DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Volume **13**, Number **1**, June **2005**

Website http://aimSciences.org

pp. 13–34

NECESSARY CONDITIONS FOR THE EXISTENCE OF WANDERING TRIANGLES FOR CUBIC LAMINATIONS

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Dedicated to the memory of Bob Kauffman

Abstract. In his 84 preprint W. Thurston proved that quadratic laminations do not admit so-called *wandering triangles* and asked a deep question concerning their existence for laminations of higher degrees. Recently it has been discovered by L. Oversteegen and the author that some closed laminations of the unit circle invariant under $z \mapsto z^d, d > 2$ admit wandering triangles. This makes the problem of describing the criteria for the existence of wandering triangles important because solving this problem would help understand the combinatorial structure of the family of all polynomials of the appropriate degree.

In this paper for a closed lamination on the unit circle invariant under $z \mapsto z^3$ (cubic lamination) we prove that if it has a wandering triangle then there must be two distinct recurrent critical points in the corresponding quotient space ("topological Julia set") J with the same limit set coinciding with the limit set of any wandering vertex (wandering vertices in J correspond to wandering gaps in the lamination).

Introduction. It is well known that in a variety of cases connected Julia sets of complex polynomials are locally connected. If so then the dynamics on them can be studied by means of the so-called **invariant laminations**, i.e. specific equivalence relations $\sim$ on the unit circle S^1 ([16], [9], [13]) invariant for $z^d : S^1 \to S^1$ (d is the degree of the polynomial). The Julia set J then can be viewed as the quotient space $J_\sim$ of S^1 under this equivalence, and the polynomial on J as the factor f of z^d induced by the quotient map. We will use the language of laminations in this paper, with the understanding that our results apply to locally connected Julia sets of polynomials as well. Saying "Julia set" we always mean the corresponding quotient space $J_\sim$ (so our Julia sets are always locally connected). Since in what follows we fix a lamination $\sim$, from now on we skip the reference to $\sim$ in the notation and talk about the (topological) Julia set J meaning $J_\sim$ for the lamination $\sim$. We also reserve the name f for the map $f : J \to J$ described above.

To state one of the main problems in the field of topological dynamics of f we need a few notions. A point c is said to be **critical** if f at c is not a homeomorphism. If $x \in J$ then by $N(x)$ we denote the number of components of the set $J \setminus \{x\})$. We call $N(x)$ the **order** of a point x in J; points x with $N(x) \geq 3$ are said to be **vertices** of J and points x of order 1 are said to be *endpoints* of J. In the language of continuum theory, vertices are **cut points** cutting the Julia set in at least three components. A point x is said to be **periodic** (of period n) if $x, f(x), \dots, f^{n-1}(x)$ are pairwise distinct points while $f^n(x) = x$. Also, a point is **preperiodic** - resp. **precritical** - if it is mapped onto a periodic - resp. critical - point by $f^k, k \geq 0$.

AMS 2000 Mathematics Subject Classification: Primary Primary 37F10; Secondary 37E25.
Key words and phrases: Julia set, lamination, growing tree, wandering triangle.
The research was partially supported by NSF grant DMS-0140349.

Finally, if a point is non-preperiodic then we call it **wandering**. The following problem was posed (and solved for quadratic laminations) in [16].

Problem 1. *Do there exist wandering non-precritical vertices of the Julia sets?*

Problem 1 is fundamental because solving it would help understand the combinatorial structure of the family of all monic polynomials with connected Julia set (so-called "connectedness locus") of the appropriate degree (see [16] for more motivation).

In what follows we adopt the following terminology: a **wa**ndering **ve**rtex of the Julia set is called a **wave**. Hence Problem 1 concerns the existence of non-precritical waves. This problem is natural if we think of J as a "graph" with infinitely many vertices and f as a continuous self-mapping of J. Indeed, by a "graph" we shall understand a compact one-dimensional branched manifold, i.e. a compact space which is locally an n-od (this includes 1-od $\equiv$ closed interval at its endpoint and 2-od $\equiv$ open interval). Clearly the notion of the order at a point, the notion of an endpoint, and the notion of a vertex can be introduced for points of a "graph". Moreover, compactness implies that a "graph" has finitely many endpoints and vertices. If J is a true "graph" (i.e. one-dimensional branched manifold) then the answer to the question in Problem 1 is clearly negative because if a vertex of a graph is non-precritical then its order in the graph cannot drop, and therefore its entire orbit is contained in a finite set of al vertices of the graph. Problem 1 extends this fact from "graphs" to polynomials on their Julia sets under the assumption that the Julia set is locally connected (Thurston proved in [16] that in the quadratic case there exist no waves of the Julia sets, so such extension holds in the quadratic case).

Related questions in degrees higher than 2 were considered by Kiwi [10] who was the first to extend some results of [16] onto laminations of degrees higher than 2. It is proven in [10] that for a lamination of degree d the number $N(x) \le d$ for any non-precritical wave $x \in J$ (this implies the result of [16]).

In [12, 2-4] further results were obtained. One of the main results of [12] was that in the *unicritical* case (i.e. when there is a unique critical point of $f|J$) the waves do not exist. In [2-3] we consider a non-empty collection Γ of non-precritical waves in J with pairwise disjoint orbits and prove upper estimates on $\sum_{x\in\Gamma}(N(x)-2)$, including $\sum_{x\in\Gamma}(N(x)-2) \le d-2$ (which implies results of [10] and [16]). We also prove in [2-3] that Fatou domains (which can be easily introduced for laminations) are preperiodic (i.e. map into periodic Fatou domains). In [4] we show that the limit set of a non-precritical wave must coincide with the limit set of a recurrent critical point.

Recently it has been discovered by L. Oversteegen and the author that some closed laminations of the unit circle invariant under $z \mapsto z^d, d > 2$ have waves (see [6]). Thus, the main problem in this field becomes to characterize all laminations which have waves. This paper can be considered as a step in this direction because here we give a dynamic necessary condition for the existence of a wave of a cubic lamination. Recall, that a **dendrite** is a locally connected continuum containing no subsets homeomorphic to a circle.

Main Theorem. *Let $\sim$ be a cubic lamination such that its quotient space J has non-precritical waves. Then J is a dendrite and the following holds:*

1. *$f|J$ has two wandering critical points c, d with distinct grand orbits, $N(c) = N(d) = 2$ and all forward images of c, d are endpoints of J;*

(2) *any two waves x', x'' have the same grand orbit, and are such that $N(x') = N(x'') = 3$;*
(3) *the points c and d are recurrent and have the same limit set coinciding with the limit set of any wave.*

1. Preliminaries. Consider an equivalence relation $\sim$ on the unit circle S^1 with the following properties ([9, 13], cf. [16]):

(E1) $\sim$ is **closed**: the graph of $\sim$ is a closed set in $S^1 \times S^1$;
(E2) $\sim$ defines a **lamination**, i.e. it is **unlinked**: if $t_1 \sim t_2 \in S^1$ and $t_3 \sim t_4 \in S^1$, but $t_2 \not\sim t_3$, then the open intervals in $\mathbb{C}$ with the endpoints t_1, t_2 and t_3, t_4 are disjoint;
(E3) each class of equivalence $\sim$ is totally disconnected.

Call $\sim$ a **closed lamination**. We assume that it is **non-degenerate** (has a class of more than one point). Equivalence classes of $\sim$ are called **($\sim$-)classes**.

Our definitions are closer to [9, 13] than to [16]. Fix an integer $d > 1$ and denote $z^d : S^1 \to S^1$ by $\sigma_d = \sigma$. Say that a subset of S^1 is **split** into classes if it contains a class of each its element. The relation $\sim$ is called **(σ-)invariant** iff:

(D1) $\sim$ is **forward invariant**: for a class g, the set $\sigma(g)$ is a class too;
(D2) $\sim$ is **backward invariant**: for a class g, its preimage $\sigma^{-1}(g) = \{x \in S^1 : \sigma(x) \in g\}$ is split into classes;
(D3) for any gap g, the map $\sigma : g \to \sigma(g)$ is a covering map with positive orientation.

Observe that in (D3) by "cover" we mean "even cover". Also, in fact (D1) implies (D2), but we put both here for the sake of convenience. Call a class g **critical** iff the map $\sigma : g \to \sigma(g)$ is not 1-to-1. Denote by $k_\sim$ the number of distinct grand orbits of critical non-preperiodic classes g such that $|\sigma(g)| = 1$. Also, call a class g a **gap** if $|g| \geq 3$ (by $|A|$ we denote the cardinality of a set A). From now on by a lamination we always mean a closed σ-invariant lamination.

Clearly, the notions above can be translated into the language of the Julia set $J = J_\sim$ associated with the lamination (we denote the factor map by p). Call a point $c \in J$ **critical** if f is not one-to-one in any neighborhood of c. Critical classes of the equivalence $\sim$ project by p onto critical points of f; the behavior of critical points is important for our investigation and is studied below in great detail. For every point $x = p(g) \in J$ the number $N(x)$ is the same as the cardinality $|g|$ of the class g. Thus, vertices of J are p-images of gaps of $\sim$. Also, if $N(x) = 1$ then x is called an **endpoint** of J; endpoints of J are p-images of degenerate classes of $\sim$. Observe, that critical wandering classes g whose all images are degenerate become in the language of J wandering critical points of $f|J$ whose all images are endpoints of J.

Let $\mathbb{D}$ be the unit open disk bounded by S^1, $L_\sim = L$ be the union of **$\sim$-hulls**, i.e. convex hulls of $\sim$-classes; by the definition $\sim$-hulls are contained in $\overline{\mathbb{D}}$ but not in $\mathbb{D}$. Define an extension $\simeq$ of $\sim$ onto $\overline{\mathbb{D}}$ as follows [9]: a $\simeq$-class is a $\sim$-hull or a point of $\overline{\mathbb{D}} \setminus L$. Extend $\simeq$ onto $\mathbb{C}$ by declaring that a point in $\mathbb{C} \setminus \overline{\mathbb{D}}$ is equivalent only to itself. Call a connected component of the complement $\mathbb{D} \setminus L$ a **($\sim$-)component**. Given an open set Ω in $\mathbb{D}$, denote by $E(\Omega)$ the set $\bar{\Omega} \cap S^1$. Call a $\sim$-component Ω **periodic** if $E(\Omega)$ is mapped back onto itself by some iteration of σ and denote the number of all orbits of periodic σ-components by k_p.

