Dynamical Systems 2, MA 761

Topological Dynamics

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1 Periodic Points 1

The main objects studied in the theory of dynamical systems are maps $f : X \to X$, where X is a space with some structure. For instance, this structure can be topological, differentiable, or there may be a measure on X. We work with the first case mostly. That is, let $f : X \to X$ be a continuous map of a compact metric space X into itself. We consider the *iterates* of f, defined by induction: $f^0 = \mathrm{id}_X$, and then $f^{n+1} = f^n \circ f$. Thus, $f^n = \underbrace{f \circ \cdots \circ f}_{n \text{ times}}$. This means that we look at the set of all iterates of f, $\{f^n\}_{n \in \mathbf{Z}_+}$.

The special case which is considered as often as the general one (and maybe even more often) is when the map f is invertible. We require then that f^{-1} is also within the class of maps we consider. That means that f and f^{-1} are continuous; then f is called a *homeomorphism*. If we consider an invertible f then we study also negative iterates of f, defined as $f^{-n} = (f^{-1})^n = (f^n)^{-1}$.

EXAMPLE 1.1. Let $X = [0, 1]$ and let $f : X \to X$ be given by the formula

$$f(x) = \begin{cases} 2x & \text{if } 0 \le x \le 1/2, \\ 2 - 2x & \text{if } 1/2 \le x \le 1. \end{cases}$$

The map f is called the (full) *tent map.* □

Call x a *fixed* point iff $f(x) = x$. Call x a *periodic* point iff $f^p(x) = x$ for some $p > 0$; all such p are called *periods* of x. If p is the minimal positive number such that $f^p(x) = x$ say that p is the *minimal period* of x.

CLAIM 1.2. *The periods of x are $\{mp : m \ge 0\}$ where p is the minimal period of x.*

Proof. Otherwise there exists $n = mp + i, 0 < i < p$ such that $f^n(x) = x$. However, $f^{mp}(x) = x$ too, so $f^i(x) = f^i(f^{mp}(x)) = x$, a contradiction with the choice of p. ■

Observe that in the arguments we do not use the continuity of f or any other specifics of the situation. Therefore the proof remains valid for a map $T : Y \to Y$ of a **set** Y into itself, that is on the set-theoretic level.

EXAMPLE 1.3. Identity map. □

A map $f : X \to X$ is called *periodic* if $f^p = \mathrm{id}_X$ for some $p > 0$; then p is called a *period* of f which allows one to define the *minimal period* of f. By Claim 1.2 and the observation after that all periods of a map f are multiples of its minimal period which remains true also on the set-theoretic level.

EXERCISE 1.4. Describe all interval homeomorphisms. Prove that all periodic maps are homeomorphisms and then describe periodic interval maps. In particular, show that their possible minimal periods are only 1 and 2. Moreover, show that if all points of the interval are periodic then the map is periodic too. □

EXAMPLE 1.5. A map $f : [0, 1] \to [0, 1]$ with $f(x) = bx(1 - x), 0 \le b \le 4$. □

EXAMPLE 1.6. Rotations of the circle, rational and irrational. Notice that irrational rotations of the circle have no periodic points while rational rotations by an angle $0 \le p/q < 1, p, q$ coprime, are periodic of period q. □

EXAMPLE 1.7. Shifts of the plane, reflections with respect to a straight line. A reflection of the plane is periodic of period 2 while a shift by a non-degenerate vector has no periodic points at all. □

CLAIM 1.8. *The set* $\mathrm{Per}_1(f) = \mathrm{Fix}(f)$ *of all fixed points of* f *is closed.*

Proof. If not then there exists a sequence of fixed points $x_n \to x$ such that x is not fixed. However this contradicts the continuity of f. ■

COROLLARY 1.9. *The set* $\mathrm{Per}_p(f)$ *of all points of period* p *is closed; the set of all points of period at most* p *is closed too. Also,* $\mathrm{Per}_p(f)$ *can be described as the set of all points whose minimal period is a divisor of* p*.*

Proof. The second half follows from 1.2. To prove the first half, apply 1.8 to f^p. ■

EXAMPLE 1.10. The following map is an example of a map for which the set of all points whose **minimal** period is 2, is not closed: $f : [0, 1] \to [0, 1], f(x) = 1 - x$. □

A point x is called *(pre)periodic* if it is mapped by some power of f into a periodic point. The sequence $\{f^i(x), i \ge 0\}$ is called the *trajectory* of x while the set $\cup_{n=0}^{\infty} f^n(x) = \mathrm{orb}(x)$ is called the *orbit* of x.

