Department of Mathematics UAB
Calculus II
MA126 Summer 2019

ASSIGNMENT 2: Solutions.

Vectors and Integrals

1. (a) The line through $(6, -7, 4)$ and parallel to $\boldsymbol{v} = \langle 1, 2, 3\rangle$ has vector equation $\langle x, y, z\rangle = \langle 6, -7, 4\rangle + t\langle 1, 2, 3\rangle$ and parametric equations

$$\begin{aligned} x &= 6 + t \\ y &= -7 + 2t \\ z &= 4 + 3t. \end{aligned}$$

(b) The line in question passes through the point $(1, -1, 1)$ and is parallel to the vector $\boldsymbol{v} = \langle 4, -3, 9\rangle$. The vector equation is $\langle x, y, z\rangle = \langle 1, -1, 1\rangle + t\langle 4, -3, 9\rangle$ and parametric equations are

$$\begin{aligned} x &= 1 + 4t \\ y &= -1 - 3t \\ z &= 1 + 9t. \end{aligned}$$

(c) The line through the points $P(1, 1, 1)$ and $Q(3, -4, 4)$ is parallel to the vector $\overrightarrow{PQ} = \langle 2, -5, 3\rangle$. The parametric equations are

$$\begin{aligned} x &= 1 + 2t \\ y &= 1 - 5t \\ z &= 1 + 3t. \end{aligned}$$

2. The line L_1 is parallel to $4\langle 2, -1, 3\rangle$ and L_2 is parallel to $8\langle 2, -1, 2.5\rangle$. As these vectors are not scalar multiples of each other, the lines are not parallel. We next see if the lines intersect, i.e. there is a point $(x, y.z)$ on both lines. If this were the case there would exist parameter values t and s with

$$\begin{aligned} (x =)\ 12 + 8t &= 4 + 16s \qquad (1) \\ (y =)\ 16 - 4t &= 12 - 8s \qquad (2) \\ (z =)\ 4 + 12t &= 16 + 20s. \end{aligned}$$

Multiplying equation (2) by 2 and adding the resulting equation to equation (1) gives $44 = 28$, which cannot be true. This contradiction tells us that no such t and s exist, and so no such point (x, y, z) can exist, and the lines are skew.

3. The line in question passes through the point $P(1,1,1)$ and is parallel to the normal vector $\boldsymbol{n} = \langle 1,4,1\rangle$ to the plane $x+4y+z=5$. The vector equation of this line is $\langle x,y,z\rangle = \langle 1,1,1\rangle + t\langle 1,4,1\rangle$ and parametric equations are

$$\begin{aligned} x &= 1+t \\ y &= 1+4t \\ z &= 1+t. \end{aligned}$$

Substituting these equations for x,y,z into the plane equation gives

$$(1+t)+4(1+4t)+(1+t)=5, \text{ or } t=-1/18.$$

So the line intersects the plane at the point $Q(1-1/18, 1-4/18, 1-1/18)$. The distance from P to the plane is the distance

$$|PQ| = \frac{\sqrt{1^2+4^2+1^2}}{18} = 1/\sqrt{18}.$$

4. (a) As the plane we whose equation we seek is parallel to the plane $5x-y-z=3$, the two planes have the same normal, $\boldsymbol{n} = \langle 5,-1,-1\rangle$. So the plane in question has equation

$$5(x-6)-(y+3)-(z-2)=0, \text{ or } 5x-y-z=31.$$

(b) Here $\overrightarrow{PQ} = \langle 6,-6,0\rangle$ and $\overrightarrow{PR} = \langle 6,0,-6\rangle$. The normal for the plane is

$$\overrightarrow{PQ}\times\overrightarrow{PR} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 6 & -6 & 0 \\ 6 & 0 & -6 \end{vmatrix} = \left\langle \begin{vmatrix} -6 & 0 \\ 0 & -6 \end{vmatrix}, -\begin{vmatrix} 6 & 0 \\ 6 & -6 \end{vmatrix}, \begin{vmatrix} 6 & -6 \\ 6 & 0 \end{vmatrix} \right\rangle = 36\langle 1,1,1\rangle.$$

The equation for the plane through $P(0,6,6)$ with normal $\boldsymbol{n} = \langle 1,1,1\rangle$ is

$$(x-0)+(y-6)+(z-6)=0 \text{ or } x+y+z=12.$$

5. The vector $\boldsymbol{a} = \langle -2,5,4\rangle$, giving the direction of the given line, is parallel to the plane we seek. Setting $t=0$ in the line equations gives a point $Q(7,3,8)$ that lies in our plane, together with the given point $P(9,0,-1)$. So the vector $\boldsymbol{b} = \overrightarrow{QP} = \langle 2,-3,-9\rangle$ is also parallel to our plane. The vector

$$\boldsymbol{a}\times\boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -2 & 5 & 4 \\ 2 & -3 & -9 \end{vmatrix} = \left\langle \begin{vmatrix} 5 & 4 \\ -3 & -9 \end{vmatrix}, -\begin{vmatrix} -2 & 4 \\ 2 & -9 \end{vmatrix}, \begin{vmatrix} -2 & 5 \\ 2 & -3 \end{vmatrix} \right\rangle = \langle -33,-10,-4\rangle.$$

