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LAMINATIONS FROM THE MAIN CUBIOID

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Abstract. Polynomials from the closure of the principal hyperbolic domain of the cubic connectedness locus have some specific properties, which were studied in a recent paper by the authors. The family of (affine conjugacy classes of) all polynomials with these properties is called the Main Cubioid. In this paper, we describe a combinatorial counterpart of the Main Cubioid — the set of invariant laminations that can be associated to polynomials from the Main Cubioid.

1. **Introduction.**

1.1. **Motivation.**

The *complex quadratic family* is the family of all polynomials $P_c(z) = z^2 + c$ (any quadratic polynomial is affinely conjugate to some P_c). An important role in studying this family is played by the *connectedness locus* $\mathcal{M}_2$ (also called the *Mandelbrot set*) consisting of all c such that the *Julia set* J_{P_c} is connected. The central part of $\mathcal{M}_2$ is the Principal Hyperbolic Domain PHD_2, i.e., the set of numbers $c \in \mathbb{C}$ such that P_c has an attracting fixed point. Its closure CA is called the *Main Cardioid* (*of the Mandelbrot set*). A combinatorial model $\mathcal{M}_2^c$ of $\mathcal{M}_2$, due to Thurston [31], implies a combinatorial model CA^c of CA; we call CA^c the *Combinatorial Main Cardioid*.

Similarly, in degree d one can consider the space of affine conjugacy classes of degree d polynomials (in the quadratic family, we made an explicit choice of a representative polynomial). The *degree d connectedness locus* (also called the *degree d Mandelbrot set*) $\mathcal{M}_d$ is the space of all degree d affine conjugacy classes, whose polynomials have connected Julia sets (equivalently, all critical points have bounded

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orbits). In what follows, by the *class* of a polynomial we will always mean the affine conjugacy class. Given a polynomial P, we denote its class by $[P]$. The *Principal Hyperbolic Domain* PHD_d in $\mathcal{M}_d$ consists of all classes of hyperbolic polynomials with Julia set homeomorphic to a circle. Equivalently, $[P]$ is in PHD_d if all critical points of P are in the immediate basin of attraction of some attracting fixed point. An important question then is to describe the set of all classes of polynomials that belong to the closure $\overline{\mathrm{PHD}}_d$ of PHD_d. Here we address this question for $d = 3$ (i.e., in the *cubic* case).

The key object in Thurston's combinatorial model is the notion of a *lamination* (full definitions are provided in Section 2, see Definitions 2.1, 2.2, 2.4, and 2.5, while in Subsection 1.2 we will only give loose descriptions). Laminations provide combinatorial models for connected polynomial Julia sets. Thurston's work [31] can be seen as consisting of two parts: he defined a set of laminations that are good candidates to be models of quadratic Julia sets, and then he described the set of such laminations, which led to a combinatorial model for the Mandelbrot set.

In this paper, we will make a similar first step for $\overline{\mathrm{PHD}}_3$ with the hope that our classification eventually yields a suitable combinatorial model. Polynomials from $\overline{\mathrm{PHD}}_3$ satisfy certain dynamical properties. These dynamical properties in turn force the corresponding laminational models to have certain properties. We will consider all laminations that satisfy these properties and provide a simple classification of these laminations which revolves around a reduction to laminations with model polynomials from $\overline{\mathrm{PHD}}_2$.

The cubic Mandelbrot set $\mathcal{M}_3$ or its parts have been studied before. P. Lavaurs in his thesis titled *Systèmes dynamiques holomorphes, Explosion de points périodiques paraboliques* (Université Paris-Sud, Orsay, 1989) proved that $\mathcal{M}_3$ is not locally connected. Epstein and Yampolsky [15] proved that the bifurcation locus in the space of real cubic polynomials is not locally connected either. This makes defining a combinatorial model of $\mathcal{M}_3$ very delicate. Buff and Henriksen [10] presented copies of quadratic Julia sets, including Julia sets which are not locally connected, in slices of $\mathcal{M}_3$; however, these copies are disjoint from the closure of PHD_3. Moreover, McMullen [20] has shown that slices of $\mathcal{M}_3$ contain lots of copies of $\mathcal{M}_2$. In addition, Gauthier [16] has shown that $\mathcal{M}_3$ contains copies of $\mathcal{M}_2 \times \mathcal{M}_2$. Observe also that in his thesis titled *Local connectivity in a family of cubic polynomials* (Cornell University, 1992) D. Faught considered the slice A of $\mathcal{M}_3$ consisting of polynomials with a fixed critical point and showed that A contains countably many homeomorphic copies of $\mathcal{M}_2$ and is locally connected everywhere else. In particular, A intersects the boundary of PHD_3 along a Jordan curve.

Roesch [30] generalized Faught's results to higher degrees. Zakeri [33] described some important Jordan curves in the boundary of PHD_3, whose points are represented by polynomials with both critical points on the boundary of the same Siegel disk. Milnor and Poirier [24] gave a classification of hyperbolic components in $\mathcal{M}_d$; in particular, he proved that the topology of many hyperbolic components can be reduced to that of PHD_3. In a recent paper [28], Petersen and Tan Lei introduced an analytic coordinate system on PHD_3 that reflects dynamical properties of the corresponding polynomials. The authors planned a sequel [29] to this paper, in which the boundary of PHD_3 would be discussed. After the main results of our paper had been obtained, we discovered that our work may have some overlap with [29].

1.2. **Introduction to laminations.** Thurston [31] gave a combinatorial model for the entire Mandelbrot set. It has been conjectured that the Mandelbrot set is homeomorphic to Thurston's model; in fact, this conjecture is equivalent to local connectivity of $\mathcal{M}_2$. Although a global homeomorphism is not known, some parts of the Mandelbrot set can be shown to be homeomorphic to the corresponding parts of the model. For example, CA is homeomorphic to CA^c, and both sets are homeomorphic to the closed disk (see, e.g., [12]).

For higher degree Mandelbrot sets $\mathcal{M}_d$ even conjectural models are missing. To begin with, it is natural to model $\overline{\mathrm{PHD}}_d$. As a first step to solving this problem in the cubic case, we study individual polynomials from $\overline{\mathrm{PHD}}_3$. Similar to Thurston [31], we use *laminations*.

We will write $\mathbb{C}$ for the plane of complex numbers, $\widehat{\mathbb{C}}$ for the Riemann sphere, and $\mathbb{D} = \{z \in \mathbb{C} \,|\, |z| < 1\}$ for the open unit disk. A lamination is a closed equivalence relation $\sim$ on $\mathbb{S}^1 = \{z \in \mathbb{C} \,|\, |z| = 1\}$, whose classes are finite sets, such that the convex hulls of different classes have disjoint relative interiors, see Definition 2.1. A lamination is (σ_d-)invariant if classes map to classes under $\sigma_d : \mathbb{S}^1 \to \mathbb{S}^1$, $z \mapsto z^d$ in a covering fashion. See Definition 2.2, which makes this more precise. If a polynomial P has a locally connected Julia set J, then there is a lamination $\sim_P$ identifying pairs of angles if the corresponding external rays land at the same point. The quotient $J_{\sim_P} = \mathbb{S}^1/\sim_P$ is homeomorphic to J, and the self-mapping $f_{\sim_P}$ of $J_{\sim_P}$ induced by σ_d is conjugate to $P|_{J_P}$; the map $f_{\sim_P}$ and the set $J_{\sim_P}$ are called a *topological polynomial* and a *topological Julia set*, respectively. Laminations can play a role for some polynomials whose Julia sets are not locally connected. Then $P|_{J_P}$ and $f_{\sim_P}|_{J_{\sim_P}}$ are not conjugate, however, they are semiconjugate by a monotone map (a continuous map, whose fibers are continua). Topological Julia sets and polynomials make sense for any σ_d-invariant lamination, not just those described above.

It is very useful to associate to each lamination $\sim$ important geometric objects defined below. We will identify $\mathbb{S}^1$ with $\mathbb{R}/\mathbb{Z}$. For a pair of angles a, $b \in \mathbb{R}/\mathbb{Z}$, we will write $\overline{ab}$ for the chord (a straight line segment in $\mathbb{C}$) connecting the points of $\mathbb{S}^1$ corresponding to angles a and b. If G is the convex hull $\mathrm{Ch}(G')$ of some closed set $G' \subset \mathbb{S}^1$, then we write $\sigma_d(G)$ for the set $\mathrm{Ch}(\sigma_d(G'))$. The set $G' = G \cap \mathbb{S}^1$ is called the *basis* of G. The boundary of G will be denoted by $\mathrm{Bd}(G)$.

Definition 1.1 (Leaves)**.** If A is a $\sim$-class, call a chord $\overline{ab}$ in $\mathrm{Bd}(\mathrm{Ch}(A))$ a *leaf* of $\sim$. All points of $\mathbb{S}^1$ are also called (*degenerate*) *leaves*. The family $\mathcal{L}_\sim$ of all leaves of $\sim$ is called the *geo-lamination* (*geometric lamination, or geodesic lamination*) *generated by* $\sim$. The union of all leaves of $\mathcal{L}_\sim$ is denoted by $\mathcal{L}^+_\sim$.

In general, collections of leaves with properties similar to those of collections $\mathcal{L}_\sim$ are also called *invariant geo-laminations*, see Definition 2.5 for a more precise formulation. In fact, it is these collections that Thurston introduced and studied in [31]. Thus, laminations (and their geo-laminations) on the one hand, and geo-laminations in general, on the other hand, are studied in complex dynamics. The motivation of this can be as follows. The direct association between polynomials and laminations was described above. However, not every polynomial P can be directly associated to a suitable lamination. Hence it is natural to consider polynomials close to P for which such association is possible and then associate to P the appropriately defined limit of their laminations. To define such a limit one has to consider geo-laminations. However, limits of geo-laminations, which were associated to laminations, are not easily associated to laminations. This motivates the usage of "abstract" geo-laminations, i.e., geo-laminations not associated with any lamination.

Definition 1.2 (Gaps)**.** Let $\mathcal{L}$ be an invariant geo-lamination, e.g., we may have $\mathcal{L} = \mathcal{L}_\sim$ for some invariant lamination $\sim$. The closure in $\mathbb{C}$ of a non-empty component of $\mathbb{D} \setminus \mathcal{L}^+$ is called a *gap* of $\mathcal{L}$. *Edges of a gap G* are defined as leaves of $\mathcal{L}$ on the boundary of G. A gap is said to be *finite* (*infinite*) if its basis is finite (infinite). Infinite gaps of $\mathcal{L}$ are also called *Fatou gaps*.

The map $\sigma_d : G' \to \sigma_d(G')$ extends to $\mathrm{Bd}(G)$ as a composition of a monotone map and a covering map of some degree m. Then m is called the *degree* of G. A Fatou gap G is *periodic* (*of period n*) if the interiors of the sets G, $\sigma_d(G)$, $\dots$, $\sigma_d^{n-1}(G)$ are disjoint while $\sigma_d^n(G) = G$. Such a gap G is said to be a periodic *Siegel* gap if the degree of $\sigma_d^n|_G$ is 1. If the degree of G is 2, then the gap G is said to be *quadratic*. If the period of G is 1, then G is said to be *invariant*.

1.3. **The cubioid.**

Our studies are based on Theorem A.

Theorem A ([9])**.** *If $[P] \in \overline{\mathrm{PHD}}_d$, then P has a fixed non-repelling point, no repelling periodic cutpoints, and at most one non-repelling periodic point with multiplier different from 1.*

A polynomial P with $[P] \in \overline{\mathrm{PHD}}_3$ has at most two non-repelling cycles. One of them must be a fixed point (as we approximate P with polynomials g, whose classes belong to PHD_3, the attracting fixed points of g converge to a non-repelling fixed point of P). If there is a non-repelling cycle of period greater than 1, then by Theorem A this cycle must have multiplier 1. Moreover, all fixed neutral points of f but one must have multiplier 1. In this paper, we will consider all polynomials which satisfy the conclusions of Theorem A.

Definition 1.3. The *main cubioid* CU is defined as the set of all classes of cubic polynomials that have a fixed non-repelling point, no repelling periodic cutpoints and at most one non-repelling periodic point with multiplier different from 1.

If $[P] \in \mathrm{CU}$ and J_P is locally connected, then Definition 1.3 forces the corresponding lamination $\sim_P$ to have certain properties (see Lemma 1.4). By an edge of a $\sim_P$-class, we mean an edge of the convex hull of that class. If an edge is periodic, its vertices may have a larger period than the period of the edge (for instance a class with vertices $\frac{1}{4}, \frac{3}{4}$ is fixed under σ_3 but its vertices are of period two). Periodic gaps/leaves of a geo-lamination $\mathcal{L}$ are naturally associated with a rotation number (unless these are periodic Fatou gaps of degree greater than 1). They are said to be *rotational* if the rotation number is not zero. A thorough treatment is given in Section 2.

Lemma 1.4. *If a polynomial P with $[P] \in \mathrm{CU}$ has locally connected Julia set, then the lamination $\sim_P$ has at most one rotational periodic set* (*hence this set must be fixed*)*. Moreover, each periodic non-degenerate $\sim_P$-class G has a cycle of edges of vertex period n, at which a Fatou gap of period n is attached to G.*

Proof. Let $\mathbf{g}$ be a rotational periodic set of $\sim_P$, and let ψ_P denote the semiconjugacy between $\sigma_3 : \mathbb{S}^1 \to \mathbb{S}^1$ and $P : J_P \to J_P$ obtained as the composition of the quotient map from $\mathbb{S}^1$ to $\mathbb{S}^1/\sim_P$ and the natural conjugacy between $f_{\sim_P}$ and $P|_{J_P}$. If $\mathbf{g}$ is finite, it maps under the semiconjugacy ψ_P to a periodic point $x \in J_P$, which must be a cutpoint of J_P. By Theorem A, the point x cannot be repelling. Hence, and since $\mathbf{g}$ is rotational, the point x is parabolic with multiplier different from 1. If $\mathbf{g}$ is infinite, then by definition $\mathbf{g}$ is a periodic Siegel gap which, since J_P is locally connected, corresponds to a periodic Siegel domain of P which contains a periodic Siegel point. Now from Theorem A it follows that $\mathbf{g}$ is unique. Indeed, by the above each rotational periodic set $\mathbf{g}$ of $\sim_P$ generates a periodic non-repelling point of multiplier distinct from 1. Yet by Theorem A the polynomial P has at most one periodic non-repelling point of multiplier distinct from 1. Hence there is at most one rotational set $\mathbf{g}$. This implies the first claim of the lemma. Since each periodic non-degenerate $\sim_P$-class G corresponds to a parabolic periodic point, the second claim follows. □

Definition 1.5 (Combinatorial Main Cubioid)**.** Define the *Combinatorial Main Cubioid* CU^c as the family of all cubic laminations $\sim$ with at most one rotational periodic set (the set must be fixed because if its period were greater than 1, then there would be at least two such sets) and such that each periodic non-degenerate $\sim$-class G has a cycle of edges of vertex period n at which Fatou gaps of period n are attached to G.

1.4. **Main results.** In this paper we classify all laminations in CU^c. In doing so it is important to keep in mind that any cubic lamination either has two critical sets of degree 2 (so that either set maps two-to-one on its image) or one critical set of degree 3 (which maps onto its image three-to-one).

To state our results, we need to discuss invariant quadratic gaps of cubic laminations. Let U be such a gap. We show that the gap U has a unique *major edge* (or simply *major*) $M = \overline{ab}$ such that the arc (a, b) is of length at least $\frac{1}{3}$ and contains no points of $\overline{U}$. It turns out that M can be either periodic (then U is said to be of *periodic type*) or critical (then U is said to be of *regular critical type*).

In either case there is a specific *canonical lamination* $\sim_U$ associated to U. In the regular critical case $\sim_U$ is defined as follows: $a \sim_U b$ if there is $N \geqslant 0$ such that $\sigma_3^N(a)$ and $\sigma_3^N(b)$ are endpoints of the same edge of U, and the set $\{\sigma_3^i(a), \sigma_3^i(b)\}$ is not separated by U for $i = 0, \dots, N-1$. Basically, this means that there is a gap V attached to U along its major and folding on top of U under σ_3; the rest of $\sim_U$ consists of well-defined pullbacks of V, which accumulate to points of $\mathbb{S}^1$.

Now, let U be of periodic type. Then one can define a periodic quadratic gap V attached to U along its major $M_U = M$. The gap V has the same period n as M and can be defined as follows. Take all points $a \in \mathbb{S}^1$ such that, for any $i \geqslant 0$, the point $\sigma_3^i(a)$ is separated from U by $\sigma_3^i(M)$ or is an endpoint of $\sigma_3^i(M)$. Then V is the convex hull of the set of all such points a. We call $V = V(U)$ the *vassal* gap of U. Now, define the lamination $\sim_U$ as follows: $a \sim_U b$ if there exists $N \geqslant 0$ such that $\sigma_3^N(a)$ and $\sigma_3^N(b)$ are endpoints of the same edge of U or the same edge of V, and the chord $\overline{\sigma_3^i(a)\sigma_3^i(b)}$ is disjoint from $U \cup V$ for $i = 0, \dots, N-1$. Basically, $\sim_U$ includes U, the orbit of V, and the rest of $\sim_U$ consists of well-defined pullbacks of V which accumulate to points of $\mathbb{S}^1$.

Let us now give a heuristic description of laminations from CU^c. Namely, a lamination $\sim$ from CU^c can be thought of as a result of an at most two-step process. First, an invariant quadratic Fatou gap U is created, together with its canonical lamination $\sim_U$. This could end the whole process. However it could also happen that afterwards the gap U is (*weakly*) *tuned* (see Definition 1.9 and Definition 1.10) by a quadratic lamination from the Combinatorial Main Cardioid (since $\sigma_3|_{U'}$ is semiconjugate to σ_2 by a map collapsing all edges of U, it is easy to define tuning in this setting). We show that basically this mechanism describes all laminations from

CU^c. Although the classification given in the Main Theorem is the main result, we do obtain as a corollary that Definition 1.5 implies an even stronger condition on attached Fatou gaps.

Corollary 1.6. *A lamination $\sim$ belongs to CU^c if and only if it has at most one rotational periodic (hence fixed) set and, for each leaf ℓ of $\sim$ of vertex period n, there is a Fatou gap of period n attached to ℓ.*

To give the precise statement of our main result, we have to introduce a few definitions. The first one relates different laminations.

Definition 1.7. Let G be a gap of some lamination. A lamination $\sim$ *coexists with* G if every leaf of $\sim$ intersecting an edge ℓ of G in $\mathbb{D}$ coincides with ℓ. A lamination $\sim$ *coexists* with a lamination $\simeq$ if no leaf of $\sim$ intersects a leaf of $\simeq$ in $\mathbb{D}$ unless the two leaves coincide.

Now we can formalize some situations in which one lamination modifies another one.

Definition 1.8. Suppose that two laminations, $\sim$ and $\simeq$, are given.

1. Say that $\sim$ *tunes* $\simeq$ if $\mathcal{L}_\sim \supset \mathcal{L}_\simeq$.
2. Let G be a Fatou gap of some lamination. If all edges of G are leaves of $\sim$ then we say that $\sim$ *tunes* the gap G.

The terminology can be better understood if we think of $\mathcal{L}_\sim$ as being obtained by adding new leaves to $\mathcal{L}_\simeq$; since these leaves can only be added *inside* gaps of $\mathcal{L}_\sim$, we can think of this as *tuning* of gaps of $\sim$ which explains the terminology.