EXERCISE 1.11. A point x is (pre)periodic iff $\mathrm{orb}(x)$ is finite. □

2 Periodic Points 2

We begin by solving some of the exercises. First we need the following simple claim.

CLAIM 2.1. *Let $h : [0, 1] \to [0, 1]$ be a homeomorphism. Then one of the following holds:*

(1) $h(0) = 0, h(1) = 1$, *any interval (a, b) complementary to the set $\mathrm{Fix}(h)$ is mapped onto itself by f in such a way that all the points inside (a, b) are mapped to the right of themselves, or all the points inside (a, b) are mapped to the left of themselves, and so $\mathrm{Per}(h) = \mathrm{Fix}(h)$;*

(2) $h(0) = 1, h(1) = 0$, *there exists a unique fixed point $a \in (0, 1)$, and the set of all intervals complementary to the set $\mathrm{Per}_2(h)$ can be partitioned into pairs of intervals (b, c) and (d, e) such that $h[b, c] = [d, e], h[d, e] = [b, c], h(b) = e, h(e) = b, h(c) = d, h(d) = b$ and there are no periodic points inside $(b, c) \cup (d, e)$, and so the set $\mathrm{Per}(h)$ of all periodic points of h equals the set $\mathrm{Per}_2(h)$ of all points of period 2.*

Proof. Since h is a homeomorphism, either $h(0) = 1, h(1) = 1$ or $h(0) = 1, h(1) = 0$. Consider the cases.

(1) $h(0) = 1, h(1) = 1$. Consider the set $\mathrm{Fix}(h)$. This is a closed subset of $[0, 1]$. If it does not coincide with $[0, 1]$ then on any interval (a, b) complementary to $\mathrm{Fix}(f)$ in $[0, 1]$ points are mapped either to the left or to the right because otherwise there will be fixed points inside (a, b), a contradiction. Moreover, because h is 1-to-1, $h(a, b) = (a, b)$. Therefore, there are no periodic points inside (a, b) - all points inside (a, b) are mapped more and more to the right or to the left all the time.

(2) $h(0) = 1, h(1) = 0$. There must be a fixed point $a \in (0, 1)$. Then $h[0, a] = [a, 1]$ and $h[a, 1] = [0, a]$; in other words, the map flips the interval over the point a. Then for $g = h^2$ we have $g(0) = 0, g(1) = 1, g(a) = a$, so the previous case applies (obviously, iterates of a homeomorphism are homeomorphisms). If we go back to h we will see, that $\mathrm{Per}(f) = \{a\} \cup B$ where B is the set of all points of minimal period 2. An interval J complementary to $\mathrm{Per}(h)$ is such that J and $h(J)$ are disjoint while $h^2(J) = J$. Also, all points inside J are mapped into the same direction by h^2. The same holds for every interval complementary to $\mathrm{Per}(fh)$. This concludes the description of homeomorphisms of $[0, 1]$. ∎

Now we can do Exercise 1.4 whose part the above claim actually is.

EXERCISE 1.4. *Describe all interval homeomorphisms. Prove that all periodic maps are homeomorphisms and then describe periodic interval maps. In particular, show that their possible minimal periods are only* 1 *and* 2*. Moreover, show that if all points of the interval are periodic then the map is periodic too.*

Proof. Observe that in general if all points of a space are periodic for a map $T : X \to X$, then this does not necessarily imply that T is a periodic map because minimal periods of T-periodic points may well be unbounded. It is not difficult to give an example of such a map on a specifically designed compact space. Indeed, consider a compact space X which is a sequence of points x_i on the interval $[0, 1]$ converging to 1 together with the point 1 itself. Divide $\{x_i\}$ into finite subsets $P_0, P_1, \dots$ so that their cardinalities increase to infinity and $P_0 < P_1 < \dots$ in the natural sense. Define a map $T : X \to X$ so that P_i is invariant and, moreover, any P_i is a periodic orbit of T. Then it is easy to see that $T : X \to X$ is not periodic while all points of X are periodic. So, the last question of the exercise makes sense.

