A MODEL OF THE CUBIC CONNECTEDNESS LOCUS

ALEXANDER BLOKH, LEX OVERSTEEGEN, AND VLADLEN TIMORIN

ABSTRACT. For the cubic connectedness locus, we define a combinatorial upper semi-continuous partition. This can be regarded as a combinatorial model of the cubic connectedness locus.

1. INTRODUCTION

The parameter space of complex degree $d$ polynomials $P$ is the space of affine conjugacy classes $[P]$ of these polynomials. The connectedness locus $\mathcal{M}_d$ consists of classes of all degree $d$ polynomials $P$, whose Julia sets $J_P$ are connected. The Mandelbrot set $\mathcal{M}_2$ has a complicated self-similar structure understood through the “pinched disk” model [8, 9, 14].

In this paper, we find a combinatorially defined upper semi-continuous (USC) partition of $\mathcal{M}_3$. A property of a polynomial is combinatorial if it can be stated based only upon knowing which pairs of rational external rays land at the same point and which pairs do not. A combinatorial USC partition of $\mathcal{M}_3$ yields a continuous map of $\mathcal{M}_3$ to a quotient space of $\mathbb{C}rP$, the space of unordered cubic critical portraits. Let us describe our approach.

Let a cubic polynomial $P$ have a connected Julia set. A point $x$ is $(P)$-stable if its forward orbit is finite and contains no critical or non-repelling periodic points. An unordered pair of rational angles $\{\alpha, \beta\}$ is $(P)$-stable if the external rays with arguments $\alpha$ and $\beta$ land at the same stable point of $P$. Write $S_P$ for the set of all $P$-stable pairs of angles; call $S_P$ the $s$-set of $P$. A cubic polynomial $P \in \mathcal{M}_3$ is visible if $S_P \neq \emptyset$ and invisible otherwise. If $P$ is visible, denote by $C_P$ the set of all critical portraits compatible with $S_P$ (i.e., no critical chord from a critical portrait in $C_P$ separates a $P$-stable pair of angles). The set $C_P$ is the combinatorial counterpart of $P$. We will show that $C_P$ depends upper semi-continuously on $P$.

We now want to define, for every $[P] \in \mathcal{M}_3$, a closed subset $A_P$ of $\mathbb{C}rP$ called an alliance. The main properties of alliances are:

1. if $P$ is visible, then $C_P \subset A_P$;

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(2) distinct alliances are disjoint;
(3) the alliances form an USC partition of $\text{CrP}$.

One special alliance is said to be prime. It contains $C_P$ for all visible polynomials $P$ with a non-repelling fixed point and some other combinatorial counterparts. Also, we associate the prime alliance with all invisible polynomials $P$. All other alliances are called regular; they are combinatorial counterparts of certain polynomials, and there are infinitely many of them.

**Main Theorem.** The sets $A_P$ form an USC partition $\{A_P\}$ of $\text{CrP}$. The map $P \mapsto A_P$ is continuous and maps $M_3$ to the quotient space $\text{CrP}/\{A_P\}$.

We will show that the union of regular alliances is open and dense in $\text{CrP}$. Even though the prime alliance is special (as it has to represent all invisible polynomials), it is small in the sense of Baire. A more detailed combinatorial study of $M_3$ will characterize combinatorial counterparts and the ways they can intersect. Thurston [14] gave a similar description of $M_2$ in terms of the Quadratic Minor Lamination $\text{QML}$. Combinatorial counterparts for $M_2$ correspond to certain gaps or leaves of $\text{QML}$. We have to pass from combinatorial counterparts to alliances in order to have a partition of $M_2$. Alliances for $M_2$ correspond to maximal baby Mandelbrot sets or single non-renormalizable quadratic polynomials, except for the central alliance, which may be called prime and which covers countable concatenations of hyperbolic components growing from the main cardioid. Our Main Theorem gives a cubic analog of this partition. On the other hand, the central part of $M_3$ is more complicated than that of $M_2$: it is not locally connected [13] and is highly nontrivial combinatorially [3, 6].

2. **Critical portraits and laminations**

We assume familiarity with complex polynomial dynamics, including Julia sets, external rays, etc. All cubic polynomials in this paper are assumed to be monic, i.e., of the form $z^3 + a$ a quadratic polynomial, and to have connected Julia sets. We can parameterize the external rays of a cubic polynomial $P$ by angles, i.e., elements of $\mathbb{R}/\mathbb{Z}$. The external ray of argument $\theta \in \mathbb{R}/\mathbb{Z}$ is denoted by $R_P(\theta)$. Clearly, $P$ maps $R_P(\theta)$ to $R_P(3\theta)$.

**Lemma 2.1** ([9], cf. Lemma B.1 [10]). Let $g$ be a polynomial, and $z$ be a stable point of $g$. If an external ray $R_g(\theta)$ with rational argument $\theta$ lands at $z$, then, for every polynomial $\tilde{g}$ sufficiently close to $g$, the ray $R_{\tilde{g}}(\theta)$ lands at a stable point $\tilde{z}$ close to $z$. Moreover, $\tilde{z}$ depends holomorphically on $\tilde{g}$.

