Department of Mathematics UAB
Calculus II

ASSIGNMENT 5

Indeterminate Limits, Improper Integrals, Area,Volume, Work

Show all necessary calculations and relevant explanations. Numerical answers with no supporting explanations will receive no credit.

1. [10 points.] Use L'Hôpital's rule to determine the following indeterminate limits.

   (a)
   $$\lim_{x\to 1} \frac{\log x}{x-1}.$$

   (b)
   $$\lim_{x\to \infty} \frac{x^3}{e^x}.$$

   (c) $\lim_{x\to 0^+} \sin x \log x$. $\left[\text{ Hint: } \sin x \log x = \frac{\log x}{1/\sin x}.\right]$

   (d) $\lim_{x\to 0^+} x^{\sqrt{x}}$.

2. Determine whether the improper integral

$$\int_2^3 \frac{1}{\sqrt{3-x}}\,dx$$

is convergent or divergent; if it is convergent, calculate the value.

3. If we define the function $F(s)$ by the improper integral

$$F(s) = \int_0^\infty f(x)e^{-sx}\,dx,$$

where $f(x) = x^2$, calculate $F(s)$ for all $s > 0$.

4. [10 points.] The volume of the region formed by rotating the area below the graph of $y = 1/x$ for $x \geq 1$ is given by the improper integral

$$\int_1^\infty \frac{\pi}{x^2}\, dx.$$

(a) Show that this integral is convergent and determine the volume of the region.

(b) The improper integral

$$\int_1^\infty 2\pi \frac{1}{x}\sqrt{1+\frac{1}{x^4}}\, dx$$

represents the surface area of this region. Using the comparison theorem determine whether or not this integral is convergent.

(c) If the region represented a futuristic building and you were a painting contractor with the knowledge that the building had finite volume, would you be still be interested in bidding for the contract to paint the exterior?

5. (a) Find the area bounded by the two curves $y = \sqrt{x}$ and $y = x^2$.

   (b) Two cars both start on a journey at time $x = 0$. If the velocity of car A at time $x \geq 0$ is $\sqrt{x}$ and that of car B at the same time is x^2, how does the area in part (a) relate to the positions of the cars at time $x = 1$?

6. The shape for a building that is to become a wheat silo is created by rotating the region bounded by the four curves

$$y = \frac{20}{x}, \quad x = 1, \quad x = 40, \quad y = 0,$$

about the x-axis.

   (a) At position x feet, where $1 \leq x \leq 40$, calculate the cross-sectional area $A(x)$ of the silo.

   (b) Find the volume of the silo.

   (c) If wheat prices at \$20 per bushel, and a bushel is 60 pounds, and the density of wheat is 50 pounds per cubic foot, how much is one silo of wheat worth?

7. An aquarium tank is 20 meters long, 4 meters wide, and 3 meters deep, and is filled with water. Assume that the density of the water is 1000 kg/m^3 and that the acceleration due to gravity is $g = 10$ m/s^2.

   (a) Imagine that the water in the tank is divided into n horizontal slices, each of depth $\Delta x = 3/n$, and with the i-th slice located at x_i, $1 \leq i \leq n$. Calculate (approximately) the work needed to move the i-th slice to the top of the tank.

   (b) How much work (again approximately) is needed to move all n slices to the top? Write you answer as a sum in sigma notation.

   (c) The limit of this sum as $n \to \infty$ is the precise amount of work needed to move all of the water in the tank to the top, and therefore out of the tank. Write this limit as a definite integral and use it to calculate the work needed to pump all of the water out of the tank.

8. Over the time period $0 \leq t \leq 1$ a spaceship travels along the the space curve

$$\boldsymbol{r}(t) = \langle 9, \frac{t^2}{2}, \frac{t^3}{3} \rangle.$$

   Find the total distance covered during this time.