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HOW LITTLE IS LITTLE ENOUGH?

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Abstract. Let f be a continuous map $f : X \to X$ of a metric space X into itself. Often the information about the map is presented in the following form: for a finite collection of compact sets $A_1, \dots, A_n$ it is known which sets have the images containing other sets, and which sets are disjoint. We study similar but weaker than usual conditions on compact sets $A_1, \dots, A_n$ assuming that the common intersection of all sets $A_1, \dots, A_n$ is empty (or making even weaker but more technical assumptions). As we show, this implies that the map is chaotic in the sense that it has positive topological entropy, and moreover, there exists an invariant compact set on which f is semiconjugate to a full one-sided shift.

1. Introduction

The modern theory of topological dynamical systems studies (continuous) maps $f : X \to X$ of topological spaces (indeed, in a more general sense it also studies flows, yet in this paper we concentrate upon maps only). An important problem here is to describe the limit behavior of trajectories of points (the **trajectory** of a point x is the sequence $x, f(x), f^2(x), \dots$). Apart from studying the trajectories of individual points, it is reasonable to also study the limit behavior of all points of X and estimate the variety of the behaviors exhibited by them.

A widely recognized parameter of a dynamical system which can serve as a quantitative counter-part of the term "variety" used above is the **topological entropy** of a map, usually denoted $h(g)$ where g is the map. Originally ([ACM]) it was introduced for compact spaces X, but can be extended onto non-compact spaces as well ([Bo]). Therefore, it is important to estimate the topological entropy of a map. Moreover, even to be able to say that the topological entropy of a map is positive is often important, and it is widely accepted that maps of positive entropy should be recognized as "chaotic". A well-known property of the topological entropy states that $h(g^n) = nh(g)$, therefore the topological entropy is positive if and only if there exists a power of the map which has positive topological entropy.

All of the above justifies the interest to conditions on the map which allow one to estimate the topological entropy or at least to be able to say that the topological entropy is positive. If we do not specify the topological space X on which the map is defined, such conditions inevitably must be very general. A rather likely language for them is that of the set theory, e.g. the language describing how various sets and their images cover each other.

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The following definition is a part of such language. Suppose that there are pairwise disjoint compact sets $A_1, \dots, A_n, n \geq 2$ such that for any $1 \leq i \leq n$ we have $f(A_i) \supset \cup_{j=1}^n A_j$. Then we say that f has an n**-horseshoe** or that $f, A_1, \dots, A_n$ **form a horseshoe** (cf. [LM, MS, S]). In this case by A' we denote the set of all points x such that $f^i(x) \in \cup_{j=1}^n A_j$ for every i. If f has an n-horseshoe for some n we simply say that f has a **horseshoe**.

It is well-known that if a power of a map has an n-horseshoe then it has positive entropy. Yet even when the suitable sets A_i are found one has to check not only the fact that their images cover their union, but also the fact that the sets are pairwise disjoint which means that one has to perform $\frac{n(n-1)}{2}$ verifications. A reasonable question then is whether the number of such verifications, necessary to ensure that the topological entropy is positive, can be decreased. On the other hand, it is also reasonable to ask if indeed *all* the conditions defining horseshoes are necessary for the conclusion that the entropy is positive. It is obvious that if we put no conditions on how the sets intersect then we will not be able to make conclusions about the dynamics at all. Indeed, we can set $A_1 = A_2 = \dots = A_n = X$ in which case the covering property $f(A_i) \supset \cup_{j=1}^n A_j$ will be satisfied. However in the situation in question we can make no conclusion about the dynamics of f, and this precisely because we make no assumptions about the intersections of sets $A_1, \dots, A_n$. Hence some conditions on the intersections of sets $A_1, \dots, A_n$ must remain for us to be able to say something about the dynamics of f.