Now we are ready to formulate in more detail the results which we have already stated in Introduction. In [10] it was proven that a wandering non-precritical gap has at most d elements. This result was extended in Theorem 1.1 which is stated below in the language of $f|J$.

Theorem 1.1 (Theorem B [2, 3]). *Let Γ be a non-empty collection of non-precritical waves of J which have pairwise disjoint orbits. Then*

$$\sum_{x\in\Gamma}(N(x)-2) \le k_\sim - 1 \le d-2-k_p \le d-2. \qquad (*)$$

We will also need to rely upon Theorem 1.2 whose Part 1 is a combinatorial version of the fundamental Sullivan No Wandering Domain Theorem ([15]).

Theorem 1.2 (Theorem C [2, 3]). *The following holds for a lamination $\sim$.*

1. *Let Ω be a $\sim$-component. Then the set $E(\Omega)\subset S^1$ is preperiodic.*
2. *If $M\subset J$ is a non-degenerate continuum then it is non-wandering.*

Theorem 1.3 establishes necessary conditions for the existence of wandering classes.

Theorem 1.3 ([4]). *Let $\sim$ be an invariant lamination. Then the limit set of a non-precritical waves of J coincides with the limit set of a recurrent critical point (and so if there are no recurrent critical points then J has no waves).*

Theorems 1.1, 1.2 and 1.3 imply Corollary 1.4.

Corollary 1.4. *Let $\sim$ be a cubic lamination such that J has non-precritical waves. Then the following facts hold:*

1. *J is a dendrite;*
2. *$f|J$ has two wandering critical points c, d with distinct grand orbits, no vertex ever maps into a critical point, $N(c) = N(d) = 2$ and all forward images of c, d are endpoints of J;*
3. *any two waves x', x'' have the same grand orbits, and are such that $N(x') = N(x'') = 3$;*
4. *there exists a recurrent critical point s of f such that $\omega(y) = \omega(s)$ for every wave y.*

Proof. (1) If there are $\sim$-components then by Theorem 1.2 they must be periodic so that $k_p \ge 1$. However then by Theorem 1.1 we would have that $d-2-k_p \le 0$ and hence non-precritical waves cannot exist, a contradiction.

(2) By Theorem 1.1, $k_\sim = 2$. Thus, $f|J$ has two wandering critical points c, d with distinct grand orbits, and all their forward images are endpoints of J. Now, the order of a critical point is at least 2; if the order of a critical point z is greater than 2 then, since the image of z must be an endpoint of J we see that f has to be at least 3-to-1 at z which implies that $c = d$, a contradiction. Hence $N(c) = N(d) = 2$ which in turn implies that no vertex ever maps into a critical point.

(3) Translating the results of Theorem 1.1 into the language of $f|J$ we see that if there are non-precritical waves x', x'' then they have the same grand orbit and also $N(x') = N(x'') = 3$.

(4) Follows from Theorem 1.3. □

2. Main Theorem. We prove our Main Theorem by establishing several facts concerning possible behavior of waves of J. In the arguments we introduce some new ideas but also rely upon the tools developed in [12, 2-4]. One of such tools is **growing trees** (see [12, 2-4]).

By a **tree** we mean a connected compact one-dimensional branched manifold with no subsets homeomorphic to a circle. In this topological context we can still use combinatorial notions (the **order** $\mathrm{ord}_T(a)$ of T at a point $a \in T$, **endpoints** (of T), **vertices** (of T), **edges** (of T)) without confusion. An **arc** (in T) is a subset of T homeomorphic to an interval. The absence in T of sets homeomorphic to circles makes the arc $[a, b]$ with endpoints $a, b \in T$ well-defined; the notation like $(a, b], [a, b), (a, b)$ is self-explanatory. The numbers of edges, endpoints, vertices of T are finite.

We also need another notion. Given a tree W and a point $a \in W$, consider all arcs $[a, b] \subset W$ such that (a, b) contains no vertices/critical points of W. Call two arcs $[a, b]$ and $[a, b']$ **equivalent** if $(a, b) \cap (a, b') \neq \emptyset$; clearly, equivalent arcs are ordered by inclusion. Classes of equivalence of arcs $[a, b]$ of W are called **germs** of W at a. One can say that a germ of a tree W at $a \in W$ is a pair (a, S), where S is an **infinitesimal** interval in W with one endpoint at a; in that sense a germ may be **contained** in a tree. On the other hand, if there are two trees $W \subset T$ then a germ in T may or may not be contained in W. The image of a germ (a, S) under a map g with finitely many critical points is defined as $g(a, S) = (g(a), g(S))$ with $g(S)$ defined as the germ at $g(a)$ containing g-images of intervals from S. In particular, we may speak of the image of a germ contained in a tree.

Let X be a metric space, $g : X \to X$ be a continuous map. Given a sequence of sets $R_0 \subset R_1 \subset R_2 \subset \dots$, denote the set $\bigcup_{i=0}^{\infty} R_i$ by R_∞. This sequence (and the set R_∞) is called a **generalized growing tree** if the following holds: (a) $R_i \subset R_{i+1} \subset R_i \cup g(R_i)$, (b) R_n is a tree for any n, and (c) there is a finite set of **critical** points $C_g = \{c_1, \dots, c_k\} \subset R_0$ with $g|R_\infty$ injective in some neighborhood of any $x \in R_\infty \setminus C_g$. Also, a point $x \in R_\infty$ is called a **vertex of** R_∞ if x is a vertex of some R_n. The definition of a **growing tree** given in [2, 3] is a bit different; namely here in (a) we only require that $R_i \subset R_{i+1} \subset R_i \cup g(R_i)$ for every i while in [2, 3] when we defined a **growing tree** T_∞ we required that $T_i \subset T_{i+1} = T_i \cup g(T_i)$ for every i. This is the only difference, and it is a subtle but important one (e.g., for growing trees we always have that $T_n = \bigcup_{i=0}^{n} g^i(T_0)$ which is not necessarily true for generalized growing trees).

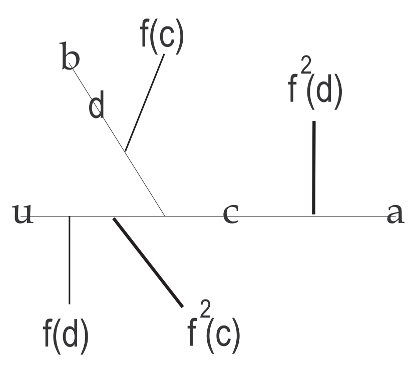

FIGURE 2.1. A growing tree

In [12, 2, 3] the specific growing tree $T_0 \subset T_1 \subset \dots$ is constructed. Since in the cubic case by Corollary 1.4 J is a dendrite, we introduce our definitions under this assumption. Also, from now we fix notation T_∞ for the tree constructed in [12, 2, 3]. Let us now introduce the construction which is defined as follows (see [12, 2, 3]):

(1) choose a non-dividing fixed point a and connect it with its preimages by arcs within J which gives the **initial** tree T_0;
(2) iterate T_0 forward thus getting $T_1 = T_0 \cup f(T_0), T_2 = T_1 \cup f(T_0)$ etc.

All properties of growing trees are easily satisfied by T_∞; in particular, it is shown in [2, 3] that all critical points of f belong to T_0. It is also worth mentioning here that despite the terminology, growing trees may happen to be finite. For example, if we consider the tree T_0 as defined above, and if all critical points of f map back into T_0 then $f(T_0) \subset T_0$ and so in fact $T_\infty = T_0$. However this is not a very interesting case for us because as it easily follows, in this case there are no non-precritical waves.

Figure 2.1 shows the tree T_2 in the cubic case. Different width of line represents different iterations of the map f; moreover, letters a, b, u, c, d are located on the picture exactly where the corresponding points are on the tree while $f(c), f(d), f^2(c)$, $f^2(d)$ are moved off the tree to avoid overloading the picture. We use the notation $[a_1, a_2, \dots, a_k]$ for the smallest connected set containing points $a_1, a_2, \dots, a_k$; then $T_0 = [a, b, u]$ while $T_1 = [a, f(c), b, u, f(d)]$ and $T_2 = [a, f^2(d), f(c), b, u, f(d), f^2(c)]$.

The following lemma is proven in [2, 3].

Lemma 2.1 ([2, 3]). *Let $T' \subset T$ be two trees. Then the set $T \setminus T'$ consists of finitely many components each of which is a tree itself. Moreover, given a component α of $T \setminus T'$, the set $\overline{\alpha} \cap T'$ consists of a single point $b(\alpha)$.*

In the future the point $b(\alpha)$ is called the **basepoint** of α, and the germ of α at $b(\alpha)$ is called the **basegerm** of α.

Lemma 2.2. *Suppose that $A_0 \subset A_1 \subset \dots$ is a sequence of trees in a dendrite (e.g., in J). Then the maximal diameter of components of $(\cup_{i=0}^\infty A_i) \setminus A_m$ converges to* 0 *as $m \to \infty$.*

Proof. Choose $\varepsilon > 0$. Denote $\cup_{i=0}^\infty A_i$ by A_∞. There may be infinitely many components of $A_\infty \setminus A_m$, yet only finitely many of them can have diameter greater then ε because our dendrite is locally connected and contains A_∞. Any component of $A_\infty \setminus A_{m+1}$ of diameter greater than ε is contained in a well-defined component of $A_\infty \setminus A_m$ of diameter greater than ε. If on each step in this process there are components of diameter greater than ε then by Ramsey type theorem there exists an infinite nested sequence of such components $S_0 \supset S_1 \supset \dots$. Since they all are of diameter greater than ε their intersection $S = \cap S_i$ is non-empty. However, $S \subset A_\infty \setminus (\cup_{i=0}^\infty A_i) = \emptyset$, a contradiction. □

We need some other general results which were obtained in [2, 3]. We also use a well-known fact (see, e.g., Lemma 3.8 [5]) according to which any fixed point a of $f|J$ has a neighborhood U in J such that every point $x \in U, x \neq a$ exits U (a point x like that may be called **mildly repelling**).

Lemma 2.3 below is applicable to laminations of all degrees. To state it we need the following definition: an endpoint of T_{n+1} which does not belong to T_n is called an **outer endpoint** of T_{n+1}.