For a pair of sets $A, B$, let $A \vee B$ denote the set of all unordered pairs $\{a, b\}$ with $a \in A$ and $b \in B$. Thus, the $s$-set $S_P$ of $P$ consists of all pairs
\{\alpha, \beta\} \in (\mathbb{Q}/\mathbb{Z}) \cup (\mathbb{Q}/\mathbb{Z}) \text{ such that } R_P(\alpha) \text{ and } R_P(\beta) \text{ land at the same stable point of } P.

A chord \overrightarrow{ab} is a closed segment connecting points \(a, b\) of the unit circle \(S = \{z \in \mathbb{C} \mid |z| = 1\}\). If \(a = b\), the chord is degenerate. Write \(\sigma_d\) for the self-map of \(S\) that takes \(z\) to \(z^d\). A chord \overrightarrow{ab} is said to be \((\sigma_d)\)-critical if \(\sigma_d(a) = \sigma_d(b)\). Let \(\text{CCh}\) be the set of all \(\sigma_d\)-critical chords equipped with the natural topology; \(\text{CCh}\) is homeomorphic to \(S\). A critical portrait is an unordered pair \(\{\overline{e}, \overline{f}\}\in \text{CCh} \lor \text{CCh}\) such that \(\overline{e}\) and \(\overline{f}\) do not intersect in the unit disk \(D = \{z \in \mathbb{C} \mid |z| < 1\}\). The space of all critical portraits is denoted by \(\text{CrP}\). It is homeomorphic to the Möbius band, cf. [15].

Let \(\sim_P\) be the equivalence relation on \(S\) defined as follows: \(e^{2\pi i \alpha} \sim_P e^{2\pi i \beta}\) if \(\{\alpha, \beta\} \in S_P\) or \(\alpha = \beta\). Let \(S'_P\) be the set of all edges of the convex hulls in \(D\) of all \(\sim_P\)-classes. Thus \(S'_P\) is a set of chords. The superscript \(l\) is from “laminational”. Two distinct chords of \(D\) cross if they intersect in \(D\). Alternatively, crossing chords are said to be linked. Two sets of chords are compatible if no chord of one set crosses a chord of the other set.

**Proposition 2.2.** The dependence \(P \mapsto \mathcal{C}_P\) is upper semi-continuous.

**Proof.** We prove that if \(P_i \to P\) and we choose \(K_i \in \mathcal{C}_{P_i}\) with \(K_i \to K\), then \(K \in \mathcal{C}_P\). Assume the contrary: \(K = \{\overline{e}, \overline{f}\}\), where \(\overline{e}\) crosses some \(\ell = \overrightarrow{ab} \in S'_P\). By Lemma 2.1, we have \(\alpha \sim_P b\) for large \(i\), and \(K_i\) contains a critical chord \(\overline{e}_i\) close to \(\overline{e}\), a contradiction, since then \(\overline{e}_i\) also crosses \(\ell\). \(\square\)

In [14] Thurston defined invariant laminations as families of chords with certain dynamical properties. We use a slightly different approach (see [2]).

**Definition 2.3** (Laminations). A **lamination** is a collection \(\mathcal{L}\) of chords called leaves such that distinct leaves are unlinked, all degenerate chords (points of \(S\)) are leaves, the set \(\mathcal{L}^+ = \bigcup_{\ell \in \mathcal{L}} \ell\) is compact. Without the last condition \(\mathcal{L}\) is called a prelamination.

From now on \(\mathcal{L}\) denotes a lamination (unless we specify that we consider prelamination). Gaps of \(\mathcal{L}\) are the closures of components of \(D \setminus \mathcal{L}^+\). A gap \(G\) is countable (finite, uncountable) if \(G \cap S\) is countable and infinite (finite, uncountable). Uncountable gaps are called Fatou gaps. For a closed convex set \(H \subset \mathbb{C}\), straight segments in \(\text{Bd}(H)\) are called edges of \(H\). In what follows, convergence of prelaminations \(\mathcal{L}_i\) to a set of chords \(\mathcal{E}\) is always understood as Hausdorff convergence of leaves of \(\mathcal{L}_i\) to chords from \(\mathcal{E}\). Evidently, \(\mathcal{E}\) is a lamination. Call \(\mathcal{L}\) nonempty if it has nondegenerate leaves, otherwise it is empty (denoted \(\mathcal{L}_e\)). Say that \(\mathcal{L}\) is countable if it has countably many nondegenerate leaves and uncountable otherwise; \(\mathcal{L}\) is perfect if it has no isolated leaves.

If \(G \subset D\) is the convex hull of \(G \cap S\), define \(\sigma_d(G)\) as the convex hull of \(\sigma_d(G \cap S)\). **Sibling** \((\sigma_d)\)-invariant laminations modify Thurston’s [14]
invariant geodesic laminations. A sibling of a leaf $\ell$ is a leaf $\ell' \neq \ell$ with $\sigma_d(\ell') = \sigma_d(\ell)$. Call a leaf $\ell^*$ such that $\sigma_d(\ell^*) = \ell$ a pullback of $\ell$.