Given a periodic Fatou gap G of some unspecified lamination, consider a map $\psi_G : \mathrm{Bd}(G) \to \mathbb{S}^1$ that collapses all edges of G to points; ψ_G maps $\mathrm{Bd}(G)$ onto the unit circle $\mathbb{S}^1$. If G is periodic of period n, then $\psi_G : \mathrm{Bd}(G) \to \mathbb{S}^1$ semiconjugates $\sigma_d^n|_{\mathrm{Bd}(G)}$ to either irrational rotation (if G is a Siegel gap) or to σ_k (if G is of degree $k > 1$). Now, suppose that $\sim$ tunes a periodic quadratic gap G. Then leaves of $\mathcal{L}_\sim$ contained in G map under ψ_G to chords of $\overline{\mathbb{D}}$. In Section 3, we show that these chords can be viewed as leaves of a geo-lamination generated by an invariant lamination; this lamination is denoted by $\psi_G(\sim)$.

Definition 1.9. Suppose that $\sim$ tunes a periodic quadratic gap G. Then we say that $\sim$ *tunes G according to the lamination $\psi_G(\sim)$*. If G is a gap of a lamination $\simeq$, and $\sim$ tunes $\simeq$, then for brevity we also say that $\sim$ *tunes $\simeq$ according to the lamination $\psi_G(\sim)$* (in general, in this last case the behavior of $\sim$ outside G, even though compatible with $\simeq$, is not completely defined by the way $\sim$ tunes G). Clearly, distinct laminations can tune the same quadratic invariant gap.

A weaker (and finer) case of tuning is described in the next definition. Let us emphasize that in it, $\sim$ only coexists with a periodic quadratic gap U (i.e., U is not necessarily a gap of $\sim$).

Definition 1.10. If $\sim$ coexists with a periodic quadratic gap U of some unspecified lamination, and $\psi_U(\sim)$ coincides with a quadratic lamination $\asymp$, then we say that $\sim$ *weakly tunes U according to the lamination $\asymp$*. If U is a periodic gap of a lamination $\simeq$ and $\sim$ coexists with $\simeq$, then for brevity we also say that $\sim$ *weakly tunes $\simeq$ (on U) according to the lamination $\asymp$*.

We are ready to state our main theorem. Recall the vassal gap $V(U)$ was defined in the second paragraph of Subsection 1.4.

Main Theorem. *Let $\sim$ be a non-empty lamination from* CU^c*. Then there exists an invariant quadratic gap* U *for which* (1) *or* (2) *takes place.*

1. *The lamination* $\sim$ *coexists with the canonical lamination* $\sim_U$ *and weakly tunes* $\sim_U$ *on* U *according to a quadratic lamination* $\asymp$ *from* CA^c *so that edges of* U *are not leaves of* $\sim$*. Moreover,* U *can be chosen to be of regular critical type* (*and, if* $\sim$ *is not canonical, then* U *must be of regular critical type*).
2. *The lamination* $\sim$ *tunes the canonical lamination* $\sim_U$ *according to a quadratic lamination* $\asymp$ *from* CA^c (*possibly empty*)*, and if* U *is of periodic type, then the vassal gap* $V(U)$ *is a gap of* $\sim$*.*

The rest of the paper is structured as follows. In Section 2, we give precise definitions of laminations and related terminology and state technical results, including Theorem 2.13 from [3], which provides a method for finding fixed objects in laminations. Section 3 investigates the structure of quadratic invariant gaps and the related notion of canonical laminations of quadratic invariant gaps, upon which the classification of cubioidal laminations rests. Tuning and weak tuning, processes which insert laminations into infinite gaps of other laminations, are also developed there. Invariant rotational sets play a prominent role in the definition of the Combinatorial Main Cubioid. In Section 4, we define and study various canonical laminations. All of these notions are combined in Section 5, where the full classification of CU^c in terms of (weak) tuning quadratic invariant gaps is given.

2. Preliminaries.

Let a, $b \in \mathbb{S}^1$. By $[a, b]$, (a, b), etc, we mean the closed, open, etc *positively oriented* circle arcs from a to b, and by $|I|$ the length of an arc I in $\mathbb{S}^1$ normalized so that the length of $\mathbb{S}^1$ is 1. In this section, we will introduce classic results due to Douady and Hubbard [13, 14] and Thurston [31]. These results allow one to study connected filled-in Julia sets by means of studying their complements in the complex plane.

2.1. Laminations.

For a compactum $X \subset \mathbb{C}$, let $U^\infty(X)$ be the unbounded component of $\widehat{\mathbb{C}} \setminus X$ containing infinity. If X is connected, there exists a Riemann mapping $\Psi_X : \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}} \to U^\infty(X)$; we always normalize it so that $\Psi_X(\infty) = \infty$ and $\Psi'_X(z)$ tends to a positive real limit as $z \to \infty$.

Consider a polynomial P of degree $d \geqslant 2$ with Julia set J_P and filled-in Julia set K_P. Extend $z^d : \mathbb{C} \to \mathbb{C}$ to a map θ_d on $\widehat{\mathbb{C}}$. If J_P is connected, then $\Psi_{J_P} = \Psi : \mathbb{C} \setminus \overline{\mathbb{D}} \to U^\infty(K_P)$ is such that $\Psi \circ \theta_d = P \circ \Psi$ on the complement of the closed unit disk [13, 14, 22]. If J_P is locally connected, then Ψ extends to a continuous function

$$\overline{\Psi} : \widehat{\mathbb{C}} \setminus \mathbb{D} \to \overline{\widehat{\mathbb{C}} \setminus K_P},$$

and $\overline{\Psi} \circ \theta_d = P \circ \overline{\Psi}$ on the complement of the open unit disk; thus, we obtain a continuous surjection $\overline{\Psi}\colon \mathrm{Bd}(\mathbb{D}) \to J_P$ (the *Carathéodory loop*, see [11]).

Let J_P be locally connected, and set $\psi = \overline{\Psi}|_{\mathbb{S}^1}$. Following Thurston [31] (see also [13, 14]), define an equivalence relation $\sim_P$ on $\mathbb{S}^1$ by $x \sim_P y$ if and only if $\psi(x) = \psi(y)$, and call it the (σ_d-invariant) *lamination of* P (since Ψ defined above semiconjugates θ_d and P, the map ψ semiconjugates σ_d and $P|_{J(P)}$ which implies that $\sim_P$ is invariant). Equivalence classes of $\sim_P$ are pairwise *unlinked*: their Euclidian convex hulls are disjoint. The topological Julia set $\mathbb{S}^1/\sim_P = J_{\sim_P}$ is homeomorphic to J_P, and the topological polynomial $f_{\sim_P} : J_{\sim_P} \to J_{\sim_P}$ is topologically conjugate to $P|_{J_P}$. One can extend the conjugacy between $P|_{J_P}$ and $f_{\sim_P} : J_{\sim_P} \to J_{\sim_P}$ to a conjugacy on the entire plane.

An equivalence relation $\sim$ on the unit circle, with similar properties to those of $\sim_P$ above, can be introduced abstractly without any reference to the Julia set of a complex polynomial (see [4]).

Definition 2.1 (Laminations)**.** An equivalence relation $\sim$ on the unit circle $\mathbb{S}^1$ is called a *lamination* if it has the following properties:
(E1) the graph of $\sim$ is a closed subset in $\mathbb{S}^1 \times \mathbb{S}^1$;
(E2) if $t_1 \sim t_2 \in \mathbb{S}^1$ and $t_3 \sim t_4 \in \mathbb{S}^1$, but $t_2 \not\sim t_3$, then the open straight line segments in $\mathbb{C}$ with endpoints t_1, t_2 and t_3, t_4 are disjoint;
(E3) each equivalence class of $\sim$ is finite.

Recall that, for a closed set $A \subset \mathbb{S}^1$, we denote its convex hull by $\mathrm{Ch}(A)$. Then by an *edge* of $\mathrm{Ch}(A)$ we mean a closed subsegment of the straight line connecting two points of the unit circle which is contained in the boundary of $\mathrm{Ch}(A)$. By an *edge* of a $\sim$-class we mean an edge of the convex hull of that class.

Definition 2.2 (Laminations and dynamics)**.** A lamination $\sim$ is called (σ_d-)*invariant* if:
(D1) $\sim$ is *forward invariant*: for a class $\mathbf{g}$, the set $\sigma_d(\mathbf{g})$ is a class too;
(D2) for any $\sim$-class $\mathbf{g}$, the map $\sigma_d : \mathbf{g} \to \sigma_d(\mathbf{g})$ extends to $\mathbb{S}^1$ as an orientation preserving covering map such that $\mathbf{g}$ is the full preimage of $\sigma_d(\mathbf{g})$ under this covering map.

Definition 2.2 (D2) has an equivalent version. Given a closed set $Q \subset \mathbb{S}^1$, a (positively oriented) *hole* (a, b) *of* Q (*or of* $\mathrm{Ch}(Q)$) is a component of $\mathbb{S}^1 \setminus Q$. Then (D2) is equivalent to the fact that for a $\sim$-class $\mathbf{g}$ either $\sigma_d(\mathbf{g})$ is a point or, for each positively oriented hole (a, b) of $\mathbf{g}$, the positively oriented arc $(\sigma_d(a), \sigma_d(b))$ is a hole of $\sigma_d(\mathbf{g})$. From now on, we assume in this paper that, unless stated otherwise, $\sim$ is a σ_d-invariant lamination.

Given $\sim$, consider the *topological Julia set* $\mathbb{S}^1/\sim = J_\sim$ and the *topological polynomial* $f_\sim : J_\sim \to J_\sim$ induced by σ_d. Using Moore's Theorem, embed $J_\sim$ into $\mathbb{C}$ and extend the quotient map $\mathrm{pr}_\sim : \mathbb{S}^1 \to J_\sim$ to $\mathbb{C}$ with the only non-trivial fibers being the convex hulls of non-degenerate $\sim$-classes. A *Fatou domain* of $J_\sim$ (or of $f_\sim$) is a bounded component of $\mathbb{C} \setminus J_\sim$. If U is a periodic Fatou domain of $f_\sim$ of period n, then $f^n_\sim|_{\mathrm{Bd}(U)}$ is conjugate either to an irrational rotation of $\mathbb{S}^1$ or to σ_k with some $k > 1$ [4]. In the case of irrational rotation, U is called a *Siegel domain*. The complement of the unbounded component of $\mathbb{C} \setminus J_\sim$ is called the *filled-in topological Julia set* and is denoted by $K_\sim$. Equivalently, $K_\sim$ is the union of $J_\sim$ and its bounded Fatou domains. If the lamination $\sim$ is fixed, we may omit $\sim$ from the notation. By default, we consider $f_\sim$ as a self-mapping of $J_\sim$. In what follows, for a collection $\mathcal{R}$ of sets, denote the union of all sets from $\mathcal{R}$ by $\mathcal{R}^+$.

In the Introduction (Definition 1.1), we defined leaves of a lamination $\sim$ and the geo-lamination $\mathcal{L}_\sim$ associated with $\sim$. Extend σ_d (keeping the notation) linearly over all *individual chords* in $\overline{\mathbb{D}}$, in particular, over leaves of $\mathcal{L}_\sim$. Note that even though the extended σ_d is not well defined on the entire disk, it is well defined on $\mathcal{L}^+_\sim$.

Recall that for a gap/leaf U we denote $U \cap \mathbb{S}^1$ by U'. A gap/leaf U of $\mathcal{L}_\sim$ is said to be (*pre*)*periodic* if $\sigma_d^{m+k}(U') = \sigma_d^m(U')$ for some $m \geqslant 0$, $k > 0$. If m can be chosen to be 0, then U is called *periodic*, otherwise U is called *preperiodic* (hence, preperiodic implies non-periodic). Also, by a (pre)periodic gap/leaf we mean gap/leaf which is either periodic or preperiodic. A *Fatou gap* is the $\mathrm{pr}_\sim$-preimage of the closure of a Fatou domain. Similarly, a *Siegel gap* is the $\mathrm{pr}_\sim$-preimage of a Siegel domain. Equivalently, these are gaps with infinite bases. By [18], a Fatou gap G is (pre)periodic under σ_d.

Definition 2.3 (Critical leaves and gaps)**.** A leaf of a lamination $\sim$ is called *critical* if its endpoints have the same image under σ_d. A gap G of $\mathcal{L}_\sim$ is said to be *critical* if $\sigma_d|_{G'}$ is at least k-to-1 for some $k > 1$. We define *precritical* and (*pre*)*critical* objects similarly to how (pre)periodic and preperiodic objects are defined above.

For example, a periodic Siegel gap is non-critical even though the first return map is not one-to-one on its basis (because there must be critical leaves in the boundaries of gaps from its orbit).

2.2. **Geometric laminations.** Laminations, understood as equivalence relations, can be described in a geometric fashion, as was done in the original approach by Thurston [31]. Thurston studied collections of chords in $\mathbb{D}$ similar to $\mathcal{L}_\sim$, for a given σ_d-invariant lamination $\sim$, with no lamination given.

Definition 2.4 (Geometric laminations, cf. [31])**.** A *geometric pre-lamination* $\mathcal{L}$ is a set of (possibly degenerate) chords in $\overline{\mathbb{D}}$ such that any two distinct chords from $\mathcal{L}$ meet at most in a common endpoint; $\mathcal{L}$ is called a *geometric lamination* (*geo-lamination*) if all points of $\mathbb{S}^1$ are elements of $\mathcal{L}$, and $\mathcal{L}^+$ is closed. Elements of $\mathcal{L}$ are called *leaves* of $\mathcal{L}$. By a *degenerate* leaf (chord) we mean a singleton in $\mathbb{S}^1$.

In the Introduction (Definition 1.2), we defined gaps of geo-laminations. Now let us discuss geo-laminations in the dynamical context. Recall that given a chord $\ell = \overline{ab}$ of the unit disk we define $\sigma_d(\ell)$ as the chord $\overline{\sigma_d(a)\sigma_d(b)}$ and extend σ_d linearly over $\overline{ab}$.

Definition 2.5 (Invariant geo-laminations, cf. [31])**.** A geometric lamination $\mathcal{L}$ is said to be an *σ_d-invariant* geo-lamination if the following conditions are satisfied:

1. (Leaf invariance) For each leaf $\ell \in \mathcal{L}$, the set $\sigma_d(\ell)$ is a leaf in $\mathcal{L}$ (if ℓ is critical, then $\sigma_d(\ell)$ is degenerate). For a non-degenerate leaf $\ell \in \mathcal{L}$, there are d pairwise disjoint leaves $\ell_1, \dots, \ell_d \in \mathcal{L}$ with $\sigma_d(\ell_i) = \ell, 1 \leqslant i \leqslant d$.
2. (Gap invariance) For a gap G of $\mathcal{L}$, the set $H = \mathrm{Ch}(\sigma_d(G'))$ is a leaf, or a gap of $\mathcal{L}$, in which case $\sigma_d : \mathrm{Bd}(G) \to \mathrm{Bd}(H)$ is a positively oriented composition of a monotone map and a covering map (thus, if G is a gap with finitely many edges, all of which are critical, then its image is a singleton).

Some invariant geo-laminations are not generated by laminations (see, e.g., [31], where Thurston considers geo-laminations with countable concatenations of leaves forming the boundary of a gap). We will use a special extension $\sigma^*_{d,\mathcal{L}} = \sigma^*_d$ of σ_d to the closed unit disk associated with $\mathcal{L}$. On $\mathbb{S}^1$ and all leaves of $\mathcal{L}$, we set $\sigma^*_d = \sigma_d$. Define σ^*_d on the interiors of gaps using a standard barycentric construction [31]. For brevity, we sometimes use σ_d instead of σ^*_d. We will mostly use the map σ^*_d if $\mathcal{L} = \mathcal{L}_\sim$ for some invariant lamination $\sim$.

2.3. **Laminational sets and their basic properties.** So far we have dealt with (geo-)laminations. However, we also consider subsets of $\overline{\mathbb{D}}$ that have the properties of leaves and gaps of geo-laminations while no actual geo-lamination is specified. A number of facts can be proven for such sets, and we establish some of them in this subsection.

Definition 2.6. Let $f : X \to X$ be a self-mapping of a set X. For a set $G \subset X$, let the *return time* (to G) of $x \in G$ be the least positive integer n_x with $f^{n_x}(x) \in G$, or infinity if there is no such integer. Set $n = \min_{x\in G} n_x$, define the set $D_G = \{x \in G : n_x = n\}$, and call the map $f^n : D_G \to G$ the *remap* (first return map of G). Also, we define *refixed* points in G as points $x \in G$ such that $f^n(x) = x$. Similarly, we talk about *reorbits* of points in G.

For example, if G is the boundary of a periodic Fatou domain of period n of a topological polynomial $f_\sim$, and the images $f^j_\sim(G), j = 0, 1, \dots, n-1$ of G are all pairwise disjoint until $f^n_\sim(G) = G$, then $D_G = G$ and the remap on $D_G = G$ is $f^n_\sim$.

By the *relative interior* of a set in the plane, we mean the interior of this set in its affine hull. Thus, the relative interior of a gap of some lamination is its interior, while the relative interior of a chord is the chord minus the endpoints.

Definition 2.7. Throughout this definition we assume that $A \subset \mathbb{S}^1$ is closed. If all the sets $\mathrm{Ch}(\sigma_d^i(A))$ are pairwise disjoint, then A is called *wandering*. If there exists $n \geqslant 1$ such that all the sets $\mathrm{Ch}(\sigma_d^i(A)), i = 0, \dots, n-1$ have pairwise disjoint relative interiors while $\sigma_d^n(A) = A$, then A is called *periodic* of period n. If there exists $m > 0$ such that all $\mathrm{Ch}(\sigma_d^i(A)), 0 \leqslant i \leqslant m+n-1$ have pairwise disjoint relative interiors and $\sigma_d^m(A)$ is periodic of period n, then we call A *preperiodic*. Observe that the above applies to sets A regardless of whether they are a part of a (geo-)lamination or not.

If A is wandering, periodic or preperiodic, and for every $i \geqslant 0$ and every hole (a, b) of $\sigma_d^i(A)$ either $\sigma_d(a) = \sigma_d(b)$, or the positively oriented arc $(\sigma_d(a), \sigma_d(b))$ is a hole of $\sigma_d^{i+1}(A)$, then we call A (and $\mathrm{Ch}(A)$) a (σ_d-)*laminational set*; we call both A and $\mathrm{Ch}(A)$ *finite* if A is finite. A (σ_d-)*stand alone gap* G is defined as a laminational set with non-empty interior (note that a gap of a geo-lamination always has non-empty interior). In other words, a stand alone G is of the form $\mathrm{Ch}(A)$ for some closed set $A \subset \mathbb{S}^1$ of more than two points such that the above listed properties hold for A.

In what follows, whenever we say that G is a "gap" we mean that at least G is a stand alone gap (it will be clear from the context if there is a lamination or a geo-lamination such that G is a part of it). Also, abusing the language, we will sometimes identify closed sets $A \subset \mathbb{S}^1$ and their convex hulls (again, it will be clear from the context what kind of set we consider).