Observe however that if $f : X \to X$ is such that all points are periodic then f must be a homeomorphism. Indeed, denote the minimal period of a point $x \in X$ by p_x. Then f is surjective because $x = f(f^{p_x-1}(x))$ for any x. Also, if $f(x) = f(y) = z$ then z belongs to the periodic orbits of x and y which implies that these two orbits are the same. Denote their common period by p. Then $f^p(x) = f^p(y) = f^{p-1}(z) = x = y$, so f is injective. So, a map f whose all points are periodic is a 1-to-1 continuous map of a compact metric space onto itself. Therefore f is a homeomorphism (that is, its inverse is continuous too).

Let us go back to the original question of the description of the interval maps whose all points are periodic. As follows from the previous paragraph, such maps are homeomorphisms. Then Claim 2.1 implies that the only two ways a homeomorphism can be a map whose all points are periodic are as follows:

(1) in the case (1) of Claim 2.1 the map f must be identity;

(2) in the case (2) of Claim 2.1 the map f must be such that f^2 is the identity and in fact there exists the unique fixed point $a \in [0, 1]$ such that $f[0, a] = [a, 1], f[a, 1] = [0, a]$ and all other points of the interval are of period 2.

■

Let us now do Exercise 1.11.

EXERCISE 1.11. A point x is (pre)periodic iff $\mathrm{orb}(x)$ is finite. □

Proof. It is enough to check that if $\mathrm{orb}(x)$ is finite then x is (pre)periodic. Indeed, by the assumption there exist two numbers $i > 0, j > 0$ such that $f^i(x) = f^{i+j}(x)$, hence $f^i(x)$ is periodic and we are done. ■

3 One example related to symbolic dynamics

EXAMPLE 3.1. *Let $X = [0,1]$ and let $f : X \to X$ be given by the formula*

$$f(x) = \begin{cases} 3x & \text{if } x \in L = [0,1/3], \\ 2-3x & \text{if } x \in N = [1/3,2/3], \\ 3x-2 & \text{if } x \in R = [2/3,1]. \end{cases}$$

This is so-called saw *bimodal map. Our space is a metric compact, and the map is continuous. Now, consider the set A of all points x whose orbits are contained in $J = L \cup R$. Show that A is the standard $1/3$-Cantor set and study $f|A$.*

The idea is to construct the set A step by step. First consider the set A_0 of all points x which belong to J. Then consider all points x such that $x \in$ and $f(x) \in J$, etc. Let us pass on to a detailed proof.

Proof. We begin by a very important construction. Let S be a finite set consisting of more than one point, for instance $S = \{1, 2, \dots, s\}$ with $s > 1$. Define $\Sigma = \prod_{-\infty}^{\infty} S$ and $\Sigma_+ = \prod_0^{\infty} S$ (for brevity we skip the dependence upon s in our notation here). More precisely, $\Sigma = \prod_{i=-\infty}^{\infty} S_i$ and $\Sigma_+ = \prod_{i=0}^{\infty} S_i$, where $S_i = S$ for each i. Thus, the elements of Σ are the doubly infinite sequences $(\dots, x_{-2}, x_{-1}, x_0, x_1, x_2, \dots)$ with $x_i \in S$ for all i, and the elements of Σ_+ are the usual one-sided sequences $(x_0, x_1, x_2, \dots)$ with $x_i \in S$ for all i. We will mostly work with Σ_+.

We define a *shift* σ on Σ and Σ_+ (we will use the same letter in both cases) as the shift by one to the left. This means that $\sigma(x_0, x_1, x_2, \dots) = (x_1, x_2, x_3, \dots)$. To write the formula for σ on Σ is more difficult. For this we have to introduce notation for the points of Σ which shows where the 0-th coordinate is. Namely, we shall write $x = (\dots, x_{-2}, x_{-1}, \overset{*}{x}_0, x_1, x_2, \dots)$ if the 0-th coordinate of x is x_0. With this notation we can write $\sigma(\dots, x_{-2}, x_{-1}, \overset{*}{x}_0, x_1, x_2, \dots) = (\dots, x_{-1}, x_0, \overset{*}{x}_1, x_2, x_3, \dots)$. Moreover, in the case of the space Σ, σ is 1-to-1. In the case of the space Σ_+, σ is s-to-one.