This justifies the following question: how can we weaken the assumption that a power of a map has a horseshoe and still be able to say that the topological entropy of the map is positive? Can we achieve this by replacing the condition of pairwise disjointness of the sets by a weaker one? According to the above given trivial example, even weakening the assumption of pairwise disjointness of the sets we should not totally remove it keeping the intersections of the sets rather small. The strategy thus is to assume that covering properties relating sets and their images are satisfied, but then make assumptions upon their intersections which should be sufficiently small, and the main question is **how little is little enough** for their intersections to warrant that the entropy is positive. This not only explains the title but also outlines the main idea of this paper thus making such non-standard for a mathematical paper title quite appropriate. So, the aim of this paper is to weaken the condition of pairwise disjointness of the sets $A_1, \dots, A_n$ and hence to decrease the number of verifications involved in the checking of this condition while still being able to conclude that the entropy is positive (we actually make a stronger conclusion which will be described in detail below).

Let us pass on to the exact statements. First we need some standard information from the field. Well-known properties which guarantee that the topological entropy of a map is positive are Theorem 1.1 and Lemma 1.2 below. To formulate them we need several definitions. Namely, we denote by Σ_n the space of all one-sided sequences of n symbols, and by σ_n the one-sided shift on Σ_n; if we do not want to specify n we call $\sigma_n|\Sigma_n$ a **full shift**. Also, let X and Y be topological spaces and let $f : X \to X$ and $g : Y \to Y$ be continuous maps. If there is a continuous onto map $h : X \to Y$ with $h \circ f = g \circ h$, we will say that f and g are **semiconjugate** (by h). The map h is called a **semiconjugacy** (between f and g), and the map g is called a **factor** of f.

Theorem 1.1. *Suppose that $f : X \to X$ is a continuous map of a topological space*

X which has a horseshoe. Then there exists an f-invariant compact set on which f is semiconjugate to a full shift.

The following lemma together with Theorem 1.1 introduces a useful machinery allowing one to prove that maps have positive topological entropy.

Lemma 1.2. *If for a continuous map F there exists an F-invariant compact set on which F is semiconjugate to a full shift, then $h(F) > 0$. In particular, if a power of a map f has a horseshoe then $h(f) > 0$.*

Notice that we deliberately avoid quantitative versions of Theorem 1.1 and Lemma 1.2 in which we would specify the number of sets forming a horseshoe, the power of f admitting this horseshoe and the numerical lower bound on the topological entropy. Indeed, our main concern in this paper is to suggest new conditions for the positivity of the topological entropy while not concentrating upon its estimates. However, in our view obtaining such estimates under the circumstances described in this paper is an interesting mathematical problem (we would like to mention here a recent paper [W] whose results can be helpful in this respect).

Before we move on, we need to introduce several new notions; they are needed to describe some results which motivated us in our study (even though they are more of measure-theoretic nature and thus are not directly relevant to the purpose of our study). A set in a measurable space is said to have **trivial measure** if the measure of this set is 0, or the measure of its complement is 0. A map is said to be **ergodic** if every invariant set for this map has trivial measure.

Another parameter which describes the variety of possible limit behaviors of points and depends on an invariant measure μ of the map g, is called the **metric entropy** (of g with respect to μ). It is denoted by $h_\mu(g)$. There is a very nice connection between the metric entropy and the topological entropy: the topological entropy is the supremum of metric entropies taken over all (ergodic) invariant measures. Thus, if the topological entropy is positive then there exists an invariant ergodic measure with respect to which the map has positive entropy in which case it is easy to see that the measure in question is **non-atomic**, i.e. vanishes on points ("atoms") of the space (non-atomic measures are also called **continuous**). In fact the existence of a non-atomic invariant measure can serve as another characteristic of the fact that the map is to some extent chaotic and that the variety of the limit behaviors of points is big, but this characteristic is weaker than the positivity of the entropy (for the positivity of the topological entropy implies the existence of a non-atomic invariant ergodic measure, but not vice versa).