Lemma 2.3 ([2, 3]). *Let $\sim$ be a lamination. Then the following holds.*

(1) *All critical points of $f|J$ belong to the initial tree T_0.*
(2) *All outer endpoints of T_n are the f^n-images of critical points of f.*
(3) *Any germ $(a, S) \subset T_\infty$ such that a is not an endpoint of J eventually maps inside T_0.*
(4) *If $x \in J$ is not an endpoint of J then for some integer n we have $f^n(x) \in T_0$.*
(5) *If y is a non-precritical vertex of J then for some n, m the point $f^n(x)$ is a vertex of T_m which has the same order in both J and T_m.*
(6) *If y is a non-precritical wave of T_{m-1} then for the least k such that $f^k(y)$ is not a vertex of T_{m-1} we have that $f^k(y)$ is either a basepoint of one of the components of $T_m \setminus T_{m-1}$, or a vertex of such component.*
(7) *For every integer there are only finitely many fixed points of $f^n|J$.*

Proof. Claims (1)-(6) are obtained in [2, 3]. Claim (7) immediately follows from Lemma 3.8 ([5]). □

From now on we prove the Main Theorem. So, in the rest of the section we assume that $\sim$ is a cubic lamination such that the quotient space J has a non-precritical wave. We rely upon Corollary 1.4 and use the notation from it. Let us consider the growing tree T_∞. By Lemma 2.3(1) $c, d \in T_0$. By Corollary 1.4(2) and Lemma 2.3(2), points $f^n(c)$ and $f^n(d)$ are the only outer endpoints of T_n. Also, by Lemma 2.3(5) and Corollary 1.4(3) we may assume that for some integer Z a point $x \in T_Z$ is a non-precritical wave of J of order 3 in both T_Z and J such that any its image $f^i(x)$ has the same order 3 in J and in T_{Z+i} with $i \geq 0$.

Consider T_m with $m > Z$ (considering growing trees below we always assume that $m > Z$, i.e. we work with trees for whom x is a vertex). Then the outer endpoints of T_m are $f^m(c) = c_m$ and $f^m(d) = d_m$. The points c_m and d_m define the components C''_m and D''_m of $T_m \setminus T_{m-1}$ which contain c_m and d_m respectively; denote their closures by C_m and D_m. By Lemma 2.1 C_m, D_m are "attached" to T_{m-1} at their basepoints u_m, v_m. Clearly, C_m and D_m have their well-defined basegerms at the basepoints u_m, v_m respectively. There are two possibilities for the sets C_m and D_m. First, it may happen that $C_m \neq D_m$ (see Figure 2.2 on which the tree T_{m-1} is shown "symbolically" as a segment of a thin straight line)).

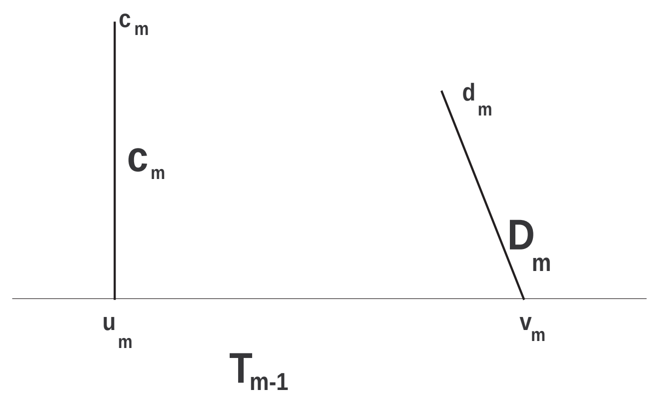

FIGURE 2.2. The case when $C_m \neq D_m$

However, the sets C_m and D_m may coincide; then there is a unique triod-shaped component of $T_m \setminus T_{m-1}$ with the basepoint $u_m = v_m$ and outer endpoints c_m and d_m (see Figure 2.3 where T_{m-1} is shown the same way as on Figure 2.2). In this case we denote the branch point of $C_m = D_m$ by z_m.

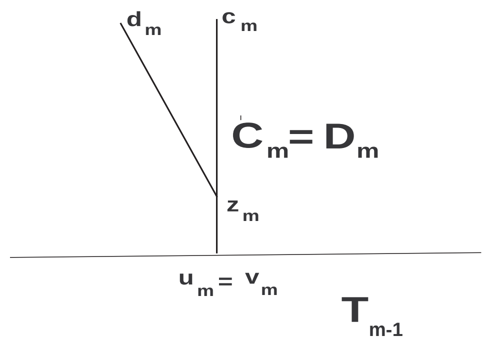

FIGURE 2.3. The case when $C_m = D_m$

By $\hat{C}_m$ we denote the closure of the component of $J \setminus \{u_m\}$ containing c_m and by $\hat{D}_m$ we denote the closure of the component of $J \setminus \{v_m\}$ containing d_m.

The following lemma is important in the proof of the Main Theorem.

Lemma 2.4. *For any $\varepsilon > 0$ and big enough m there exists $k \geq 0$ such that $f^k(d_m) \in \hat{C}_m$ and $d(f^k(d_m), c_m) < \varepsilon$.*

Proof of Lemma 2.4. We prove the lemma by establishing a series of claims. Observe that Figures 2.2 and 2.3 can be considered as illustrations to some of them.

Claim A. *Points u_m and v_m do not belong to the orbits of critical points, are not endpoints of T_i for any i and hence are vertices of T_m.*

Proof of Claim A. Indeed, if u_m is an endpoint of T_i for some i then u_m is an image of a critical point. However, u_m is not an endpoint of T_m because at least one more germ of T_m grows out of u_m compare to the germs of T_{m-1}, namely the basegerm of C_m. Since all the orbits of critical points consist of endpoints of J, this is a contradiction. Similarly, v_m is not an endpoint of T_i for any i. Clearly this implies the rest of the lemma. □

Assume that $M > Z$ is chosen big enough to guarantee that the maximal diameter of a component of $T_\infty \setminus T_m$ for any $m > M$ is less than $\varepsilon/3$ (this is possible by Lemma 2.2).

Claim B. *If $C_m \cap D_m \neq \emptyset$ then Lemma 2.4 holds.*

Proof of Claim B. Left to the reader. □

By Claim B from now on we assume that C_m and D_m are disjoint (and hence $C_m = [u_m, c_m], D_m = [v_m, d_m]$ and $u_m \neq v_m$).

Claim C. *At least one of the points u_m, v_m belongs to the forward orbit of x.*

Proof of Claim C. Follows from Lemma 2.3(6) and the fact that C_m, D_m are arcs. □

Now we continue the proof by way of contradiction. Observe that by the choice of M if the former claim of the lemma holds then so does the latter. Thus from now on in the proof of the lemma we make the following assumption.

Assumption Z. *For some $m > M$ there exists no k such that $f^k(d_m) \in \hat{C}_m$.*

Below we will use the following a bit non-conventional terminology: given two trees which have a unique point a in common we say that one of them (usually perceived as the smaller one) **sticks out** of the other one (at the point a). Now we introduce a useful for the future construction. Define a sequence of sets $T'_r, r \ge m-1$ inductively as follows. Set $T'_{m-1} = T_{m-1}$ and $T'_m = T'_{m-1} \cup D_m$. Then T'_m is a tree which coincides with the union of T'_{m-1} and an arc $D'_m = D_m = [v_m, d_m]$ which sticks out of T'_{m-1}. Now, for every $j \ge m$ we set $T'_{j+1} = T'_j \cup f(T'_j \setminus T'_{j-1})$. In particular, $T'_{m+1} = T'_m \cup f(D'_m) = T'_{m_1} \cup D'_m \cup f(D'_m)$, and so on. In Claim D we show that $T'_{m-1} \subset T'_m \subset \dots$ is a generalized growing tree with specific properties.

Claim D. *The following facts hold.*

1. *The sequence $T'_{m-1} \subset T'_m \subset \dots$ is a generalized growing tree: for any two trees $T'_{i-1} \subset T'_i$ there is an arc $D'_i = [v'_i, d_i]$ sticking out of T'_{i-1} with an endpoint $d_i = f^i(d)$ such that $T'_i = T'_{i-1} \cup D'_i$ and $T'_{i+1} = T'_i \cup f(D'_i)$.*
2. *$T'_{i+1} = T'_{m-1} \cup (\cup_{j=m}^i D'_j) \cup f(D'_i) = T'_{m-1} \cup (\cup_{j=m}^{i+1} D'_j))$.*
3. *The basepoint v'_{m+i} of sets D'_{m+i} do not belong to the orbits of critical points and are vertices of T'_{m+k} with $k \ge i$.*

Proof of Claim D. Observe first that the claim (2) of the lemma follows from the claim (1) and the construction. Observe also that if (1) holds then repeating the arguments from the proof of Claim A we can easily prove (3). Thus it remains to establish (1) which we do by induction.

First notice that the base of induction holds. Now, suppose that all the properties listed in (1) are satisfied for all numbers $m-1, m, \dots, m+k$ and show that they are then satisfied for $m+k+1$. By induction $T'_{m+k} = T'_{m+k-1} \cup D'_{m+k}$ and by the construction we define T'_{m+k+1} as $T'_{m+k} \cup f(D'_{m+k})$. Consider the point $v'_{m+k} \in T'_{m+k-1}$ and show that $f(v'_{m+k}) \in T'_{m+k}$. By (3) and induction v_{m+k} does not belong to the orbit of a critical point and is not an endpoint of T'_r for any r. Now, since $v'_{m+k} \in T'_{m+k-1}$ then either $f(v'_{m+k}) \in T'_{m+k-1}$, or $f(v'_{m+k}) \in C_m$, or $f(v'_{m+k}) \in D'_{m+k}$ (by induction the only two places where points can exit T'_{m+k-1} are D'_{m+k} and C_m). Let us show that $f(v'_{m+k})$ cannot belong to C_m. Indeed, if v'_{m+k} is mapped into C_m then $f(D'_{m+k})$ sticks out of C_m because v'_m is not a critical point and a neighborhood of v'_{m+k} in T'_{m+k-1} maps onto a neighborhood of $f(v'_{m+k})$ in C_m thus "occupying" all available germs of C_m at $f(v'_m)$ and "forcing" $f(D'_{m+k})$ to stick out of C_m. Since the point d_{m+k+1} is obviously an endpoint of $f(D'_{m+k})$ this implies that $d_{m+k+1} \in \hat{C}_m$ and contradicts Assumption Z. So, $f(v'_{m+k}) \notin C_m$ and hence $f(v'_{m+k}) \in T'_{m+k}$.