**Definition 2.4** ([2]). A (pre)lamination $\mathcal{L}$ is sibling ($\sigma_d$)-invariant if

1. for each $\ell \in \mathcal{L}$, we have $\sigma_d(\ell) \in \mathcal{L}$,
2. for each $\ell \in \mathcal{L}$ there exists $\ell^* \in \mathcal{L}$ with $\sigma_d(\ell^*) = \ell$,
3. for each $\ell \in \mathcal{L}$ such that $\sigma_d(\ell)$ is a nondegenerate leaf, there exist $d$ pairwise disjoint leaves $\ell_1, \ldots, \ell_d$ in $\mathcal{L}$ such that $\ell_1 = \ell$ and $\sigma_d(\ell_1) = \cdots = \sigma_d(\ell_d)$.

Collections of leaves from (3) above are full sibling collections. Their leaves cannot intersect even on $\mathbb{S}$. By cubic (resp., quadratic) laminations, we always mean sibling $\sigma_3$-(resp., $\sigma_2$-) invariant laminations. When dealing with cubic laminations, we write $\sigma$ instead of $\sigma_3$. From now on $\mathcal{L}$ (possibly with sub- and superscripts) denotes a cubic sibling invariant lamination.

These are properties of cubic sibling invariant laminations [2]:

- **gap invariance**: if $G$ is a gap of $\mathcal{L}$, then $H = \sigma(G)$ is a leaf of $\mathcal{L}$ (possibly degenerate), or a gap of $\mathcal{L}$, and in the latter case, the map $\sigma : G \cap \mathbb{S} \to H \cap \mathbb{S}$ extends to a map of the boundary of $G$ onto the boundary of $H$ so that the extended map is an orientation preserving composition of a monotone map and a covering map;

- **compactness**: if a sequence of sibling invariant prelaminations converges to a set of chords, this set of chords is a sibling invariant lamination.

Gap invariance is a part of Thurston’s original definition [14].

A chord $\ell$ is inside a gap $G$ if $\ell$ is, except for the endpoints, in the interior of $G$. A gap $G$ of $\mathcal{L}$ is critical if either all edges of $G$ are critical, or there is a critical chord inside $G$. A critical set of $\mathcal{L}$ is a critical leaf or a critical gap. We also define a lap of $\mathcal{L}$ as either a finite gap of $\mathcal{L}$ or a nondegenerate leaf of $\mathcal{L}$ not on the boundary of any gap.

**Lemma 2.5.** Suppose that $\mathcal{L}_i \to \mathcal{L}$ are $\sigma_d$-invariant laminations, and let $G$ be a periodic lap of $\mathcal{L}$. Then $G$ is also a lap of $\mathcal{L}_i$ for all sufficiently large $i$.

**Proof.** Let $G$ be a lap and $\ell$ an edge of $G$; we write $k$ for the minimal period of $\ell$. Then $\mathcal{L}_i$, for large $i$, must have a lap $G_i$ with $G_i \to G$. Choose an edge $\ell_i$ of $G_i$ so that $\ell_i \to \ell$. Then $\ell_i$ does not cross $\ell$ for large $i$ as otherwise the leaves $\sigma^k_d(\ell_i)$ and $\ell_i$ would cross. Moreover, $\ell_i$ is disjoint from the interior of $G$ for large $i$ as otherwise $\sigma^k_d(\ell_i)$ would intersect the interior of $G_i$ (note that $\ell_i$ maps farther away from $\ell$ under $\sigma^k_d$). By way of contradiction assume that $\mathcal{L}_i$ do not contain $G$. Then $G_i \not\supset G$ and $\ell_i \neq \ell$ for at least one edge $\ell$ of $G$. It follows that $\sigma^k_d(G_i) \not\supset G_i$, a contradiction. \qed
We now discuss some special classes of laminations. A lamination $L$ is **clean** if any pair of non-disjoint leaves of $L$ is on the boundary of a finite gap. Clean laminations give rise to equivalence relations: $a \sim_L b$ if either $a = b$ or $a, b$ are in the same lap of $L$. By Lemma 3.16 of [4], any clean lamination has the following period matching property: if one endpoint of a leaf is periodic, then the other endpoint is also periodic with the same minimal period. Limits of clean $\sigma_d$-invariant laminations are called **limit laminations**, cf. [5]. Chords are always considered in the Hausdorff metric.

**Definition 2.6** (Perfect laminations [7]). A closed set of chords is **perfect** if it has no isolated chords. The maximal perfect subset $L^p$ of $L$ is called the **perfect part** of $L$. Clearly, a lamination $L$ is **perfect** if $L = L^p$.

One can define the perfect part of $L$ as the set of all leaves $\ell \in L$ such that arbitrarily close to $\ell$ there are uncountably many leaves of $L$. By Lemma 3.12 of [7], the set $L^p$ is an invariant lamination.