is normal to our plane, which therefore has equation

$$-33(x-9)-10(y-0)-4(z+1)=0 \text{ or } 33x+10y+4z=-293.$$

6. If we set $z = 0$ in the two given plane equations we see that $x + y = 4$ and $x - y = 4$ and thus $x = 4$ and $y = 0$. This means that the point $Q(4, 0, 0)$ lies in both planes and thus lies on the line of intersection of the two planes. Using the given point $P(-1, 3, 2)$ the vector $\boldsymbol{b} = \overrightarrow{QP} = \langle -5, 3, 2 \rangle$ is parallel to the plane whose equation we seek. The direction of the line of intersection of the given planes is perpendicular to each of the normals $\langle 1, 1, -1 \rangle$ and $\langle 1, -1, 5 \rangle$ to these planes. So the vector

$$\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 1 & -1 \\ 1 & -1 & 5 \end{vmatrix} = \left\langle \begin{vmatrix} 1 & -1 \\ -1 & 5 \end{vmatrix}, - \begin{vmatrix} 1 & -1 \\ 1 & 5 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} \right\rangle = 2\langle 2, -3, -1 \rangle,$$

as well as $\boldsymbol{a} = \langle 2, -3, 1 \rangle$ is also parallel to the plane we seek. The normal to this plane is

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & -3 & -1 \\ -5 & 3 & 2 \end{vmatrix} = \left\langle \begin{vmatrix} -3 & -1 \\ 3 & 2 \end{vmatrix}, - \begin{vmatrix} 2 & -1 \\ -5 & 2 \end{vmatrix}, \begin{vmatrix} 2 & -3 \\ -5 & 3 \end{vmatrix} \right\rangle = \langle -3, 1, -9 \rangle.$$

and hence we can take the normal to be $\langle -3, 1, -9 \rangle$ and the equation is

$$-3(x + 1) + (y - 3) - 9(z - 2) = 0 \text{ or } -3x + y - 9z = -12.$$

7. Substitute the equations $x = t$, $y = 0$, and $z = 4t - t^2$ into the surface equation $z = x^2 + y^2$. This gives $4t - t^2 = t^2 + 0^2$, or $2t(t - 2) = 0$. So $t = 0$ or $t = 2$. The space curve meets the paraboloid in the points $(0, 0, 0)$ and $(2, 0, 4)$.

8. If we choose $x = 4 \cos t$ and $y = 4 \sin t$ then

$$x^2 + y^2 = (4 \cos t)^2 + (4 \sin t)^2 = 16(\cos^2 t + \sin^2 t) = 16,$$

and the equation $x^2 + y^2 = 16$ is satisfied. Next, choose

$$z = (4 \cos t).(4 \sin t) = 16 \cos t \sin t.$$

The curve of intersection has parametric equations

$$\begin{aligned} x &= 4 \cos t \\ y &= 4 \sin t \\ z &= 16 \cos t \sin t. \end{aligned}$$

9. The vector function for this space curve is given by

$$\boldsymbol{r}(t) = \langle 1 + 4t^{1/2}, t^5 - t, t^5 + t \rangle.$$

and its derivative is

$$\boldsymbol{r}'(t) = \langle 2t^{-1/2}, 5t^4 - 1, 5t^4 + 1 \rangle.$$

Notice that the parameter value t for the point $P(5,0,2)$ on this space curve is $t = 1$, so that $\boldsymbol{r}(1) = \langle 5, 0, 2\rangle$. From a theorem in class we know that a tangent to the space curve at P is given by the vector

$$\boldsymbol{r}'(1) = \langle 2, 4, 6\rangle.$$

A vector equation for the tangent line is given by $\langle x, y, z\rangle = \langle 5, 0, 2\rangle + t\langle 2, 3, 5\rangle$ and we have parametric equations

$$\begin{aligned} x &= 5 + 2t \\ y &= 4t \\ z &= 2 + 6t \end{aligned}$$

for the tangent line.

10. A Riemann sum approximation for the area under the graph of $f(x) = 10\cos x$ over the interval $[0, \pi/2]$ using right-hand ordinates is found as follows. Choose the sub-division points for the four sub-intervals to be $x_0 = 0$, $x_1 = \pi/8$, $x_2 = \pi/4$, $x_3 = 3\pi/8$ and $x_4 = \pi/2$. Here the step-size is $h = \Delta x = \pi/8$. The Riemann sum approximation is given by

$$\begin{aligned} \sum_{i=1}^{i=4} f(x_i)\Delta x &= \frac{\pi}{8}\left[f\left(\frac{\pi}{8}\right) + f\left(\frac{\pi}{4}\right) + f\left(\frac{3\pi}{8}\right) + f\left(\frac{\pi}{2}\right)\right] \\ &= \frac{10\pi}{8}\left[\cos\left(\frac{\pi}{8}\right) + \cos\left(\frac{\pi}{4}\right) + \cos\left(\frac{3\pi}{8}\right) + \cos\left(\frac{\pi}{2}\right)\right] \approx 7.91, \end{aligned}$$

using Maple. By the way, using our "area function" anti-derivative approach the precise area is 10.