the above given description. Let U be of degree one and let ℓ be a critical edge of U. Then either ℓ coincides with, say, $\overline{y}$, or $\overline{y}$, $\overline{c}$, and ℓ form a triangle. Thus, we may assume that, say, $\overline{y}$ is non-disjoint from U and $T \subset \mathrm{Bd}(U)$. Since the degree of U is one, $\overline{c}$ is disjoint from the interior of U. Clearly, $U \cap \mathbb{S}$ cannot be contained in $\overline{I(\overline{c})}$ as then there is no room for $\overline{y}$ there; hence $U \cap \mathbb{S} \subset \overline{L(\overline{c})}$, we have $T \cap I(\overline{c}) = \varnothing$, and $\mathcal{K}$ is weak. □

Some laminations must have compatible weak critical portraits.

Theorem 4.4. *Suppose that a nonempty cubic $\mathcal{L}$ has an infinite periodic gap U and either $\sigma(U) = U$ or U shares an edge with a finite rotational lap of $\mathcal{L}$. Then there is a weak critical portrait compatible with $\mathcal{L}$.*

Proof. By Lemma 4.3 we may assume that U shares an edge ℓ with a finite rotational lap of $\mathcal{L}$. Choose a critical chord $\overline{c}$ in a gap from the orbit of U with an endpoint coinciding with an endpoint of the corresponding image of ℓ. Choose a critical chord $\overline{y} \neq \overline{c}$ compatible with $\mathcal{L}$ and not crossing $\overline{c}$. Then $\{\overline{c}, \overline{y}\}$ is compatible with a quadratic invariant gap and, by Lemma 4.3, $\{\overline{c}, \overline{y}\}$ is weak. □

From now on, by a chief, we mean a chief of some ***limit*** *lamination.* A gap G of a lamination is *invariant* if $\sigma(G) = G$ (with "$=$" rather than "$\subset$").

Lemma 4.5. *Any chief has an invariant lap or an invariant infinite gap.*

Proof. By Lemma 3.7 of [15], any clean lamination has a lap or an infinite gap G such that $\sigma(G) = G$. Passing to the limit, we conclude that any limit lamination has the same property (even though the limit of finite invariant gaps may be infinite). A chief of a limit lamination must then also have the above mentioned property. □

Definition 4.6 (Friends, prime critical portraits)**.** Critical portraits $\mathcal{K}_1, \mathcal{K}_2 \in \mathrm{CrP}$ are *friends* (through a lamination $\mathcal{L}$) if $\mathcal{K}_1, \mathcal{K}_2 \in \mathcal{C}(\mathcal{L})$. A critical portrait $\mathcal{K}$ is *prime* if a friend of $\mathcal{K}$ has a weak friend. A lamination is *prime* if it has a prime critical portrait.

The next lemma follows from definitions and compactness of the family of all cubic sibling invariant laminations.

Lemma 4.7. *If $\mathcal{K}_1$, $\mathcal{K}_2$ are friends, then they are compatible with some chief $\mathcal{L}$ (thus, $\mathcal{K}_1, \mathcal{K}_2 \in \mathcal{C}(\mathcal{L})$). Friendship is a closed relation: if $\mathcal{K}_i \to \mathcal{K}$ and $\mathcal{K}'_i \to \mathcal{K}'$ and $\mathcal{K}_i$ and $\mathcal{K}'_i$ are friends for all i, then so are $\mathcal{K}$ and $\mathcal{K}'$.*

Proof. The former claim follows from definitions. To prove the latter claim, choose laminations $\mathcal{L}_i$ such that $\mathcal{K}_i, \mathcal{K}'_i \in \mathcal{C}(\mathcal{L}_i)$. Passing to a subsequence, arrange that $\mathcal{L}_i \to \mathcal{L}$. It follows that $\mathcal{K}$, $\mathcal{K}' \in \mathcal{C}(\mathcal{L})$ and, hence, $\mathcal{K}$ and $\mathcal{K}'$ are friends. Observe that $\mathcal{L}$ is non-empty because by Lemma 3.8 each lamination $\mathcal{L}_i$ contains a leaf of length at least $\frac{1}{d+1}$. □

Given $\mathcal{L}$ and a nondegenerate leaf $\ell \in \mathcal{L}$, let $\mathcal{G}(\ell)$ be the set of iterated pullbacks of the nondegenerate iterated images of ℓ.