**Definition 2.7** (Oldest ancestors). If $L$ is nonempty, an **oldest ancestor** of $L$ is defined as a minimal by inclusion nonempty sublamination of $L$.

An oldest ancestor is perfect or countable: if $L$ is uncountable, then $L^p \subset L$ is nonempty. Given a chord $\ell = ab$ denote by $|\ell|$ the length of the smaller arc with endpoints $a$ and $b$ (computed with respect to the Lebesgue measure on $S$ such that the total length of $S$ is 1); call $|\ell|$ the **length** of $\ell$.

**Lemma 2.8.** If $L$ is nonempty, then $L$ contains an oldest ancestor.

**Proof.** Let $L_{\alpha}$ be a nested family. Definition 2.4 implies that then $\bigcap L_{\alpha}$ is a sibling invariant lamination too. Any nonempty lamination contains leaves of length $\geq \frac{1}{d+1}$. Indeed, for a nondegenerate leaf $\ell$ either $|\sigma_d(\ell)| = d|\ell|$ or $|\ell| \geq \frac{1}{d+1}$. Thus, if all $L_{\alpha}$ above are nonempty, then $\bigcap L_{\alpha}$ is nonempty. Now the desired statement follows from the Zorn lemma. \hfill $\square$

The next lemma follows from the definitions and the compactness property of invariant laminations.

**Lemma 2.9.** Let $L$ be an oldest ancestor. If $\ell \in L$ is a nondegenerate leaf, then the iterated pullbacks of the nondegenerate iterated images of $\ell$ are dense in $L$.

### 3. Invariant Gaps and Prime Portraits

An **invariant gap** is an invariant gap of a cubic $L$, not necessarily specifying the latter. An infinite invariant gap is **quadratic** if it has degree 2. By Section 3 of [4], any quadratic invariant gap can be obtained as follows. A critical chord $c$ gives rise to the complementary circle arc $L(c)$ of length $2/3$ with the same endpoints as $c$. The set $\Pi(\bar{c})$ of all points with orbits in
$L(\tau)$ is nonempty, closed and forward invariant. The convex hull $G(\tau)$ of $\Pi(\tau)$ is an invariant quadratic gap, and any invariant quadratic gap is of this form. For any invariant gap $G$, finite or infinite, a major of $G$ is an edge $M = ab$ of $G$, for which there is a critical chord $\overline{ax}$ or $\overline{by}$ disjoint from the interior of $G$. By Section 4.3 of [4], a degree 1 invariant gap has one or two majors; every edge of $G$ eventually maps to a major and if $G$ is infinite, at least one of its majors is critical. An invariant gap $G$ is rotational if $\sigma$ acts on its vertices (i.e., $G \cap \mathbb{S}$) as a combinatorial rotation. Recall that $\sigma = \sigma_3$. For brevity say that a chord is compatible with a finite collection of gaps if it does not cross edges of these gaps.

Define $I(\tau)$ as the complement of $L(\tau)$, define $I(\overline{y})$ similarly. Call $K$ weak if the forward orbit of $\tau$ is disjoint from $I(\overline{y})$, or the forward orbit of $\overline{y}$ is disjoint from $I(\tau)$; otherwise call $K$ strong.

**Lemma 3.1.** The set of strong critical portraits is open and dense in $\text{CrP}$ while the set of weak critical portraits is closed and nowhere dense in $\text{CrP}$.

Lemma 3.1 is left to the reader.

**Lemma 3.2.** A critical portrait $\{\tau, \overline{y}\}$ is compatible with an invariant quadratic gap if and only if it is weak.

**Proof.** Let $T$ be the forward orbit of $\overline{y}$. If $T$ is disjoint from $I(\tau)$, then $T \subset \Pi(\tau)$; hence, $\{\tau, \overline{y}\}$ is compatible with $G(\tau)$. Assume now that $\{\tau, \overline{y}\}$ is compatible with an invariant quadratic gap $U$. Then $U$ contains $\tau$ or $\overline{y}$, say, $\overline{y} \subset U$. Since $U$ is quadratic, $\tau$ is disjoint from the interior of $U$, thus $I(\overline{y}) \cap U = \emptyset$. We conclude that $T$ never visits $I(\overline{y})$. □

Observe that a strong critical portrait $K$ is not compatible with an infinite invariant gap $G$ as otherwise, by [4], at least one chord from $K$ is non-disjoint from $G$. Thus, a critical portrait is weak if and only if it is compatible with an infinite invariant gap and strong otherwise.

**Theorem 3.3.** Suppose that a nonempty cubic $\mathcal{L}$ has an infinite periodic gap $U$ and either $\sigma(U) = U$ or $U$ shares an edge with a finite rotational lap of $\mathcal{L}$. Then there is a weak critical portrait compatible with $\mathcal{L}$.