A condition guaranteeing that a map f admits a non-atomic invariant measure is obtained in [LP]. Before we state it, let us introduce some useful terminology. Suppose that $f : X \to X$ is a continuous map of a topological space X. Suppose that there are $n \geq 2$ compact non-empty sets $A_1, \dots, A_n$ such that $f(A_i) \supset \cup_{j=1}^n A_j$ and $\cap_{k=1}^n A_k = \emptyset$. Then we say that f has a **weak n-horseshoe** and that $f, A_1, \dots, A_k$ **form a weak n-horseshoe**. If we do not want to specify the number n we simply say that f has a **weak horseshoe**. Observe that a weak 2-horseshoe is simply a 2-horseshoe. For $n \geq 3$, it is easy to find examples for which a weak n-horseshoe is different from an n-horseshoe. Thus, if the condition of pairwise disjointness of our sets is replaced by a weaker condition of their common intersection being empty, then the sets are said to form a weak horseshoe. Observe that it is easier to verify whether a collection of sets forms a weak horseshoe because we only need to check n intersections of our sets (we intersect the first set with the second, then their intersection with the third etc) rather than $\frac{n(n-1)}{2}$ intersections of pairs of the sets.

The following theorem was proven in [LP].

Theorem 1.3. *Suppose that $f : X \to X$ is a continuous map of a topological space X and it has a weak n-horseshoe. Then f admits a non-atomic invariant measure.*

It turns out that the assumptions about the sets made in [LP] serve our purposes too. Namely, the following theorem is the main theorem of our paper.

Theorem 1.4. *Suppose that $f : X \to X$ is a continuous map of a topological space X. Suppose that a power of f admits a weak horseshoe. Then $h(f) > 0$ and there exists a set B and a power g of f such that B is g-invariant and $g|B$ is semiconjugate to a full shift.*

A question asked in [LP] is whether in the situation of Theorem 1.3 f has an invariant non-atomic *ergodic* measure. Theorem 1.4 implies Corollary 1.5 answering this question.

Corollary 1.5. *Suppose that $f : X \to X$ is a continuous map of a topological space X. Suppose that a power of f has a weak horseshoe. Moreover, there exists an invariant non-atomic measure ergodic measure μ of f such that $h_\mu(f) > 0$.*

In a recent paper [BGL] some weaker conditions implying the existence of a weak horseshoe for a map are discovered. Combined with Theorem 1.4 they imply the same conclusions as this theorem. However since the conditions from [BGL] are of technical nature we postpone their exact formulations as well as the statement of the theorem which combines the results of [BGL] and our Theorem 1.4 until the third section of this paper.

Let us now make some final remarks. We show that under some circumstances a certain power f^k of the map f has an f^k-invariant subset on which f^k is semiconjugate to a full shift which implies the positivity of the entropy. In other words, in our case the positive entropy is assumed on an invariant for some power of f horseshoe-like set. This is related to a deep question whether for certain classes of dynamical systems the topological entropy of the map can be approximated on horseshoes admitted by its powers. Even a relevant but weaker question below does not have a general answer: for what classes of non-invertible dynamical systems is it true that the existence of horseshoes in the above sense is *equivalent* to the positive entropy of the map? This problem is that of topological dynamics rather than theory of smooth dynamical systems. It was answered in dimension one (it is proven in [LM, MS] that the topological entropy is approximated on horseshoes for all maps of graphs), yet we are not aware of general topological results in higher dimensions.

The number of papers devoted to various definitions of "chaotic" maps is enormous. It is virtually impossible, and actually unnecessary, to list them in a short paper devoted to a very specific aspect of this notion. Therefore we decided to refer the reader of this paper only to works which are relevant to our topic while the choice of more general sources of information open (we recommend the book [ALM] a good source). In fact the advantage of our approach is that even though the corollaries of the main result use non-trivial and highly non-elementary notions such as the metric and topological entropy, the main argument is quite elementary and available to the reader with minimal experience in the dynamical systems theory.

One of papers containing results somewhat related to ours is [KY]. The main result of [KY] is indeed that under some circumstances a topological dynamical system admits an invariant subset on which it is semiconjugate to a full shift over M symbols where $M > 1$. However, the assumptions made in [KY] are much more complicated than ours and topological in its nature (in particular, they involve connected sets and a specific partition of the space Q into connected subsets connecting two chosen subsets of Q and mapping onto each other in a specific way). Moreover, the part of the conditions in [KY] which slightly resembles what is being done in our paper, is in fact assuming pairwise disjointness of the sets involved whereas the whole point of our paper is to weaken this assumption and replace it by the fact that the *common intersection of all sets* involved is empty. However it may well be so that combining the tools developed in [KY] and our tools one can get new results, and we thank the referee for drawing our attention to the paper [KY] as well as for other useful remarks.