Lemma 4.8. *Let $\mathcal{L}$ be a countable chief. Then:*

1. *for any nondegenerate leaf $\ell \in \mathcal{L}$, the set of all nondegenerate leaves in $\mathcal{L}$ coincides with $\mathcal{G}(\ell)$;*
2. *all nondegenerate leaves of $\mathcal{L}$ are isolated;*
3. *there is a weak critical portrait in $\mathcal{C}(\mathcal{L})$.*

Proof. (1) Choose an isolated leaf $\ell_0 \in \mathcal{L}$. We claim that $\mathcal{G}(\ell_0)$ is the set of all nondegenerate leaves of $\mathcal{L}$. Indeed, otherwise choose a nondegenerate leaf $\ell \in \mathcal{L} \setminus \mathcal{G}(\ell_0)$; then leaves of $\mathcal{G}(\ell)$ cannot approximate ℓ_0 or coincide with ℓ_0, a contradiction with Lemma 3.10. If now ℓ is a nondegenerate leaf of $\mathcal{L}$, then $\ell \in \mathcal{G}(\ell_0)$, and hence $\mathcal{G}(\ell) = \mathcal{G}(\ell_0)$ is the set of all nondegenerate leaves of $\mathcal{L}$, as desired.

(2) All non-isolated leaves in $\mathcal{L}$ form a forward invariant closed family of leaves. If ℓ is non-isolated, choose leaves $\ell_i \to \ell$, choose their pullbacks $\overline{q}_i$, and choose a converging subsequence of these pullbacks; in the end we will find a non-isolated leaf $\overline{q}$ with $\sigma(\overline{q}) = \ell$. Now, let ℓ be non-isolated and non-critical. Choose a sequence $\ell_i \to \ell$ so that each ℓ_i is not on the boundary of a critical polygon. Then the siblings ℓ_i', ℓ_i'' of ℓ_i are well defined, and $\ell_i' \to \ell'$ while $\ell_i'' \to \ell''$. Clearly, $\sigma(\ell) = \sigma(\ell') = \sigma(\ell'')$. We claim that ℓ, ℓ' and ℓ'' are pairwise disjoint. Indeed, if, say, $\ell = \overline{ab}$ and $\ell' = \overline{bc}$, where $\sigma(c) = \sigma(a)$, then ℓ_i and ℓ_i' have *distinct* endpoints close to b and mapping to the same point; a contradiction. Hence by definition the set of all non-isolated leaves of $\mathcal{L}$ is itself a sibling-invariant lamination, a contradiction with $\mathcal{L}$ being a chief.

(3) By Lemma 4.5, we can find an invariant lap or infinite gap T of $\mathcal{L}$. If T is infinite, our claim follows from Theorem 4.4. Hence we may assume that T is finite. Let ℓ be an edge of T; it is isolated by (2). Let H be a gap of $\mathcal{L}$ attached to T along ℓ. If H is infinite, the desired statement follows from Theorem 4.4. Assume that H is finite. If n is the minimal period of edges of T, then there are two cases: $\sigma^n(H) = H$ and $\sigma^n(H) = \ell$. The former case contradicts (1), hence $\sigma^n(H) = \ell$, and we may assume that $\sigma(H) = \sigma(\ell)$. Choose a critical chord $\overline{y} \subset H$ that shares an endpoint with ℓ, and a critical chord $\overline{c}$ in a critical gap or leaf of $\mathcal{L}$ disjoint from H. (If H has degree 3, then simply take a critical portrait $\{\overline{c}, \overline{y}\}$ in H.) The critical portrait $\{\overline{c}, \overline{y}\}$ is compatible with $G(\overline{c})$, hence weak by Lemma 4.3, as desired. □

The next definition complements Definition 4.6.

Definition 4.9 (Regular critical portraits, laminations, and chiefs)**.** A chief is *regular* if all its critical portraits have only strong friends. A lamination is *regular* if it has a regular chief. A critical portrait is *regular* if it is compatible with a regular chief.

Regular chiefs have nice properties.

Lemma 4.10. *Suppose that $\mathcal{L}$ is a regular chief. Then $\mathcal{L}$ is perfect. Also, $\mathcal{C}(\mathcal{L})$ is disjoint from any set $\mathcal{C}(\mathcal{L}')$ where $\mathcal{L}' \neq \mathcal{L}$ is a regular chief.*

Proof. By Lemma 4.8, the lamination $\mathcal{L}$ is uncountable; hence by Lemma 3.7 it is perfect. Let $\mathcal{K} = \{\overline{c}, \overline{y}\}$ be a critical portrait compatible with $\mathcal{L}$ and a chief $\mathcal{L}' \neq \mathcal{L}$. Since $\mathcal{K}$ is strong and has only strong friends, by Lemma 4.3 invariant sets of $\mathcal{L}$ and $\mathcal{L}'$ are finite, and $\mathcal{L}'$ is perfect by Lemma 4.8.