**Proof.** Choose a critical chord $\tau$ in a gap from the orbit of $U$ with an endpoint of the same period as $U$. Choose a critical chord $\overline{y} \neq \tau$ compatible with $\mathcal{L}$ and not crossing $\tau$. Then $\{\tau, \overline{y}\}$ is compatible with a quadratic invariant gap by Lemma 3.2. □

From now on, by an oldest ancestor, we mean an oldest ancestor of some limit lamination. A gap $G$ of a lamination is invariant if $\sigma(G) = G$ (with “=” rather than “$\subset$”).

**Lemma 3.4.** Any oldest ancestor has an invariant lap or infinite gap.
Proof. By Lemma 3.7 of [10], any clean lamination has a lap or an infinite gap $G$ such that $\sigma(G) = G$. Passing to the limit, we conclude that any limit lamination has the same property (even though the limit of finite invariant gaps may be infinite). An oldest ancestor of a limit lamination must then also have the above mentioned property. □

Definition 3.5 (Friends, prime critical portraits). Critical portraits $K_1, K_2 \in \text{CrP}$ are friends (through an oldest ancestor $L$) if $K_1, K_2$ are compatible with $L$. A critical portrait $K$ is prime if some friend of $K$ has a weak friend.

The next lemma is straightforward; we leave it to the reader.

Lemma 3.6. If $K_1, K_2$ are friends, then there is an oldest ancestor with whom they are compatible. Also friendship is a closed relation: if $K_i \to K$ and $K'_i \to K'$ and $K_i$ and $K'_i$ are friends for all $i$, then so are $K$ and $K'$.

Given $L$ and a nondegenerate leaf $\ell \in L$, let $G(\ell)$ be the set of iterated pullbacks of the nondegenerate iterated images of $\ell$.

Lemma 3.7. Let $L$ be a countable oldest ancestor. Then

1. for any nondegenerate leaf $\ell \in L$, the set of all nondegenerate leaves in $L$ coincides with $G(\ell)$;
2. all nondegenerate leaves of $L$ are isolated;
3. at least one weak critical portrait is compatible with $L$.

Proof. (1) Choose an isolated leaf $\ell_0 \in L$. If $G(\ell_0)$ does not coincide with the set of all nondegenerate leaves of $L$, choose a nondegenerate leaf $\ell \in L \setminus G(\ell_0)$. Then leaves of $G(\ell)$ cannot approximate $\ell_0$ or coincide with $\ell_0$, a contradiction with Lemma 2.9. Let now $\ell$ be any nondegenerate leaf of $L$; we proved that $\ell \in G(\ell_0)$. Therefore, $G(\ell) = G(\ell_0)$ is the set of all nondegenerate leaves of $L$, as desired.

(2) All non-isolated leaves in $L$ form a forward invariant closed family of leaves. If $\ell$ is non-isolated, choose leaves $\ell_i \to \ell$, choose their pullbacks $\bar{\ell}_i$, and choose a converging subsequence of these pullbacks; in the end we will find a non-isolated leaf $\bar{\ell}$ with $\sigma(\bar{\ell}) = \ell$. Now, let $\ell$ be non-isolated and non-critical. Choose a sequence $\ell_i \to \ell$ so that each $\ell_i$ has exactly two siblings in $L$ (the only way a leaf $\ell$ can have more siblings is when there is a critical 4-gon or 6-gon that maps onto $\sigma(\ell)$). We may assume that these siblings are $\ell'_1, \ell''_1$ and that $\ell_i \to \ell'$ while $\ell''_i \to \ell''$. Clearly, $\sigma(\ell) = \sigma(\ell') = \sigma(\ell'')$. We claim that $\ell, \ell'$ and $\ell''$ are pairwise disjoint. Indeed, if, say, $\ell = \overline{ab}$ and $\ell' = \overline{bc}$, where $\sigma(c) = \sigma(a)$, then $\ell_i$ and $\ell'_i$ have distinct endpoints close to $b$ and mapping to the same point; a contradiction. Hence by definition the set of all non-isolated leaves of $L$ is itself a sibling-invariant lamination, a contradiction with $L$ being an oldest ancestor.
(3) By Lemma 3.4, we can find an invariant lap or infinite gap \( G \) of \( \mathcal{L} \). If \( G \) is infinite, our claim follows from Theorem 3.3. Hence we may assume that \( G \) is finite. Let \( \ell \) be an edge of \( G \); it is isolated by (2). Let \( H \) be a gap of \( \mathcal{L} \) attached to \( G \) along \( \ell \). If \( H \) is infinite, the desired statement follows from Theorem 3.3. Assume that \( H \) is finite. Let \( \ell \) be an edge of \( G \); it is isolated by (2). Let \( H \) be a gap of \( \mathcal{L} \) attached to \( G \) along \( \ell \). If \( H \) is infinite, the desired statement follows from Theorem 3.3. Assume that \( H \) is finite. Let \( n \) be the minimal period of edges of \( G \), and there are two cases: \( \sigma^n(H) = H \) and \( \sigma^n(H) = \ell \). The former case contradicts (1), hence \( \sigma(H) = \sigma(\ell) \). Choose a critical chord \( \bar{y} \in H \) that shares an endpoint with \( \ell \), and a critical chord \( c \) in a critical gap or leaf of \( \mathcal{L} \) disjoint from \( H \). By Lemma 3.2, the critical portrait \( \{c, \bar{y}\} \) is compatible with \( G(c) \) as desired. \( \square \)

An oldest ancestor is regular if all its critical portraits have only strong friends.