2. Main Theorem

It will be convenient to refer to sets which form a weak horseshoe as **allowed sets**. Even though the conclusions of Theorem 1.4 deal with a power of f, we may assume, without loss of generality, that the map itself admits a weak horseshoe. We prove Theorem 1.4 by mathematical induction. The case involving a weak 2-horseshoe is covered by Theorem 1.1 and Lemma 1.2, thus establishing the base of mathematical induction. We clarify the inductive proof by proving the theorem for the case of a weak 3-horseshoe based on the fact that the theorem holds for a weak-2 horseshoe.

So, let us look at the case of three allowed sets A_1, A_2, A_3. In this case, each set may intersect with all others, but the intersection of all sets is empty. Since the union of all three sets (which we denote by $\tilde{A}$) is contained in the image of any set, then for any $i = 1, 2, 3$ and $j = 1, 2, 3$ we have that $f(f^{-1}(A_i) \cap A_j) = A_i$.

A **finite itinerary** is an m-tuple, $I_m(x) = (i_0, i_1, ..., i_m)$, which describes the location of a given x in $\tilde{A}$ under finitely many iterations of f. In this notation, i_j indicates that $f^j(x) \in A_{i_j}$. The fact that the image of any allowed set under f contains $\tilde{A}$ enables us to construct an itinerary for any point in $\tilde{A}$ for as long as the point and its forward images remain in $\tilde{A}$. For example, suppose that we have a point x that lies in A_1 and is mapped into a point $f(x) \in A_2$, and then into a point $f^2(x) \in A_1$, and then into a point $f^3(x) \in A_3$. Then $I_3(x) = (1, 2, 1, 3)$.

Observe that in our situation we should talk of **an** itinerary, not **the** itinerary because such itineraries are not necessarily unique. For example, suppose the point x considered above lies in the intersection of A_1 and A_2. Then $I'_3(x) = (2, 2, 1, 3)$ is also a correct description of the location of x under iterations of f. Such a condition may occur for any iterate of f. So, generally speaking for a given point x we may have a **set** of itineraries of given length. Also, in the case when a point is mapped outside of $\tilde{A}$ we consider the corresponding finite itinerary of the point not defined and do not work with such cases from now on.

Similarly we can define an **infinite itinerary**, $I(x)$, of a point x as a sequence $(i_0, i_1, \dots)$ such that for any j we have $f^j(x) \in A_{i_j}$. Again, since we allow for intersections between sets, we may have a collection (maybe even infinite) of infinite itineraries of x, and $I(x)$ is only defined for the set A' of points x whose entire orbit is contained in $\tilde{A}$.

Let us define a **finite cylinder** as the set of all points exhibiting a given finite itinerary under f. Similarly, an infinite itinerary determines an **infinite cylinder** as the set of all points which exhibit it. It is easy to see from the definition that if $I_m = (i_0, i_1, \dots, i_m)$ is a finite itinerary then the corresponding cylinder is $A_{I_m} = A_{i_0} \cap f^{-1}(A_{i_1}) \cap f^{-2}(A_{i_2}) \cap \dots \cap f^{-m}(A_{i_m})$. Similarly, if $I = (i_0, i_1, \dots)$ is an infinite itinerary then the corresponding infinite cylinder is $\cap_{j=0}^{\infty} f^{-j}(A_{i_j})$. The following is an easy but useful lemma.

Lemma 2.1. *All cylinders, finite or infinite, are non-empty compact sets. The m-th image of the cylinder generated by a finite itinerary $I = (i_0, i_1, i_2, ..., i_m)$, is A_{i_m}, and so $f^{m+1}(A_I) \supset \tilde{A}$.*

Proof. Left to the reader. ∎

We now consider two cylinders and examine the intersection of these two cylinders.