Let G and G' be invariant laps of $\mathcal{L}$ and $\mathcal{L}'$, resp., located in the same component of $\overline{\mathbb{D}} \setminus \bigcup \mathcal{K}$. No leaves of $\mathcal{L}$ intersect the interior of G' since otherwise uncountably many leaves of $\mathcal{L}$ would intersect edges of G' contradicting the fact that $\mathcal{K}$ is compatible with $\mathcal{L}$ and $\mathcal{L}'$, and [11, Lemma 3.53]. Therefore, $G' \subset G$. Similarly, $G \subset G'$, hence $G = G'$.

If iterated images of $\overline{c}$ and $\overline{y}$ avoid G, then iterated $\mathcal{L}$-pullbacks of G and iterated $\mathcal{L}'$-pullbacks of G' are the same. Hence $\mathcal{L} = \mathcal{L}'$ since the iterated pullbacks of G are dense in both $\mathcal{L}$ and $\mathcal{L}'$ by Lemma 3.10. Let for some minimal $n \geqslant 0$ the point $\sigma_3^n(\overline{c})$ be a vertex of G. Let C, C' be the critical sets of $\mathcal{L}$, resp., $\mathcal{L}'$ containing $\overline{c}$. Since infinite gaps of $\mathcal{L}$ and $\mathcal{L}'$ are disjoint from G by Theorem 4.4, the sets C, C' are laps. Since $\mathcal{L}$ and $\mathcal{L}'$ are compatible and perfect, $C = C'$ by [11, Theorem 3.57]. Similarly, we see that either $\overline{y}$ never maps to G or the critical sets of $\mathcal{L}$, $\mathcal{L}'$ containing $\overline{y}$ coincide. Thus, pullbacks of G in $\mathcal{L}$ are the same as pullbacks of G in $\mathcal{L}'$, and $\mathcal{L} = \mathcal{L}'$. □

Let us establish a few useful facts concerning regular chiefs.

Lemma 4.11. *Suppose that $\mathcal{L}$ is a regular chief. Then $\mathcal{L}$ has infinitely many periodic laps. Moreover, for any periodic leaf ℓ of $\mathcal{L}$ the family of its pullbacks is dense in $\mathcal{L}$.*

Proof. Since $\mathcal{L}$ is perfect, there are uncountably many grand orbits of non-degenerate leaves of $\mathcal{L}$ none of which contains a leaf from a critical set of $\mathcal{L}$. Choose a leaf ℓ from one of these grand orbits. Since $\mathcal{L}$ is perfect, it is clean. Consider the associated topological polynomial $f_{\mathcal{L}} : J_{\mathcal{L}} \to J_{\mathcal{L}}$ to which $\sigma_3|_{\mathbb{S}}$ semiconjugate by a map, say, φ; then $\varphi(\ell) = x$ is a cutpoint of $J_{\mathcal{L}}$ such that all points of the $f_\sim$-orbit of x are cutpoints of $J_{\mathcal{L}}$. Such dynamics was studied in [6] where, in Theorem 3.8, it was proven that then $f_{\mathcal{L}}$ has infinitely many periodic cutpoints. Taking their φ-preimages, we see that $\mathcal{L}$ has infinitely many periodic laps. Now, take a periodic leaf ℓ of $\mathcal{L}$, consider its grand orbit and then its closure. By the compactness of the family of laminations, this grand orbit is dense in $\mathcal{L}$ as desired. □

Corollary 4.12 follows from definitions and Lemma 4.10.

Corollary 4.12. *A friend of a regular critical portrait is regular. All critical portraits of a regular chief $\mathcal{L}$ form a closed subset $\mathcal{C}(\mathcal{L})$ of* CrP *consisting of friends, and no other critical portrait can be their friend. A regular lamination has a unique chief that is regular too.*

One can define regular critical portraits through the concept of a friend.