**Lemma 3.8.** A regular oldest ancestor \( \mathcal{L} \) does not share a critical portrait with another oldest ancestor.

**Proof.** The lamination \( \mathcal{L} \) is uncountable by Lemma 3.7; hence it is perfect. Let \( \mathcal{K} = \{\bar{c}, \bar{y}\} \) be a critical portrait compatible with \( \mathcal{L} \) and an oldest ancestor \( \mathcal{L}' \neq \mathcal{L} \). Since \( \mathcal{K} \) is strong, invariant sets of \( \mathcal{L} \) and \( \mathcal{L}' \) are finite. Let \( G \) be an invariant lap of \( \mathcal{L} \). Let \( G' \) be an invariant leaf or gap \( \mathcal{L}' \) located in the same component of \( \mathcal{D} \setminus \bigcup \mathcal{K} \) as \( G \). Since all friends of \( \mathcal{K} \) are strong, \( \mathcal{L}' \) is uncountable (and hence perfect) by Lemma 3.7. Moreover, by Theorem 3.3, any gap of \( \mathcal{L}' \) non-disjoint from \( G' \) is finite. There are no leaves of \( \mathcal{L} \) intersecting the interior of \( G' \) since otherwise uncountably many leaves of \( \mathcal{L} \) would intersect edges of \( G' \). Since \( \mathcal{K} \) is compatible with both \( \mathcal{L} \) and \( \mathcal{L}' \), this is impossible by [7, Lemma 3.53]. Therefore, \( G' \subset G \). A similar argument shows that \( G' \subset G' \), hence \( G = G' \).

If iterated images of \( \bar{c} \) and \( \bar{y} \) avoid \( G \), then iterated \( \mathcal{L} \)-pullbacks of \( G \) and iterated \( \mathcal{L}' \)-pullbacks of \( G' \) are the same. Hence \( \mathcal{L} = \mathcal{L}' \) since the iterated pullbacks of \( G \) are dense in both \( \mathcal{L} \) and \( \mathcal{L}' \). Let for some minimal \( n \geq 0 \) the point \( \sigma^n_3(\bar{c}) \) be a vertex of \( G \). Let \( C, C' \) be the critical sets of \( \mathcal{L} \), resp., \( \mathcal{L}' \) containing \( \bar{c} \). Infinite gaps of \( \mathcal{L} \) and \( \mathcal{L}' \) are disjoint from \( G \) by Theorem 3.3; so, \( n \geq 1 \) and \( C, C' \) are finite. Since \( \mathcal{L} \) and \( \mathcal{L}' \) are compatible and perfect, \( C = C' \) by [7, Theorem 3.57]. Similarly, we see that either \( \bar{y} \) never maps to \( G \) or the critical sets of \( \mathcal{L}, \mathcal{L}' \) containing \( \bar{y} \) coincide. In both cases pullbacks of \( G \) in \( \mathcal{L} \) are the same as pullbacks of \( G \) in \( \mathcal{L}' \), hence \( \mathcal{L} = \mathcal{L}' \). \( \square \)

A critical portrait is regular if it is compatible with a regular oldest ancestor. Corollary 3.9 follows immediately from definitions and Lemma 3.8.

**Corollary 3.9.** A friend of a regular critical portrait is regular. All its critical portraits of a regular oldest ancestor form a closed subset of \( \text{CrP} \) consisting of friends, and no other critical portrait can be their friend.
One can define regular critical portraits through the concept of a friend.

**Lemma 3.10.** A critical portrait $K$ is regular if and only if all friends of friends of $K$ are strong (i.e., if $K$ is not prime).

**Proof.** Let $K$ be a regular critical portrait. Then there is a unique regular oldest ancestor $L$ compatible with $K$. All critical portraits of $L$ are strong and have only strong friends; by Lemma 3.8, none of them is compatible with an oldest ancestor $L' \neq L$. Hence all friends of $K$ are compatible with $L$, i.e. are regular. Repeating this, we see that friends of friends of $K$ are compatible with $L$, and, hence, strong. On the other hand, suppose that all friends of friends of a critical portrait $K$ are strong. Take an oldest ancestor $L$ compatible with $K$. Then all its critical portraits have only strong friends. By definition $L$ is regular which implies that $K$ is regular as desired. □

**Lemma 3.11.** A limit of prime critical portraits is prime.

**Proof.** If $K_i$ are prime and $K_i \rightarrow K$, then, by definition, some friends $K'_i$ of $K_i$ have weak friends $K''_i$. Passing to a subsequence we can arrange that $K'_i \rightarrow K'$ and $K''_i \rightarrow K''$. By Lemma 3.6, the portrait $K'$ is a friend of $K$, and $K''$ is a friend of $K'$. By Lemma 3.1, the portrait $K''$ is weak. By definition, $K$ is prime. □