The basis $G' = G \cap \mathbb{S}^1$ of a gap G coincides with the union $A \cup B$ of two well-defined sets, where A is a maximal Cantor subset of G' or an empty set and B is countable. Assume that $A \neq \varnothing$, and define a map $\psi_G : \mathbb{S}^1 \to \mathbb{S}^1$ that collapses all holes of A to points. Suppose that G is m-periodic. It is well-known that ψ_G can be chosen so that it semiconjugates $\sigma_d^m|_{G'}$ to an irrational rotation of the circle or to the map σ_k, where $k \geqslant 2$. Indeed, by Definition 2.7, the map of the circle, to which σ_d is semi-conjugate under ψ_G, is locally 1-to-1 and orientation preserving. On the other hand, the fact that σ_d is locally expanding implies that the induced map of the circle does not have wandering arcs (see, e.g., [4] where similar arguments were used in classifying different types of gaps of laminations). This implies the above claim.

Accordingly, if $A \neq \varnothing$, we call a stand alone periodic gap G a *stand alone periodic Fatou gap of degree* k if in the above construction the map ψ_G semiconjugates $\sigma_d^m|_{G'}$ to $\sigma_k, k \geqslant 2$. Also, we call a stand alone periodic gap G a *stand alone periodic Siegel gap* if, in the above construction, the map ψ_G semiconjugates $\sigma_d^m|_{G'}$ to an irrational rotation. Moreover, for periodic laminational sets G with finite basis G' and for periodic stand alone Siegel gaps G we can define the *rotation number* τ_G. If the rotation number is not equal to zero, the set G is said to be *rotational*. If such G is invariant, we call it an *invariant rotational set*.

Lemma 2.8 (Lemma 2.16 [8])**.** *Suppose that $\ell = \overline{xy}$ is a laminational chord such that there exists a component Q of the complement of its orbit in the disk $\mathbb{D}$ whose closure contains $\sigma_d^n(\ell)$ for all $n \geqslant 0$. Then the chord ℓ is either* (*pre*)*critical or* (*pre*)*periodic.*

Let U be the convex hull of a closed subset A of $\mathbb{S}^1$. For every edge ℓ of U, let $H_U(\ell)$ denote the hole of U whose endpoints coincide with the endpoints of ℓ. In this situation, we define $|\ell|_U$ as $|H_U(\ell)|$. Notice that if U is a chord, then it has two holes (on opposite sides of U).

Suppose in addition that $A \subset \mathbb{S}^1$ is a laminational set. Mostly, the holes of A map increasingly onto the holes of $\sigma_d(A)$. However, if the length of a hole is at least $\frac{1}{d}$, then the map σ_d wraps the hole around the circle one or more times. Thus, holes H of U such that $|H| \geqslant \frac{1}{d}$ are the only holes which the map σ_d does not take one-to-one onto their images.

Definition 2.9. Let G be the convex hull of a closed subset of $\mathbb{S}^1$. If ℓ is an edge of G such that its hole $H_G(\ell)$ is not shorter than $\frac{1}{d}$, then ℓ is called a (σ_d-)*major edge of* G (or simply a (σ_d-)*major of* G), and $H_G(\ell)$ is called a (σ_d-)*major hole of* G.

It is useful to work with a wider class of sets, which we now introduce.

Definition 2.10. A closed set $A \subset \mathbb{S}^1$ (and its convex hull) is said to be (σ_d-)*semi-laminational* if, for every hole (x, y) of A, we have $\sigma_d(x) = \sigma_d(y)$, or the open arc $(\sigma_d(x), \sigma_d(y))$ is a hole of A (the set A is not assumed to satisfy $\sigma_d(A) = A$).

A set is said to be *invariant* if it maps into itself. Clearly, a σ_d-invariant laminational set is σ_d-semi-laminational.

For a chord ℓ, let $\mathrm{orb}_{\sigma_d}(\ell)$ denote the union of all chords in the forward orbit of ℓ under σ_d.

Lemma 2.11. *If $\ell = \overline{xy}$ is a σ_2-periodic chord of period r with σ_2^r-fixed endpoints and if there is a unique component Z of $\overline{\mathbb{D}} \setminus \mathrm{orb}_{\sigma_2}(\ell)$ such that all images of ℓ are edges of Z, then $\overline{Z}$ is a finite σ_2-invariant stand alone gap.*

Proof. We begin by making an observation which applies to all maps σ_d (the situation of the lemma can be described for σ_d instead of σ_2). If $I = (\sigma_d^k(x), \sigma_d^k(y))$ is a hole of $\overline{Z}$ while $(\sigma_d^{k+1}(y), \sigma_d^{k+1}(x))$ is a hole of $\overline{Z}$, then we say that σ_d *changes orientation* on I. Let us show that if $I = (\sigma_d^k(x), \sigma_d^k(y))$ is a hole of $\overline{Z}$ such that σ_d changes orientation on I, then I contains a σ_d-fixed point. Indeed, the image of I is an arc which connects $\sigma_d^{k+1}(x)$ to $\sigma_d^{k+1}(y)$ and potentially wraps around the circle a few (zero or more) number of times. If $(\sigma_d^{k+1}(y), \sigma_d^{k+1}(x))$ is a hole of $\overline{Z}$, it follows that $I \subset \sigma_d(I)$ and that the endpoints of $\sigma_d(I)$ do not belong to I. Hence I contains a σ_d-fixed point.

Let us now prove the lemma. By the above, σ_2 changes orientation on at most one hole. Since ℓ is periodic and its endpoints are refixed, σ_2 changes orientation an even number of times. Hence σ_2 never changes orientation on holes of $\overline{Z}$. This implies that Z is semi-laminational. Now, if a hole of $\overline{Z}$ is shorter than $\frac{1}{2}$, then it doubles in length under σ_2 while still being mapped onto its image one-to-one. Hence there exists a hole H of Z whose length is at least $\frac{1}{2}$. If we draw a critical leaf c with endpoints in H, we see that the entire orbit of ℓ consists of leaves with endpoints in the complement of H.

Now, it is well known (see, e.g., [31]) that a periodic orbit of σ_2 contained in a given half-circle (in our case, this is any half-circle containing $\mathbb{S}^1 \setminus H$) is the point 0, or the pair $\{1/3, 2/3\}$, or the set of vertices of a finite invariant stand alone σ_2-gap. Moreover, a given half-circle contains exactly one periodic σ_2-orbit. It follows that the endpoints of ℓ are vertices of an invariant σ_2-gap $\overline{Z}$. Clearly, ℓ then has to be an edge of $\overline{Z}$ (if ℓ is a diagonal of $\overline{Z}$, then distinct images of ℓ intersect inside $\mathbb{D}$). □

The class of semi-laminational sets is wider than the class of invariant laminational sets because Definition 2.10 allows for circle arcs to be parts of semi-laminational sets. For example, take a σ_d-invariant stand alone Fatou gap G of degree $k > 1$ such that there is a periodic orbit Q of edges of G. Let $H_1, \dots, H_n$ be the holes of G behind edges from Q. Then $A = \mathbb{S}^1 \setminus \bigcup_{i=1}^n H_i$ is a semi-laminational set. The assumption that a Fatou gap like G exists means that d must be greater than 2. In Lemma 3.9 we study semi-laminational sets for cubic laminations.

Majors of semi-laminational sets play an important role because, as we see below in Lemma 2.12, all edges of semi-laminational sets have majors of these semi-laminational sets in their forward orbits.

Lemma 2.12. *An edge of a semi-laminational set $G = \mathrm{Ch}(A)$ (where $A \subset \mathbb{S}^1$ is closed) is a major if and only if the closure of its hole contains a fixed point. Any edge of G eventually maps to a major of G.*

Proof. Let ℓ be an edge of G. The case when ℓ is invariant (i.e. such that $\sigma_d(\ell) = \ell$) is left to the reader. Otherwise if $|H_G(\ell)| < 1/d$, then $H_G(\ell)$ maps onto the hole $\sigma_d(H_G(\ell))$ one-to-one. The fact that ℓ is not invariant implies that $\sigma_d(H_G(\ell))$ is disjoint from $H_G(\ell)$. Hence $H_G(\ell)$ contains no fixed points. On the other hand, suppose that $|H_G(\ell)| \geqslant 1/d$. Then $\sigma_d(H_G(\ell))$ covers the entire $\mathbb{S}^1$ while the images of the endpoints of $H_G(\ell)$ are outside $H_G(\ell)$. This implies that there exists a fixed point $a \in \overline{H_G(\ell)}$. To prove the second claim, choose an edge ℓ of G. For any i set $T_i = H_G(\sigma_d^i(\ell))$. As long as $|T_i| < \frac{1}{d}$, we have $|T_{i+1}| = |\sigma_d(T_i)| = d|T_i|$. Hence there exists the least n such that $|T_n| \geqslant \frac{1}{d}$. Then the leaf $\sigma_d^n(\ell)$ is a major of G, as desired. □

2.4. **Fixed points and invariant sets.** Theorem 2.13 allows one to find specific invariant sets in some parts of the disk. We state it in the language of laminations.

Theorem 2.13 ([3])**.** *Let $\sim$ be a σ_d-invariant lamination. Suppose that $e_1, \dots, e_m$ are some leaves of $\sim$ and X is a component of $\mathbb{D} \setminus \bigcup_{i=1}^m e_i$ such that for each i*

1. *the leaf e_i lies on the boundary of X,*
2. *there exists no finite gap of $\sim$ inside X with an edge e_i, and*
3. *either σ_d fixes each endpoint of e_i, or $\sigma_d(e_i)$ is contained in the component of $\overline{\mathbb{D}} \setminus e_i$ that contains X.*

Then at least one of the following claims holds:

1. *X contains an invariant gap of $\sim$ of degree $k > 1$;*
2. *X contains an invariant rotational set.*

3. **Invariant quadratic gaps and their canonical laminations.** By cubic laminations we mean σ_3-invariant laminations. By a *quadratic* gap we mean a stand alone periodic Fatou gap U of degree 2. In this section, we assume that U is a σ_3-invariant quadratic gap and study its properties. We then define *canonical* laminations, which correspond to these gaps and describe other laminations that *refine* the canonical ones. Throughout the rest of the paper we will often write σ instead of σ_3.

3.1. **Invariant quadratic gaps.** Recall that, given a gap U with an edge ℓ, we write $|\ell|_U$ for $|H_U(\ell)|$. If U is given, we may drop the subscript U from the notation.

Lemma 3.1. *Let U be a σ-invariant stand alone quadratic gap. Then there exists a unique major edge ℓ of U, on all holes $\widetilde{H} \neq H(\ell)$ of U the map σ is a homeomorphism onto its image, $|\sigma(\widetilde{H})| = 3|\widetilde{H}|$, and the following cases are possible:*

1. *we have $|\ell|_U = \frac{1}{3}$, the leaf ℓ is not periodic, and all holes $\widetilde{H} \neq H(\ell)$ of U are of length at most $\frac{1}{9}$;*
2. *the leaf ℓ is periodic of some period k, we have $\frac{1}{3} < |\ell|_U \leqslant \frac{1}{2}$, and $|\ell|_U = \frac{1}{2}$ only if $\ell = \overline{0\frac{1}{2}}$.*

Proof. The existence of a major ℓ follows from Lemma 2.12. Observe that, if a set $A \subset \mathbb{S}^1$ lies in the complement of two disjoint closed arcs in $\mathbb{S}^1$ of length $\geqslant 1/3$ each, then the restriction of σ to A is injective. This implies that all holes $\widetilde{H} \neq H(\ell)$ of U are shorter than $\frac{1}{3}$ and that $|\ell|_U < \frac{2}{3}$ (here we use the fact that $\sigma|_{U'}$ is two-to-one).

Clearly, $|\ell|_U$ can be equal to $\frac{1}{3}$ (just take $\ell = \overline{\frac{1}{3}\frac{2}{3}}$ and assume that U is the convex hull of the set of all points $x \in \mathbb{S}^1$ with orbits outside the arc $(\frac{1}{3}, \frac{2}{3})$). This situation corresponds to case (1) of the lemma. Since σ expands the length by the factor of 3, it follows that all holes $\widetilde{H} \neq H(\ell)$ of U are shorter than $\frac{1}{9}$.

Suppose that $|\ell|_U > \frac{1}{3}$. Then $\sigma(\ell)$ is eventually mapped to ℓ by Lemma 2.12, hence ℓ is periodic. Since $\frac{1}{3} < |\ell|_U = x < \frac{2}{3}$, then $|\sigma(\ell)|_U = 3x - 1$. If the period of ℓ is k, then $3^{k-1}(3x-1) = x$ and so $x = |\ell|_U = 3^{k-1}(3^k - 1)^{-1} \leqslant \frac{1}{2}$ with equality possible only if $k = 1$ in which case clearly $\ell = \overline{0\frac{1}{2}}$ (observe that under the assumptions of the lemma the leaf ℓ maps onto itself by σ_3^k so that each endpoint of ℓ maps to itself). This corresponds to case (2) of the lemma. □

Lemma 3.2. *In case (2) of Lemma 3.1, there exists a unique leaf ℓ^* disjoint from ℓ with endpoints in $H(\ell)$ such that $\sigma(\ell^*) = \sigma(\ell)$ and such that one of the following holds:*

1. *we have $\ell = \overline{0\frac{1}{2}}$, the only possible holes of $U \cup \ell^*$ of length $\frac{1}{6}$ are*
$$\left(0, \frac{1}{6}\right), \left(\frac{1}{6}, \frac{1}{3}\right), \left(\frac{1}{3}, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{2}{3}\right), \left(\frac{2}{3}, \frac{5}{6}\right), \left(\frac{5}{6}, 1\right),$$
and all other holes of $U \cup \ell^*$ are of length at most $\frac{1}{18}$;
2. *we have $\ell \neq \overline{0\frac{1}{2}}$, every hole of $U \cup \ell^*$ is shorter than $\frac{1}{6}$ except for the hole of ℓ^* that is disjoint from U and has length greater than $\frac{5}{18}$.*

Proof. Let $\ell = \overline{ab}$ and observe first that $\ell^* = \overline{b^*a^*}$, where $b^* = b - \frac{1}{3}, a^* = a + \frac{1}{3}$. Consider now two cases. First, assume that $\ell = \overline{0\frac{1}{2}}$. Then it is easy to see that the claims of the lemma hold. Since $\overline{0\frac{1}{2}}$ is a unique major edge ℓ of an invariant quadratic gap with $|\ell| = \frac{1}{2}$, from now we may assume that $\ell < \frac{1}{2}$.

Consider a hole $\widetilde{H}$ of $U \cup \ell^*$. If $\widetilde{H}$ is a hole of U, then it maps forward monotonically (and hence expanding by the factor of 3) a few times before it maps onto $H(\ell)$. Since, by the above, $|H(\ell)| \leqslant \frac{1}{2}$, we have $|\widetilde{H}| < \frac{1}{6}$. Now, $|(a, b^*)| = |H(\ell)| - |\frac{1}{3} < \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$, and, similarly, $|(a^*, b)| < \frac{1}{6}$. Finally, consider the arc (b^*, a^*). Its image is the arc complementary to the arc $(\sigma(a), \sigma(b))$. Since we assume that $\ell \neq \overline{0\frac{1}{2}}$, it follows that $(\sigma(a), \sigma(b))$ is a hole of U distinct from $H(\ell)$. Hence the length of $(\sigma(a), \sigma(b))$ is less than $\frac{1}{6}$, which implies that the length of $(\sigma(b), \sigma(a))$ is more than $\frac{5}{6}$, and hence the length of (b^*, a^*) is more than $\frac{5}{18}$. $\square$

From now on, the major ℓ of U is denoted by $M(U)$. Lemma 3.1 implies a simple description of the basis U' of U if $M(U)$ is given.

Lemma 3.3. *The basis U' of U is the set of all points $x \in \mathbb{S}^1$, whose orbits are disjoint from $H(M(U))$. All edges of U are preimages of $M(U)$.*

Proof. Orbits of all points of U' are disjoint from $H(M(U))$. Also, if $x \in \mathbb{S}^1 \setminus U'$, then x lies in a hole of U behind a leaf ℓ. By Lemmas 2.12 and 3.1, the orbit of ℓ contains $M(U)$. Hence for some n we have $\sigma^n(\ell) = M(U)$, $\sigma^n(H(\ell)) = H(M(U))$, and $\sigma^n(x) \in H(M(U))$. $\square$

It is natural to consider the two cases from Lemma 3.1 separately. We begin with the case, when an invariant quadratic gap U has a periodic major $M(U) = \overline{ab}$ of period k and $H(M(U)) = (a, b)$. Set $a^* = a + \frac{1}{3}$ and $b^* = b - \frac{1}{3}$. Set $M^*(U) = \overline{b^*a^*}$. Consider the set $N(U)$ of all points of $\mathbb{S}^1$ with σ^k-orbits contained in $[a, b^*] \cup [a^*, b]$ and its convex hull $V(U) = \mathrm{Ch}(N(U))$ (this notation is used in several lemmas below). We call $V(U)$ the *vassal* (gap) of U (see Figure 1).

Lemma 3.4. *Assume that $M(U)$ is periodic of period k. Then $N(U)$ is a Cantor set; $V(U)$ is a periodic quadratic gap of period k.*

Proof. Under the action of σ on $[a, b]$ the arc $[a, a^*]$ wraps around the circle once, and the arc $[a^*, b]$ maps onto the arc $[\sigma(a), \sigma(b)]$ homeomorphically. Similarly, we can think of $\sigma|_{[a,b]}$ as homeomorphically mapping $[a, b^*]$ onto the arc $[\sigma(a), \sigma(b)]$ and wrapping $[b^*, b]$ around the circle once. Thus, first the arcs $[a, b^*]$ and $[a^*, b]$ map homeomorphically to the arc $[\sigma(a), \sigma(b)]$ (which, by definition, is the closure of a hole of U). Then, under further iterations of σ, the arc $[\sigma(a), \sigma(b)]$ maps homeomorphically onto closures of distinct holes of U until σ^{k-1} sends it, homeomorphically, onto $[a, b]$ (all this follows from Lemma 3.1). This generates the quadratic gap $V(U)$ contained in the strip between $\overline{ab}$ and $\overline{a^*b^*}$. In the language of one-dimensional dynamics (see [26, 27, 25, 32] and the book [1]) one can say that closed intervals $[a, b^*]$ and $[a^*, b]$ form a 2-horseshoe of period k. A standard argument shows that $N(U)$ is a Cantor set. The remaining claim easily follows. $\square$

We also define another type of gap called a *caterpillar* (gap). This is a periodic gap Q with the following properties:

- The boundary of Q consists of a periodic (possibly degenerate) leaf $\ell_0 = \overline{xy}$ of period k called a *head of the caterpillar gap Q* , a σ^k-critical leaf $\ell_{-1} = \overline{yz}$ concatenated to it, and a countable concatenation of leaves ℓ_{-n} accumulating at x (the leaf ℓ_{-r-1} is concatenated to the leaf ℓ_{-r}, for every $r = 1, 2, \dots$).
- We have $\sigma^k(x) = x$, $\sigma^k(\{y, z\}) = \{y\}$, and σ^k maps each ℓ_{-r-1} to ℓ_{-r} (all leaves are shifted by one towards ℓ_0 except for ℓ_0, which maps to itself, and ℓ_{-1}, which collapses to the point y).