Lemma 4.13. *A critical portrait $\mathcal{K}$ is regular if and only if all friends of friends of $\mathcal{K}$ are strong (i.e., if $\mathcal{K}$ is not prime).*

Proof. Let $\mathcal{K}$ be a regular critical portrait. Then there is a unique regular chief $\mathcal{L}$ compatible with $\mathcal{K}$. All critical portraits of $\mathcal{L}$ are strong and have only strong friends; by Lemma 4.10, none of them is compatible with a chief $\mathcal{L}' \neq \mathcal{L}$. Hence all friends of $\mathcal{K}$ are compatible with $\mathcal{L}$, i.e. are regular. Repeating this, we see that friends of friends of $\mathcal{K}$ are compatible with $\mathcal{L}$ and, hence, strong. On the other hand, suppose that all friends of friends of a critical portrait $\mathcal{K}$ are strong. Take a chief $\mathcal{L}$ compatible with $\mathcal{K}$. Then all its critical portraits have only strong friends. By definition $\mathcal{L}$ is regular which implies that $\mathcal{K}$ is regular as desired. □

The terminology is partially self-evident and partially explained by the fact that if $\mathcal{L}$ is a dendritic regular chief then $J_\mathcal{L}$ is a dendrite.

Lemma 4.14. *A limit of prime critical portraits is prime.*

Proof. If $\mathcal{K}_i$ are prime and $\mathcal{K}_i \to \mathcal{K}$, then, by definition, some friends $\mathcal{K}'_i$ of $\mathcal{K}_i$ have weak friends $\mathcal{K}''_i$. Passing to a subsequence we can arrange that $\mathcal{K}'_i \to \mathcal{K}'$ and $\mathcal{K}''_i \to \mathcal{K}''$. By Lemma 4.7, the portrait $\mathcal{K}'$ is a friend of $\mathcal{K}$, and $\mathcal{K}''$ is a friend of $\mathcal{K}'$. By Lemma 4.1, the portrait $\mathcal{K}''$ is weak. By definition, $\mathcal{K}$ is prime. □

Let us define alliances.

Definition 4.15. The *prime alliance* $\mathcal{A}_0$ is the set of all prime critical portraits. A *regular alliance* is the set $\mathcal{C}(\mathcal{L})$ where $\mathcal{L}$ is a regular chief.

We are ready to prove a part of the Main Theorem that can be viewed as its combinatorial analog.

Lemma 4.16. *Alliances form a USC-partition of the set* CrP. *The union of regular alliances is open and dense in* CrP.

Proof. That alliances form a partition of CrP follows from definitions and Lemma 4.10. Hence if two critical portraits are friends, then they belong to the same alliance. Suppose that $\mathcal{K}_i \to \mathcal{K}$ and $\mathcal{K}'_i \to \mathcal{K}'$ where $\mathcal{K}_i, \mathcal{K}'_i$ are friends. We may assume that either all $\mathcal{K}_i, \mathcal{K}'_i$ are prime, or all $\mathcal{K}_i, \mathcal{K}'_i$ are strong. In the former case Lemma 4.14 implies that both $\mathcal{K}$ and $\mathcal{K}'$ are prime, and we are done. In the latter case, $\mathcal{K}$ and $\mathcal{K}'$ are friends by Lemma 4.7, and we are done too. This proves the first claim.

To prove the second claim, observe that the union of all regular alliances is open. Let $\mathcal{K} = \{\overline{c}, \overline{y}\}$ be a critical portrait such that the orbits of $\sigma(\overline{c})$ and $\sigma(\overline{y})$ are dense in $\mathbb{S}$. Such portraits are dense in CrP. We prove that $\mathcal{K}$ is regular by proving that, for any friend $\mathcal{K}' = \{\overline{c}', \overline{y}'\}$ of $\mathcal{K}$, the orbits of $\sigma(\overline{c}')$ and $\sigma(\overline{y}')$ are dense in $\mathbb{S}$.

Let $\mathcal{L}$ be a chief compatible with $\mathcal{K}$ and $\mathcal{K}'$. Let C be the leaf $\overline{c}$ if $\overline{c} \in \mathcal{L}$ or the critical gap of $\mathcal{L}$ containing $\overline{c}$ otherwise. Define Y similarly. Arrange that $\overline{c}' \subset C$ and $\overline{y}' \subset Y$, possibly renaming $\overline{c}'$ and $\overline{y}'$. We claim that the orbit of $\sigma(\overline{c}')$ is dense in $\mathbb{S}$. Otherwise consider the nondegenerate chord $\overline{q} = \overline{xx'}$, where $x = \sigma(\overline{c})$ and $x' = \sigma(\overline{c}')$. There is $\varepsilon > 0$ and an arc $I \subset \mathbb{S}$ such that $\sigma^n(x')$ is never ε-close to I. On the other hand, iterated images of x are dense in I; the corresponding images of $\overline{q}$ have length $\geqslant \varepsilon$. Therefore, all leaves of $\mathcal{L}$ originating in I have length ε or more.