4. **ALLIANCES**

The **prime alliance** $A_0$ is the set of all prime critical portraits. Section 3 implies that the prime alliance is closed topologically and under friendship. For a visible polynomial $P$ such that $S_P$ is not compatible with prime critical portraits, define the **regular alliance** $A_P$ as the set of friends of critical portraits from $C_P$. If $S_P$ is compatible with a prime critical portrait (e.g., if $S_P = \emptyset$), then $A_P$ is defined as the prime alliance. This defines $A_P$ for any polynomial $P$ with $[P] \in M_3$. The prime alliance is special as it serves all invisible polynomials, however diverse they are. It also serves all polynomials with non-repelling fixed points and some other polynomials.

For $[P] \in M_3$, define a clean lamination $L_P^s$ (“s” from stable) as follows. First, define an equivalence relation $\approx_P$ on $S$ by declaring $e^{2\pi i \alpha} \approx_P e^{2\pi i \beta}$ if $R_P(\alpha)$ and $R_P(\beta)$ land at the same point eventually mapped to a stable point of $P$; $a \sim_P b$ implies $a \approx_P b$ but not vice versa. Then $L_P^s$ is defined as the set of all edges of the convex hulls of all $\approx_P$-classes and the limits of these edges. Clearly, $L_P^s$ is a cubic lamination containing $S_P$ (cf [2]).

Recall some results of [12]. A polynomial $P$ with no neutral periodic point defines a clean lamination $L_P^R$ called the **real lamination** of $P$. Here $e^{2\pi i \alpha}$ and $e^{2\pi i \beta}$ are vertices of a lap of $L_P^R$ iff there are angles $\alpha_0 = \alpha, \alpha_1, \ldots, \alpha_k = \beta$ such that the impressions of angles $\alpha_i$ and $\alpha_{i+1}$ intersect. A clean lamination without infinite degree 1 gaps has the form $L_P^R$ for some
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For a periodic lap $G$ of $L_P$, external rays corresponding to vertices of $G$ land at the same stable point.

**Lemma 4.1.** A regular alliance $A_P$ has the form $C_{P_0}$ for some visible polynomial $P_0$, possibly different from $P$.

**Proof.** Suppose that $S'_P$ is not compatible with a prime critical portrait. Then $L'_P$ cannot be compatible with a prime critical portrait either. Consider an oldest ancestor $L$ of $L'_P$; it is compatible with some $K \in C_P$. By Lemma 3.8, the lamination $L$ is not compatible with any other oldest ancestor. By [12], there is a polynomial $P_0$ without neutral periodic points such that $L_{P_0} = L$. We want to prove that $A_P = C_{P_0}$, that is, if $K \in C_P$ and $K'$ is a friend of $K$, then $K' \in C_{P_0}$. Indeed, $K$ is compatible with $L'_P$, hence also with $L$. If $K'$ is a friend of $K$, then there is an oldest ancestor $L'$ compatible with both $K'$ and $K$. Since $L$ is not compatible with any other oldest ancestor, $L = L'$. Hence $K' \in C_{P_0}$ as desired. \(\square\)

A regular alliance is closed topologically (because $C_{P_0}$ is closed by definition) and under friendship (by Corollary 3.9).

**Lemma 4.2.** For any visible $P$ we have $C_P \subset A_P$.

**Proof.** If there are no prime portraits in $C_P$, then $C_P \subset A_P$ by definition of a regular alliance. If $C_P$ has a prime portrait, then $A_P$ is the prime alliance. In this case $C_P \subset A_P$ since $A_P$ is closed under friendship. \(\square\)

**Lemma 4.3.** If $P$ has no neutral cycles, and $G$ is a periodic lap of $L_P$, then the external rays for $P$ corresponding to vertices of $G$ land at the same point.

**Proof.** Indeed, since $L_P^\infty \supset L_P^*$, the set $G$ is also a lap of $L_P^\infty$. The conclusion now follows from [12]. \(\square\)

Theorem 4.4 implies the Main Theorem.

**Theorem 4.4.** The map $P \mapsto A_P$ from $M_3$ to the quotient space of $\text{Cr}P$ generated by alliances is continuous.

**Proof.** Consider a sequence $P_i \to P$ of polynomials, and set $A_i = A_{P_i}$. Suppose that $K_i \to K$, where $K_i \in A_i$; we claim that $K \in A_P$. Passing to a subsequence, assume that either all $K_i$ are prime, or all $K_i$ are regular. If $K_i$ are prime, then $K$ is prime by Lemma 3.11. Assume that all $K_i$ are regular. By Proposition 2.2, we have $K \in C_P$ and hence $K \in A_P$ by Lemma 4.2 if $P$ is visible. It remains to show that if $S_P = \emptyset$ then $K$ is prime.