We conclude that $f(D'_{m+k})$ is an arc connecting $f(v'_{m+k}) \in T'_{m+k}$ and d_{m+k+1}. Since J is a dendrite we see that in general $f(D'_{m+k})$ is the union of two concatenated arcs, one of which is the arc $[f(v'_{m+k}), v'_{m+k+1}]$ contained in T'_m while the other is $D'_{m+k+1} = [v'_{m+k+1}, d_{m+k+1}]$. Observe that if $f(D'_{m+k})$ sticks out of T'_{m+k} then $D'_{m+k+1} = f(D_{m+k})$ and $f(v'_m) = v'_{m+1}$. Otherwise (i.e. if $f(D'_m)$ turns inside T'_m) the arc $[f'(v_{m+k}), v'_{m+k+1}]$ is non-degenerate and $D'_{m+k+1} \subsetneq f(D'_{m+k})$. In any case, this inductively proves the first claim of the lemma and therefore the entire lemma. □

Essentially, Claim D follows from Assumption Z. If Assumption Z failed then some v'_{m+i} could belong to C_m, and the corresponding T'_{m+i} defined as above would not be a tree because it would then be disconnected.

In what follows we will need another non-conventional term. Suppose that we are given two sequences of trees, A_i and B_i, such that for all $0 \le i \le k-1$ the tree A_i sticks out of B_i while A_k has more than one common point with B_k (usually these trees will be dynamically defined). Then we say that at the moment k the tree A_{k-1} **turns (inside B_k)**, and the moment k when it happens is said to be the **first turning moment** (for A_k). We think that this terminology helps visualize the proofs which justifies its introduction.

In fact one of important and general (applicable to laminations of all degrees) observations concerning the growing tree $T_0 \subset T_1 \subset T_2 \subset \dots$ deals exactly with the phenomenon of turning. Indeed, fix m and consider the tree $T_m = B_0$ and a component $A = A_0$ of $T_{m+1} \setminus T_m$. Then as B_i we take T_{m+i} and as A_i we take $f^i(A)$. Consider the basepoint a of A and the basegerm (a, S) of A. Then by Lemma 2.3(3) the germ (a, S) eventually maps into T_0, so definitely there will be the first moment j when it will map into B_j. This is the first turning moment we introduced in the previous paragraph. More explicitly - and without our terminology - one can say that j is the least such number i that $f^i(A)$ and T_{m+i} are non-disjoint (observe, that when $i = 0$ the sets $T_m = B_0$ and $A = A_0$ are disjoint and that for every i we have $f^i(a) \in T_{m+i}$).

In general at the first turning moment of A a number of combinatorial (in the dynamical sense) events may take place. We illustrate only one simple way in which this can happen because the picture is applicable in the cubic case. Namely, assume that a is not an endpoint of T_m and does not pass through a critical point before A turns. Assume also that $f^{i-1}(A)$ is an arc and so are all other components of $T_{m+i} \setminus T_{m+i-1}$. Then the fact that i is the first turning moment of A implies that actually $f^i(a)$ is a vertex of T_{m+i}. Indeed, under f the basegerm of $f^{i-1}(A)$ maps into T_{m+i}, but the germs of T_{m+i-1} at a do so as well, and since by the assumptions on a there are at least two of them we see that there are at least 3 germs of T_{m+i} at $f^i(a)$ as claimed. This is sketched on Figure 2.4 (thick lines show the images of A).

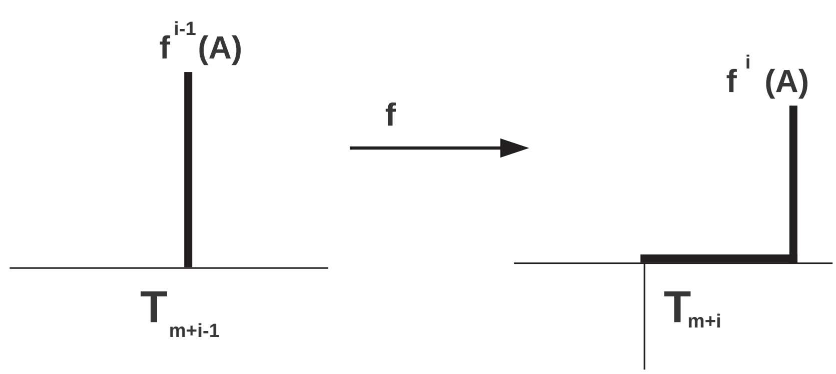

FIGURE 2.4. The turning moment

Denote by S_k the set of all basepoints of sets D'_{m+i}. By Claim D all these basepoints are vertices of T'_{m+k}. Moreover, by Claim D the set S_k together with the vertices of T_{m-1} form the set of all vertices of T'_{m+k}. The set S_k can be divided into orbit segments of some particular vertices from S_k. Indeed, first these are images of $v_m = v'_m$ taken over the period of time when arcs $f^s(D'_m)$ do not turn inside trees T'_{m+s-1}. At the **first turning moment** (when $f^{s_1}(D'_m)$ turns inside T'_{m+s_1-1}) a new basepoint v'_{m+s_1} is created and then an initial segment of its orbit is included in the set S_k of vertices, namely the initial segment until the **next turning** moment $m+s_1+s_2$, etc. We summarize these observations in Claim E below.

Claim E. *The set S_k can be divided into orbit segments of points*

$$\{v'_m, f(v'_m) = v'_{m+1}, \dots, f^{s_1-1}(v'_m)\} = I_1; \{v'_{m+s_1}, \dots, f^{s_2-1}(v'_{m+s_1})\} = I_2; \dots$$

where $m+s_1, m+s_1+s_2, \dots$ are the turning moments as defined above.

Now we can describe the strategy of the proof of the lemma. We will show that the set V'_{m+k} of all vertices of any tree $T'_{m+k}, k \ge 0$ consists of points which are preperiodic or preimages of u_m (in particular v_m is preperiodic or preimage of u_m). On the other hand, if k is big enough the analysis of the behavior of u_m shows that either u_m is preperiodic or it is eventually mapped onto one of the vertices from V'_{m+k}. Since by Claim C the wave x passes through either u_m or v_m we see that x cannot be wandering, a contradiction.

Claim F. *The point $f^{m+l}(d) = d_{m+l}$ does not belong to T'_{m+l-1}.*

Proof of Claim F. Indeed, by Corollary 1.4(2) all images of d are endpoints of J, hence d_{m+l} is an endpoint of T'_{m+l}. Assume by way of contradiction that d_{m+l} belongs to T'_{m+l-1}. Then it is an endpoint of T'_{m+l-1}. By the construction the endpoints of T'_{m+l-1} are either endpoints of T_0 (all of which are preperiodic), or images of c (with which d_{m+l} cannot coincide by Corollary 1.4(2)), or points d_i with $i < m+l$ (so that if $d_{m+l} = d_i$ then d is preperiodic, a contradiction to Corollary 1.4(2)). Thus $d_{m+l} \notin T'_{m+l-1}$. □

The next claim is one of the major ingredients of the proof of Lemma 2.4.

Claim G. *All vertices of T'_{m+k} are preperiodic or preimages of u_m.*

Proof of Claim G. First consider vertices of $T'_{m-1} = T_{m-1}$; by Corollary 1.4(2) none of them is precritical. By way of contradiction consider a wave z of T_{m-1}. Then by Lemma 2.3(6) z is eventually mapped onto u_m or v_m. In the former case we are done with respect to z, in the latter case it is enough to consider points of the set S_k which are not vertices of T_{m-1}. So let us now assume that z is a non-precritical wave in S_k. First let $z = v_m$. Let us follow the orbit of v_m. Observe that if v_m ever maps onto a vertex of T_{m-1} then by Lemma 2.3(6) some future image of v_m coincides with either v_m or u_m as desired. Assume that v_m is never mapped onto a vertex of T_{m-1}. Take the first moment s_1 when $f^{s_1}(D'_m)$ turns inside T'_{m+s_1-1}. At this moment $z = v_m$ maps onto a vertex of T'_{m+s_1-1} which by assumption is not a vertex of T'_{m-1}. However as explained above all such vertices are images of $v_m = z$. Hence $v_m = z$ is preperiodic as desired.

Consider now the case of z being a vertex of T'_{m+k} which is never mapped onto a vertex of T'_{m-1} or onto v_m. By Claim E it is enough to consider the case when z is the initial point $z = v'_{m+s_1+\dots+s_i}$ of an orbit segment I_{i+1}, one of the orbits segments into which S_k is divided by Claim E. Then the point $f^{s_{i+1}}(z)$ is a vertex of $T_{m+s_1+\dots+s_{i+1}-1}$ which either belongs to the same orbit segment I_{i+1} (and so z is preperiodic) or belongs to another orbit segment I_t with $t < i+1$. Clearly, after finitely many steps the point z is "forced" to either become a preperiodic point or be mapped onto a vertex of T'_{m-1} or v_m which as we saw above leads to the desired conclusion. □

To finish the proof of Lemma 2.4 we need the formula for the tree T_{m+j} obtained in Claim H below.

Claim H. $T_{m+j} = T'_{m+j} \cup (\cup_{i=0}^{j} f^i(C_m))$.

Proof of Claim H. By the construction of the standard growing tree $T_i \subset T_{i+1} \subset \dots$ we know that $T_{m+j} = T_{m-1} \cup (\cup_{i=0}^{j} f^i(C_m \cup D_m))$. To prove the claim it is enough to show that $T_{m+j} \subset T'_{m+j} \cup (\cup_{i=0}^{j} f^i(C_m))$ (the opposite containment is obvious). Because of the above formula for T_{m+j} it is sufficient to show that $\cup_{i=0}^{j} f^i(D_m) \subset T'_{m+j} \cup (\cup_{i=0}^{j} f^i(C_m))$. Let us prove it by induction. Clearly, this containment holds for $j = 0$. Assume that it holds for j, i.e. that $\cup_{i=0}^{j} f^i(D_m) \subset T'_{m+j} \cup (\cup_{i=0}^{j} f^i(C_m))$. This implies that $f(\cup_{i=0}^{j} f^i(D_m)) \subset f(T'_{m+j} \cup (\cup_{i=1}^{j+1} f^i(C_m)))$, and since by the construction $f(T'_{m+j}) \subset C_m \cup T'_{m+j+1}$ we see that indeed $\cup_{i=0}^{j+1} f^i(D_m) \subset T'_{m+j+1} \cup (\cup_{i=0}^{j+1} f^i(C_m))$. So we finally conclude that $T_{m+j+1} = T'_{m+j+1} \cup (\cup_{i=0}^{j+1} f^i(C_m))$ as desired. □

The next claim effectively completes the proof of Lemma 2.4.