Note that C and Y are not periodic, therefore, no σ-periodic point of $\mathbb{S}$ is an eventual image of x or x'. There is a positive integer N with $\sigma^N(I) = \mathbb{S}$. Since any σ-periodic point a of $\mathbb{S}$ has a σ^N-preimage in I, we have $\overline{ab} \in \mathcal{L}$ for some $b \neq a$. Since $\mathcal{L}$ is perfect and hence clean, endpoints of periodic leaves must have the same period. Thus, the horizontal diameter $\overline{\mathrm{Di}}$ connecting the two σ-fixed points of $\mathbb{S}$ is a leaf of $\mathcal{L}$. Consider a nondegenerate chord ℓ with endpoint $i = e^{2\pi i(1/4)}$. Then $\sigma^n(\ell)$ crosses $\overline{c}$ or $\overline{y}$ for some $n \geqslant 0$. Thus $\ell \notin \mathcal{L}$, a contradiction. We conclude that the orbit of $\sigma(\overline{c}')$ is dense. Similarly, the orbit of $\sigma(\overline{y}')$ is dense. □

5. THE MODEL

Let P be a polynomial with $[P] \in \mathcal{M}_3$. Let $\sim_P$ be the equivalence relation on $\mathbb{S}$ defined by $e^{2\pi i\alpha} \sim_P e^{2\pi i\beta}$ if $\{\alpha, \beta\} \in \mathcal{Z}_P$ or $\alpha = \beta$. Let $\mathcal{Z}_P^{lam}$ (from "lamination") be the set of all edges of the convex hulls in $\overline{\mathbb{D}}$ of all $\sim_P$-classes. Define $\mathcal{L}_P^r$ as the set of all edges of the convex hulls of all $\sim_P$-classes and the limits of these edges. By the compactness of laminations (see Section 3), $\mathcal{L}_P^r$ is a clean cubic lamination (cf. [3]).

The lamination $\mathcal{L}_P^r$ is associated with an equivalence relation $\sim_{\mathcal{L}_P^r}$ so that all laps of $\mathcal{L}_P^r$ are convex hulls of $\sim_{\mathcal{L}_P^r}$-classes (see discussion in Section 3 right after Definition 3.4). It is easy to see that $\sim_{\mathcal{L}_P^r}$ is the closure of $\sim_P$, so in what follows we simply denote it by $\sim_P$.

Suppose that P has no neutral periodic points. Then $\mathcal{Z}_P^{lam}$ coincides with the *rational* lamination [16] of P while $\mathcal{L}_P^r$ coincides with the *real* lamination [18] of P. The next lemma summarizes some results of [16, 18].

Lemma 5.1. *Suppose that P has no neutral cycles. Then the restriction $P|_{J_P}$ is monotonically semiconjugate to the topological polynomial $f_{\sim_P} : J_{\sim_P} \to J_{\sim_P}$ on its topological Julia set so that fibers of this semiconjugacy are trivial at all (pre)periodic points of $J_{\sim_P}$ (thus, for a periodic lap G of $\mathcal{L}_P^r$, external rays corresponding to vertices of G land at the same legal point). Also, any clean lamination without infinite periodic degree 1 gaps has the form $\mathcal{L}_P^r$ for some P.*

If $\mathcal{L}_P^r$ is regular, then by Corollary 4.12 it has a unique regular chief denoted by $\mathcal{L}_P^c$; set $\mathcal{A}_P = \mathcal{C}(\mathcal{L}_P^c)$ (observe that then $\mathcal{A}_P$ is a regular alliance). Equivalently, $\mathcal{A}_P$ can be defined as the set of all friends of critical portraits from $\mathcal{C}_P$. However, if $\mathcal{L}^r_P$ is prime (e.g., if P is invisible and, hence, $\mathcal{L}^r_P$ is empty) set $\mathcal{A}_P = \mathcal{A}_0$ to be the prime alliance. The prime alliance is special as it serves all invisible polynomials, however diverse they are. It also serves all polynomials with non-repelling fixed points and some other polynomials. By Section 4, the prime alliance is closed topologically and under friendship. This defines $\mathcal{A}_P$ for any polynomial P with $[P] \in \mathcal{M}_3$.