Assume that $K$ is regular. Then $K$ is compatible with a regular oldest ancestor $L^0$; in particular, $L^0$ is perfect. By Lemma 3.21 of [7], the equivalence on $S$ that collapses all laps and infinite gaps of $L^0$ semiconjugates $\sigma$ to
an induced map on a dendrite. This map satisfies assumptions of Theorem 7.2.6 of [1], which implies that it has infinitely many periodic points. Since $\mathcal{L}^0$ has only finitely many infinite periodic gaps, lifting the self-map of the dendrite back to $\mathcal{L}^0$ and $\mathbb{S}$, we obtain infinitely many periodic laps of $\mathcal{L}^0$.

Thus, there is a periodic lap $G$ of $\mathcal{L}^0$ with the following property. For every vertex $e^{2\pi i\alpha}$ of $G$, the ray $R_P(\alpha)$ lands at a stable periodic point of $P$. Suppose that, for two vertices $e^{2\pi i\alpha}, e^{2\pi i\beta}$ of $G$, the rays $R_P(\alpha), R_P(\beta)$ land at distinct points. By Lemma 2.1, the rays $R_{P_i}(\alpha), R_{P_i}(\beta)$ land at distinct stable points of $P_i$, for large $i$. However, by Lemma 2.5, the set $G$ is a lap of $\mathcal{L}_i$ for all large $i$, a contradiction with Lemma 4.3. The conclusion is that the external rays of $P$ corresponding to the vertices of $G$ land at the same stable point of $P$. A contradiction with the assumption that $P$ is invisible. □

Finally, let us show that the prime alliance is a small subset of $\text{CrP}$.

**Lemma 4.5.** An open and dense subset of $\text{CrP}$ consists of non-prime critical portraits.

**Proof.** The union of all regular alliances is open. Let $K = \{\overline{c, y}\}$ be a critical portrait such that the orbits of $\sigma(\overline{c})$ and $\sigma(\overline{y})$ are dense in $\mathbb{S}$. Such portraits are dense in $\text{CrP}$. We prove that $K$ is regular by proving that, for any friend $K' = \{\overline{c', y'}\}$ of $K$, the orbits of $\sigma(\overline{c'})$ and $\sigma(\overline{y'})$ are dense in $\mathbb{S}$.

Let $\mathcal{L}$ be an oldest ancestor compatible with $K$ and $K'$. Let $C$ be the leaf $\overline{c}$ if $\overline{c} \in \mathcal{L}$ or the critical gap of $\mathcal{L}$ containing $\overline{c}$ otherwise. Define $Y$ similarly. Arrange that $\overline{c'} \subset C$ and $\overline{y'} \subset Y$, possibly renaming $\overline{c'}$ and $\overline{y'}$. We claim that the orbit of $\sigma(\overline{c'})$ is dense in $\mathbb{S}$. Otherwise consider the nondegenerate chord $\overline{y} = \overline{xx'}$, where $x = \sigma(\overline{c})$ and $x' = \sigma(\overline{c'}).$. There is $\varepsilon > 0$ and an arc $I \subset \mathbb{S}$ such that $\sigma^n(x')$ is never $\varepsilon$-close to $I$. On the other hand, iterated images of $x$ are dense in $I$; the corresponding images of $\overline{y}$ have length $\geq \varepsilon$. Therefore, all leaves of $\mathcal{L}$ originating in $I$ have length $\varepsilon$ or more.

Note that $C'$ and $Y$ are not periodic, therefore, no $\sigma$-periodic point of $\mathbb{S}$ is an eventual image of $x$ or $x'$. There is a positive integer $N$ with $\sigma^N(I) = \mathbb{S}$. Since any $\sigma$-periodic point $a$ of $\mathbb{S}$ has a $\sigma^N$-preimage in $I$, we have $\overline{ab} \in \mathcal{L}$ for some $b \neq a$. Thus, the horizontal diameter $\overline{D_1}$ connecting the two $\sigma$-fixed points of $\mathbb{S}$ is a leaf of $\mathcal{L}$. Consider a nondegenerate chord $\ell$ with endpoint $i = e^{2\pi i(1/4)}$. Then $\sigma^n(\ell)$ crosses $\overline{c}$ or $\overline{y}$ for some $n \geq 0$. Thus $\ell \notin \mathcal{L}$, a contradiction. We conclude that the orbit of $\sigma(\overline{c'})$ is dense. Similarly, the orbit of $\sigma(\overline{y'})$ is dense. □

**References**


(Alexander Blokh and Lex Oversteegen) DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ALABAMA AT BIRMINGHAM, BIRMINGHAM, AL 35294

(Vladlen Timorin) FACULTY OF MATHEMATICS, HSE UNIVERSITY, 6 USACHEVA STR., MOSCOW, RUSSIA, 119048

Email address, Alexander Blokh: ablokh@math.uab.edu
Email address, Lex Oversteegen: overstee@uab.edu
Email address, Vladlen Timorin: vtimorin@hse.ru