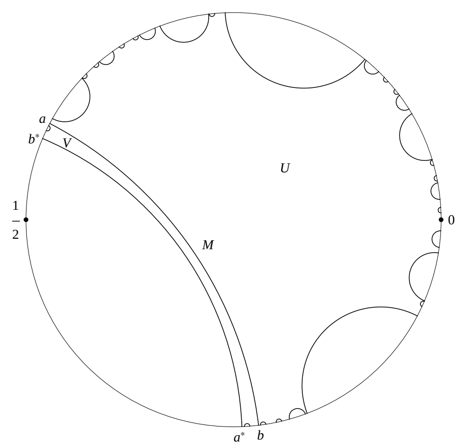

FIGURE 1. A quadratic invariant gap U of periodic type and its vassal V. We draw geodesics in the Poincaré metric instead of straight chords to make the pictures look better.

Similar gaps are already useful for quadratic laminations (see [31] where the invariant gap with edges $\overline{0\frac{1}{2}}, \overline{\frac{1}{2}\frac{1}{4}}, \dots, \overline{\frac{1}{2^n}\frac{1}{2^{n+1}}}, \dots$ is considered). Lemma 3.5 is left to the reader.

Lemma 3.5. *Suppose that $M(U) = \overline{ab}$ is periodic of period k. Then one can construct two caterpillar gaps with head $M(U)$ such that their bases are contained in $\overline{H(M(U))}$. In the first of them, the critical leaf is $\overline{aa^*}$, and in the second one the critical leaf is $\overline{bb^*}$.*

Proof. Let us describe one of the caterpillar gaps in question. Since the the hole (a, b) of U behind $M(U)$ has length between $\frac{1}{3}$ and $\frac{2}{3}$, there exists a unique point $a^* \in (a, b)$ such that $\sigma_3(a^*) = \sigma_3(a)$ and there exists a unique point $b^* \in (a, b)$ such that $\sigma_3(b^*) = \sigma_3(b)$. Let $\ell_0 = M(U)$ and $\ell_{-1} = \overline{aa^*}$ (clearly, $\overline{aa^*}$ is a σ_3-critical leaf). Choose the unique σ_3^k-preimage a_{-2} of $a_{-1} = a^*$ such that $a < a_{-1} < a_{-2} < b$. Put $\ell_{-2} = \overline{a_{-1}a_{-2}}$. Consider further pullbacks ℓ_{-n} of ℓ_0 so that they form a concatenation of leaves converging to the point b. Then the required σ_3^k-fixed caterpillar gap is the convex hull of the union of leaves $\bigcup_{i=0}^{\infty} \ell_{-i}$. The other desired caterpillar gap with critical leaf $\overline{bb^*}$ can be constructed similarly. □

We call the caterpillar gaps from Lemma 3.5 *canonical caterpillar gaps of U*. A critical edge c of a canonical caterpillar gap defines it, so this caterpillar gap is denoted by $C(c)$. We denote its basis by $C'(c)$. To study related invariant quadratic gaps we first prove the following general lemma, in which we adopt a different point of view. Namely, any leaf ℓ that is not a diameter defines an open arc $L(\ell)$ (a component of $\mathbb{S}^1 \setminus \{\ell\}$) of length greater than $\frac{1}{2}$ (in particular, a critical leaf c defines an arc $L(c)$ of length $\frac{2}{3}$). Let $\Pi(\ell)$ be the set of all points with orbits in $\overline{L(\ell)}$.

Lemma 3.6. *Suppose that c is a critical leaf. The set $\Pi(c)$ is non-empty, closed and forward invariant. A point $x \in \Pi(c)$ has two preimages in $\Pi(c)$ if $x \neq \sigma(c)$, and three preimages in $\Pi(c)$ if $x = \sigma(c)$. The convex hull $G(c)$ of $\Pi(c)$ is a stand *alone invariant quadratic gap. If $\sigma(c) \in \Pi(c)$, we have the situation of case* (1) *of Lemma 3.1 and c is the major of $G(c)$.*

Observe that, since c is a critical leaf, its σ-image is a point.

Proof. It is easy to see that the set $\Pi(c)$ is closed and forward invariant; it is non-empty because it contains at least one fixed point (indeed, as was noticed before the lemma, the circle arc $L(c)$ in the case at hand is of length $\frac{2}{3}$, and each circle arc of length $\frac{2}{3}$ contains at least one σ_3-fixed point). Let $x \in \Pi(c)$. If $x \neq \sigma(c)$, then, of its three preimages, one belongs to $\mathbb{S}^1 \setminus \overline{L(c)}$ while two others are in $\overline{L(c)}$, and hence, by definition, also in $\Pi(c)$. Suppose that $x = \sigma(c)$ (and so, since by the assumption $x \in \Pi(c)$, the orbit of c is contained in $\overline{L(c)}$). Then the entire triple $\sigma^{-1}(\sigma(c))$ is contained in $\overline{L(c)}$ and, again by definition, in $\Pi(c)$.

To prove the next claim of the lemma, we prove that any hole I of $\Pi(c)$ except for the hole T, whose closure contains the endpoints of c, maps to a hole of $\Pi(c)$. Indeed, otherwise there is a point $y \in I$ such that $\sigma(y)$ is a point of $\Pi(c)$. Since $I \subset \overline{L(c)}$, we have $y \in \Pi(c)$, a contradiction. Consider the hole $T = (a, b)$ defined above. If $T = \mathbb{S}^1 \setminus \overline{L(c)}$, there is nothing to prove as in this case the leaf c is a critical edge of $G(c)$ that maps to a point of $\Pi(c)$. Suppose now that $T \neq \mathbb{S}^1 \setminus \overline{L(c)}$ and that there is a point $z \in (\sigma(a), \sigma(b)) \cap \Pi(c)$. Then there is a point $t \in T \cap L(c)$ with $\sigma(t) = z$ and hence $t \in \Pi(c)$, a contradiction. Thus, $(\sigma(a), \sigma(b))$ is a hole of $\Pi(c)$, and $\overline{\sigma(a)\sigma(b)}$ is an edge of $G(c)$. All this implies that $G(c)$ is an invariant gap, and it follows from the definition that it is quadratic. The last claim easily follows. $\square$

Below we will use the notation $G(c) = \mathrm{Ch}(\Pi(c))$. Let us relate invariant quadratic gaps defined in terms of periodic majors (see Lemma 3.4) and caterpillar gaps $C(c)$ (see Lemma 3.5) to the gaps $G(c)$. To state Lemma 3.7, we need the following concept. By Lemmas 3.1 and 3.3, all holes of U map onto $H(M(U))$ after finitely many steps in a monotone fashion. Suppose that I is a hole of U and $n \geqslant 0$ is a positive integer such that $\sigma^n(I) = H(M(U))$ and $\sigma^n|_I$ is monotone. Then for any point $x \in H(M(U))$ we call a unique point $\widetilde{x} \in I$ with $\sigma^n(\widetilde{x}) = x$ the *first pullback of x to I*.

Lemma 3.7. *Let $M(U) = \overline{ab}$ be periodic of period k and $c = \overline{xy}$ be a critical chord with endpoints in $\overline{H(M(U))}$. Then:*

1. *if $x, y \in H(M(U))$, then $U' = \Pi(c)$,*
2. *otherwise $\Pi(c)$ consists of U', $C'(c)$, and all first pullbacks of points of $C'(c)$ to holes of U.*

Proof. Clearly $U' \subset \Pi(c)$. We may assume that the circle arc $[a, b]$ is ordered in the positive direction from a to b. Let $\ell = \overline{b^* a^*}$ be the non-periodic edge of $V(U)$ such that $\sigma(a^*) = \sigma(a)$, $\sigma(b^*) = \sigma(b)$; suppose that $\ell \cap c = \varnothing$. The order of points on the circle arc $[a, b]$ is as follows: $a < b^* < a^* < b$. By construction of $V(U)$, we have that $I = H_{V(U)}(\ell)$ coincides with (b^*, a^*), which in particular implies that the length of I is less than $\frac{1}{3}$ (because the length of (a, a^*) is $\frac{1}{3}$). Moreover, $\overline{I}$ contains no points of $\Pi(c)$. Since c is disjoint from ℓ and has endpoints in $\overline{H(M(U))}$, we may assume that $x \in (a, b^*)$ and $y \in (a^*, b)$. Thus, we have $J = (x, y) \supset I$. Note that the restriction of σ^k to $H(M(U)) \setminus \overline{J}$ is a one-to-one expanding map to $H(M(U))$. It follows that all points of $H(M(U))$ eventually map to $\overline{J}$, therefore, they do not belong to $\Pi(c)$

either. By Lemma 3.3, any point $x \notin U'$ eventually maps to $H(M(U))$. Thus, $\Pi(c) = U'$. The case, where $c = \overline{aa^*}$ or $c = \overline{b^*b}$, is left to the reader. □

Lemma 3.8 complements Lemma 3.7.

Lemma 3.8. *If $c = \overline{xy}$ is a critical leaf with a periodic endpoint, say, y such that $x, y \in \Pi(c)$ then there exists a quadratic invariant gap W with a periodic major $M(W) = \overline{zy}$ of period k, the point z is the closest to x in $L(c) = (y, x)$ point that is σ^k-fixed, and z is of period k.*

Proof. Assume that the period of y is k. Consider the gap $G(c)$. By the invariance of $G(c)$, there is an edge c_{-1} of $G(c)$ attached to c at x which maps to c under σ^k. Moreover, $|c_{-1}|_{G(c)} = 3^{-k-1}$. This can be continued infinitely many times so that the m-th edge of $G(c)$, which maps to c under σ^{mk}, is a leaf c_{-m} such that its hole is of length 3^{-mk-1}. Clearly, the concatenation A of leaves c, c_{-1}, … converges to a point $z \in \mathbb{S}^1$ which is σ^k-fixed. Set $\overline{zy} = M$.

Since, by Lemma 3.6, the gap $G(c)$ is quadratic, there are many preimages of $\overline{A}$ on the boundary of $G(c)$. Replace them all by the corresponding preimages of M (e.g., replace $\overline{A}$ by M). It follows that the newly constructed gap W is a quadratic invariant stand alone Fatou gap with major $M = M(W)$ as desired. The last claim of the lemma easily follows. □

Let us summarize the above results. Let c be a critical leaf. We will now define an invariant stand alone quadratic Fatou gap $U(c)$ determined by c. Even though in the beginning of this section we announced that we will consider a given invariant stand alone quadratic Fatou gap U, we choose the notation $U(c)$ for the gap determined by c to reflect the fact that $U(c)$ has the same properties (i.e., that it is an invariant stand alone quadratic Fatou gap). In what follows we will use the notation $L(c), \Pi(c)$ and $G(c)$ introduced above.

The orbit of a critical leaf c can be of three types. First, the orbit of c can be contained in $L(c)$ so that no endpoint of c is periodic. Then we set $U(c) = G(c)$ and call $U(c)$ and the leaf c *regular critical*. Second, an endpoint of c can be periodic with the orbit of c contained in $\overline{L(c)}$. By Lemmas 3.7 and 3.8, then $\Pi(c)$ consists of U', $C'(c)$, and first pullbacks of $C'(c)$ to holes of U for some invariant quadratic gap $U = U(c)$ with a periodic major $M(U)$ (the gap U can be defined as the convex hull of all non-isolated points in $\Pi(c)$). This defines the gap $U(c)$ in that case. Then we call the gap $G(c)$ an *extended caterpillar gap*, and the critical leaf c a *caterpillar* critical leaf. Third, there can be $n > 0$ with $\sigma^n(c) \notin \overline{L(c)}$. Denote by n_c the smallest such n. Then $G(c)$ has a periodic major of period n_c and we set $U(c) = G(c)$ and call this gap a *gap of periodic type*. Only regular critical gaps or gaps of periodic type can be invariant quadratic gaps of laminations (a critical leaf with a periodic endpoint would imply that the corresponding infinite canonical caterpillar is contained in one class, a contradiction with condition (E3) of Definition 2.1).

We show below in Lemma 3.9 that properties listed in item (2) of Lemma 3.1 are basically sufficient for a periodic leaf ℓ to be the major of a quadratic invariant gap of periodic type. The arguments in Lemma 3.9 are related to those in Lemma 2.11.

Lemma 3.9. *Let $\ell = \overline{xy}$ be a σ-periodic leaf of period r with σ^r-fixed endpoints for which there is a unique component Z of $\overline{\mathbb{D}} \setminus \mathrm{orb}_\sigma(\ell)$ such that any two iterated images of ℓ are disjoint or coincident edges of Z. Then a hole of Z with length greater than $\frac{1}{3}$ exists if and only if Z is a semi-laminational set. In that case an eventual σ-image* *of ℓ that corresponds to the major hole of Z is a major of a quadratic invariant gap of periodic type.*

Observe that Z has r holes each of which is located behind an image of ℓ and has length equal to neither $\frac{1}{3}$ nor $\frac{2}{3}$. Recall that if a Fatou gap G is invariant, then the quotient map $\psi_G : \mathrm{Bd}(G) \to \mathbb{S}^1$ is defined as a map collapsing all edges of G to points and mapping G to the unit circle.

Proof. First assume that Z is semi-laminational. Then, by the above remark and Lemma 2.12, at least one hole H of Z must be longer than $\frac{1}{3}$. We may assume that $H = H_Z(\ell)$. Choose a critical leaf c whose non-periodic endpoints are in (x, y). Let us show that $G(c)$ is of periodic type and ℓ coincides with the major M of the gap $G(c)$. Indeed, suppose otherwise. By definition the endpoints of ℓ belong to $G'(c)$ and $\psi_{G(c)}$ maps ℓ to a leaf $\psi_{G(c)}(\ell)$ such that the leaf $\psi_{G(c)}(\ell)$ and its σ_2-images satisfy conditions of Lemma 2.11. Hence $\psi_{G(c)}(\ell)$ and its σ_2-images are the edges of a finite σ_2-invariant gap. In particular, they are not pairwise disjoint.

Now, $\psi_{G(c)}$ collapses only preimages of M. If $G(c)$ is of regular critical type (i.e., $M = c$ has no periodic endpoints), it will follow that ℓ and its σ-images are not pairwise disjoint, a contradiction. If $G(c)$ is of periodic type (i.e., M is a periodic leaf) then, if ℓ and its σ-images miss endpoints of M, we have that $\psi_{G(c)}$ is one-to-one on the endpoints of ℓ and its σ-images. Hence ℓ and its σ-images are not pairwise disjoint, a contradiction.

Suppose finally that, say, $\ell \neq M$ shares an endpoint x with $M = \overline{xz}, z \neq y$. Since ℓ and its σ-images are pairwise disjoint, y does not belong to the same periodic orbit as x. On the other hand, $\psi_{G(c)}$-images of leaves from the σ-orbit of ℓ are the edges of a finite σ_2-invariant gap. Thus, y belongs to some σ-image of M and so the orbit of y coincides with the orbit of z. However this is impossible as by the construction $z \in H$ while H cannot contain points of σ-images of ℓ. The rest of the lemma follows from the above and the already obtained description of quadratic invariant gaps of σ. □

By the above proven lemmas, each gap $W = G(c)$ of periodic type has a periodic major $M(W) = \overline{xy}$ of period n_c with endpoints in $L(c)$; moreover, x and y are the closest in $L(c)$ points to the endpoints of c that are σ^{n_c}-fixed (in fact, they are periodic of period n_c).

Lemma 3.10. *If an invariant quadratic gap W is either of regular critical or of periodic type, then W' is a Cantor set. If $W = G(c)$ is an extended caterpillar gap, then W' is the union of a Cantor set and a countable set of isolated points, all of which are preimages of the endpoints of c.*

Proof. In the regular critical and periodic cases, it suffices to prove that the set W' has no isolated points. Indeed, by Lemma 2.12, an isolated point in W' must eventually map to an endpoint of $M(W)$. Thus it remains to show that the endpoints of $M(W)$ are not isolated. This follows because the endpoints of $M(W)$ are periodic, and suitably chosen pullbacks of points in W' to W' under the iterates of the remap of W' will converge to the endpoints of $M(W)$. The case of an extended caterpillar gap follows from Lemma 3.7. □

3.2. **Canonical laminations of invariant quadratic gaps.** Let us associate a specific lamination with each invariant quadratic gap. We do this in the spirit of [31], where pullback laminations are defined for maximal collections of critical leaves.

Since $\sim$-classes are finite, an invariant lamination cannot contain any caterpillar gaps. Hence we consider only regular critical gaps and gaps of periodic type.

Let U be a stand alone quadratic invariant gap of regular critical type with critical major $M(U)$. Edges of U have uniquely defined iterated pullbacks disjoint from U which define an invariant lamination. More precisely, we define a lamination $\sim_U$ as follows: $a \sim_U b$ if there is $N \geqslant 0$ such that $\sigma^N(a)$ and $\sigma^N(b)$ are endpoints of the same edge of U, and the set $\{\sigma^i(a), \sigma^i(b)\}$ is not separated by U for $i = 0, \dots, N-1$. Loosely, one can say that points a, b "travel" together visiting the same holes of U until at some moment they simultaneously map to the endpoints of an edge of U. Clearly, all $\sim_U$-classes are either points or leaves and $\sim_U$ is an invariant lamination. Now, let U be of periodic type and V be its vassal. Define a lamination $\sim_U$ as follows: $a \sim_U b$ if there exists $N \geqslant 0$ such that $\sigma^N(a)$ and $\sigma^N(b)$ are endpoints of the same edge of U or the same edge of V, and the chord $\overline{\sigma^i(a)\sigma^i(b)}$ is disjoint from $U \cup V$ for $i = 0, \dots, N-1$. Note that V is a gap of $\sim_U$. It is easy to check that the canonical lamination of a quadratic periodic gap U does not have periodic non-degenerate classes that are not edges of U.

Lemma 3.11. *If U is a stand alone invariant quadratic gap of regular critical type, then $\sim_U$ is the unique invariant lamination such that U is one of its gaps.*

In Definition 1.7, we defined coexistence of a gap (of some unspecified lamination) and a lamination. The definition holds verbatim if a gap of some lamination is replaced with a stand alone gap. We also defined coexistence of two laminations.

Lemma 3.12. *Suppose that a cubic invariant lamination $\sim$ coexists with a stand alone invariant quadratic gap U of regular critical type. Then $\sim$ also coexists with the canonical lamination $\sim_U$ of U.*

Proof. Suppose that a leaf ℓ of $\sim$ crosses a leaf ℓ_U of $\sim_U$ in $\mathbb{D}$. By the assumption of the lemma, both ℓ and ℓ_U must have their endpoints in the closure of some hole of U. Every hole of U maps one-to-one onto its image. It follows that $\sigma(\ell)$ and $\sigma(\ell_U)$ also intersect in $\mathbb{D}$. However, any leaf of $\sim_U$ eventually maps to an edge of U, a contradiction. ☐

Proof of Lemma 3.11. Suppose that $\sim$ is a cubic invariant lamination and U is a gap of $\sim$; then $\sim$ coexists with U, hence, by Lemma 3.12, the lamination $\sim$ coexists with the canonical lamination $\sim_U$. If a leaf ℓ of $\sim$ is not a leaf of $\sim_U$, then ℓ is in some pullback of U with respect to $\sim_U$. Hence the leaf ℓ eventually maps to U. Since U is a gap of $\sim$, the leaf ℓ eventually maps to an edge of U. By definition it follows that ℓ is a leaf of $\sim_U$. The definition of a lamination now implies that all leaves of $\sim_U$ are leaves of $\sim$. ☐

The proof of Lemma 3.13 is similar to that of Lemmas 3.11 and 3.12.