Lemma 5.2. *A regular alliance $\mathcal{A}_P$ has the form $\mathcal{C}_{P_0}$ for some visible polynomial P_0, possibly different from P.*

Proof. Let $\mathcal{L}$ be a chief of $\mathcal{L}^r_P$. Then it has no infinite periodic gaps of degree 1. Indeed, otherwise by Lemma 4.2 it has an infinite gap with a critical edge. By properties of laminations this edge is isolated, a contradiction with $\mathcal{L}$ being perfect by Lemma 4.10. By Lemma 5.1, there is a polynomial P_0 without neutral periodic points such that $\mathcal{L}^r_{P_0} = \mathcal{L}$. Then by Lemma 4.10 $\mathcal{A}_P = \mathcal{C}(\mathcal{L}) = \mathcal{C}_{P_0}$ as desired. □

A regular alliance is closed topologically (because $\mathcal{C}_{P_0}$ is closed) and under friendship (by Corollary 4.12).

Lemma 5.3. *For any visible P we have $\mathcal{C}_P \subset \mathcal{A}_P$.*

Proof. If $\mathcal{L}^r_P$ is regular, $\mathcal{C}_P \subset \mathcal{A}_P$ by definition. If $\mathcal{L}^r_P$ is prime, then $\mathcal{A}_P = \mathcal{A}_0$ is prime, and $\mathcal{C}_P \subset \mathcal{A}_P = \mathcal{A}_0$ since $\mathcal{A}_0$ is closed under friendship. □

To prove the Main Theorem we need a couple of new concepts. Let P be such that $[P] \in \mathcal{M}_3$. A point x is *(P-)stable* if its forward orbit is finite and contains no critical or non-repelling periodic points. The next lemma shows how stable points can be applied.

Lemma 5.4 ([14], cf. Lemma B.1 [15])**.** *Let g be a polynomial, and z be a stable point of g. If an external ray $R_g(\theta)$ with rational argument θ lands at z, then, for every polynomial $\tilde{g}$ sufficiently close to g, the ray $R_{\tilde{g}}(\theta)$ lands at a stable point $\tilde{z}$ close to z. Moreover, $\tilde{z}$ depends holomorphically on $\tilde{g}$.*

An unordered pair of rational angles $\{\alpha, \beta\}$ is *(P-)stable* if the external rays with arguments α and β land at the same stable point of P. Write $\mathcal{S}_P$ for the set of all P-stable pairs of angles; call $\mathcal{S}_P$ the *s-set* of P. Denote by $\mathcal{S}^{lam}_P$ the set of chords connecting $e^{2\pi i\alpha}$ with $e^{2\pi i\beta}$, where the pair $\{\alpha, \beta\} \in \mathcal{S}_P$ is not separated in $\mathbb{R}/\mathbb{Z}$ by any other P-stable pair of angles. Evidently, $\mathcal{S}_P \subset \mathcal{Z}_P$ and $\mathcal{S}^{lam}_P \subset \mathcal{Z}^{lam}_P$; these sets do not have to coincide as some legal points may be non-stable because their orbits pass through critical points before they map to repelling periodic points.

If $\mathcal{S}_P \neq \varnothing$, let $\mathcal{C}^s_P$ be the set of all critical portraits *compatible* with $\mathcal{S}_P$ (i.e., no critical chord from a critical portrait in $\mathcal{C}^s_P$ crosses a leaf from $\mathcal{S}^{lam}_P$). It follows that $\mathcal{C}_P \subset \mathcal{C}^s_P$.

Proposition 5.5. *The dependence $P \mapsto \mathcal{C}^s_P$ is upper semi-continuous.*

Proof. We prove that if $P_i \to P$ and we choose $\mathcal{K}_i \in \mathcal{C}^s_{P_i}$ with $\mathcal{K}_i \to \mathcal{K}$, then $\mathcal{K} \in \mathcal{C}^s_P$. Assume the contrary: $\mathcal{K} = \{\overline{c}, \overline{y}\}$, where $\overline{c}$ crosses some $\ell = \overline{ab} \in \mathcal{S}^{lam}_P$. By Lemma 5.4, $a \sim_{P_i} b$ for large i, and $\mathcal{K}_i$ contains a critical chord $\overline{c}_i$ close to $\overline{c}$, a contradiction, since then $\overline{c}_i$ also crosses ℓ. □

The next lemma relates $\mathcal{C}^s_P$ and the alliance $\mathcal{A}_P$.