Claim I. *The point u_m is preperiodic.*

Proof of Claim I. Let us consider the arc C_m until its own **first turning moment** by which we as always mean the least k such that $f^k(C_m)$ turns inside T_{m+k-1}. For each $j \le k-1$ we have the following: the arc C_{m+j} sticks out of T_{m+j-1} and has the image $f^j(u_m)$ of u_m as the basepoint. Then $f^k(u_m)$ is a vertex of $T_{m+k-1} = T'_{m+k-1} \cup (\cup_{i=0}^{k-1} f^i(C_m))$. If $f^k(u_m)$ is a vertex of T'_{m+k-1} then by Claim G it is either preperiodic or mapped onto u_m; either way u_m is preperiodic. Otherwise $f^k(u_m)$ is a vertex of T_{m+k-1} which is not a vertex of T'_{m+k-1}. As follows from the choice of k and Claim H, in fact all the vertices of T_{m+k-1} are divided into two subsets: A which is the set of all vertices of T'_{m+k-1} and $B = \{u_m, f(u_m), \dots, f^{k-1}(u_m)\}$. Hence the only remaining case is when $f^k(u_m) \in B$ which again implies that u_m is preperiodic as desired. □

Finally we are ready to finish the proof of Lemma 2.4. Indeed, by Claim C the forward orbit of the wave x passes through either u_m or v_m. However, u_m is preperiodic by Claim I and v_m is preperiodic by Claim G and Claim I, a contradiction. This shows that Assumption Z leads to a contradiction. Therefore for any $m > M$ there exists k such that $f^k(d_m) \in \hat{C}_m$ and $d(f^k(d_m), c_m)\varepsilon$. □

Observe that Lemma 2.4 is symmetric with respect to the critical points c and d. With respect to the limit behavior of these points it tells us that given ε from some time on any forward image of c can be approximated by some forward image of d, and vice versa. This implies the claim of the Main Theorem which states that $\omega(c) = \omega(d) = A$. By Corollary 1.4(4) we conclude that in fact $\omega(y) = A$ for any wave y.

The second half of the Main Theorem is the claim that both critical points of f are recurrent. We prove that in fact for the wave x chosen above we have $d \in \omega(x)$

(it follows similarly that $c \in \omega(y)$). Clearly together with the conclusions of the preceding paragraph this would complete the proof of the Main Theorem. Observe that our proof of the fact that $d \in \omega(x)$ does not use Corollary 1.4(4). Since by Corollary 1.4(2) no vertex is ever mapped into a critical point we conclude that x does not come closer to d than a certain positive number ε.

Now we need a couple of general properties which can be considered as extensions of well-known properties of interval maps onto certain maps of dendrites. Basically these properties follow from the fact that $f|J$ has no wandering continua (by Theorem 1.2(2)) and that for any n the set of fixed points of f^n is zero-dimensional (Lemma 2.3(7)). First we need the following lemma.

Lemma 2.5. *For every $\varepsilon > 0$ there exists a number $\sigma(\varepsilon) > 0$ such that for any continuum $K \subset J$ with* $\mathrm{diam}(K) \geq \varepsilon$ *we have that* $\mathrm{diam}(f^i(K)) > \sigma(\varepsilon)$ *for any $i \geq 0$.*

Proof. First observe that there exists a finite collection of continua $K_1, \dots, K_m$ in J such that every continuum of diameter greater than ε contains one of K_i's. Indeed, by Theorem 10.27 from [14] there exists a finite tree $T \subset J$ such that the diameter of any component of $J \setminus T$ is less than $\varepsilon/4$. Clearly, there exist finitely many arcs $K_1, \dots, K_m$ in T such that any subcontinuum of T of diameter greater than $\varepsilon/2$ contains a set K_i with $1 \leq i \leq m$. Consider a continuum $K \subset J$ such that $\mathrm{diam}(K) \geq \varepsilon$. Then $K \cap T$ is a continuum itself. Let us show that $\mathrm{diam}(K \cap T) > \varepsilon/2$. Indeed, suppose otherwise. Choose two points $y, y' \in K$ such that $d(y, y') = \mathrm{diam}(K) > \varepsilon$ and also the component Y of $J \setminus T$ containing y and the component Y' of $J \setminus T$ containing y'. Then $\mathrm{diam}(Y) < \varepsilon/4$, $\mathrm{diam}(Y') < \varepsilon/4$ and on the other hand there are points $a \in \overline{Y} \cap K \cap T$, $b \in \overline{Y'} \cap K \cap T$. By the triangle inequality we conclude that $d(a, b) > d(y, y') - d(y, a) - d(b, y') > \varepsilon - \varepsilon/2 = \varepsilon/2$ which implies that $\mathrm{diam}(K \cap T > \varepsilon/2$. Hence there exists $i, 1 \leq i \leq m$ such that $K \supset K \cap T \supset K_i$ as desired.

Now, by Lemma 3.8 ([5]) given a non-degenerate continuum $K \subset J$ there exists $\delta > 0$ such that $\mathrm{diam}(f^j(K)) > \delta$ for any $j \geq 0$. In particular this holds for $K_1, \dots, K_m$. Thus if for every i we choose the lower bound on $\mathrm{diam}(f^j(K_i)), j \geq 0$ and then choose the maximum of these lower bounds we will get the desired number $\sigma(\varepsilon) > 0$. □

Next we discuss another general fact which now deals with preperiodic and precritical points. A set A such that every non-degenerate continuum in S contains a point from A is said to be **condense** in S ("continuum" + "dense"); this term was introduced in [7] where it was used in a totally different context. In the case of finite graphs the fact that a set is condense is equivalent to the fact that it is dense. In general it is not so; the next lemma specifies the situation for dendrites.

Lemma 2.6. *If S is a dendrite then a set A is condense in S if and only if every non-degenerate arc in S contains a point from A.*

Proof. It is enough to show that if every non-degenerate arc in S contains a point of A then so does every non-degenerate continuum. To see that observe that every non-degenerate continuum in S is arcwise connected (see [11]) and therefore contains non-degenerate arcs; since they contain points of A by assumption, we are done. □

In the next lemma we draw one more parallel between $f|J$ and one-dimensional maps (i.e. maps of one-dimensional branched manifolds, or "graphs") and show that some properties of one-dimensional maps hold for maps of dendrites without wandering continua (which then applies to a factor map $f|J$ of z^d under a lamination by Theorem 1.2(2)). A map is said to be **topologically exact** if for any open set U there is a number n such that $f^n(U) = J$. Also, by a **critical point** of a continuous map we mean a point at which the map is not a local embedding.

Lemma 2.7. *Suppose that $f : J' \to J'$ is a continuous self-mapping of a dendrite J' without wandering continua. Then the following holds.*

1. *Preperiodic points are condense in J.*
2. *If the set of fixed point of f^n is zero-dimensional for every n then precritical points are condense in J.*
3. *If $f : J \to J$ is a factor of z^d under a lamination then it is topologically exact.*

Proof. (1) By way of contradiction assume otherwise. Then there exists an arc I whose forward orbit avoids periodic points. On the other hand by assumptions I is not wandering. So we may assume that there exists $k > 0$ such that $f^k(I) \cap I \neq \emptyset$. Consider the set $A' = \cup_{i=0}^{\infty} f^{ik} I$ and then the set $A = \overline{A'}$. By the construction $f^k(A) \subset A$. We want to study the restriction $f^k|_A$. By the assumption A' contains no periodic points. On the other hand there must exist f^k-fixed points in A because A is a dendrite itself (see, e.g., [14], Theorem 10.31). We conclude that all f^k-fixed points in A are endpoints of A.

Choose a fixed endpoint a of A and consider dynamics in its small neighborhood. Pick a point $b \in A$ very close to a and consider the arc $[b, a]$ and the set $f^k([b, a])$. Since a is an endpoint of A we see that these sets have a non-trivial intersection which has to be an arc $[a, y]$. We can choose a smaller arc $[a, z] \subset [a, y]$ such that all its points map into $[a, y]$. Then since by the assumption all fixed points of $f^k|_A$ are endpoints of A and there are no wandering continua we see that all points of $(a, z_a]$ map farther away from a. Choose a point $z_a \in (a, z)$ which is not a vertex of A (by [14], Theorem 10.23, the set of all vertices of A is countable). Then z_a cuts A into two components; denote that one of them which contains a by R_a. We can always choose z_a so close to a that $\text{diam}(R_a)$ is smaller than $\text{diam}(A)/3$. The set R_a is open and its endpoint z_a does not map into R_a. Repeating this argument for all points from the set F of fixed points of $f^k|_A$ and using the fact that F is compact we can find a finite set B of fixed points of f^k such that $\cup_{a \in B} R_a = W \supset F$.

It is easy to see that if $u, v \in B$ then either R_u and R_v are disjoint, or one of these sets contains the other. Indeed, suppose that $u \notin R_v$ and show that then $z_u \notin R_v$. Indeed, otherwise we get that $R_u \cup R_v = A$ which impossible because $\text{diam}(R_u) < \text{diam}(A)/3, \text{diam}(R_v) < \text{diam}(A)/3$. So, $z_u \notin R_v$ which implies that R_u and R_v are disjoint. Now, suppose that $u \in R_v$. Then there are two possibilities: it may happen that $z_u \in R_v$ in which case $R_u \subset R_v$, or it may happen that $z_u \notin R_v$ in which case $R_v \subset R_u$. Either way the claim is proven, and so we can refine our collection B and assume that all sets $R_t, t \in B$ are pairwise disjoint.

Define a map $h : A \to A$ which maps each $R_a, a \in B$ onto a and is identity elsewhere. Next, consider a dendrite $D = A \setminus W$ and define a new map $g = hf^k$ of D into itself. Clearly, g is a continuous map of a dendrite D into itself, so it must have at least one fixed point. However it cannot be a point of D not coinciding with z_a for some $a \in B$ because of the assumption that all fixed points of $f^k|_A$ are endpoints of A. On the other hand it cannot be z_a with some $a \in F$ because of the choice of points z_a, a contradiction which completes the proof.