We will regard Σ and Σ_+ as metric spaces. We need the following definition: *a cylinder* is a set of the form $C_{y_{-n}, y_{-n+1}, \dots, y_{n-1}, y_n} = \{(\dots, x_{-2}, x_{-1}, x_0, x_1, x_2, \dots) \in \Sigma : x_i = y_i \text{ for all } i \in \{-n, -n+1, \dots, n-1, n\}\}$ in the space Σ and of the form $C_{y_0, y_1, \dots, y_n} = \{(x_0, x_1, x_2, \dots) \in \Sigma : x_i = y_i \text{ for all } i \in \{0, 1, \dots, n\}\}$ in the space Σ_+. We define a metric in these spaces by setting $d(x, y) = 2^{-k}$, where k is the smallest non-negative integer such that there is m with $|m| = k$ and such that the m-th terms of the sequences x and y are different.

Denote by S a two-symbol set $S = \{L, R\}$ and also by Σ_+ the set of all one-sided sequences of elements of S. Thus, the elements of Σ_+ are the usual one-sided sequences

$(x_0, x_1, x_2, \ldots)$ with $x_i \in S$ for all i (for brevity we skip the dependence on $s = 2$).

The map f maps each of the intervals L, R onto the whole $[0, 1]$ in an increasing way. Now we apply the coding procedure to the system (A, f) with the partition $\{L, R\}$ of $J = L \cup R$. Denote the set of points $x \in [0, 1]$ such that the point $f^i(x), 0 \le i \le n-1$ lies in J, by X_{n-1}. For any $x \in X_{n-1}$ we look at the set L or R in which the point $f^i(x)$ lies for $i = 0, 1, 2, \ldots, n$ and call this set A_i. In such a way we get for every point $x \in A$ a code $(A_0, A_1, A_2, \ldots) \in \Sigma_+$, sometimes also called the *itinerary* of x. Observe, that while the finite segment of the sequence $(A_0, A_1, A_2, \ldots, A_{n-1})$ is defined for all points of X_{n-1}, the infinite sequence $(A_0, A_1, A_2, \ldots) \in \Sigma_+$ is defined for all $x \in A$ only. We claim that given a code there is a unique point with this code.

Let $K = (A_0, A_1, A_2, \ldots) \in \Sigma_+$. For a given n the set of points whose code begins with $(A_0, A_1, \ldots, A_{n-1})$ is equal to $I_n(K) = \bigcap_{i=0}^{n-1} f^{-i}(A_i)$ (this immediately follows from the definitions). We show by induction that for every code K such a set is an interval of length 3^{-n} whose homeomorphic f^n-image is $[0, 1]$. This is definitely true for $n = 1$ because then the set of points whose code begins with $L(R)$ is simply the interval $L(R)$ itself. If it is true for some n then, since $I_{n+1}(K) = A_0 \cap f^{-1}(I_n(\sigma(K)))$ and by the induction hypothesis $I_n(\sigma(K))$ is an interval of length 3^{-n}, we get that $I_{n+1}(K)$ is an interval of length 3^{-n-1} whose homeomorphic f^{n+1}-image is $[0, 1]$. This completes the induction step.

Now we have a descending sequence $(I_n(K))_{n=0}^{\infty}$ of closed non-empty subsets of $[0, 1]$. Since $[0, 1]$ is compact, the intersection of all $I_n(K)$ is non-empty. Since the length of $I_n(K)$ goes to 0 as $n \to \infty$, this intersection consists of one point. We call this point $\varphi(K)$. In such a way we define a map $\varphi : \Sigma_+ \to A$. Since every point of A has a code, this map is onto. On the other hand, since the itinerary of any point $x \in A$ is well-defined, we conclude that this is a 1-to-1 map. We shall show that φ is continuous. Let $\lim_{n\to\infty} K_n = M = (B_0, B_1, B_2, \ldots)$. As $n \to \infty$, longer and longer initial pieces of K_n are the same as the initial pieces of M, so if we choose any m then $I_m(K_n) = I_m(M)$ if n is sufficiently large. This implies that $|\varphi(K_n) - \varphi(M)| \le 3^{-m}$ if n is sufficiently large. Therefore $\lim_{n\to\infty} \varphi(K_n) = \varphi(M)$, so φ is continuous.