Lemma 5.6. *Take $[P] \in \mathcal{M}_3$. If $\mathcal{L}^r_P$ is regular, then $\mathcal{C}^s_P \subset \mathcal{A}_P$.*

Proof. Let $\mathcal{L}$ be the chief of $\mathcal{L}^r_P$. By Lemma 4.10, it is perfect. Now, let $\mathcal{K}$ be a critical portrait from $\mathcal{C}^s_P$ (i.e., compatible with $\mathcal{S}_P$). We claim that then $\mathcal{K}$ is compatible with $\mathcal{L}$. Indeed, otherwise a critical leaf $\overline{c} \in \mathcal{K}$ crosses a leaf $\ell \in \mathcal{L}$. By Lemma 4.11, arbitrarily Hausdorff-close to ℓ, there are iterated preimages of a periodic lap of $\mathcal{L}$ that is not an eventual image of a critical lap of $\mathcal{L}$; it follows that edges of these preimages are leaves from $\mathcal{S}^{lam}_P$, a contradiction with the assumption that $\mathcal{K} \subset \mathcal{C}^s_P$. Thus, $\mathcal{C}^s_P \subset \mathcal{A}_P$. □

We are ready to prove Theorem 5.7 which implies the Main Theorem.

Theorem 5.7. *The map $P \mapsto \mathcal{A}_P$ from $\mathcal{M}_3$ to the quotient space of* CrP *generated by alliances is continuous.*

For a critical portrait $\mathcal{K}$, consider the corresponding Thurston pullback lamination $\mathcal{T}(\mathcal{K})$ (see [20]).

Lemma 5.8. *Let G be a periodic lap of some invariant lamination, whose iterated images are disjoint from $\bigcup \mathcal{K}$. Then leaves of $\mathcal{T}(\mathcal{K})$ cannot cross edges of G.*

This is a straightforward corollary of [20].

Proof of Theorem 5.7. Consider a sequence $P_i \to P$ of polynomials, and set $\mathcal{A}_i = \mathcal{A}_{P_i}$. We need to show that sets $\mathcal{A}_i$ converge into $\mathcal{A}_P$. To this end it suffices to consider critical portraits $\mathcal{K}_i \to \mathcal{K}$, where $\mathcal{K}_i \in \mathcal{A}_i$, and show that $\mathcal{K} \in \mathcal{A}_P$. We may assume that either all $\mathcal{K}_i$ are prime, or all $\mathcal{K}_i$ are regular. If $\mathcal{K}_i$ are prime, then $\mathcal{K}$ is prime by Lemma 4.14. Assume that all $\mathcal{K}_i$ are regular. By Proposition 5.5, we have $\mathcal{K} \in \mathcal{C}^s_P$. If $\mathcal{L}^r_P$ is regular, then by Lemma 5.6 $\mathcal{K} \in \mathcal{A}_P$, and we are done.

Assume that $\mathcal{L}^r_P$ is prime. We need to show that $\mathcal{K}$ is prime. Assume that $\mathcal{K}$ is regular. Then $\mathcal{K}$ is compatible with a regular chief $\mathcal{L}^\circ$. By Lemma 4.11 there are infinitely many periodic laps of $\mathcal{L}^\circ$ whose first return is onto. Let G be one of them, of period n, so that $\sigma^n(G) = G$. We may assume that the (finite) orbit of G is disjoint from the chords in $\mathcal{K}$. Then the orbit of G is disjoint from the chords in $\mathcal{K}_i$ with large i.

Set $\mathcal{L}_i = \mathcal{T}(\mathcal{K}_i)$. By Lemma 5.8, the lap G is contained in a lap G_i of $\mathcal{L}_i$. Since G is periodic, so is G_i. The first return map of $\mathrm{Bd}(G_i)$ has degree 1 as any iterated image of G_i is contained in a complementary component of $\bigcup \mathcal{K}$. Hence all periodic vertices of G_i are of the same period. In particular, it implies that if G' and G'' are periodic laps of $\mathcal{L}^\circ$ such that periods of their vertices are distinct, then the gap G'_i cannot contain G''_i, and vice versa.

It is easy to see that an infinite periodic gap of degree 1 must have at least one (pre-)critical edge, each critical edge serving at most two such gaps. Thus, there may exist at most six cycles of infinite periodic degree 1 gaps of a cubic lamination. Choose a periodic lap G of $\mathcal{L}^\circ$. Passing to a subsequence, we may assume that all G_i are finite, or all are infinite. In the latter case, we *reject* G and replace it with another lap G' whose vertex period is greater than that of G. By the above, there will be at most six rejected cycles of periodic gaps. Refine our sequence of polynomials and choose a sequence of finite periodic laps $G^1, \dots, G^j, \dots$ of $\mathcal{L}^\circ$ such that periods of their vertices grow and G^j_i are finite for all i and j. If $\mathcal{L}^\circ$ has Fatou gaps, we may assume that all G^j_i are disjoint from them.