Lemma 2.2. *Suppose that there exist finite itineraries $I \neq J$ such that corresponding cylinders A_I and A_J, are disjoint. Then the following holds:*

(1) *a power g of f admits a horseshoe;*
(2) *there exists a g-invariant compact set on which g is semiconjugate to a full shift;*
(3) $h(f) > 0$.

Proof. By Theorem 1.1 and Lemma 1.2 it is enough to prove the first claim of the lemma. Without loss of generality we may assume that I, J are of the same length m. Let $g = f^{m+1}$. Then by Lemma 2.1 we have $g(A_I) \cap g(A_J) \supset \tilde{A} \supset A_I \cup A_J$. Since A_I, A_J are disjoint we see that claim (1) is verified as desired. ∎

Recall that our aim is to prove Theorem 1.4 in the case when a power of the map has a weak 3-horseshoe. According to Lemma 2.2, if there are two finite cylinders which are disjoint then the conclusions of Theorem 1.4 hold. Therefore it remains to consider the case when all finite cylinders are pairwise non-disjoint. Since by Lemma 2.1 they are compact sets in this case any two infinite cylinders are also non-disjoint.

Let us introduce new sets and itineraries with respect to them. Namely, set $P^{(i)} = \cap_{i \neq j} A_j$. Clearly, there are 3 sets $P^{(1)}, P^{(2)}, P^{(3)}$ which we get this way. It is important to note that for every $i \neq j$, $P^{(i)} \cap P^{(j)} = \emptyset$, otherwise $\cap_{i=1}^{3} A_i \neq \emptyset$ contrary to the assumption that A_1, A_2 and A_3 form a weak horseshoe.

Set $\cup_{i=1}^{3} P^{(i)} = R$. Denote by R' the set of all points y such that the orbit of y is contained in R. As before, to every point $y \in R'$ we can associate its itinerary, $I_P(y)$, in the sense of sets $P^{(1)}, P^{(2)}, P^{(3)}$ which we will simply call its **P-itinerary**. Observe that since P-sets are pairwise disjoint then for any point for which there exists its P-itinerary, this itinerary is unique.

Let us now show that actually for any P-itinerary there exists a point x which exhibits it. It will be useful to introduce new notation. Namely, let $\{1, 2, 3\} \setminus \{l\} = \{b_1(l), b_2(l)\}$ with $b_1(l) < b_2(l)$. Set $B_1(l) = A_{b_1(l)}$ and $B_2(l) = A_{b_2(l)}$. In other words, given l we remove the set A_l from the collection of sets A_1, A_2, A_3, and consider two remaining sets denoting them $B_1(l), B_2(l)$. Observe that $P^{(l)} = B_1(l) \cap B_2(l)$.

Now, let us consider a P-itinerary $(i_0, i_1, \dots, i_n, \dots)$. Define two sequences of sets

$$K_1 = (B_1(i_0), B_1(i_1), \dots, B_1(i_n), \dots)$$

and

$$K_2 = (B_2(i_0), B_2(i_1), \dots, B_2(i_n), \dots)$$

and consider them as usual itineraries in the sense of the allowed sets A_1, A_2, A_3. This allows us to define the infinite cylinders $A_{K_1} = \cap_{j=0}^{\infty} f^{-j}(B_1(i_j))$ and $A_{K_2} = \cap_{j=0}^{\infty} f^{-j}(B_2(i_j))$. By the assumptions these cylinders have a non-empty intersection. On the other hand it is easy to see that this intersection can be written as follows:

$$A_{K_1} \cap A_{K_2} = \bigcap_{j=0}^{\infty} (f^{-j}(B_1(i_j)) \cap f^{-j}(B_2(i_j))) = \bigcap_{j=0}^{\infty} f^{-j}(P^{(i_j)})$$

which implies that there is a point on which the P-itinerary $(i_0, i_1, \dots, i_n, \dots)$ is realized. In other words, if $x \in A_{K_1} \cap A_{K_2}$, then for the l-th iteration of f at the point x we have $f^l(x) \in B_1(i_l) \cap B_2(i_l) = P^{(i_l)}$. Therefore, the point x exhibits the P-itinerary $(i_0, i_1, \dots, i_n, \dots)$ as desired.