(2) By way of contradiction assume otherwise. Then there exists an arc I whose

forward orbit avoids critical points. Applying to I and its iterates the same construction as before we may assume that $A \subset J$ is a dendrite such that $f^k(A) \subset A$ for some k and $f^k : A \to A$ is an embedding (in principle, critical points of f may belong to A but only as endpoints). Let us show that this is impossible.

Indeed, by (1) there are two periodic points x, y of $f^k|_A$. Consider a power g of f such that $g(x) = x, g(y) = y$. Since g (being a power of $f^k|_A$) is an embedding we see that g maps the arc $[x, y]$ onto itself in a homeomorphic fashion. Since by the assumption the set of all f^n-fixed points is zero-dimensional, this implies that there exists a g-fixed point $z \in [x, y]$ attracting points on at least on side in $[x, y]$, a contradiction with the non-existence of wandering continua.

(3) Immediately follows from the fact that $z^d : S^1 \to S^1$ is topologically exact and properties of laminations. □

The assumption that J is a dendrite is necessary here - otherwise the Julia set may contain a Siegel type closed curve, and there are not periodic points in such curves. Also, one can think of claim (1) of Theorem 2.7 as an extension of a well-known fact according to which periodic points are dense in the Julia set of a polynomial. For the interval maps results similar to claims (1) and (2) of Theorem 2.7 are known (see, e.g., [1]).

The last general technical result which we need here is the backward stability of f. We define it as follows: if X is a compact metric space then f is said to be **backward stable** if for any δ there is ε such that for any continuum K with $\text{diam}(K) \le \varepsilon$, any $n \ge 0$ and any component M of $f^{-n}(K)$, $\text{diam}(M) \le \delta$ (similarly, the backward stability at a point can be defined).

The notion extends the classical Lyapunov stability onto backward orbits of non-invertible maps. Essentially, it was first introduced by Fatou who showed that a polynomial $P : \mathbb{C} \to \mathbb{C}$ is backward stable at points not belonging to the limit sets of critical points. Other facts concerning backward stability which follow from classical results (in particular, from the description of the local dynamics at periodic points - see, e.g.,[8]) are that $P : \mathbb{C} \to \mathbb{C}$ is not backward stable at any parabolic periodic point which lies in the Julia set. Obviously, P is not backward stable at attracting periodic points. Thus, the well-known obstacle for the backward stability of a polynomial at a point of its non-wandering set is that the point could be an attracting or neutral periodic point, and in [5] we prove that if $J(P)$ is locally connected then this is the *only* obstacle for backward stability at such a point. Let us also point out that the above discussed way of defining backward stability was introduced in [12].

In the preceding paragraph the map is considered at points on the plane while we are interested in the backward stability of the entire $f|J$ as defined above. This problem was partially solved in [12, 2] and then solved in [5].

Theorem 2.8 [5]. *The map $f|J$ is backward stable.*

Now we are ready to pass on to the proof of the rest of the Main Theorem which states that both critical points of f are recurrent. By Corollary 1.4 there are two possible cases for the sets C_m, D_m.

Case 1. *The sets C_m, D_m are disjoint arcs.*

This was the main case considered in the proof of the fact that with the assumption of the existence of a non- precritical wave we have $\omega(c) = \omega(d)$.

Case 2. *The sets C_m and D_m coincide and are homeomorphic to a triod.*

Below we will prove, that in fact Case 2 is impossible. However to begin with we will have to argue assuming that Case 2 can take place. In order to consider possibilities we need to introduce notation. Recall that the basepoints of C_m, D_m are denoted u_m, v_m respectively. In Case 2 the vertex of $C_m = D_m$ is denoted by z_m. In our arguments an important role is played by the arc I_m^m defined as follows: if Case 1 takes place then $I_m^m = (v_m, d_m]$, if Case 2 takes place then $I_m^m = (z_m, d_m]$. So in any case I_m^m is an arc in D_m whose one endpoint is d_m and who extends inside D_m until it hits a vertex of T_m. We want to pull I_m^m back m times along the orbit of d inside T_{m-1}. By a **pull back** of a continuum $K \subset T_m$ inside T_{m-1} we understand a component of $f^{-1}(K) \cap T_{m-1}$, and if we specify a point from the pull back then we mean the pull back containing this point. For example, I_{m-1}^m is defined as the pull back of I_m^m inside T_{m-1} containing d_{m-1}, I_{m-2}^m is defined as the pull back of I_{m-1}^m in side T_{m-1} containing d_{m-2}, and so on. This defines continua $I_0^m, I_1^m, \dots, I_m^m$, and explains our notation.

Let us now make a couple of useful observations. First, it follows from Lemma 2.2 that $\operatorname{diam}(I_m^m) \to 0$ as $m \to \infty$. By Theorem 2.8 this implies that in fact $\operatorname{diam}(I_i^m) \to 0$ as $m \to \infty$ uniformly over $i, 0 \le i \le m$. Second, I_0^m is a continuum containing d inside which maps onto I_1^m in at most 2-to-1 fashion (the point d_1 has a unique preimage inside I_0^m though), the set I_1^m has d_1 as an endpoint, the set I_2^m has d_2 as an endpoint, etc. The entire collection of sets $I_m^m, \dots, I_0^m$ is denoted by $\mathcal{I}$.

The main step now is to show that in some cases the sets $I_j^m, 0 \le j \le m$ are arcs. Also, we describe the trajectory of a point $z \in T_{m-1}$ assuming that it stays in T_{m-1} for a while and then exits T_{m-1} and gets mapped into I_m^m. Basically, we show that this can happen only if the point z passes through I_0^m.

We want to remind the reader that we call $k+1$ the **turning moment** for C_k if f maps the basegerm of C_k into T_k. Let us show that if $k+1$ is a turning moment for C_k then $f(u_k)$ is a vertex of T_k. Indeed, by Corollary 1.4(2) u_k is not a critical point, so distinct germs at u_k map onto distinct germs. Now, there are at least 2 germs of T_{k-1} at u_k which are *not* the basegerm of C_k (observe that if there is a basegerm of $D_k \neq C_k$ at u_k then as follows from Corollary 1.4(3) $u_k = v_k$ is preperiodic whereas x must pass through u_k or v_k by Lemma 2.3(6), a contradiction with the assumption that x is wandering). The images of all the germs of T_k at u_k are in T_k, hence indeed $f(u_k)$ is a vertex of T_k.

Lemma 2.9. *Let $m+1$ be a turning moment for C_m. Then sets from $\mathcal{I}$ contain no vertices of T_m.*

Proof. First we prove that sets $I_m^m, I_{m-1}^m, \dots, I_0^m$ contain no vertices of T_{m-1}. Even though the pictures are different depending on whether Case 1 or Case 2 take place for m, to begin with a common argument can be suggested which covers both cases. Before we proceed recall that none of the sets I_r^m contains a critical point of f, and hence if a germ/tree sticks out of some I_j^m then its images stick out of images of I_j^m on each step through the interval I_m^m.

By way of contradiction assume that there is a vertex e_j of T_{m-1} in I_j^m, and that j is the greatest number between 0 and m with this property. Then $j < m$ because no part of T_{m-1} sticks out of I_m^m. We denote a component of $T_{m-1} \setminus \{e_j\}$ which sticks out of I_j^m at e_j by E_j and denote the basegerm of E_j at e_j by (e_j, R). Since (e_j, R) is contained in T_{m-1}, its image $f(e_j, R)$ is contained in T_m. This shows that in fact $j < m-1$. Indeed, otherwise $f(e_j, R)$ sticks out of I^m_m while no germ of T_m sticks out of I^m_m, a contradiction. Hence $j < m-1$ and therefore $f(e_j, R)$ sticks out of I^m_{j+1} where $j+1 < m$.

Next we prove the following claim.

Claim J. *The set D_m cannot stick out of the set I^m_r with $0 \le r \le m$.*

Proof of Claim J. We may assume that $r < m$. It follows that if D_m sticks out of I^m_r then the set $f^{m-r}(D_m)$ sticks out of $f^{m-r}(I^m_r) \subset I^m_m$. This implies that $\hat{D}_m$ is mapped by f^{m-r} inside itself which contradicts topological exactness of f. □

Consider the germ $f(e_j, R)$. On the one hand, $f(e_j) \in T_{m-1}$, on the other hand $f(e_j, R)$ is a germ of T_m which sticks out of T_{m-1} (by the choice of j the germ $f(e_j, R)$ cannot belong to T_{m-1}). Hence it is the basegerm of either C_m or D_m. By Claim J the case when $f(e_j, R)$ is the basegerm of D_m which sticks out of I^m_{j+1} is impossible. Therefore under our assumptions Case 2 cannot hold for m because otherwise $f(e_j, R)$ must be the basegerm of D_m (recall that in this case $C_m = D_m$).

Thus we may assume that Case 1 takes place for m and that $f(e_j, R)$ is the basegerm of C_m which sticks out of $I^m_{j+1}, j+1 < m$. By the choice of m if $f(e_j, R)$ is the basegerm of C_m then on the next step the germ $f(e_j, R)$ has to map into T_m (because $m+1$ is a turning moment for C_m). So, the germ $f^2(e_j, R)$ belongs to T_m and sticks out of I^m_{j+2} with $j+2 \le m$. If $j+2 = m$ then we get that a germ of T_m sticks out of I^m_m which is impossible. If $j+2 < m$ then there are several possibilities, and we will show that none of them can take place. First, $f^2(e_j, R)$ can belong to T_{m-1}. In this case $f^2(e_j)$ is a vertex of T_{m-1} because there are at least three germs of T_{m-1} at $f^2(e_j)$, namely the two germs of I^m_{j+2} and $f^2(e_j, R)$. This is impossible by the choice of j as the maximal number between 0 and m such that I^m_j contains a vertex of T_{m-1}. Next, the germ $f^2(e_j, R)$ can be the basegerm of D_m which then sticks out of I^m_{j+2} with $j+2 < m$. By Claim D, this is impossible either. Finally, $f^2(e_j, R)$ can be the basegerm of C_m. However, $f(e_j, R)$ is the basegerm of C_m too. Thus in this case $f(\hat{C}_m) \subset \hat{C}_m$ which contradicts the topological exactness of f. We conclude that none of the three possibilities can take place and so no vertices of T_{m-1} belong to the sets from $\mathcal{I}$.

