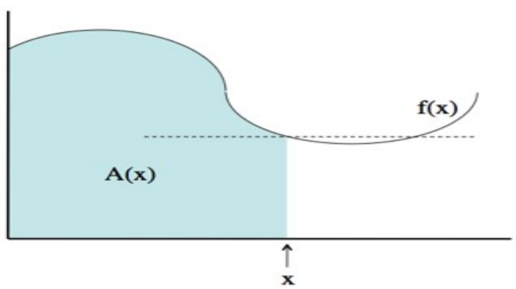
Department of Mathematics UAB
Calculus II

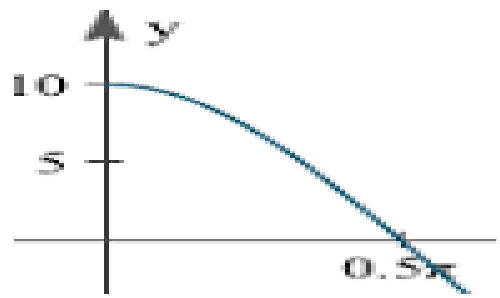
ASSIGNMENT 3

Integrals

Show all necessary calculations and relevant explanations. Numerical answers with no supporting explanations will receive no credit.

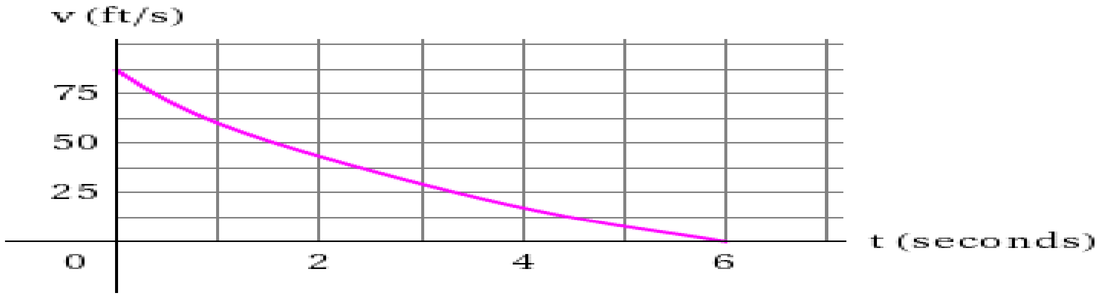
1. We proved in class that the area function $A(x)$ (defined below, left) for the function $f(x)$ has the property that at each x we have $A'(x) = f(x)$, so that the function $A(x)$ is





an anti-derivative of (i.e. differentiates to) $f(x)$. Find a formula for all anti-derivatives of the function $f(x) = 10\cos x$ and use this theorem to find the precise area under the graph (see above, right) of $y = f(x) = 10\cos x$ over the interval $[0, \pi/2]$.