Set $G = G^j$. Since $\mathcal{K}_i$ is compatible with $\mathcal{L}^r_{P_i}$, the lamination $\mathcal{L}^r_{P_i}$ consists of leaves that do not cross leaves of $\mathcal{L}_i$. Indeed, if a leaf $\ell' \in \mathcal{L}^r_{P_i}$ crosses a leaf $\ell \in \mathcal{L}_i$, then we may assume that ℓ is a pullback of a leaf of $\mathcal{K}_i$. Then, as above with G, the crossing of the two leaves is kept until ℓ maps to a leaf of $\mathcal{K}_i$ which forces the corresponding image of ℓ' to cross the same leaf of $\mathcal{K}_i$, a contradiction. Hence leaves of $\mathcal{L}^r_{P_i}$ do not cross leaves of $\mathcal{L}_i$.

All G_i have the same period n. By Kiwi [17], vertices (edges) of gaps from the orbit of any finite periodic gap of a cubic lamination form one or two orbits. Hence either $G = G_i$, or $G \subsetneqq G_i$ in which case G_i has two orbits of edges. Since edges of G_i do not cross leaves of $\mathcal{L}^r_{P_i}$ then there are at most finitely many leaves of $\mathcal{L}^r_{P_i}$ intersecting the interior of G; all these leaves are diagonals of G_i. It follows that here are two cases listed below.

(1) A finite period n lap H_i of $\mathcal{L}^r_{P_i}$ is non-disjoint from the interior of G. Then H_i is contained in G_i (indeed, by construction the edges of G_i are approximated by distinct pullbacks of leaves of $\mathcal{K}_i$ which are disjoint from leaves of $\mathcal{L}^r_{P_i}$).

(2) An infinite period $\leqslant n$ gap U_i of $\mathcal{L}^r_{P_i}$ contains $G_i \supset G$.

Accordingly, consider two cases.

(a) There are infinitely many gaps G^j for which case (1) above holds. We can find any number of distinct periodic laps which are shared by all laminations $\mathcal{L}^r_{P_i}$. Let H be one of them chosen so that all P-external rays whose arguments are vertices of H land at repelling periodic points. We claim that in fact all P-external rays whose arguments are vertices of H land at the same point. Indeed, suppose otherwise. Then by Lemma 5.4 there are

P_i-external rays with vertices of H as arguments that land at distinct points, a contradiction. Thus, H is (non-strictly) contained in a finite lap H' of $\mathcal{L}^r_P$. We conclude that $\mathcal{L}^r_P$ has infinitely many finite periodic laps.

Thus, $\mathcal{L}^r_P \neq \varnothing$, and we can always choose a periodic lap H' of $\mathcal{L}^r_P$ so that H' corresponds to a point z' which is not an eventual image of a critical point of P. It follows that the entire P-grand orbit of z' consists of stable points. Consider the closure $\mathcal{L}'$ of the grand orbit of H'. Evidently, $\mathcal{L}'$ is a lamination, and $\mathcal{L}' \subset \mathcal{L}^r_P$. If $\mathcal{L}'$ is incompatible with $\mathcal{K}$, an iterated pullback ℓ of an edge of H' crosses a chord in $\mathcal{K}$. By Lemma 5.4, there are leaves of $\mathcal{L}^r_{P_i}$ that converge to ℓ as $i \to \infty$. Since $\mathcal{K}_i \to \mathcal{K}$, for large i some leaves of $\mathcal{L}^r_{P_i}$ cross chords in $\mathcal{K}_i$, a contradiction. Hence $\mathcal{L}'$ is compatible with $\mathcal{K}$. However, since $\mathcal{K}$ is regular, it implies that $\mathcal{L}'$ and $\mathcal{L}^r_P \supset \mathcal{L}'$ are regular, a contradiction.

(b) For all but finitely many gaps G^j and all but finitely many i case (2) above holds and a periodic Fatou gap $U^j_i \supset G^j$ of period at most n_j exists. Moreover, since there are infinitely many gaps G^j, Fatou gaps in question will contain infinitely many gaps G^j. A priori, as $i \to \infty$, these gaps may change. We may assume that $\mathcal{L}^r_{P_i} \to \mathcal{L}''$. Since $\mathcal{K}_i \to \mathcal{K}$ and $\mathcal{K}_i$'s are compatible with $\mathcal{L}^r_{P_i}$'s, it follows that $\mathcal{L}''$ is compatible with $\mathcal{K}$. Hence $\mathcal{L}^\circ \subset \mathcal{L}''$. In particular, the limit U^j of the gaps U^j_i must be contained in a Fatou gap of $\mathcal{L}^\circ$ while containing G^j. However no gap G^j can be contained in a Fatou gap of $\mathcal{L}^\circ$, a contradiction. □

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