We can summarize the proven as follows: in our situation there are three pairwise disjoint compact sets $P^{(1)}, P^{(2)}, P^{(3)}$ such that *every* P-itinerary is exhibited by some point in their union. To finish the proof of Theorem 1.4 in this case define the map φ which associates to every point $x \in R'$ its P-itinerary. It remains to prove that in this situation the map φ is continuous and semiconjugates $f|R'$ to the full shift $\sigma_3|\Sigma_3$.

Let us start with a point $x \in R'$. Let $I^P(x) = (i_0, i_1, i_2, ...)$ be its P-itinerary. Then $f(x) \in P^{(i_1)}$ by definition of the P-itinerary. So $(\varphi \circ f)(x) = (i_1, i_2, ...)$. Now we compute $(\sigma_3 \circ \varphi)(x)$. This is simply the one-sided shift on $I^P(x)$, which equals $(i_1, i_2, ...)$. Thus φ semiconjugates $f|R'$ to the full shift $\sigma_3|\Sigma_3$ as desired.

Let us now prove that φ is a continuous map. Let $\{x_n\} \to x$ be a convergent sequence in R'. By the continuity of f we easily observe that for a given l we have $\{f^l(x_n)\} \to f^l(x)$. Let us fix an integer l. Since $P^{(1)}, P^{(2)}, P^{(3)}$ are pairwise disjoint compact sets, then if $y, z \in R'$ are sufficiently close then $y, z \in P^{(i)}$ for some i. Moreover, $f^r(y)$ and $f^r(z)$ belong to the same P-set for all $1 \le r \le l$. So given l we see that for all sufficiently large n we have $f^r(x_n), f^r(x) \in P^{(i_r)}$ for all $r \le l$. This implies that φ-images of x and x_n all have the same first l entries and therefore are close. So it follows that $\{\varphi(x_n)\} \to \varphi(x)$ as desired.

Now we can prove Theorem 1.4 in general by means of mathematical induction. Let us make the inductive hypothesis by supposing that given a weak k-horseshoe, the conclusions for Theorem 1.4 hold. Suppose we are given a weak $(k+1)$-horseshoe (this will be a standing assumption for the rest of the paper). We may again define finite or infinite itineraries and cylinders relating the given allowed sets with f as we did in the case when f admits a weak 3-horseshoe. We also let $\tilde{A} = \cup_{i=1}^{k+1} A_i$.

Lemma 2.3. *All cylinders, finite or infinite, are non-empty compact sets. The m-th image of the cylinder generated by a finite itinerary $I = (i_0, i_1, i_2, ..., i_m)$, is A_{i_m}, and so $f^{m+1}(A_I) \supset \tilde{A}$.*

Proof. Left to the reader. ∎

Lemma 2.4. *If there exist k finite itineraries $I_1, \dots, I_k$ in terms of $A_1, \dots, A_{k+1}$ such that $\cap_{j=1}^{k} A_{I_j} = \emptyset$, then the following holds:*

(1) *a power g of f admits a weak k-horseshoe;*

(2) *there exists a g-invariant compact set on which g is semiconjugate to a full shift;*
(3) $h(f) > 0$.

Proof. Without loss of generality, we may again assume that itineraries $I_1, I_2, \dots, I_k$ are of length m. Let $g = f^{m+1}$. Then by Lemma 2.3, we have $\cap_{j=1}^k g(A_{I_j}) \supset \tilde{A} \supset \cup_{j=1}^k A_{I_j}$. Since $\cap_{j=1}^k A_{I_j} = \emptyset$, we see that we do have a weak k-horseshoe. By the inductive hypothesis, there exists a set B such that B is g-invariant and $g|B$ is semiconjugate to a full shift. This completes the proof of Theorem 1.4 in this case. ■

By Lemma 2.4, we may assume that for any collection of k itineraries, the intersection of their corresponding cylinders is non-empty(and therefore the intersection of any k infinite cylinders is also non-empty). We may again introduce the sets defined as $P^{(j)} = \cap_{i \neq j} A_i$. This way we will get $k+1$ sets $P^{(1)}, \dots, P^{(k+1)}$. Observe that these $k+1$ sets are pairwise disjoint, otherwise $\cap_{j=1}^{k+1} A_j \neq \emptyset$, contrary to the assumption that the sets $A_1, A_2, \dots, A_{k+1}$ and f form a weak $(k+1)$-horseshoe.