Look at $I_n(K)$. We know that $f^n(I_n(K)) = [0, 1]$ and f^n on $I_n(K)$ is linear and increasing. On the other hand, $I_{n+1}(K) = f^{-n}(A_n) \cap I_n(K)$. Thus, depending on whether $A_n = L$ or $A_n = R$, the interval $I_{n+1}(K)$ is either the left of the right third of $I_n(K)$. This implies that A is constructed the same way as the standard Cantor set is, and thus A coincides with the standard Cantor set. If $x, y \in A, d(x, y) \le \epsilon$ then $d(f^n(x), f^n(y)) \le 3^n(\epsilon)$. So if $d(x, y) < 3^{-n-1}$ then the first n entries of the itineraries of x and y coincide because $f^i(x), f^i(y), 0 \le i \le n-1$ cannot belong to L, R or R, L respectively. Thus, φ^{-1} is continuous. Hence φ is a homeomorphism and Σ_+ is compact. ∎

4 Structural stability

In math objects which look similarly are treated as the same object; the same is done in dynamical systems. Let X and Y be metric spaces and let $f : X \to X$ and $g : Y \to Y$ be continuous maps. If there is a homeomorphism $h : X \to X$ with $h \circ f = g \circ h$ (in other words, the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ \Big\downarrow h & & \Big\downarrow h \\ Y & \xrightarrow{g} & Y \end{array}$$

commutes), we will say that f and g are *conjugate*. Notice that in this case X and Y are homeomorphic. The homeomorphism h is called a *conjugacy* (between f and g). In this situation, if h is not necessarily a homeomorphism, but just a continuous map of X *onto* Y, we say that g is a *factor* of f and h is a *semiconjugacy* of f with g.

Another way of looking at the maps above is as follows: one of them can serve as a model for the other one providing a system of coordinate in which the original map is easier to deal with. Major questions in dynamical systems theory are whether two maps are conjugate or not and what extra properties the conjugacy φ has (smoothness and the like). Clearly, finding invariants defined in topological terms (*topological invariants*) is very important for figuring out if two maps are conjugate: if the value of an invariant is different for the two maps we consider they cannot by conjugate. The ideal situation is when one can come up with a *complete* collection of invariants so that two maps are conjugate if and only if they have the same invariants (which is why the system is called complete).

A major topic in dynamical systems is to figure out how the dynamics of a map changes under small perturbations. Here "small" means "small in the sense of a certain topology/distance" introduced in the space of all maps. Usually and because C^0-metric allows rather wild perturbations of a map, the metric which is considered is smooth and could be C^n with $n \geq 1$ (recall that two maps are C^n-close if the maps and all their derivatives up to the n-th derivative are close).

Now the above problem can be restated as follows: what properties does a map have to have in order to guarantee that its small perturbations are conjugate to itself? A map f which has a C^n-neighborhood U such that all maps from U are conjugate to f is called *C^n-structurally stable.* Similar definitions can be given if we restrict our attention onto homeomorphisms only.

EXAMPLE 4.1. *Consider an orientation preserving C^1-homeomorphism $f : [0, 1] \to$*

$[0, 1]$ *such that* $f'(0) > 1 > f'(1)$ *and there are no other fixed points of* f *and show that* f *is structurally stable in the class of homeomorphisms.*

Proof. Consider a map g which is very close to f in the C^1-sense and show that f and g are conjugate. To do this let us show that if g is very close then $g(0) = 0, g(1) = 1$ and $g'(0) > 1 > g'(1)$ and there are no other fixed points of g either. Indeed, the first part of the claim follows from the fact that g is a homeomorphism while the second follows from the choice of g very close to f in C^1-topology.

Let us show that $0, 1$ are the only fixed points of g. To this end first choose small $\varepsilon > 0$ and $\delta > 0$ such that for every $x \in [0, \varepsilon]$ we have $f'(x) > 1 + \delta$ and for every $x \in [1 - \varepsilon, 1]$ we have $f'(x) < 1 - \delta$. Then choose g so close to f that $g'(x) > 1 + \delta/2$ for $x \in [0, \varepsilon]$ and $g'(x) \leq 1 - \delta/2$ for $x \in [1 - \varepsilon/2, 1]$. This implies that g has no fixed points inside $J = [0, \varepsilon] \cup [1 - \varepsilon, 1]$ because otherwise there would be points $z \in J$ with $g'(z) = 1$, a contradiction.