Lemma 3.13. *Let U be an invariant quadratic gap of periodic type. Then $\sim_U$ is the unique invariant lamination such that U and the vassal $V(U)$ are its gaps. If a cubic invariant lamination $\sim$ coexists with U and $V(U)$, then $\sim$ coexists with the canonical lamination $\sim_U$ of U.*

3.3. **Tuning.** In this subsection we discuss the notion of coexistence of two laminations and make it more explicit. We will define the notion of *tuning* which is stronger than coexistence of laminations. Right after Definition 2.7 we introduced a monotone map $\psi_G : \mathrm{Bd}(G) \to \mathbb{S}^1$; for an invariant stand alone gap G of degree

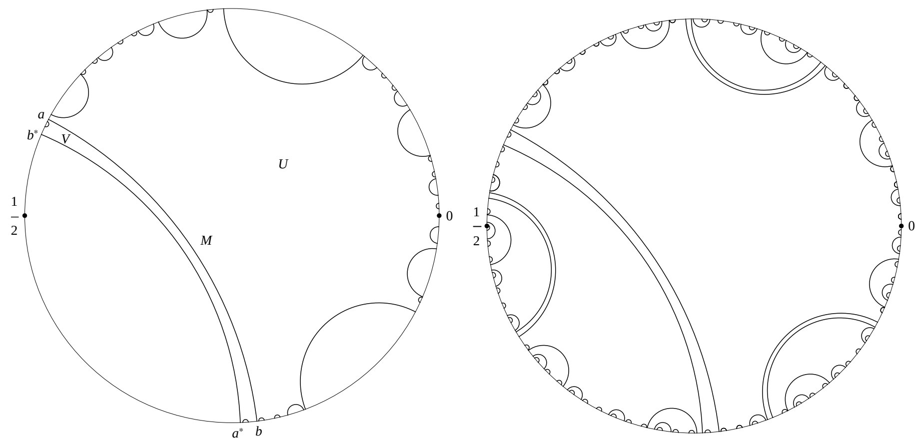

FIGURE 2. Left: a quadratic invariant gap U of periodic type and its vassal V. Right: its canonical lamination.

$k > 1$, we have that $\sigma_d|_{\mathrm{Bd}(G)}$ is semiconjugate to σ_k by means of ψ_G. If a lamination $\sim$ coexists with G, then we want to obtain an induced lamination $\psi_G(\sim)$ which is invariant under σ_k. We need the following notions.

Definition 3.14 ([6])**.** Let $\mathcal{L}$ be a geo-lamination such that the σ_d-images of its leaves are again leaves of the same geo-lamination. Any two disjoint leaves of $\mathcal{L}$ with the same σ_d-image are called *sibling leaves*, or *siblings*. Suppose that any leaf of $\mathcal{L}$ has at least one σ_d-preimage, and, for any leaf ℓ_1 of $\mathcal{L}$ with non-degenerate σ_d-image, there are $d-1$ leaves $\ell_2, \dots, \ell_d$ of $\mathcal{L}$ such that the leaves ℓ_i, $i = 1, \dots, d$ are pairwise disjoint and map to $\sigma_d(\ell_1)$. This property is called the *Sibling Property* and $\mathcal{L}$ is then called a *sibling invariant lamination*.

Let us explain the difference between Definition 3.14 and the so-called *leaf invariance* (see Definition 2.5(1)). In Definition 2.5(1), one begins with a non-degenerate leaf ℓ and postulates the existence of d pairwise disjoint preimage-leaves of ℓ. In Definition 3.14 we begin with any leaf ℓ_1 whose image is a non-degenerate leaf and postulate the existence of $d-1$ pairwise disjoint (and disjoint from ℓ_1) leaves with the same image $\sigma_d(\ell_1)$. This is a little stronger than Definition 2.5(1) as it does not follow from Definition 2.5(1) that we will be able to find siblings of *any* leaf mapped to ℓ. A surprising fact is that this subtle difference proves to be sufficient to imply all other properties of invariant geo-laminations.

More precisely, the main result of [6] is that *sibling invariant laminations are invariant geo-laminations*. An advantage of using sibling invariant laminations is that checking if a geo-lamination is sibling invariant requires considering only leaves of the lamination. Also, geo-laminations generated by laminations are sibling invariant [6], and it is proved in [6] that if sibling invariant laminations $\mathcal{L}_i$ are such that continua $\mathcal{L}_i^+$ converge (in the Hausdorff metric) to a continuum K, then in fact there exists a sibling invariant geo-lamination $\mathcal{L}$ such that $\mathcal{L}^+ = K$.

Now we consider the union of all leaves of two coexisting laminations $\sim$ and $\simeq$. By the above, this gives rise to a sibling invariant lamination $\mathcal{L}_\sim \cup \mathcal{L}_\simeq$. To show that this generates a lamination, we need more tools.

Definition 3.15 (Proper lamination [6])**.** Two leaves with a common endpoint v and the same non-degenerate image are said to form a *critical wedge* (*with vertex* v). A lamination $\mathcal{L}$ is *proper* if it contains no critical leaf with a periodic endpoint and no critical wedge with a periodic vertex.

A geo-lamination $\mathcal{L}$ has *period matching property* if any leaf of $\mathcal{L}$ with a periodic endpoint of period n is such that its other endpoint is also of period n. Lemma 3.16 follows from the definitions.

Lemma 3.16. *Suppose that $\sim$ is a lamination and $\mathcal{L}_\sim$ is the geo-lamination generated by $\sim$. Then $\mathcal{L}_\sim$ has period matching property. Also, any geo-lamination with period matching property is proper.*

By Lemma 3.16, given two coexisting laminations $\sim$ and $\simeq$, their geo-laminations $\mathcal{L}_\sim$ and $\mathcal{L}_\simeq$ have period matching property. Then $\mathcal{L}_\sim \cup \mathcal{L}_\simeq$ also has period matching property and, hence, is a proper geo-lamination. Conversely, suppose that $\mathcal{L}$ is an invariant geo-lamination. Define an equivalence relation $\approx_\mathcal{L}$ by declaring that $x \approx_\mathcal{L} y$ if and only if there exists a *finite* concatenation of leaves of $\mathcal{L}$ connecting x and y. Theorem 3.17 specifies properties of $\approx_\mathcal{L}$.

Theorem 3.17 ([6])**.** *Let $\mathcal{L}$ be a proper invariant geo-lamination. Then $\approx_\mathcal{L}$ is an invariant lamination.*

We defined tuning in Definition 1.8. Take an invariant lamination $\simeq$ that coexists with an invariant stand alone quadratic gap U; e.g., it may be that $\simeq$ tunes U. We want to show that then ψ_U transports $\simeq$ to a quadratic invariant lamination, which we will denote $\psi_U(\simeq)$. Take a non-critical leaf ℓ of $\simeq$ inside U. It has two sibling leaves which are disjoint. Clearly, one of them, say, $\overline{a}$, is contained in U. If $\psi_U(\overline{a})$ and $\psi_U(\ell)$ are non-disjoint, then either $\psi_U(\ell) = \psi_U(\overline{a})$ is a critical leaf or the two chords $\psi_U(\ell)$, $\psi_U(\overline{a})$ form a critical wedge, a wedge subtended by a diameter. We need to show that the latter case is impossible. In this case, ℓ and $\overline{a}$ share endpoints with an edge $\overline{e}$ of U. There are no leaves of $\simeq$ separating $\ell \setminus \mathbb{S}^1$ from $\overline{a} \setminus \mathbb{S}^1$ in $\mathbb{D}$, since any such leaf would have to cross $\overline{e}$. Therefore, ℓ and $\overline{a}$ are edges of the same gap G of $\simeq$. Since some edges of G are sibling leaves, the gap G must be quadratic. The sibling $\overline{e}^*$ of $\overline{e}$ in G is clearly an edge of U connecting the endpoints of ℓ and $\overline{a}$ different from the endpoints of $\overline{e}$.

But then we have $\psi_U(\ell) = \psi_U(\overline{a})$. Thus, ψ_U-images of leaves of $\simeq$ inside U form a sibling σ_2-invariant geo-lamination, cf. [6, Section 6].

As $\mathcal{L}_\simeq$ is proper, it follows that $\psi_U(\mathcal{L}_\simeq)$ is proper. Indeed, by Lemma 3.16, if an endpoint of a leaf ℓ of $\mathcal{L}_\simeq$ is periodic, then ℓ is periodic. Consider a leaf $\widetilde{\ell}$ of $\mathcal{L}_\simeq$ and the leaf $\psi_U(\widetilde{\ell})$. If an endpoint of $\psi_U(\widetilde{\ell})$ is periodic, then $\widetilde{\ell}$ has a σ_3-periodic endpoint, hence $\widetilde{\ell}$ is periodic, hence $\psi_U(\widetilde{\ell})$ is periodic. Thus the lamination $\psi_U(\mathcal{L}_\simeq) = \mathcal{L}$ is proper and, by Theorem 3.17, one can construct the lamination $\approx_\mathcal{L}$ which generates a geo-lamination $\mathcal{L}_{\approx_\mathcal{L}}$.

Let us compare $\mathcal{L}$ with $\mathcal{L}_{\approx_\mathcal{L}}$ and show that they almost coincide.

Lemma 3.18. *Let $\mathcal{L}$ be a proper quadratic geo-lamination. Then $\mathcal{L} \supset \mathcal{L}_{\approx_\mathcal{L}}$. Moreover, $\mathcal{L} \setminus \mathcal{L}_{\approx_\mathcal{L}}$ consists of the grand orbit of a critical leaf and/or the grand orbit of a critical quadrilateral of $\mathcal{L}$ that is strictly inside a finite critical gap of $\mathcal{L}_{\approx_\mathcal{L}}$.*

Proof. To prove that $\mathcal{L} \supset \mathcal{L}_{\approx_\mathcal{L}}$, it suffices to show that the equivalence relation $\approx_\mathcal{L}$ produces leaves which the geo-lamination $\mathcal{L}$ already contains. By Thurston's No Wandering Triangle theorem [31], any finite non-precritical gap of a quadratic lamination is pre-periodic or periodic. Moreover, by [31], the vertices of a periodic gap must form one cycle. Hence any chord inside such a gap will cross itself and cannot be a leaf of any lamination. Therefore, if a gap G of $\approx_{\mathcal{L}}$ is not a gap of $\mathcal{L}$, then it must be critical or pre-critical. Suppose that G is critical. If $\mathcal{L}$ has more than a critical leaf or/and a critical quadrilateral inside G, then the image of G is still a gap and has at least one chord inside. However, $\sigma_2(G)$ cannot be pre-critical, and, by the previous case, this is impossible. □

Hence, any quadratic proper sibling lamination $\mathcal{L}$ can be cleaned (if necessary) by means of removing critical leaves/quadrilaterals of it contained inside appropriate finite gaps of $\mathcal{L}_{\approx_{\mathcal{L}}}$ (as described above) as well as removing their pullbacks. This results into the geo-lamination generated by $\approx_{\mathcal{L}}$. In particular, we can clean the geo-lamination $\psi_U(\mathcal{L}_{\simeq})$ constructed above and in this way relate it to a certain quadratic lamination $\asymp$. Strictly speaking, the lamination $\asymp$ coincides with $\approx_{\psi_U(\mathcal{L}_{\simeq})}$ (first the lamination $\simeq$ generates the geo-lamination $\mathcal{L}_{\simeq}$, then the geo-lamination $\mathcal{L}_{\simeq}$ is transported to the circle by the map ψ_U, and then the geo-lamination $\psi_U(\mathcal{L}_{\simeq})$ generates the lamination $\approx_{\psi_U(\mathcal{L}_{\simeq})}$), however for brevity in what follows we will simply denote $\asymp$ by $\psi_U(\simeq)$.

The above arguments allowed us to define the lamination $\psi_U(\simeq)$. They were based on the fact that U is an invariant *quadratic* stand alone gap. Literally the same arguments apply if U is a stand alone periodic quadratic gap (i.e., a periodic Fatou gap of degree two). Hence, in the periodic case, given a lamination $\simeq$ that coexists with U, we can also define the lamination $\psi_U(\simeq)$.

Lemma 3.19 proves in the case of tuning the claims established in Lemmas 3.11, 3.12 and 3.13. The proof is analogous to the proofs of Lemma 3.11 and 3.12 and is left to the reader.

Lemma 3.19. *Suppose that $\approx$ is a lamination which tunes an invariant quadratic gap U. Then the following holds.*

1. *If U is of regular critical type, then $\approx$ tunes the canonical lamination $\sim_U$.*
2. *If U is of periodic type and $\approx$ tunes $V(U)$ as well, then in fact $\approx$ tunes the canonical lamination $\sim_U$.*

This provides a more explicit description of how a lamination can tune an invariant quadratic gap.

3.4. **Cubic laminations with no rotational gaps or leaves.** We now describe all cubic laminations with no periodic rotational gaps or leaves (a *rotational leaf* is a periodic leaf with non-refixed endpoints). Recall that for an invariant quadratic gap of periodic type U we let $M^*(U)$ denote the sibling leaf of $M(U)$ in $V(U)$. Recall from [6] that, for any invariant lamination $\sim$, the corresponding geo-lamination $\mathcal{L}_\sim$ is sibling invariant.

Lemma 3.20. *Let U be a stand alone invariant quadratic gap of periodic type with major $M = M(U)$ of period k. Suppose that M is a leaf of a lamination $\sim$. Then the leaf $M^*(U) = M^*$ is a leaf of $\sim$ too. Moreover, if both M and M^* are edges of a single gap of $\sim$, then this gap coincides with $V(U)$, and if M and M^* are not on the boundary of a single gap of $\sim$, then $\sim$ has a rotational gap or leaf in $V(U)$.*

Observe that we do not assume U to be a gap of $\sim$.

Proof. Since M is a leaf of $\sim$, then by the Sibling Property so is M^*. Now, if there is a gap G of $\sim$ such that M and M^* are edges of G, then, by definition of $V(U)$, we see that $G \subset V(U)$ is a σ_3^k-invariant gap. Applying the map $\psi_{V(U)}$, we get a σ_2-invariant gap $\psi_{V(U)}(G)$ that contains the angle 0. Clearly, then $\psi_{V(U)}(\mathrm{Bd}(G)) = \mathbb{S}^1$ and hence $G = V(U)$.

Suppose that M and M^* are not on the boundary of a single gap of $\sim$. Then there must be a leaf ℓ of $\mathcal{L}_\sim$ that separates $M \setminus \mathbb{S}^1$ from $M^* \setminus \mathbb{S}^1$ in $\mathbb{D}$. Note that one of its siblings ℓ^* is also contained in the strip between M and M^*. We claim that $(\ell \cup \ell^*) \cap (M \cup M^*) = \varnothing$. To see this, note that if ℓ and M share an endpoint, then ℓ is not critical because $\mathcal{L}_\sim$ is proper. Hence $\sigma^k(\ell) \cup \ell \cup M$ is a tripod, which is impossible. Indeed, $\sigma^k(\ell)$ cannot coincide with ℓ, since there are no k-periodic points between M and M^*. Hence M is approached by leaves of $\sim$ separating M from M^* (as such we can choose pullbacks of ℓ or ℓ^*). Choose one such leaf $\bar{q}$ of $\sim$ close enough to M and then choose the leaf $\bar{q}^*$ with the same σ-image as $\bar{q}$, separating $\bar{q}$ from M^* ($\bar{q}^*$ is a leaf of $\sim$ by the Sibling Property). By Theorem 2.13, there exists a σ^k-fixed gap or leaf Q in the closed strip $\overline{S}$ where $S = S(\bar{q}, \bar{q}^*)$ is the open strip between $\bar{q}$ and $\bar{q}^*$. It follows that the entire orbit of Q is located in the same parts of the circle, where the orbit of $V(U)$ is located, which implies that $Q \subset V(U)$. Applying $\psi_{V(U)}$ to Q and using well-known facts about quadratic laminations and their invariant sets, we see that Q is rotational, as desired. □

Let us characterize laminations with no rotational sets.

Lemma 3.21. *Suppose that a cubic invariant lamination $\sim$ has no periodic rotational gaps or leaves. Then either $\sim$ is empty, or it coincides with the canonical lamination of an invariant quadratic gap.*

Proof. Suppose that a non-empty lamination $\sim$ has no rotational sets. Then by Theorem 2.13 there is an invariant gap U of degree $1 < k \leqslant 3$. If $k = 3$, then $\sim$ is empty, hence $k = 2$. If U is of regular critical type, then, by Lemma 3.11, the lamination $\sim$ is the canonical lamination $\sim_U$. Let U be of periodic type. Since $M(U)$ is a leaf of $\mathcal{L}_\sim$, then by Lemma 3.20, so is $M^*(U)$. Now, if there is a gap G of $\sim$ such that $M(U)$ and $M^*(U)$ are edges of G, then, by Lemma 3.20, we have $G = V(U)$, and, by Lemma 3.13, the lamination $\sim$ coincides with the canonical lamination $\sim_U$. Suppose that $M(U)$ and $M^*(U)$ are not contained in the same gap of $\sim$. Then, by Lemma 3.20, there exists a rotational gap or leaf of $\sim$, a contradiction. □

3.5. Coexistence of quadratic invariant gaps and other laminational sets.

Here we show how invariant quadratic gaps can coexist with each other as well as how they can coexist with other laminational sets (we consider gaps of laminations, i.e., invariant quadratic gaps which are regular critical or of periodic type). We are motivated here by the desire to provide a model for specific families of laminations and laminational sets (such as the family of all quadratic invariant gaps) which should be helpful in the description of the entire cubic Mandelbrot set $\mathcal{M}_3$. Some of these lemmas are used in [9].

Let us discuss two special quadratic invariant gaps which often play the role of exceptions to the claims proven below. Let $\overline{0\frac{1}{2}} = \overline{\mathrm{Di}}$ be the unique chord in $\mathbb{D}$ with σ-invariant endpoints. Let FG_a be the convex hull of all points with orbits **a**bove $\overline{\mathrm{Di}}$ and FG_b be the convex hull of all points with orbits **b**elow $\overline{\mathrm{Di}}$. Then $\overline{\mathrm{Di}}$ is the major of both gaps. However as a major of FG_b it should be viewed so that the positive direction on $\overline{\mathrm{Di}}$ is from 0 to $\frac{1}{2}$, and if $\overline{\mathrm{Di}}$ is considered as the major of FG_a, then the positive direction on $\overline{\mathrm{Di}}$ is from $\frac{1}{2}$ to 0. Recall that when talking about a

Jordan curve K which encloses a simply connected domain W on the plane, by the *positive direction* on K one means the counterclockwise direction with respect to W, i.e., the direction of a particle moving along K so that W remains on its left.

Let U be an invariant quadratic gap. If U is of regular critical type, then we set $M^*(U) = M(U)$; if U is of periodic type, then we set $M^*(U)$ to be the leaf that is not an edge of U and that has the property $\sigma(M^*(U)) = \sigma(M(U))$. We summarize a few simple facts

If a lamination has the gap U, it must have the leaf $M^*(U)$. Let S_U be the closed strip in $\mathbb{D}$ between $M(U)$ and $M^*(U)$, and set $H(M(U)) = H(U)$. In the regular critical case, we have $\frac{1}{3} = |H(U)|$; in the periodic case, by Lemma 3.1, we have $\frac{1}{3} < |H(U)| \leqslant \frac{1}{2}$ with $|H(U)| = \frac{1}{2}$ only if $M(U) = \overline{\mathrm{Di}}$ (and hence only if $U = \mathrm{FG}_a$ or $U = \mathrm{FG}_b$). For $M(U) \neq \overline{\mathrm{Di}}$, the arc $H(U)$ has the same endpoints as $M(U)$ and is shorter than $\frac{1}{2}$; the basis of the gap U is contained in $\mathbb{S}^1 \setminus H(U)$.