Let $Q = \cup_{i=1}^{k+1} P^{(i)}$. Denote by Q' the set of all points $y \in Q$ such that the orbit of y is contained in Q. Notice that to every point $y \in Q'$ we can associate its itinerary in terms of the sets $P^{(1)}, P^{(2)}, \dots, P^{(k+1)}$, which we again refer to as its P**-itinerary**. This itinerary is unique since the sets $P^{(1)}, P^{(2)}, \dots, P^{(k+1)}$ are pairwise disjoint.

Now let us show that for any P-itinerary, there exists a point y which exhibits it. We again introduce a useful notation. Let $B(l) = \{1, 2, \dots, k+1\} \setminus \{l\} = \{b_1(l) < b_2(l) < \cdots < b_k(l)\}$. As before, set $B_i(l) = A_{b_i(l)}$. In other words, given l we remove the set A_l from the collection $A_1, A_2, \dots, A_{k+1}$ of sets and consider the k remaining sets denoting them $B_1(l), B_2(l), \dots, B_k(l)$. Observe that $P^{(l)} = \cap_{i=1}^k B_i(l)$.

Let us consider a P-itinerary $(i_0, i_1, ..., i_n, ...)$. Define k sequences of sets

$$K_j = (B_j(i_0), B_j(i_1), \dots, B_j(i_n), \dots), 1 \le j \le k$$

and consider them as usual itineraries in the sense of the $k+1$ sets $A_1, A_2, \dots, A_{k+1}$. This allows us to define the infinite cylinders $A_{K_h} = \cap_{j=0}^{\infty} f^{-j}(B_h(i_j))$, $1 \le h \le k$. By our assumptions, $\cap_{h=1}^k A_{K_h} \neq \emptyset$. It is easy to again see that this intersection can be written as follows:

$$\bigcap_{h=1}^{k} A_{K_h} = \bigcap_{j=0}^{\infty} \cap_{h=1}^k f^{-j}(B_h(i_j)) = \bigcap_{j=0}^{\infty} f^{-j}(\cap_{h=1}^k B_h(i_j)) = \bigcap^{\infty} f^{-j}(P^{(i_j)})$$

which implies that there is a point on which the P-itinerary $(i_o, i_1, \dots, i_n, \dots)$ is realized. In other words, if $x \in \cap_{h=1}^k A_{K_h}$, then for the l-th iteration of f at the point x we have $f^l(x) \in \cap_{h=1}^k B_h(i_l) = P^{(i_l)}$. Therefore, the point x exhibits the P-itinerary $(i_0, i_1, \dots, i_n, \dots)$ as desired.

We can summarize the proven as follows: in this situation there are k pairwise disjoint compact sets $P^{(i)}, 1 \le i \le k+1$ such that *every* P-itinerary is exhibited by some point in their union. To finish the proof of Theorem 1.4 in this general case, we again define the map φ which associates to every point $x \in Q'$ its P-itinerary. It remains to prove that in this situation the map φ is continuous and semiconjugates $f|Q'$ to the full shift $\sigma_{k+1}|\Sigma_{k+1}$.

The proof that φ semiconjugates $f|Q'$ to the full shift $\sigma_{k+1}|\Sigma_{k+1}$ is analogous to its proof in the case of 3 allowed sets and is left to the reader.

Let us now prove that φ is a continuous map. Let $\{x_n\} \to x$ be a convergent sequence in Q'. By the continuity of f we easily observe that for a given l we have $\{f^l(x_n)\} \to f^l(x)$. Let us fix an integer l. Since $P^{(1)}, P^{(2)}, \dots, P^{(k+1)}$ are pairwise disjoint compact sets, then if $y, z \in Q'$ are sufficiently close then $y, z \in P^{(i)}$ for some i. Moreover, $f^r(y)$ and $f^r(z)$ belong to the same P-set for all $1 \le r \le l$. So given l we see that for all sufficiently large n we have $f^r(x_n), f^r(x) \in P^{(i_r)}$ for all $r \le l$. This implies that φ-images of x and x_n all have the same first l entries and therefore are close. So it follows that $\{\varphi(x_n)\} \to \varphi(x)$ as desired.