2. The velocity of a car undergoing braking over the time interval $[0, 6]$ is shown below.

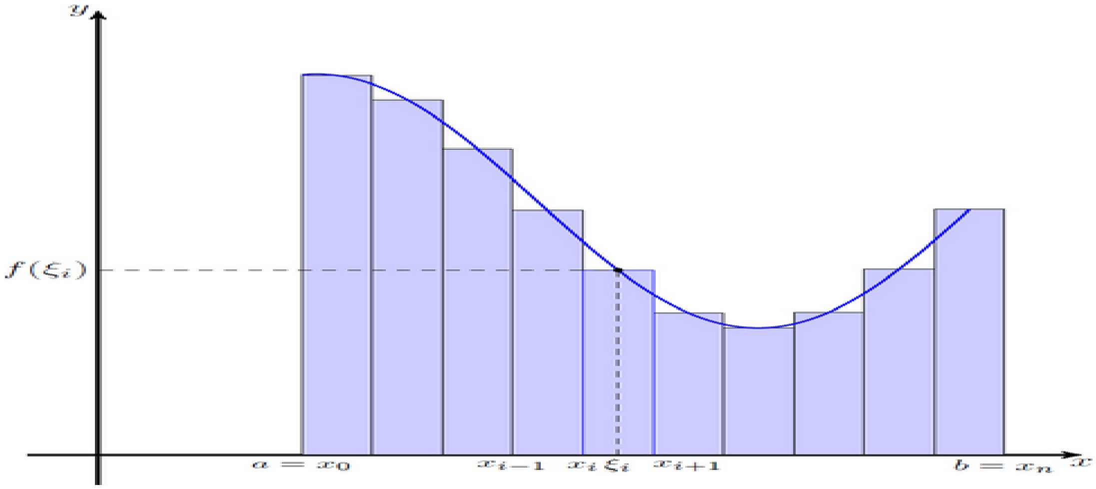


(a) Explain why the area function for the velocity function $v(t)$ graphed here represents the distance traveled by the car under braking. (Hint: why does the area of one Riemann rectangle represent a distance).

(b) Use a Riemann sum with six rectangles and right-side ordinates to estimate the total distance traveled by the car while braking.

(c) Explain why the anti-derivative approach used above is no good here.

3. The exact area A under the graph of $y = f(x)$ over $[a, b]$ is given by the following limit



of Riemann sums (also called the definite integral of f over $[a, b]$):

$$A = \int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{i=n} f(\xi_i)\Delta x = \lim_{n\to\infty} [f(\xi_1)\Delta x + f(\xi_2)\Delta x + \cdots + f(\xi_n)\Delta x],$$

where $\Delta x = x_i - x_{i-1}$ is the width of each rectangle. Use this formula to find the precise area under the graph of $y = f(x) = x^3$ over the interval $[0, 1]$, using $a = 0$, $b = 1$, $x_i = i/n$ for $0 \le i \le n$, and hence $\Delta x = 1/n$, and use right-hand ordinates (so $\xi_i = x_i$ for $1 \le i \le n$). Hint: you may use the formula

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \sum_{i=1}^{i=n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

4. Using the notation from the figure in problem 3, and setting $a = 0$ and $b = 2$ express the limit below as a definite integral over the interval $[0, 2]$:

$$\lim_{n\to\infty} \sum_{i=1}^{i=n} x_i \log(2 + x_i^2)\ \Delta x.$$

5. Sketch the graph of the function $y = f(x) = 3 + \sqrt{4 - x^2}$ over $[-2, 2]$ and use this to evaluate the definite integral below by using known areas:

$$\int_{-2}^{2} \left[3 + \sqrt{4 - x^2}\right]\ dx.$$

6. Let $\int_0^1 e^{x^2}\ dx = A$. Use the properties of integrals to express the definite integral

$$\int_{-1}^{1} [3e^{x^2} + x + 2]\ dx$$

in terms of A.

7. Evaluate the definite integrals below using anti-derivatives and, if needed, the integral rules:

(a)
$$\int_0^1 x\left(4\sqrt[3]{x}+\frac{2}{\sqrt{x}}\right)^2\,dx.$$

(b)
$$\int_0^{1/\sqrt{3}} 2\cdot\frac{t^2-1}{t^4-1}\,dt.$$

8. (a) If $w'(t)$ is the rate of growth of a child in pounds per year what does the number $\int_4^7 w'(t)\,dt$ represent?

(b) A bacteria population is 4000 at time $t=0$ and its rate of growth is $1000\cdot 7^t$ bacteria per hour after t hours. What is the population after 2 hours? Hint: by definition $7^t = e^{t\log 7}$ and the anti-derivative of e^{at}, for any constant a, is in the "basic" anti-derivative list handed out in class that you should commit to memory!

9. Evaluate the indefinite integral (anti-derivative)

$$\int x^3(9+x^4)^5\,dx$$

by using the substitution $u = 9 + x^4$. Don't forget to add the constant of integration.

10. Evaluate the indefinite integral

$$\int \frac{\sin 2x}{1+\cos^2 x}\,dx.$$

Hint: check out the trig formulae in your derivative/integral sheet (these should also be memorized) and look for a suitable substitution; in particular, whenever you spot an integrand of the form $f(g(x))g'(x)$, the substitution $u = g(x)$ is likely helpful.