Now, since $f|[\varepsilon, 1-\varepsilon]$ has no fixed points then there exists $\gamma > 0$ such that $f(x)-x > \gamma$ for all $\varepsilon \leq x \leq 1-\varepsilon]$. If g is chosen close enough to f this implies that $g(x) - x > \gamma/2$ for all $\varepsilon \leq x \leq 1-\varepsilon]$ and completes the proof of the fact that g has only two fixed points, 0 and 1.

It remains to prove that any two maps F and G of $[0, 1]$ into itself such that the end-points are fixed and the points inside map to the right are conjugate. To show that this is indeed the case let us choose a point $x \in (0, 1)$ and denote $[x, F(x))$ by I. Then $F(I)$ is an interval to the right of I whose left endpoint is $F(x)$ (so it is "attached" to I from the right) and $F^{-1}(I)$ is an interval to the left of I whose right endpoint is x (so it is "attached" to I from the left). The union of the sets $F^n(I), -\infty < n < \infty$ covers the entire $(0, 1)$ (to prove that one needs the assumption about the fixed points of F being only 0 or 1). Choose a point $y \in (0, 1)$, consider the interval $J = [y, G(y))$ and perform the same construction as above but now for G and J.

Construct an increasing homeomorphism $\varphi : I \to J$. Then extend it onto $[0, 1]$ as follows: 1) for every point $x \in (0, 1)$ choose the unique $n_x = n$ such that $F^n(x) \in I$; 2) define $\varphi(x)$ as $G^{-n}\varphi F^n(x)$ (it is well-defined since $F^n(x) \in I$). Thus, iterates $F^n(I)$ will be mapped onto their counterparts $G^n(J)$ in a way compatible with the way φ maps I onto J. Moreover, φ is continuous and can be extended onto $[0, 1]$ by declaring that $\varphi(0) = 0, \varphi(1) = 1$.

To finish the proof one needs to do Exercise 4.1 below. ∎

EXERCISE 4.1. Finish the proof, i.e., show that $\varphi \circ f = g \circ \varphi$. □

EXERCISE 4.2. For the class of increasing interval homeomorphisms of class C^1 whose all fixed points are such that the derivative at them has the absolute value greater than 1 or less than 1 describe the complete topological invariant. □

5 Non-wandering Points

Let X be a compact metric space and $f : X \to X$ a continuous map. We call a point $x \in X$ *wandering* if there exists a neighborhood U of x such that $U \cap f^n(U) = \emptyset$ for every $n > 0$. A point is called *non-wandering* if it is not wandering. The set of non-wandering points of f is denoted $\Omega(f)$. Recall that a point is called *isolated* if it is open as a set.

The following proposition lists the basic properties of the set of non-wandering points.

PROPOSITION 5.1. *Let $f : X \to X$ be a continuous map of a compact metric space X into itself. Then the following properties hold.*

(a) $\Omega(f)$ is compact and contains the closure of all periodic points of f.

(b) $\Omega(f)$ is invariant, that is $f(\Omega(f)) \subset \Omega(f)$.

(c) If f is a homeomorphism then $f(\Omega(f)) = \Omega(f)$.

(d) $\Omega(f^n) \subset \Omega(f)$ for any $n > 0$.

(e) If f is a homeomorphism then $\Omega(f^{-1}) = \Omega(f)$.

Proof. (a) By the definition, the set of wandering points is open. Therefore the set of non-wandering points is closed, and hence compact (since X is compact).

(b) Assume that $x \in \Omega(f)$. Let U be a neighborhood of $f(x)$. Then $V = f^{-1}(U)$ is a neighborhood of x. Since x is non-wandering, there exists $n > 0$ such that $V \cap f^n(V) \neq \emptyset$. This means that there is $y \in V$ such that $f^n(y) \in V$. Then $f(y) \in f(V) = U$ and $f^n(f(y)) = f(f^n(y)) \in f(V) = U$. Therefore $U \cap f^n(U) \neq \emptyset$. Therefore, $f(x) \in \Omega(f)$.

(d) Assume that $x \in \Omega(f^n)$. If U is a neighborhood of x then there exists $m > 0$ such that $U \cap (f^m)^n(U) \neq \emptyset$. Since $(f^m)^n = f^{mn}$, this shows that $x \in \Omega(f)$.