We have now shown that if Theorem 1.4 holds in the case of a weak k-horseshoe, then it holds when the map has a weak $(k+1)$-horseshoe. By the principle of mathematical induction this completes the proof of Theorem 1.4 in general.

3. Concluding remarks

It is easy to see that Theorem 1.4 can be generalized even more. Just like the existence of a weak horseshoe implies that there is a set on which a map is semiconjugate to a full one-sided shift, there are conditions which on the surface of it look weaker than that of existence of a weak horseshoe while imply such, perhaps for some power of f. For example, consider a collection $\mathcal{A}$ of compact sets $A_1, \dots, A_n$ such that $f(A_i) \supset \cup_{i=1}^n A_i$. Also, denote the intersection $\cap_{i=1}^n A_i = B_{\mathcal{A}}$ and assume that there are no points whose entire forward orbits are contained in $B_{\mathcal{A}}$. Clearly this assumption is weaker than that of the set $B_{\mathcal{A}}$ being empty (i.e. than that of the collection $A_1, \dots, A_n$ forming a weak horseshoe. However it easily implies that a power of f does have a weak horseshoe. Indeed, since there are no points with forward orbits in $B_{\mathcal{A}}$ we see that for the set $\cup_{i=0}^{\infty} f^{-i}(B_{\mathcal{A}}) = B_{\mathcal{A}}^{\cap}$ we have that $B_{\mathcal{A}}^{\cap} = \emptyset$, and since $B_{\mathcal{A}}$ is compact we see that there exists a finite N such that $\cup_{i=0}^{N} f^{-i}(B_{\mathcal{A}}) = \emptyset$. Now, let us consider finite itineraries of length $N+1$ defined as $I_1 = (1, 1, \dots, 1), I_2 = (2, 2, \dots, 2), \dots, I_n = (n, n, \dots, n)$. Then consider the cylinders $A_{I_1}, \dots, A_{I_n}$. By Lemma 2.3 we have that $f^{N+2}(A_{I_k}) \supset \cup_{i=1}^n A_i$ for any $k = 1, \dots, n$. On the other hand the fact that $\cup_{i=0}^{N} f^{-i}(B_{\mathcal{A}}) = \emptyset$ implies that $\cap_{k=1}^n A_{I_k} = \emptyset$. Therefore $A_{I_1}, \dots, A_{I_n}$ form a weak horseshoe for f^{N+2} and the conclusions of Theorem 1.4 hold.

Similar ideas are employed in a recent paper [BGL] in which an even weaker than above set of conditions implying the existence of a weak horseshoe is suggested. Following [BGL] and using the notation introduced above, let us say that a map $T : X \to X$ is *strict* with respect to finite collection $\mathcal{A}$ of non-empty compact sets $A_1, \dots, A_N$ if the set $B_{\mathcal{A}}^{\cap}$ has a neighborhood U such that $\cap_{k=1}^{N} A_k \not\subset \cup_{k=1}^{\infty} T^k(U)$. The following theorem is proven in [BGL].

Theorem 3.1. *Suppose that there exists a collection of compact sets $A_1, \dots, A_N$ such that f is strict with respect to this collection and also $f(A_i) \supset \cup_{k=1}^{\infty} A_k$ for any $i = 1, \dots, N$. Then f has a weak horseshoe.*

Combining Theorem 3.1 and Theorem 1.4 we immediately get the following Theorem 3.2.

Theorem 3.2. *Suppose that there exists a collection of compact sets $A_1, \dots, A_N$ such that some power f^m of a map f is strict with respect to this collection and also $f^m(A_i) \supset \cup_{k=1}^{N} A_k$ for any $i = 1, \dots, N$. Then $h(f) > 0$ and there exists a set*

R and a power g of f such that R is g-invariant and $g|R$ is semiconjugate to a full shift.

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