Denoting $M(U)$ by $\overline{ab}$, we *always* mean that the direction from a to b along $H(U)$ is positive. Denote the closed circle arcs from $\mathrm{Bd}(S_U)$ by L_U, R_U with positive direction on $H(U)$ being from R_U to L_U. If $M(U)$ is critical, $S_U = M(U)$ is a chord. Clearly, $M(U)$ determines R_U, and R_U determines $M(U)$: if $R_U = [a, b^*]$, where $b^* = b - \frac{1}{3}$, then $M(U) = \overline{ab}$. By Lemma 3.3, the orbit of an endpoint of $M(U)$ avoids $H(U)$; thus, the orbit of a cannot enter $(a, b] \supset (a, a^*]$ (as before, we write a^* for $a + \frac{1}{3}$). Observe that a is periodic and b^* is pre-periodic. Clearly, $|R_U| = |L_U| = |H(U)| - \frac{1}{3}$. Hence $|R_U| \leqslant \frac{1}{6}$ with $|R_U| = \frac{1}{6}$ only if $M(U) = \overline{\mathrm{Di}}$.

Lemma 3.22. *Let $U \neq W$ be invariant quadratic gaps of two different laminations. Then $R_U \cap R_W = L_U \cap L_W = \varnothing$.*

Proof. If U, W are of regular critical type, the claim follows. Let U be of periodic type such that $M(U)$ is of period m. Consider a point t in the interior of R_U. Then the analysis of the dynamics of σ^m on S_U (similar to that of the dynamics of σ_2) implies that $\sigma^m(t) \in (t, t + \frac{1}{3})$ (similar to the statement that $\sigma_2(s) \in (s, s + \frac{1}{2})$ for every $s \in (0, \frac{1}{2})$). Hence t cannot be the initial endpoint of a major of regular critical type or of periodic type. Now consider the endpoints of R_U. Let $R_U = [a, b^*]$ and $H(U) = (a, b)$, where $b^* = b - \frac{1}{3}$. Let us show that a is not the initial point of the major of a quadratic invariant gap $W \neq U$. Without loss of generality we may assume that $H(W) = (a, \widetilde{b}^*)$ and $\widetilde{b}^* \in (a^*, b)$, where $a^* = a + \frac{1}{3}$. Then, as above, $\sigma^m(\widetilde{b}^*) \in (a, \widetilde{b}^*)$, a contradiction. Also, neither a nor b^* can be the initial point of a major of regular critical type because both points are initial points of majors of extended caterpillar type. Since a is periodic, it is impossible that $[t, a] = R_W$ for some invariant quadratic gap W. This exhausts all possibilities and shows that $R_U \cap R_W = \varnothing$ if $U \neq W$. Similarly, $L_U \cap L_W = \varnothing$. □

Lemma 3.22 describes a generic type of intersection between two majors. The remaining case is when two majors meet at one point and are oriented so that their holes are disjoint. In this case majors must meet at their common endpoint, and pairs of such majors are rather specific.

Lemma 3.23. *Let $M(U) = \overline{xy}$, $M(W) = \overline{zx}$ be majors of invariant quadratic gaps U, W. Assume that $x \in [\frac{1}{2}, 0]$. Then there are the following cases.*

1. *Both $M(U)$ and $M(W)$ coincide with $\overline{\mathrm{Di}}$ oriented in opposite directions.*
2. *Both U, W are of regular critical type, in which case $\sigma(x) = \sigma(y) = \sigma(z)$, the forward orbit of $\sigma(x)$ is in $[y, z] \cap \mathrm{Bd}(\mathrm{FG}_a)$, and the convex hull of the closure of this orbit is a Siegel gap.*

3. *Both U, W are of periodic type, points $y \in (0, \frac{1}{6})$, $z \in (\frac{1}{3}, \frac{1}{2})$ belong to the same periodic orbit $P \subset \mathrm{Bd}(\mathrm{FG}_a)$ on which the map acts as a rational rotation.*

Proof. First let U, W be of regular critical type. Then the orbit of $\sigma(x)$ is located in the circle arc $[y, z]$ of length $\frac{1}{3}$. By [5], the closure T of the orbit of x is such that by collapsing arcs complementary to T we will semiconjugate $\sigma|_T$ to an irrational rotation. By the properties of majors, $0 \in (x, y)$ and $\frac{1}{2} \in (z, x)$. Hence the orbit of $\sigma(x)$ is contained in $(0, \frac{1}{2})$. This completes case (2).

Now let U and W be of periodic type. Observe that if $x = 0$ or $x = \frac{1}{2}$, then case (1) takes place. So we may assume that $x \in (\frac{1}{2}, 0)$. Then $0 \in (x, y)$ and $\frac{1}{2} \in (z, x)$. Suppose that $\sigma^k(y) \in [\frac{1}{2}, x]$ for some k. Then, since the orbit of x never enters $[z, x) \cup (x, y]$ and $\sigma^k(y) \neq y$, it follows that $\sigma^k(x) \in (y, z)$. Hence $\sigma^k(M(U))$ separates $\frac{1}{2}$ from y. Since the orbit of $M(U)$ is on the boundary of U and $\frac{1}{2} \in \mathrm{Bd}(U)$ by Lemma 2.12 and Lemma 3.1, this leads to a contradiction and shows that the orbit of y is contained in the circle arc $[0, \frac{1}{2}]$, and hence $y \in \mathrm{Bd}(\mathrm{FG}_a)$. Moreover, recall that the orbit of y is contained in $[y, x]$; if $y \in [\frac{1}{3}, \frac{1}{2}]$ then $\sigma(y) \in (x, y)$, a contradiction. Hence $y \in [0, \frac{1}{6})$.

If $y = 0$, then $x = \frac{1}{2}$ and $z = 0$ so that case (1) holds. If $y \in (0, \frac{1}{6})$ then $y^* = y + \frac{1}{3} \in (\frac{1}{3}, \frac{1}{2})$, and $z \in (y, y^*)$. Set $x^{**} = x + \frac{2}{3}$; then $\overline{x^{**}y^*}$ is an edge of U. Clearly, $z \in (y, x^{**})$ because, by Lemma 3.1, the length of the arc (z, x) is between $\frac{1}{3}$ and $\frac{1}{2}$. Suppose that $\sigma^k(y) \in (y^*, \frac{1}{2})$ for some k. Clearly, $\sigma^k(x) \notin (x, x^{**})$. Hence $\sigma^k(x) \in (y^*, \sigma^k(y)) \subset (z, x)$, a contradiction. Thus, the orbit of y is contained in $[y, y^*]$. By [5], then the order among points of the orbit of y is the same as the order of points in a periodic orbit under a rational circle rotation. Since the orbit of $M(U)$ consists of edges of U, it follows that the same order is maintained among the points from the orbit of x. Literally the same can be said about the orbit of z and the orbit of x. This implies that the rotation numbers associated with the orbits of y and z are the same. Since both orbits are contained in FG_a, and $\sigma|_{\mathrm{FG}_a}$ is semiconjugate to σ_2, well-known properties of σ_2 imply that y and z belong to the same periodic orbit. So, under the current assumption, case (3) holds. □

Lemma 3.24. *Let A be a non-degenerate class or an infinite gap of a cubic lamination $\sim$ that has a quadratic invariant gap U. Suppose that A never maps to U, $M(U)$ or $V(U)$. Then U is of periodic type, no image of A intersects $M^*(U)$, there exists a number $q \geqslant 0$ and an edge e of $\sigma_3^q(A)$ such that either $e = M(U)$, or e separates U from $M^*(U)$. Moreover, there is a leaf $\hat{e}$ (possibly equal to e) such that $\sigma_3(\hat{e}) = \sigma_3(e)$, and $\hat{e}$ separates U from the second critical set $D \neq U$ of $\sim$. If A is periodic of period m, and k is the period of $M(U)$, then $m > k$. If we have $\sigma_3^n(A) = U$, and A is located between $M(U)$ and $M^*(U)$, then $n > k$.*

Proof. Suppose that U is of regular critical type. Then A is a preimage of U or a preimage of $M(U)$, an edge of A maps to $M(U)$, and we can set $e = \hat{e} = M(U)$. The last claim of the lemma follows because if U is of regular critical type, A can only be periodic or critical if $A = U$ or $A = M(U)$.

Let U be of periodic type with vassal V. If $\sigma^l(A) = U$, or $\sigma^l(A) = M(U)$, or $\sigma^l(A) = V$ (by Lemma 3.13, this is true for a suitable choice of l if $\sim$ is the canonical lamination of U), then set $q = l, e = \hat{e} = M(U)$. Assume now that $\sim$ is not the canonical lamination of U, and A never maps to U, V or $M(U)$. Let us show that no image of A intersects $M^*(U)$. Indeed, if an image T of A contains an endpoint of $M^*(U)$ then the properties of laminations imply that either $T = M^*$, or $T = V$, or T is a gap with an edge $M^*(U)$ which maps onto U after k more steps. Since this contradicts our assumptions, it remains to consider the case when an image T of A will cross $M^*(U)$. However then it is easy to see that $\sigma^K(T)$ will cross M, a contradiction. So, no image of A intersects $M^*(U)$.

Then an edge ℓ of A connects two points t_1, t_2 of the boundary of a gap T of the canonical lamination $\sim_U$ of U and passes through the interior of T. This includes the possibility that ℓ crosses two edges of T in $\mathbb{D}$. We show that $\sigma^q(A)$ separates $M(U)$ from $M^*(U)$ for some q. Indeed, as long as T maps onto its image one-to-one, the images of ℓ connect the corresponding images of t_1, t_2. By the properties of $\sim_U$ there is the least n with $\sigma^n(T) = U$ or $\sigma^n(T) = V$. The leaf $\sigma^n(\ell)$ connects two points of $\mathrm{Bd}(\sigma^n(T)))$ and passes through the interior of $\sigma^n(T)$. Hence $\sigma^n(T) \neq U$, the leaf $\sigma^n(\ell)$ connects two points of $\mathrm{Bd}(V)$ and passes through the interior of V. Let ψ_V collapse all edges of V; then ψ_V semiconjugates $\sigma^k|_{\mathrm{Bd}(V)}$ to σ_2 and maps $\sigma^n(\ell)$ to a chord inside $\mathbb{D}$. By properties of σ_2, the chord $\sigma^q(\ell) = e$ separates $M(U)$ and $M^*(U)$ for some $q \geqslant n$. The claim about $\hat{e}$ immediately follows (the critical set $D \neq U$ of $\sim$ is located between e and $\hat{e}$).

Let us prove the last claims of the lemma. Let A be periodic of period m and $A \notin \{U, M(U), V(U)\}$. We claim that then for all i the set $\sigma_3^i(A)$ cannot have $M(U)$ as an edge. Indeed, suppose otherwise. Then $\sigma_3^i(A)$ must be periodic of period k. It follow from the definition of $V(U)$ that $A \subset V(U)$. Applying the map ψ_V defined above we see that $\psi_V(A)$ must be the unique fixed point of σ_2. Hence $\sigma_3^i(A) = M(U)$, a contradiction.

By the above there exists an image B of A and an edge e of B such that e separates $M(U)$ from $M^*(U)$. Hence for k steps images of B will be located "behind" the corresponding images of $M(U)$ and $\sigma_3^k(B) \neq B$ as otherwise $\sigma_V(B)$ will have to be the unique fixed point of σ_2, a contradiction. Therefore $k < m$ as desired. A similar argument proves the very last claim of the lemma. □

Now let us study which laminations with invariant quadratic gaps can share a non-degenerate class or an infinite gap.

Lemma 3.25. *Let $\sim_i$ $(i = 1, 2)$ be two laminations. Suppose that A is a non-degenerate class or an infinite gap of both laminations, and that each $\sim_i$ has a quadratic invariant gap U_i, $i = 1, 2$. Then $U_1 = U_2$ except for the case when A is a gap or a class of the lamination that has both* FG_a *and* FG_b *as gaps; in the latter case, U_1 and U_2 may coincide with* FG_a *and* FG_b *(in any order).*

Proof. Suppose that A is a non-degenerate class or an infinite gap of a lamination $\sim$ that has an invariant quadratic gap U. We show how to recover U knowing A, except in the case, where both FG_a and FG_b are gaps of $\sim$. This will imply the statement of the lemma.

For every leaf or gap T of $\sim$, let $||T||$ denote the length of the largest hole of T. Set $\mu = \inf_{n\geqslant 0} ||\sigma^n(A)||$. There may or may not be a leaf or gap B in the forward orbit of A with the property $||B|| = \mu$. If there is no such B, then there exists a leaf ℓ of $\sim$ such that $||\ell|| = \mu$, and there is a sequence of leaves or gaps in the forward orbit of A accumulating on ℓ. In this latter case, we set $B = \ell$. Suppose that no image of A has $\overline{0\frac{1}{2}}$ as its edge, and let H be the largest hole of B. We claim that, in this case, U' can be recovered as the set of all points in H, whose forward orbits stay in H.

To prove the claim, let M denote the major of U and, as before, let M^* denote its sibling not in U. Observe that, by our assumptions and by Lemma 3.20, both M

and M^* are leaves of $\sim$. Let us show that either $B = M$ or B is in the strip between M and M^*. Indeed, by Lemma 3.24, images of A are located in non-major holes of U, or in holes of $U \cup M^*(U)$, or separate $M(U)$ from $M^*(U)$. If an image T of A is located in a non-major hole of U, then, by Lemma 3.2, we have $||T|| > \frac{5}{6}$. If an image T of A is located in one of the three remaining holes of $U \cup \mathcal{M}^*(U)$ but not in the hole of $U \cup M^*(U)$ separated from U by $M^*(U)$, then, again by Lemma 3.2, we have $||T|| > \frac{5}{6}$. Finally, since, by Lemma 3.24, there are images of A separating $M(U)$ from $M^*(U)$, it follows that B cannot be separated from U by $M^*(U)$. We conclude that either $B = M$ or B is in the strip between M and M^*. In either case it follows that U' coincides with the set of points in the largest hole H of B, whose forward orbits stay in H. Observe that even if $||B|| = \frac{1}{2}$, the set U' is well-defined. The remaining case when some image of A equals $\overline{0\frac{1}{2}}$ is immediate. □

4. **Invariant rotational sets.** Fix an invariant rotational laminational set G. There are one or two majors of G. We classify invariant rotational gaps by types. This classification mimics Milnor's classification of hyperbolic components in slices of cubic polynomials and quadratic rational functions [23, 21]. Namely, we say that

- the gap G is of type A (from "Adjacent") if G has only one major (whose length is at least $\frac{2}{3}$);
- the gap G is of type B (from "Bi-transitive") if G has two majors that belong to the same periodic cycle;
- the gap G is of type C (from "Capture") if it is not type B, and one major of G eventually maps to the other major of G;
- the gap G is of type D (from "Disjoint") if there are two majors of G, whose orbits are disjoint.

Clearly, it follows from the definitions that finite rotational gaps cannot be of type C (since then $\sigma|_{\mathrm{Bd}(G)}$ is not one-to-one). Also, if G is of type B, then $\sigma|_{\mathrm{Bd}(G)}$ is one-to-one, and hence G must be finite.

4.1. **Finite rotational sets.** A classification of finite rotational sets (under the name of fixed point portraits) can be found in [17]. We now give some examples illustrating a part of this classification concerning the degree 3 case.

Let G be a finite invariant rotational set (as we fix G in this section, we may omit using G in the notation). By [18], there are at most two periodic orbits (of the same period denoted in this section by k) forming the set of vertices of G. If vertices of G form two periodic orbits, points of these orbits alternate on $\mathbb{S}^1$.

Example 4.1. Consider the invariant rotational gap G with vertices $\frac{7}{26}$, $\frac{4}{13}$, $\frac{11}{26}$, $\frac{10}{13}$, $\frac{21}{26}$ and $\frac{12}{13}$. This is a gap of type D. The major leaf M_1 connects $\frac{12}{13}$ with $\frac{7}{26}$ and the major leaf M_2 connects $\frac{11}{26}$ with $\frac{10}{13}$. These major leaves belong to two distinct periodic orbits of edges of G. The major hole $H_G(M_1)$ contains 0 and the major hole $H_G(M_2)$ contains $\frac{1}{2}$.

The next example can be obtained by considering *one* periodic orbit of vertices in the boundary of the gap from Example 4.1

Example 4.2. Consider the finite gap G with vertices $\frac{7}{26}$, $\frac{11}{26}$ and $\frac{21}{26}$. This is a gap of type B. The first major leaf M_1 connects $\frac{21}{26}$ with $\frac{7}{26}$ and the second major leaf M_2 connects $\frac{11}{26}$ with $\frac{21}{26}$. The edges of G form one periodic orbit to which both M_1 and M_2 belong. The major hole $H_G(M_1)$ contains 0 and the major hole $H_G(M_2)$ contains $\frac{1}{2}$.

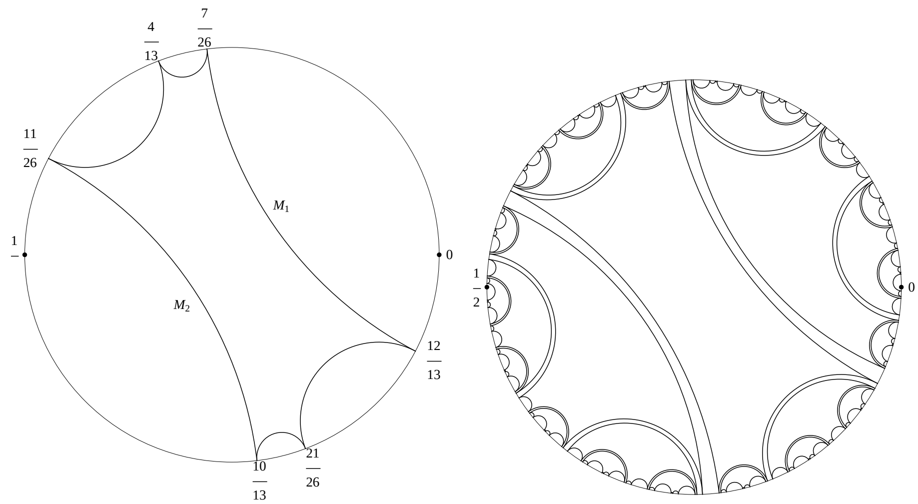

FIGURE 3. The rotational gap described in Example 4.1 and its canonical lamination.