(e) Assume that f is a homeomorphism and $x \in \Omega(f)$. If U is a neighborhood of x then there exists $n > 0$ such that $U \cap f^n(U) \neq \emptyset$. This means that there is $y \in U$ such that $f^n(y) \in U$. Therefore $U \cap f^{-n}(U) \neq \emptyset$. This shows that $\Omega(f) \subset \Omega(f^{-1})$. This is true for all homeomorphisms of X, in particular for f^{-1}. Therefore $\Omega(f^{-1}) \subset \Omega((f^{-1})^{-1}) = \Omega(f)$.

(c) From (e) and (b) it follows that if f is a homeomorphism then

$$f^{-1}(\Omega(f)) = f^{-1}(\Omega(f^{-1})) \subset \Omega(f^{-1}) = \Omega(f),$$

so $\Omega(f) \subset f(\Omega(f))$. Together with (b) this gives (c). ■

EXAMPLE 5.2. Consider the space

$$X = \{0, 2\} \cup \{-2^{-n} : n \geq 0\} \cup \{2^{-n} : n \geq 0\} \cup \{2 + 2^{-n} : n \geq 0\}.$$

Define the map $f : X \to X$ by

$$f(x) = \begin{cases} -2x & \text{if } x < 0, \\ -x & \text{if } 0 \leq x \leq 1, \\ x - 2 & \text{if } x \geq 2. \end{cases}$$

Clearly, X is compact and f is continuous. All the points of X, except 0 and 2, are isolated and not periodic. Therefore they are wandering. The point 0 is a fixed point, so it is non-wandering. In any neighborhood of the point 2 there are points of the form $2 + 2^{-n}$. The image of such a point is 2^{-n} and the $2n + 2$-nd image of this point is 2. Therefore 2 is non-wandering. Thus, $\Omega(f) = \{0, 2\}$. However, we have $f(\{0, 2\}) = \{0\}$. This shows that the assumption in (c) that f is a homeomorphism is essential.

The set $U = \{2\} \cup \{2+2^{-n} : n \geq 0\}$ is a neighborhood of 2. Set $V = \{0\} \cup \{-2^{-n} : n \geq 0\}$. We have $f^2(U) \subset V$ and $f^2(V) \subset V$. Since V is disjoint from U, we get $U \cap f^{2m}(U) = \emptyset$ for every $m > 0$. So, 2 is wandering for f^2, and in (d) we might not have the equality. Also, the set $\Omega(f|_{\Omega(f)})$ (which is $\{0\}$ here) can be smaller than $\Omega(f)$. □

The next proposition gives a reason why the notion of the set of non-wandering points is important. We will use the notation $\operatorname{dist}(x, y)$ for the distance between two points x and y, and $\operatorname{dist}(x, A)$ for the distance of a point x from the set A (that is, $\operatorname{dist}(x, A) = \inf_{y \in A} \operatorname{dist}(x, y)$).

Proposition 5.3. *Let $f : X \to X$ be a continuous map of a compact metric space X into itself. Then every point is attracted by the set of non-wandering points, that is*

$$\lim_{n \to \infty} \operatorname{dist}(f^n(x), \Omega(f)) = 0$$

for every $x \in X$.

Proof. Suppose that there is a point $x \in X$ such that $\operatorname{dist}(f^n(x), \Omega(f))$ does not tend to zero. Then there is an $\varepsilon > 0$ and a subsequence of the orbit $(f^n(x))_{n=0}^{\infty}$ of x such that the distance of every point of this subsequence from $\Omega(f)$ is greater than or equal to ε. Since X is compact, there is a subsequence of this subsequence that converges to some point $y \in X$. Clearly, $\operatorname{dist}(y, \Omega(f)) \geq \varepsilon$. For every neighborhood U of y there are arbitrarily large integers n with $f^n(x) \in U$. We choose two of them, $n < m$. We have $f^m(x) \in U$, and since $f^n(x) \in U$, we get $f^m(x) = f^{m-n}(f^n(x)) \in f^{m-n}(U)$. Thus, $U \cap f^{m-n}(U) \neq \emptyset$. This proves that the point y is non-wandering, contrary to the property $\operatorname{dist}(y, \Omega(f)) \geq \varepsilon > 0$. This contradiction completes the proof. ■

We can interpret the above proposition as follows. The trajectory of every point approaches $\Omega(f)$, so on longer and longer pieces it looks like trajectories of some points of $\Omega(f)$. Thus, all interesting dynamics can be found in $\Omega(f)$.