It remains to show that u_m or v_m cannot belong to the sets $I^m_m, I^m_{m-1}, \dots, I^m_0$. There are several cases when it follows from the previous results. Namely, by Claim J the basepoint $u_m = v_m$ of $C_m = D_m$ cannot belong to sets from $\mathcal{I}$, so we may assume that Case 1 takes place. Moreover, $u_m \notin I^m_m$ simply by the definition.

By Claim J it remains to consider the case when $C_m \ne D_m, u_m \in I^m_r, r < m$. Then there are at least three germs of T_m at u_m: the basegerm of C_m and two germs of I^m_r. Since u_m is not a critical point then these three germs have distinct images. Moreover, the image of the basegerm of C_m is contained in T_m because C_m turns, and the images of germs of I^m_r at u_m are contained in T_m because $I^m_r \subset T_{m-1}$. Hence $f(u_m)$ is a vertex of T_m, and since it belongs to $I^m_{r+1}, r+1 \le m$ it cannot be a vertex of T_{m-1} by the proven above. By Claim J $f(u_m) \ne v_m$ either. So the only remaining case is when $f(u_m) = u_m$. However then u_m is a dividing fixed point of f which has to belong to T_0 by Lemma 2.3(4) while the set I^m_{r+1} is disjoint from T_0 (if it is not then applying f^{m-r-1} we see that there are points of T_{m-1} is I^m_m which is impossible). Thus we see that indeed v_m or u_m cannot belong to the intervals $I^m_m, I^m_{m-1}, \dots, I^m_0$. □

Let us now prove the following lemma which uses the notation from the previous lemma (e.g., $m+1$ is the turning moment for C_m).

Lemma 2.10. *Suppose that for a point y there exists a number k such that $y \in T_{m-1}$, $f(y) \in T_{m-1}, \dots, f^{k-1}(y) \in T_{m-1}, f^k(y) \in I^m_m$. Then $f^{k-1}(y) \in I^m_{m-1}$, $f^{k-2}(y) \in I^m_{m-2}, \dots$ and in general $f^{k-i}(y) \in I^m_{m-i}$ for any $i \le \min\{k, m\}$.*

Proof. We prove the lemma by way of contradiction. Consider the initial segment of the orbit of the point y. If the conclusions of the lemma fail then there exists $0 < j \le k$ such that $f^i(y), 0 \le i \le j-1$ does not belong to the union of the arcs $I^m_m, I^m_{m-1}, \dots, I^m_0$ while $f^j(y)$ belongs to an interval I^m_r with some $r > 0$. In other words, the first time y enters the intervals $I^m_m, I^m_{m-1}, \dots, I^m_0$ takes place outside I^m_0.

Consider the point $f^{j-1}(y) \in T_{m-1}$. Since by Lemma 2.9 no parts of T_m stick out of the intervals $I^m_m, I^m_{m-1}, \dots, I^m_0$ then a small neighborhood U of $f^{j-1}(y)$ in T_{m-1} is mapped by f onto a small neighborhood of $f^j(y)$ in I^m_r. This implies that U is an arc; we can think of it as a "vector" starting at its initial point, ending at its terminal point and pointing in the direction which, when mapped forward by f, corresponds to the direction **toward** d_r. Extend the "vector" U beyond its terminal point until it hits an endpoint of T_{m-1} or a critical point (whichever comes first) and denote the new vector U' and its new (compare to U) terminal point by w. Observe that by Lemma 2.10 there are no vertices of T_{m-1} in U'. The f-image of U' must end within I^m_r, and since $f(w)$ is an image of a critical point we see that $f(w) = d_r$ (otherwise we contradict Lemma 2.9). However this implies that $w = d_{r-1}$ and that $f^{j-1}(y)$ already belongs to I^m_{r-1}, a contradiction. □

Consider now Case 1 and Case 2 with respect to the behavior of sets C_m, D_m as $m \to \infty$. To do so we need more notation. Namely, the endpoint of I^m_k which is not equal to an image of d will be denoted by w_{m-k} and the germ of I^m_k at w_{m-k} will be denoted by (w_{m-k}, A).

Lemma 2.11. *Let m be a large number such that $C_m \ne D_m = I^m_m$ and C_m turns. Then the f-image of the basegerm of C_m cannot be a germ (w_{m-k}, A) for some $0 \le k < m$.*

Proof. We prove the lemma by way of contradiction. Assume that the basegerm of C_m is (u_m, B) and that $f(u_m, B) = (w_{m-k}, A)$ for some $0 \le k < m$. Then the map f^k maps (w_{m-k}, A) onto the basegerm (v_m, F) of D_m. Let us follow the forward orbit of D_m until it turns for the first time; denote the turning moment for D_m by $m+s$ (so that s is the least number such that f^s maps the basegerm of D_m inside T_{m+s-1}). By Lemma 2.3(6) and because $f^{k+1}(u_m) = v_m$ we see that actually both u_m and v_m are waves, and so v_m never maps into u_m. Thus at the moment $m+s$ the point $f^{m+s}(v_m)$ is not equal to u_m or v_m.

Next we need the following claim.

Claim K. *For any $i, 0 \le i \le s$, we have that the point $f^i(v_m)$ belongs to T_{m-1} and if $i < s$ then the set $f^i(D_m) = D_{m+i}$ sticks out of T_{m-1}.*

Proof of Claim K. Indeed, if v_{m+i} does not belong to T_{m-1} for the first time then one of the following two cases takes place: a) $v_{m+i} \in C_m$, b) $v_{m+i} \in D_m$. In the case a) $f^i(D_m)$ sticks out of C_m because the other germs of T_{m-1} at v_{m+i-1} have images which occupy germs of C_m at v_{m+i} while v_{m+i-1} is not critical (by Corollary 1.4(2)). Then the assumptions on the behavior of C_m imply that f^{k+i+1} maps $\hat{C}_m$ into itself, a contradiction. If Case b) holds then similarly f^i maps $\hat{D}_m$ into itself, a contradiction. The other statement of the claim is obvious. One can visualize the dynamics by thinking of D_m as "sliding" on the "surface" of T_{m-1}. □

The fact the point $f^{m+s}(v_m)$ is not equal to u_m or v_m and Claim K imply that $f^{m+s}(v_m)$ is a vertex of T_{m-1}. Then because it is wandering Lemma 2.3(6) and our assumptions concerning the behavior of u_m and v_m imply that some forward image of v_m coincides with v_m, a contradiction. □

For the sake of completeness we now prove the following lemma.

Lemma 2.12. *There exists Q such that for any $i \geq Q$ Case1 takes place.*

Proof. We prove the lemma by way of contradiction. First suppose that Case 1 and Case 2 take place for infinitely many values of m. Then there are moments when Case 1 is replaced by Case 2 and vice versa. Consider a moment m when Case 1 is replaced by Case 2, and assume that m is sufficiently large. Then C_m and D_m are disjoint arcs while $C_{m+1} = D_{m+1}$ is a triod. This implies that there are points $s_m, t_m \in C_m, p_m, q_m \in D_m$ such that $f(s_m) = f(p_m) = u_{m+1} = v_{m+1}$ and $f(t_m) = f(q_m) = z_{m+1}$. Now, let us follow the set $C_{m+1} = D_{m+1}$ until its turning moment. That is, let $k \geq m+1$ be the least such number that $C_k = D_k$ turns (the basegerm of $C_k = D_k$ maps inside T_k). By the proven above, this implies that for the interval $I^k_k = (z_k, d_k]$ its backward orbit in T_{k-1} is $I^k_{k-1}, \dots, I^k_0$ continued by preimages of I^k_0. Now we continue by proving a series of claims. Claim K1 below is very close to Claim K from the proof of Lemma 2.11; in Claim K1 we look at the behavior of the set D_{m+1} from the moment $m+1$ through the moment k (that is, within the segment of the orbit when no turns take place).

Claim K1. *For any $i, 0 \leq i \leq k-m-1$, we have that the point $f^i(u_{m+1})$ belongs to T_m and the set $f^i(D_{m+1}) = D_{m+i+1}$ sticks out of T_m.*

Proof of Claim K1. Indeed, if u_{m+i+1} does not belong to T_m for the first time then it must belong to D_{m+1}. By the choice of k as the first turning moment after $m+1$ and because $m+i+1 \leq k$ this implies that $f^i(D_{m+1}) = D_{m+i+1}$ sticks out of D_{m+1}. Therefore $f^i(\hat{D}_{m+1}) = \hat{D}_{m+i} \subset \hat{D}_{m+1}$, a contradiction with the topological exactness of f. So, we have $f^i(u_{m+1}) \in T_m$ for $0 \leq i \leq k-m-1$, and by choice of k the set $f^i(D_{m+1}) = D_{m+i+1}$ sticks out of T_m. □

The next claim studies the behavior of a wave $x \in T_{k-1}$.

Claim L. *If $x \in T_k$ is a wave then $x = f^s(z_{m+1}), 0 \leq s \leq k-m-1$.*

Proof of Claim L. Suppose that x is a vertex of T_{k-1}. Consider the first time x stops being a vertex of T_{k-1}. By Lemma 2.3(6) it can only happen when $f^r(x) = u_k$ or when $f^r(x) = z_k$. Moreover, germs of T_{k-1} at x map by f^r into germs of T_k at u_k or z_k respectively because otherwise we would have that $N(f^r(x)) > 3$, a contradiction with Corollary 1.4(3). Suppose that $f^r(x) = z_k$ and consider a tiny interval $U = (x, y] \subset T_{k-1}$ such that $f^r(x, y] \subset I^k_k = (z_k, d_k]$ and for any $j < r$ we have $f^j(U) \subset T_{k-1}$. Such interval U exists because otherwise there is a smaller than r power of f which maps x into a non-vertex of T_{k-1}. Then by Lemma 2.10, by Claim K and by the assumptions from above about creation of $C_{m+1} = D_{m+1}$ we see that it can happen only if $x = f^s(z_{m+1}), 0 \leq s \leq k-m-1$.

It remains to consider the case when $f^r(x) = u_k$. We prove below that u_k is not a wave which excludes this case. Moreover, this shows that of the vertices of T_k which are not vertices of T_{k-1} only z_k could be a wave and thus completes the proof of the claim.