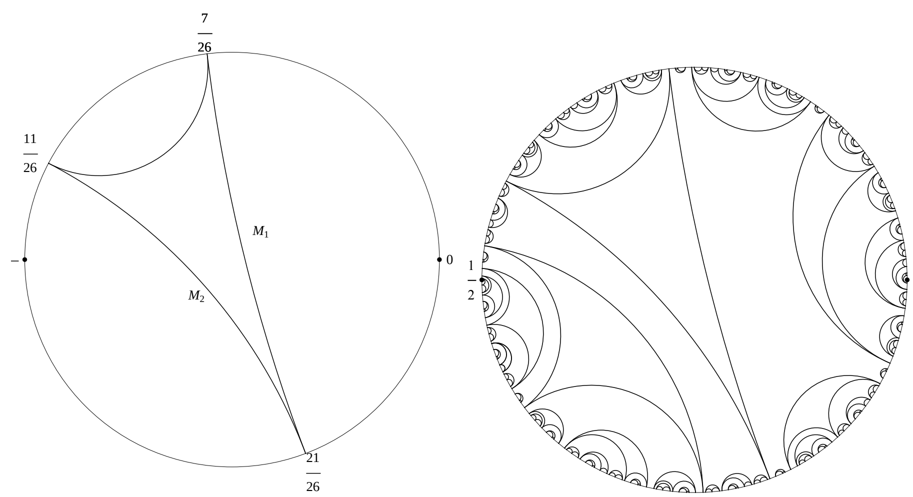

FIGURE 4. The rotational gap described in Example 4.2 and its canonical lamination.

Example 4.3. Consider the finite gap G with vertices $\frac{1}{26}$, $\frac{3}{26}$ and $\frac{9}{26}$. This is a gap of type A. The only major leaf $M = M_1 = M_2$ connects $\frac{9}{26}$ with $\frac{1}{26}$. The edges of G form one periodic orbit to which M belongs. The major hole $H_G(M)$ contains 0 and $\frac{1}{2}$ and is longer than $\frac{2}{3}$.

Let $G = \ell = \overline{ab}$ be an invariant leaf. We can think of G as a gap with empty interior and two edges $\overline{ab}$ and $\overline{ba}$, and deal with all finite invariant sets in a unified way. Let us list all non-degenerate invariant leaves $\overline{ab}$. Either points a, b are fixed, or they form a two-periodic orbit. In the first case, we have the leaf $\overline{0\frac{1}{2}} = \overline{\mathrm{Di}}$, in the second case, we have one of the leaves $\overline{\frac{1}{8}\frac{3}{8}}$, $\overline{\frac{1}{4}\frac{3}{4}}$, $\overline{\frac{5}{8}\frac{7}{8}}$. Informally, we regard $\overline{\mathrm{Di}}$ as an invariant rotational set of type D (even though its rotation number is 0).

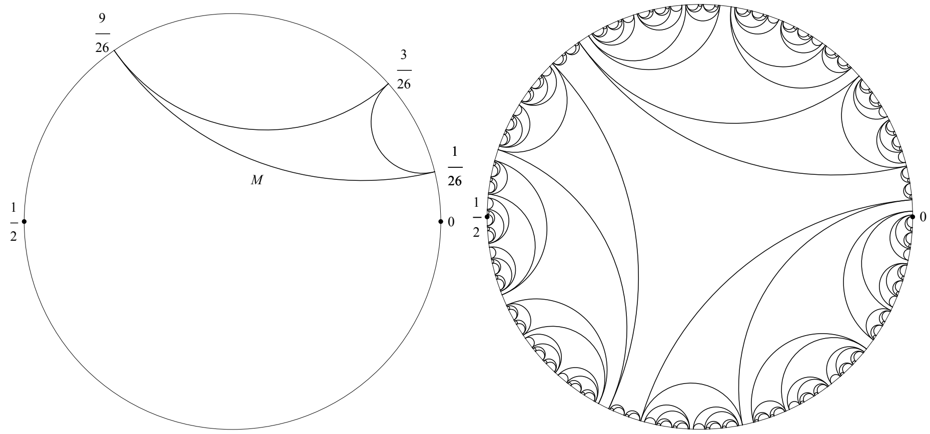

FIGURE 5. The rotational gap described in Example 4.3 and its canonical lamination.

Let G be a finite invariant laminational set with m edges $\ell_0, \ldots, \ell_{m-1}$. For each i, let FG_i be the convex hull of all points $x \in \overline{H_G(\ell_i)}$ with $\sigma^j(x) \in \overline{H_G(\sigma^j(\ell_i))}$ for every $j \geqslant 0$ (compare this to the definition of a vassal in Section 3). It is straightforward to see that FG_i are infinite stand alone gaps such that FG_i maps to FG_j if $\ell_j = \sigma(\ell_i)$. These gaps are called the *canonical Fatou gaps attached to* G. The gap FG_i is critical if and only if the corresponding edge ℓ_i is a major.

4.2. **Canonical laminations of finite invariant rotational sets.** To every finite invariant rotational set G, we associate its *canonical* lamination $\sim_G$.

Suppose first that G is of type B or D. Then by definition there are two major edges of G, which are denoted by M_1 and M_2. Let H_1 and H_2 be the corresponding holes. By Lemma 2.12, we may assume that $0 \in H_1$ and $\frac{1}{2} \in H_2$. Since M_1 and M_2 are periodic, their lengths are strictly greater than $\frac{1}{3}$. Let U_1 and U_2 be the canonical Fatou gaps attached to M_1 and M_2, respectively. Thurston's pullback construction [31] yields an invariant lamination formed by the pullbacks of G disjoint from the interiors of U_1 and U_2. More precisely, we can define a lamination $\sim_G$ as follows: two points a and b on the unit circle are equivalent if there exists $N \geqslant 0$ such that $\sigma^N(a)$ and $\sigma^N(b)$ are vertices of G, and the chords $\overline{\sigma^i(a)\sigma^i(b)}$ are disjoint from G and from the interior of $U_1 \cup U_2$ for $i = 0, \ldots, N-1$. It is straightforward to check that $\sim_G$ is indeed an invariant lamination. This lamination is called the *canonical lamination associated with* G.

Assume now that G is of type A. Let M be the major edge of G, and U the corresponding canonical Fatou gap attached to G at M. The canonical lamination $\sim_G$ of G is defined similarly to those for types B and D. Namely, two points a and b on the unit circle are equivalent with respect to $\sim_G$ if there exists $N \geqslant 0$ such that $\sigma^N(a)$ and $\sigma^N(b)$ are vertices of G, and the chord $\overline{\sigma^i(a)\sigma^i(b)}$ is disjoint from G and from the interior of U for $i = 0, \ldots, N-1$.

Lemma 4.4 is similar to Lemma 3.11. It is based on Thurston's pullback construction of laminations.

Lemma 4.4. *Suppose that* $\sim$ *has a finite invariant gap* G *and all the canonical Fatou gaps attached to* G *are gaps of* $\mathcal{L}_\sim$. *Then* $\sim$ *coincides with the canonical lamination of* G.

Lemma 4.5. *Suppose that a cubic invariant lamination $\sim$ has a finite invariant gap G of type D. If a canonical Fatou gap U of G is not a gap of $\sim$, then $\sim$ has a rotational gap or leaf in U. Thus, if $\sim$ is not the canonical lamination of G, then $\sim$ has a rotational periodic gap or leaf in a canonical Fatou gap attached to G.*

Proof. Clearly, a major of a gap G of type D satisfies the conditions of Lemma 3.9 and can be viewed as the major of some invariant quadratic gap W of periodic type. Then the canonical Fatou gap U attached to G coincides with the vassal gap $V(W)$ of W. Hence the rest of the lemma follows from Lemmas 3.20 and 4.4. □

4.3. **Irrational invariant gaps and their canonical laminations.** The description of irrational gaps is close to that of finite laminational sets. In Subsection 4.3 we fix an irrational rotation number τ.

Let G be an invariant Siegel gap of rotation number τ. Then G may have one or two critical majors of length $\frac{1}{3}$, or one critical major of length $\frac{2}{3}$. It is also possible that G has a non-critical major. However, a non-critical major eventually maps to the critical major by Lemma 2.8 (in this case, G is of type C). Thus an infinite rotational gap G can have type A, C or D.

We now construct the canonical lamination for a Siegel gap G of type D with critical edges L and M. Consider well-defined pullbacks of G attached to G at L and M. Then apply Thurston's pullback procedure to these gaps. As holes in the union of bases of these gaps are shorter than $\frac{1}{3}$, the pullbacks of the gaps converge in diameter to 0. Alternatively, we can define $\sim_G$ as follows: two points a and b in the unit circle are equivalent if there exists $N > 0$ such that $\sigma^N(a)$ and $\sigma^N(b)$ lie on the same edge of G, and the chords $\overline{\sigma^i(a)\sigma^i(b)}$ are disjoint from G for $i = 0, \dots, N-1$.

We will not define canonical laminations of type A or C Siegel gaps.

Lemma 4.6. *Suppose that G is a type D stand alone invariant gap of Siegel type, and $\sim$ is an invariant cubic lamination with gap G. Then $\sim$ coincides with $\sim_G$.*

The proof goes almost verbatim as in Lemma 3.11. It is based on Thurston's pullback construction of laminations.

5. **The description of the Combinatorial Main Cubioid.** If G is a finite σ_2-invariant rotational set, we define the *canonical lamination* of G as the only quadratic invariant lamination with a cycle of Fatou gaps attached to edges of G (it represents a parabolic quadratic polynomial $z^2 + c$, whose parameter c belongs to the Main Cardioid). Similarly, if G is a stand alone invariant Siegel gap with respect to σ_2, we define the *canonical lamination* of G as the unique quadratic invariant lamination that has G as its gap.

Proposition 5.1. *A non-empty quadratic lamination $\sim$ with at most one periodic* (*hence fixed*) *rotational set G coincides with the canonical lamination of G. The Combinatorial Main Cardioid* CA^c *consists of quadratic laminations with at most one periodic rotational set.*

Proof. By Theorem 2.13, the lamination $\sim$ has an invariant rotational set G_2. If G_2 is a Siegel gap, then $\sim$ is the canonical lamination of G_2 because all pullbacks of G_2 are uniquely defined (cf. Lemma 3.11). Let G_2 be a finite gap or leaf. By [18], the laminational set G_2 has a unique major M_2 separating G_2 from 0, and the edges of G_2 form one cycle (of period r). Let V_2 be the Fatou gap of the canonical lamination of G_2 which has M_2 as one of its edges. Let M_2^* be the sibling of M_2;

then M_2^* is an edge of V_2. By Theorem 2.13 and because $\sim$ has at most one periodic rotational gap or leaf, the strip between M_2 and M_2^* contains a σ_2^r-invariant Fatou gap U of $\sim$ of degree greater than 1. In fact, r is the period of U as U passes through every hole of G_2 before returning. Hence $U \subset V_2$, which immediately implies that $U = V_2$, and $\sim$ is the canonical lamination of G_2. □

Let us go back to cubic laminations $\sim$ (recall that σ_3 is denoted by σ). Gaps $U(c)$ for critical chords c are defined right after Lemma 3.8.

Lemma 5.2. *Suppose that $\sim$ is a lamination that coexists with two disjoint critical chords, c and d such that c has non-periodic endpoints, no leaf of $\sim$ contains an endpoint of d, and d intersects no edge of $U(c)$ in $\mathbb{D}$. Then $\sim$ coexists with the gap $U(c)$. Moreover, either $\sim$ tunes $U(c)$, or no edge of $U(c)$ belongs to $\sim$.*

Proof. We need to show that a leaf of $\sim$ does not intersect an edge of $U(c) = U$ in $\mathbb{D}$ unless it coincides with it. By way of contradiction, suppose that a leaf ℓ of $\sim$ intersects an edge $\bar{j} \neq \ell$ of U in $\mathbb{D}$. Let us show that then $\sigma(\ell)$ intersects $\sigma(\bar{j})$ in $\mathbb{D}$. Indeed, there exists a component of $\overline{\mathbb{D}} \setminus (c \cup d)$ whose closure contains both ℓ and $\bar{j}$. Hence, $\sigma(\bar{j}) \neq \sigma(\ell)$. The only case when $\sigma(\ell)$ and $\sigma(\bar{j})$ do not intersect in $\mathbb{D}$ under the circumstances is as follows: ℓ and $\bar{j}$ contain distinct endpoints of one of the critical chords, c or d. Let us show that this is impossible.

By the assumptions on d the leaf ℓ does not contain an endpoint of d. Suppose that each of the chords ℓ and $\bar{j}$ contains an endpoint of c. Then an endpoint of c belongs to $U(c)$, and since c has no periodic endpoints, c is regular critical. Hence the only edge of $U(c)$ containing an endpoint of c is c, and $\bar{j} = c$. But then c and ℓ do not intersect in $\mathbb{D}$ because c and leaves of $\sim$ distinct from c are disjoint inside $\mathbb{D}$ by the assumption. Thus $\sigma(\ell)$ and $\sigma(\bar{j}) \neq \sigma(\ell)$ intersect in $\mathbb{D}$. By induction this implies that for any $n \geqslant 0$, $\sigma^n(\bar{j}) \neq \sigma^n(\ell)$ intersect in $\mathbb{D}$.

Since no image of ℓ can intersect c while not coinciding with c, this implies that c cannot be regular critical. Then the whole orbit of $\bar{j}$ stays strictly on one side of c which implies that so does the whole orbit of ℓ. Hence (see Subsection 3.1) the endpoints of ℓ belong to $U'(c)$, and $\ell \neq \bar{j}$ cannot intersect $\bar{j}$ in $\mathbb{D}$, a contradiction. Denote the major of $U(c)$ by M. The last claim of the lemma follows from the fact that if M is a leaf of $\sim$ then $\sim$ tunes $U(c)$ (because $\sim$ is backward invariant and coexists with d) while if M is not a leaf of $\sim$ then no edge of $U(c)$ can be a leaf of $\sim$ (because $\sim$ is forward invariant and by Lemma 3.3 which states that all edges of $U(c)$ are preimages of M). □

We also need the following lemma. Observe that a cubic lamination that has a critical set of degree two must have a second critical set, also of degree two.

Lemma 5.3. *Let $\sim$ be a lamination from CU^c with a finite invariant gap G of type A or B such that a cycle $\mathcal{F}$ of Fatou gaps attached to G at each edge consists of Fatou gaps of degree two. Suppose that the second critical set W of $\sim$ is infinite. Then the following holds.*

1. *The set W is a periodic Fatou gap of degree 2, and the refixed edge ℓ of W separates the rest of W from G.*
2. *The leaf ℓ is the major of a unique quadratic invariant gap U; the lamination $\sim$ tunes the canonical lamination $\sim_U$ according to a quadratic lamination $\asymp$ from CA^c (possibly empty); and $W = V(U)$.*

3. *The gap U is a unique quadratic invariant gap which coexists with $\sim$ except for the case when $G = \overline{\mathrm{Di}}$ in which case either $V = \mathrm{FG}_a$ and $U = \mathrm{FG}_b$, or $V = \mathrm{FG}_b$ and $U = \mathrm{FG}_a$.*

Proof. Clearly, W is a periodic Fatou gap of degree 2. We claim that the orbit of W is contained in $E \cup W$, where E is the component of $\overline{\mathbb{D}} \setminus W$ containing G. Indeed, otherwise there is i with $\sigma^i(W)$ contained in the closure of a component $F \neq E$ of $\mathbb{D} \setminus W$. If $\sigma^i(W)$ touches W at a vertex, an edge of W and an edge of $\sigma^i(W)$ must have a common vertex. This implies that these two edges in fact are edges of a finite rotational set of $\sim$ distinct from G, a contradiction with $\sim$ being from CU^c. If $\sigma^i(W)$ and W share an edge, then this edge is rotational, again a contradiction. Finally, if $\sigma^i(W)$ is disjoint from W, then, by Theorem 2.13, the component F contains a σ^i-invariant rotational gap or leaf of $\sim$ or a σ^i-invariant Fatou domain of $\sim$. The former is impossible because $\sim \in \mathrm{CU}^c$, and the latter is impossible because the only two cycles of Fatou domains of $\sim$ are $\mathcal{F}$ and the cycle of W. This proves the claim that the orbit of W is contained in $E \cup W$.

The gap W has a unique edge ℓ that separates the rest of W from G. Denote the sibling of ℓ in W by $\hat{\ell}$. We claim that ℓ is the refixed edge of W (being quadratic, W has a unique refixed edge). Assume that ℓ is not refixed in W. Choose a critical chord c of W with strictly preperiodic endpoints. Then c divides W in two halves, A and B, while $\psi_W(c)$ divides $\mathbb{S}^1$ in two halves, $\psi_W(A)$ and $\psi_W(B)$. By the properties of σ_2, each of the two sets $\psi_W(A)$, $\psi_W(B)$ contains an invariant rotational gap, leaf, or point $\{0\}$, denoted by $T(A)$, $T(B)$, and one of the sets $T(A)$, $T(B)$ is $\{0\}$.

Since c has strictly preperiodic endpoints, the forward orbit of c intersects both A and B. In particular, c cannot be a regular critical major, and the gap $U(c)$ is of periodic type. Let M be the major of $U(c)$. Then M separates c from G. Clearly, if M is an edge of W, then $M = \ell$, and if M is not an edge of W, then M separates c from ℓ. In both cases, we have $M \subset W$.

Assume that $\ell \subset A$. Then by definition $\psi_W^{-1}(T(A)) \subset U(c)$. Consider the images $M_0, \dots, M_k$ of M in W, where $M_0 = M$. Their endpoints are all located in A. Hence, by the properties of σ_2, their ψ_W-images form either the cycle of vertices of $T(A)$ or the cycle of edges of $T(A)$. Thus, $M_0, \dots, M_k$ either form a cycle of edges of W, or a cycle of chords of W projected to $T(A)$ under ψ_W. Since in the former case M_0 cannot separate c from ℓ, either $M = M_0 = \ell$ or $M_0, \dots, M_{k-1}$ project onto edges of $T(A)$. In the former case, ℓ is refixed as otherwise M_0 separates M_1 from G, and cannot be the major of a quadratic invariant gap of periodic type. By way of contradiction, assume that $M_0, \dots, M_{k-1}$ project onto edges of $T(A)$.

The above arguments apply to all choices of c with strictly preperiodic endpoints. If $T(A) = \{0\}$, then M is a refixed edge of W, which as before implies that $M = \ell$. If $T(A) \neq \{0\}$, then there are at least two images of M in W that separate G from c (two coinciding leaves with different orientations are considered as different). We see that one of the chords $M_0, \dots, M_k$ separates G from another one, contradicting properties of majors of periodic type. This contradiction finally proves that $\ell = M$ is the major of an invariant quadratic gap of periodic type.

Now, we have already shown the existence of a quadratic invariant gap $U = U(c)$ coexisting with $\sim$; moreover, we proved that the periodic leaf ℓ is the major of U and that U is tuned by $\sim$ (by Lemma 5.2, since ℓ is a leaf of $\sim$, all edges of U are leaves of $\sim$). Clearly, then W is the vassal of U. It remains to prove that U is the unique quadratic invariant gap coexisting with $\sim$. Suppose that $Q \neq U$ is a quadratic invariant gap with major m coexisting with $\sim$.

We will write V for the critical Fatou gap in $\mathcal{F}$. If m is critical, then it is contained in V or W (since Q and $\sim$ coexist, m cannot cross leaves of $\sim$). However this is impossible because then the remap of V (or of W) will push m away from the refixed edge of V (or of W) and hence from G resulting in images of m separated from the $\frac{2}{3}$-arc created by m, a contradiction with the dynamics of regular critical gaps.

Suppose that m is periodic. Since there are only two cycles of periodic Fatou gaps of $\sim$ (namely, the orbit of W and the orbit of V), the images of m are contained in W or in V. The argument from above with m instead of M and a suitable choice of c proves that either $m = \ell$ is the refixed edge of W, or m is the refixed edge of V (and hence an edge of G). Clearly, the latter is impossible (the images of m must all be pairwise disjoint) except when $G = \overline{\mathrm{Di}}$. Thus, if $G \neq \overline{\mathrm{Di}}$, then $m = \ell$ is the refixed edge of W and $Q = U$. The remaining easy case $G = \overline{\mathrm{Di}}$ is left to the reader. □

Lemma 5.4 gives a preliminary description of laminations from CU^c. Recall that by Definition 1.5 a lamination from CU^c has at most one rotational set (which then by necessity is fixed).

Lemma 5.4. *Let $\sim$ be a lamination from CU^c. Then $\sim$ coexists with an invariant quadratic gap U, and if G is the unique rotational set of $\sim$, then $G \subset U$. Moreover, either $\sim$ tunes U or no edge of U belongs to $\sim$. In the latter case, U can be chosen to be of regular critical type, and if $\sim$ is not canonical, then U must be of regular critical type. If G is of type A or B, then:*

1. *if $\sim$ is canonical, then U can be chosen to be regular critical and weakly tuned by $\sim$;*
2. *if $\sim$ is not canonical and has two critical sets, a critical gap V attached to G and the second critical set C not attached to G, then U can be chosen as $U(c)$ where $c \subset C$ is **any** critical chord with non-periodic endpoints.*

Proof. By Lemma 3.21, if $\sim$ has no rotational gap or leaf, $\sim$ is empty or the canonical lamination of an invariant quadratic gap as desired. Assume that $\sim$ has a rotational gap or leaf G. By Lemmas 4.5 and 4.6, and because $\sim$ comes from CU^c (and hence has at most one rotational set), if G is of type D, then it follows that $\sim = \sim_G$ tunes an invariant quadratic gap U, whose major is one of the two majors of G, and $G \subset U$. Hence from now on we assume that G is of type A, B or C.

First assume that G is finite and has n edges. By Definition 1.5, there exists a cycle $\mathcal{F}$ of Fatou gaps attached to G. Let G be of type B. Suppose first that $\mathcal{F}$ has two gaps, V and W, on that the map σ is two-to-one. Let M be the major of G that is an edge of V, and $\sigma^k(M)$ be the major of G that is an edge of W. Denote by $N \neq M$ the edge of V such that $\sigma(N) = \sigma(M)$, and by $T \neq \sigma^k(M)$ the edge of W with $\sigma(T) = \sigma(\sigma^k(M))$. We want to find a regular critical major separating M from N. To do so, consider the model map for $\sigma^n|_V$ which is σ_4 (as always, the modeling map is the map collapsing all edges of gaps to points). Clearly, we can find σ_4-critical diameters ℓ, whose orbits are contained in the half-circle bounded by ℓ and containing 0. By definition, the critical chord L inside V corresponding to ℓ is a major of regular critical type of an invariant quadratic gap U.

Since the orbits of the endpoints of $\sigma^k(M)$ and T are contained in the circle arc of length $\frac{2}{3}$, whose endpoints are the endpoints of L, these endpoints belong to U'. Hence the edges of U are disjoint from the convex hull Q of $\sigma^k(M) \cup T$. This in turn implies that edges of U and leaves of $\sim$ do not intersect. Indeed, otherwise we can map such intersecting leaves forward, and their images intersect too (because by the above the intersecting leaves must be such that their endpoints belong to an arc of length less than $\frac{1}{3}$). In the end an edge of U will map to L, a contradiction since we know that L is disjoint from all leaves of $\sim$. Hence in this case we can always choose U to be of a regular critical type. Clearly, $G \subset U$.

Now, if G is of type A, then there is only one gap V of $\mathcal{F}$ which does not map forward one-to-one; V is attached to the unique major edge M of G. If V is cubic, then, similar to the above, we can choose a regular critical chord L inside V so that L is a major of regular critical type of an invariant quadratic gap U. To show that $\sim$ and U coexist, consider the major M of G. Then there are two edges of V having the same image as M. Let N be one of them chosen so that L does not separate M from N. The existence of N is derived from the following observation: for a σ-critical chord ℓ of regular critical type such that $0 \in U'(\ell)$, we have $\frac{1}{3} \notin H_{U(\ell)}(\ell)$ or $\frac{2}{3} \notin H_{U(\ell)}(\ell)$. As before, let Q be the convex hull of $M \cup N$. Then literally repeating the arguments from the previous case, we can show that U and $\sim$ coexist.

This completes our consideration of the canonical laminations in the case when G is finite. Thus from now on we may assume that G is either of type A or of type B, and there is a unique Fatou gap V from $\mathcal{F}$ attached to some edge of G such that $\sigma|_{\mathrm{Bd}(V)}$ is not one-to-one; moreover, in the remaining cases we may assume that the remap on V is two-to-one. Then clearly there exists a critical leaf or gap C which is not a gap from $\mathcal{F}$. If C is infinite, then all the claims follow from Lemma 5.3. Hence we may assume that C is finite; in particular, all vertices of C are non-periodic.

Choose a critical chord c in C. Consider the arc I of length $\frac{2}{3}$, one of the two arcs into which c divides $\mathbb{S}^1$. The vertices of G belong to I; the bases of G and of the gaps of $\mathcal{F}$ consist of points of $U'(c)$. We may take d to be a critical chord of V whose endpoints are not the endpoints of any leaf of $\sim$ (the basis of V is a Cantor set, so we can choose d satisfying this property). Clearly c and d satisfy the conditions of Lemma 5.2, which implies the existence of a quadratic invariant gap U coexisting with $\sim$ and such that either $\sim$ tunes U, or no edge of U is a leaf of $\sim$. Let us show that in the latter case U *must* be of regular critical type.

Indeed, otherwise the major M of U is of periodic type. Since M is not a leaf of $\sim$, it is contained in a periodic gap H, which is at least a quadrilateral. If H is finite, then, by [18], the remap on H is not the identity map. Thus, H is the second finite periodic gap of $\sim$ on which the remap is not the identity, a contradiction with the definition of CU^c. This implies that H is a periodic Fatou gap. Moreover, by Lemma 5.3, the gap H comes from the orbit of V. As in the proof of Lemma 5.3, this yields that M must be an edge of G, a contradiction with M not being a leaf of $\sim$.

A similar argument holds in the case, where G is a Siegel gap. Let d be a critical edge of G. There is some other critical leaf or gap C. Let c be a critical chord in C. As before, c may be chosen to have non-periodic endpoints and $G \subset U(c)$. Since no leaves of $\sim$ other than d intersect d, then c and d satisfy the conditions of Lemma 5.2. □

We are ready to prove the Main Theorem stated in the Introduction. The following statement makes it more precise (we refer to the notation introduced in the Main Theorem). Let $\sim$ be a lamination from CU^c. Then by Lemma 5.4 $\sim$ coexists with a quadratic invariant gap U. The gap U in the Main Theorem can be chosen as in Lemma 5.4.

Theorem 5.5. *Assume the conditions of Lemma 5.4 and adopt the notation from its conclusion. If the ψ_U-image of the major M of U does not eventually map (by σ_2) to a periodic Fatou gap of $\asymp = \psi_U(\sim)$, then case (2) of the Main Theorem holds. This is also the case, when U is of periodic type and the lamination $\sim$ is not the canonical lamination of a finite invariant rotational gap or leaf of type A or B.*

Proof of the Main Theorem and of Theorem 5.5. Suppose that $\sim$ is not a canonical lamination of an invariant quadratic gap. Then, by Theorem 2.13, there exists an invariant rotational gap or leaf G. Suppose that G is of type D (finite or Siegel). Then, by Lemma 4.5 and Lemma 4.6, the lamination $\sim$ must be the canonical lamination of G. Choose a major M of G. It follows that the same major M defines also an invariant quadratic gap U. It is easy to see that then $\sim$ tunes the canonical lamination $\sim_U$ according to an appropriate quadratic invariant lamination from CA^c, which corresponds to case (2). From now on, we may assume that G is not of type D.

By Lemma 5.4, the lamination $\sim$ coexists with a quadratic invariant stand-alone gap U, either $\sim$ tunes U or no edge of U belongs to $\sim$ and U can be chosen to be of regular critical type (moreover, if $\sim$ is not canonical, then U must be of regular critical type). Furthermore, $G \subset U$. Suppose that the map ψ_U projects the restriction of $\sim$ onto U to a quadratic invariant lamination $\asymp = \psi_U(\sim)$. By Definition 1.10, the lamination $\sim$ weakly tunes U according to the lamination $\asymp$. Since $\sim$ has a unique rotational set, so does $\asymp$. By Proposition 5.1, the lamination $\asymp$ comes from the Combinatorial Main Cardioid CA^c.

If U is of periodic type and its canonical lamination is tuned by $\sim$ then, if the gap $V(U)$ is of period greater than 1, it cannot be tuned by $\sim$ as this would create a rotational set of period greater than 1. Now, the only cases when $V(U)$ is of period 1 are when $U = \mathrm{FG}_a, V(U) = \mathrm{FG}_b$, or $U = \mathrm{FG}_b, V(U) = \mathrm{FG}_a$. In the former case, if $\sim$ is not empty inside U, then again $V(U)$ cannot be non-trivially tuned by $\sim$ as this would create two rotational sets of $\sim$. Suppose now that $\sim$ is empty inside U. Then it may happen that $V(U)$ is non-trivially tuned by $\sim$. Similarly to the above, this tuning must be according to some quadratic lamination $\asymp$ from CA^c. In that case we simply declare that $\widehat{U} = V(U) = \mathrm{FG}_b$ and $V(\widehat{U}) = \mathrm{FG}_a$. Clearly, this is possible, and with this choice of the gap tuned by $\sim$, the lemma holds. The case $U = \mathrm{FG}_b, V(U) = \mathrm{FG}_a$ is similar.

Now we need to prove the remaining claims of the theorem. By Lemma 5.4, to see whether $\sim$ tunes U, we need to see whether the major M of U belongs to $\sim$. Suppose that the point $\psi_U(M)$ does not eventually map (by σ_2) to a periodic infinite gap of $\asymp$. Then well-known properties of quadratic laminations from the Combinatorial Main Cardioid CA^c imply that $\psi_U(M)$ is separated from the rest of the circle by a sequence of leaves of $\asymp$. Hence M is the limit of appropriately chosen leaves of $\sim$, and so M itself is a leaf of $\sim$. Thus, if $\psi_U(M)$ does not eventually map (by σ_2) to a periodic infinite gap of $\asymp$, then $\sim$ tunes U.

Finally, let us prove the last claim of the theorem. We need to prove that if U is of periodic type and $\sim$ is not canonical, then case (2) must hold. However, this immediately follows from Lemma 5.4 as in the case of weak tuning and non-canonical lamination this lemma states that U must be of regular critical type. This completes the proof. ☐

The statement of the Main Theorem is somewhat involved. However it leads to a more explicit description if one thinks of constructing a non-empty lamination $\sim$

from CU^c. Indeed, for definiteness assume that $\sim$ has a finite rotational gap G. Observe that for canonical laminations of type D the explanation as how $\sim$ fits into the description from the Main Theorem is given in the proof. Otherwise, just like in the arguments of some of our theorems, consider both critical sets of $\sim$. One of them is attached to G. The other one can be either (a) a vassal gap of some invariant quadratic gap U of periodic type, or (b) a critical leaf which is a major of regular critical type of some quadratic invariant gap U, or (c) the same as the first one (canonical lamination of type A), or (d) an infinite gap-preimage of the first one (canonical laminations of type B or C), or (e) a finite gap-preimage of G.

In cases (a) or (b) the lamination $\sim$ tunes the canonical lamination of U according to a lamination from the Main Cardioid. In the other cases the basis of the second critical set contains endpoints of a critical leaf which is itself a major of regular critical type of some quadratic invariant gap U; moreover, all other edges of U are also present as diagonals (but not as edges) of other gaps of $\sim$. The construction of such $\sim$ can be viewed as a three step process: first, we take the canonical lamination of an invariant quadratic gap U, then U is tuned according to a quadratic lamination from the Main Cardioid, and then finally edges of U and their preimages are erased giving rise to $\sim$ (whether we get a lamination described in (c), (d) or (e) above depends on the relation between the major of U, the gap G and gaps of the canonical lamination of G attached to G).

In conclusion, we prove Corollary 1.6 from the Introduction, which allows for a shorter definition of laminations from the Combinatorial Main Cubioid CU^c.

Proof of Corollary 1.6. The "if" part of the claim follows immediately from definitions. To prove the "only if" part of this corollary one simply has to go over different types of laminations listed in Theorem 5.5 (or in our explanation right after this theorem). Indeed, first we observe that by definition if $\sim$ belongs to CU^c then it has at most one rotational periodic (hence fixed) set. Now, consider the second part of the claim. It is obvious for canonical laminations of quadratic invariant gaps or for canonical laminations of finite gaps of type D. In the case when the lamination $\sim$ is obtained as described in Case (1) of Theorem 5.5 — or, equivalently, in cases (c), (d) or (e) above — the only periodic leaves of $\sim$ are edges of the periodic rotational gap G (and only in the case when G is finite), so the claim follows. Finally, if Case (2) of Theorem 5.5 applies, then, in addition to the edges of G, the lamination $\sim$ may also have periodic leaves ℓ that form the orbit of a major M of periodic type generating an invariant quadratic gap U from Theorem 5.5. However, in that case, there must exist an infinite gap attached to each such leaf ℓ that itself belongs to the orbit of the vassal gap attached to M. Thus the claim holds in this case too. $\square$

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REFERENCES

[1] L. Alseda, J. Llibre and M. Misiurewicz, *Combinatorial Dynamics and Entropy in Dimension One*, World Scientific (Advanced Series in Nonlinear Dynamics, vol. 5), Second Edition, 2000.

[2] A. Blokh, C. Curry and L. Oversteegen, Locally connected models for Julia sets, *Advances in Mathematics,* **226** (2011), 1621–1661.

[3] A. Blokh, R. Fokkink, J. Mayer, L. Oversteegen and E. Tymchatyn, Fixed point theorems in plane continua with applications, *Memoirs of the American Mathematical Society,* **224** (2013), xiv+97 pp.

[4] A. Blokh and G. Levin, Growing trees, laminations and the dynamics on the Julia set, *Ergod. Th. and Dynam. Sys.,* **22** (2002), 63–97.

[5] A. Blokh, J. Malaugh, J. Mayer, L. Oversteegen and D. Parris, Rotational subsets of the circle under z^n, *Topology and its Appl.,* **153** (2006), 1540–1570.

[6] A. Blokh, D. Mimbs, L. Oversteegen and K. Valkenburg, Laminations in the language of leaves, *Trans. of the Amer. Math. Soc.,* **365** (2013), 5367–5391.

[7] A. Blokh and L. Oversteegen, Monotone images of Cremer Julia sets, *Houston Journal of Mathematics,* **36** (2010), 469–476.

[8] A. Blokh, L. Oversteegen, R. Ptacek and V. Timorin, Dynamical cores of topological polynomials, *Frontiers in complex dynamics, Princeton Math. Ser.,* Princeton Univ. Press, Princeton, NJ, **51** (2014), 27–48.

[9] A. Blokh, L. Oversteegen, R. Ptacek and V. Timorin, The main cubioid, *Nonlinearity,* **27** (2014), 1879–1897.

[10] X. Buff and C. Henriksen, Julia Sets in Parameter Spaces, *Commun. Math. Phys.,* **220** (2001), 333–375.

[11] C. Carathéodory, Über die Begrenzung einfach zusammenhängender Gebiete (German), *Math. Ann.,* **73** (1913), 323–370.

[12] L. Carleson and T. W. Gamelin, *Complex Dynamics*, Springer, 1993.

[13] A. Douady and J. H. Hubbard, Étude dynamique des polynômes complexes I, Publications Mathématiques d'Orsay, 1984.

[14] A. Douady and J. H. Hubbard, Étude dynamique des polynômes complexes II, Publications Mathématiques d'Orsay, **85-04**, 1985.

[15] A. Epstein and M. Yampolsky, Geography of the Cubic Connectedness Locus: Intertwining Surgery, *Ann. Sci. Éc. Norm. Sup.,* **32** (1999), 151–185.

[16] T. Gauthier, Higher bifurcation currents, neutral cycles, and the Mandelbrot set, *Indiana Univ. Math. J.,* **63** (2014), 917–937.

[17] L. Goldberg and J. Milnor, Fixed points of polynomial maps. II. Fixed point portraits, *Ann. Sci. École Norm. Sup. (4),* **26** (1993), 51–98.

[18] J. Kiwi, Wandering orbit portraits, *Trans. of the Amer. Math. Soc.,* **354** (2002), 1473–1485.

[19] J. Kiwi, $\mathbb{R}$eal laminations and the topological dynamics of complex polynomials, *Advances in Mathematics,* **184** (2004), 207–267.

[20] C. McMullen, The Mandelbrot set is universal, in: *The Mandelbrot Set, Theme and Variations,* ed. T. Lei, Cambridge U.K. Cambridge Univ. Press. Revised, **274** (2007), 1–17.

[21] J. Milnor, Geometry and dynamics of quadratic rational maps, *Experimental Math.,* **2** (1993), 37–83.

[22] J. Milnor, *Dynamics in One Complex Variable*, Annals of Mathematical Studies, **160**, Princeton, 2006.

[23] J. Milnor, Cubic polynomial maps with periodic critical orbit I, in: *Complex Dynamics, Families and Friends,* ed. D. Schleicher, A.K. Peters (2009), 333–411.

[24] J. Milnor and A. Poirier, Hyperbolic components in spaces of polynomial maps, *Contemp. Math.,* Conformal dynamics and hyperbolic geometry, Amer. Math. Soc., Providence, RI, **573** (2012), 183–232. arXiv:math/9202210

[25] J. Milnor and W. Thurston, On iterated maps of the interval, in *Dynamical systems, Lecture Notes in Math.,* **1342**, Springer, Berlin, (1988), 465–563.

[26] M. Misiurewicz, Horseshoes for mappings of the interval, *Bull. Acad. Pol. Sci., Ser. sci. math., astr. et phys.,* **27** (1979), 167–169.

[27] M. Misiurewicz and W. Szlenk, Entropy of piecewise monotone mappings, *Studia Math.,* **67** (1980), 45–63.

[28] C. L. Petersen and T. Lei, Analytic coordinates recording cubic dynamics, In: *Complex Dynamics: Families and Friends, ed. Dierk Schleicher,* Wellesley Massachusetts: A. K. Peters, Limited, (2009), 413–449.

[29] C. L. Petersen, P. Roesch and T. Lei, Parabolic slices on the boundary of $\mathcal{H}$, work in progress.

[30] P. Roesch, Hyperbolic components of polynomials with a fixed critical point of maximal order, *Ann. Sci. cole Norm. Sup. (4),* **40** (2007), 901–949.

[31] W. Thurston, On the geometry and dynamics of iterated rational maps, in: *Complex dynamics: Families and Friends,* ed. by D. Schleicher, A K Peters, (2009), 3–137.

[32] L.-S. Young, On the prevalence of horseshoes, *Trans. Amer. Math. Soc.,* **263** (1981), 75–88.

[33] S. Zakeri, Dynamics of cubic Siegel polynomials, *Comm. Math. Phys.,* **206** (1999), 185–233.

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