So, let us show that u_k is not a wave. By the choice of k on the next step $f(u_k)$ remains a vertex of T_k (the image of the basegerm of D_k belongs to T_k, and so are the images of the two germs of T_{k-1} at u_k). If this vertex $f(u_k)$ is z_k then it is easy to see that the basegerm (s, A) of D_k cannot map into I_k^k. Indeed, otherwise $f(\hat{D}_m) \subset \hat{D}_m$ which contradicts the topological exactness of f. Hence it is a germ of T_{k-1} at u_k which maps into I_k^k by f. By Lemma 2.10 it implies that $u_k = z_{k-1}$. However by the shown above $u_k \in T_m$ while $z_{k-1} \notin T_m$ because $z_{k-1} \in D_{k-1}$ and D_{k-1} sticks out of T_m. This contradiction shows that $f(u_k) \neq z_k$. Then $f(u_k)$ is a vertex of T_{k-1} which will have to exit the set of vertices of T_{k-1} at a later moment. When it happens, it cannot happen at u_k because then u_k is periodic. So $f(u_k)$ is a wave in T_{k-1}. By the previous paragraph $f(u_k) = f^s(z_{m+1}) = z_{m+s+1}, 0 \leq s \leq k-m-1$.

Consider this situation in detail. If the basegerm of D_k maps into the germ (z_{m+s+1}, A) of $[z_{m+s+1}, d_{m+s+1}]$ by f then as before we see that f^{k-m-s} maps $\hat{D}_{m+s+1}$ into itself which is impossible because f is topologically exact. Hence it is a germ of T_m at u_k which maps into (z_{m+s+1}, A). By Lemma 2.10 if $s > 0$ then $u_k = z_{m+s}$ which is impossible because $u_k \in T_m$ while points $z_{m+s} \notin T_m$ if $s > 0$. Hence $s = 0$. Now, the only parts of T_m which map into $C_{m+1} = D_{m+1}$ are points of C_m, D_m. In fact, we introduced the appropriate notation before; according to it, either $u_k = t_m \in C_m$ or $u_k = q_m \in D_m$. However in either case the image of a small interval connecting u_k with u_m or v_m respectively covers a small semineighborhood of z_{m+1} in the interval $[z_{m+1}, u_{m+1}]$. In the language of germs we can say that the f-image of one of the two germs of T_m at u_k must be the germ of $[z_{m+1}, u_{m+1}]$ at z_{m+1}. Therefore, the f-image of the basegerm of D_k at u_k must be either the germ of $[z_{m+1}, c_{m+1}]$ at z_{m+1}, or the germ of $[z_{m+1}, d_{m+1}]$ at z_{m+1}. This shows that the f-image of $\hat{D}_k$ is the component of $J \setminus \{z_{m+1}\}$ containing either $(z_{m+1}, c_{m+1}]$ or $(z_{m+1}, d_{m+1}]$ respectively. If we now apply f^{k-m-1} to this picture then we see that $f^{k-m}(\hat{D}_k) \subset \hat{D}_k$, a contradiction with the topological exactness of f. This finally shows that u_k is not a wave and completes the proof of the claim. □

To summarize: we have proven that in the situation as above (i.e. if Case 1 is replaced by Case 2 at some moment m and then the first turning moment of D_{m+1} is $k+1$) we have that waves of T_{k-1} are points $f^s(z_{m+1}) = z_{m+s+1}), 0 \leq s \leq k-m-1$ and all other vertices of T_{k-1} are periodic. Let us now show that z_k is preperiodic too. This will imply that all vertices of T_k are preperiodic.

Lemma 2.3(4) shows that there exists a natural number t such that $f^t(z_k) \in T_0$. Then choose numbers $K > M > k+t$ such that at the moment M Case 1 is replaced by Case 2, and $K \geq M+1$ is the least such number that $C_K = D_K$ turns (the basegerm of $C_K = D_K$ maps inside T_K). All that is possible i n particular because by the assumption Case 1 and Case 2 take place for infinitely many values of m. Then similarly to the above we can define points $Z_{M+1}, \dots, Z_K$. The fact that $M > k+t$ implies that the point $f^t(z_k)$ cannot be one of the points $Z_{M+1}, \dots, Z_K$ and therefore by the proven above it is preperiodic. Thus we see that Case 1 and Case 2 take place infinitely many times then for a wave x we can find a big number m such that x is a vertex of T_m and m is a moment when Case 1 is replaced by Case 2. By the proven above it would imply that x is not wandering, a contradiction.

Hence the dynamics cannot infinitely many times switch from Case 1 to Case 2. On the other hand it is easy to show that Case 2 cannot take place all the time from some time on. Indeed, if so then for some number m and for all $i \geq 0$ we will have that $C_{m+i} = D_{m+i}$ is a triod. However if we choose the least i such that $f^i(z_m) \in T_{m+i-1}$ (such i exists by Lemma 2.3(3)) then we see that $C_{m+i} = D_{m+i}$ being a triod is impossible for this i, a contradiction. This shows that the only remaining case is when Case 1 takes place from some time on, i.e. when $C_m \neq D_m$

for all $m > Q$ for some $Q \geq 0$ and proves the lemma. $\square$

Let us now prove that the critical point d belongs to $\omega(x)$. To this end we choose a big number $N > Q$ for which there exists a triod $Y \subset T_N$ "centered" at x and such that:

(1) for some k we have $Y \subset T_N, f(Y) \subset T_N, \dots, f^k(Y) \subset T_N$,
(2) $f^k|_Y$ is 1-to-1,
(3) $f^k(x)$ is not the basepoint of C_N or D_N,
(4) each branch $Y_i, i = 1, 2, 3$ of Y maps forward by f^k in such a way that at some moment $j_i < k$ its f^{j_i}-image is an arc whose endpoints are $f^{j_i}(x)$ and one of the critical points (this can be done by Lemma 2.7(2));
(5) $N + 1$ is a turning moment for C_N (i.e., the f-image of the basegerm of C_N maps into T_N).

It is easy to see that a number N with properties (1)-(5) exists (just fix k for which (2) and (4) hold, and then choose N which is big enough so that C_N turns).

Let us study the orbit of x. By Lemma 2.3(6) there exists R such that $f^R(x)$ is either u_N or v_N and before that the images of x are vertices of T_{N-1}. By (3) we see that $R > N$. Assume to begin with that $f^R(x) = v_N$ and that along the way the orbit of x does not pass through u_N. Then there exists a germ (x, A) of Y which has the orbit contained in T_{N-1} until f^R maps it into the basegerm of D_N. By Lemma 2.10 this implies that (x, A) is contained in the set $\overline{I^N_k}$ for some $k > 0$ or is mapped into $\overline{I^N_0}$ on a non-negative step.

Let us show that the former is impossible. Indeed, if it holds then the branch of Y which contains (x, A) is contained into images of $\overline{I^N_k}$ and hence cannot be cut by a critical point on a step before R, a contradiction with the assumptions. Hence x passes through $\overline{I^N_0}$ on a non-negative step. Suppose that the described above phenomenon (when $f^R(x) = v_N$) takes place for infinitely many N_i's satisfying (1)-(5) above. Observe that $\text{diam}(D_n) \to 0$ as $n \to \infty$ by Lemma 2.2. Then by the backward stability of f (Theorem 2.8) we see that $\text{diam}(I^{N_i}_0) \to 0$ as $N_i \to \infty$. Since we know by the above that x passes through $I^{N_i}_0$ for every i we conclude that $d \in \overline{\text{orb}(x)}$ and since x never maps onto d by Corollary 1.4(2) we conclude that $d \in \omega(x)$. On the other hand we know that $\omega(x) = \omega(d)$ by the first part of the Main Theorem. Hence $d \in \omega(d) = \omega(x)$ is a recurrent point.

It remains to consider the case when from some time on for any N satisfying (1)-(5) we have $f^R(x) = u_N$ (which implies that u_N is wandering itself) while all preceding images of x are vertices of T_{N-1}. Recall that N is the time when C_N turns. Then $f(u_N)$ is either a vertex of T_{N-1} or v_N. In the former case we can apply f yet several (say, q) times to see by Lemma 2.3(6) that $f^q(f(u_N))$ equals v_N. So in general (regardless of which case takes place) we can say that there exists a unique $q \geq 0$ such that $f^q(f(u_N)) = v_N$. Moreover, for any $0 \leq i < q$ (so if $q = 0$ such i does not exist) the point $f^i(f(u_N))$ is a vertex of T_{N-1} because otherwise it will have to pass through u_N or v_N thus making u_N periodic, a contradiction. By Lemma 2.10 this implies that either $f(u_N)$ passes through $\overline{I^N_0}$ or $f(u_N)$ belongs to $\overline{I^N_k}$ with some $k > 0$. If the former takes place for infinitely many N's then similarly to the previous paragraph one can easily show that d is recurrent. So from now on we may assume that $f(u_N)$ belongs to I^N_k with some $k > 0$. Since I^N_k by Lemma 2.9 contains no vertices of T_{N-1} we see that in fact $f(u_N)$ is the endpoint w_k of I^N_k not equal to d_k. Observe that all these arguments are necessary if $q > 0$; if $q = 0$ then $f(u_N) = v_N$ is the endpoint of I^N_N not equal to d_N, so essentially the same picture holds. Our aim now is to show that this picture still implies that the orbit of x **taken from the very beginning** passes through $\overline{I_0^N}$ which as before implies that d is recurrent.

An important step of the proof now is to observe that Lemma 2.11 is applicable to our situation. It implies that the basegerm of C_N cannot map to the germ (w_k, A) of I_k^N at w_k. Hence it is a germ of T_{N-1} which maps to (w_k, A). If we now follow the orbit of x from x to u_N we see that along the way it cannot pass through either u_N or v_N (because then it is not wandering) and therefore the entire segment of the orbit of x from x to u_N (i.e. the points $x, f(x), \dots, f^{R-1}(x)$) consists of vertices of T_{N-1}. Denote by (x, B) the germ of Y at x which maps into (w_k, A) by f^{R+1}. Then we see that in fact this germ maps into the basegerm of D_N by f^{R+q} and stays inside T_{N-1} throughout this orbit segment. This is the situation where Lemma 2.10 is applicable to (x, B). By Lemma 2.10 we have that either x passes through $\overline{I_0^N}$ on its way to v_N, or it does not. In the former case the same arguments as before imply that d is recurrent. Now, in the latter case we see by Lemma 2.10 that in fact $x \in I_k^N$ with some $k > 0$. However this implies that there is a branch of Y contained in I_k^N which thus does not get cut at a critical point as it should according to our choice of Y and related numbers. This shows that $x \in I_k^N, k < N$ is impossible and completes the proof of the second claim of the Main Theorem.

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Received March 3, 2004; revised November 24, 2004.