6 Limit Sets

For any point $x \in X$ we define the *ω-limit (omega limit) set* $\omega(x)$ of x (or of the trajectory of x) as the set of limits of all convergent subsequences of $(f^n(x))_{n=0}^{\infty}$. Thus another way of stating Proposition 5.3 is that $\omega(x) \subset \Omega(f)$ for every $x \in X$. Let us list some properties of limit sets.

EXAMPLE 6.1. Let x be a periodic point. Then $\omega(x) = \mathrm{orb}(x)$; in particular, if a map is periodic then the limit set of any point is its periodic orbit. Similarly, if x is preperiodic and $y \in \mathrm{orb}(x)$ is periodic then $\omega(x) = \mathrm{orb}(y)$. □

Let us prove the following simple but useful claim concerning limit sets.

PROPOSITION 6.2. *If $\omega(x)$ is a singleton then it is a fixed point.*

Proof. If $f^n(x) \to y$ then by continuity $f(y) = y$ as desired. ■

The cases described in Example 6.1 are not the only ones which describe the trajectories of points whose limit sets are periodic orbits. Another series of examples is presented in Example 4.1 and actually in earlier given Exercise 1.4. Indeed, we can easily prove the following proposition.

PROPOSITION 6.3. *For any orientation preserving interval homeomorphism f and any point x the limit set $\omega(x)$ is a fixed point; in the orientation reversing case $\omega(x)$ could be a fixed point or a periodic orbit of period* 2.

Proof. Without loss of generality we may assume that $f : [0, 1] \to [0, 1]$ is orientation preserving so that $f(0) = 0$ and $f(1) = 1$. It is enough to consider a point x which is not fixed. Then $x \in I$ where $I = (a, b)$ is invariant under f, $f(a) = a, f(b) = b$ and all points of I are mapped in the same direction. For the sake of definiteness let all $y \in I$ be mapped to the right. Then $x < f(x) < \dots$ and therefore $\omega(x)$ is the unique limit point of the increasing sequence $(x, f(x), \dots)$. By Proposition 6.2 $\omega(x)$ is a fixed point which must belong to $\bar{I}$ and be greater than x. Hence $\omega(x) = b$. ■

PROPOSITION 6.4. *Let $f : X \to X$ be a continuous map of a compact metric space X into itself, $x \in X$. Then $\omega(x)$ is closed, $f(\omega(x)) = \omega(x)$ and $\omega(x) \subset \Omega(f)$. Moreover, if the orbit of x is finite then x is preperiodic and $\omega(x)$ is a periodic orbit.*

Proof. The first claim follows from the fact that the set A of all limit points of any sequence $\{y_i\}$ in X is closed. Indeed, if $z \notin A$ then there exists an open $U \ni z$ and N such that $U \cap \{y_i\}_{i>N} = \emptyset$. This implies that $U \cap A = \emptyset$ and thus $X \setminus A$ is open. Since

X is compact, $\omega(x) \neq \emptyset$. Observe that the notion of the limit set can be introduced in the non-compact case as well and in a non-compact metric space $\omega(x)$ may be empty; however it must be closed in any case (e.g., for the shift on the plane the limit sets of all points are empty).

It is easy to see that $\omega(x) \subset \Omega(f)$. Indeed, if $y \in \omega(x)$ then for some $\{n_k\}$ we have $f^{n_k}(x) \to y$. If U is a neighborhood of y then there exists k such that $f^{n_k}(x) \in U$ and $f^{n_{k+1}} \in U$, hence $f^{n_{k+1}-n_k}(U) \cap U \neq \emptyset$ and so $y \in \Omega(f)$ as desired.

To prove that $f(\omega(x)) = \omega(x)$ we prove that $f(\omega(x)) \supset \omega(x)$ and that $f(\omega(x)) \subset \omega(x)$. Indeed, if $y \in \omega(x)$ then $y = \lim_{k\to\infty} f^{n_k}(x)$ for some subsequence n_k. Then by continuity of f we get $f(y) = \lim f^{n_k+1}(x) \in \omega(x)$ which implies $f(\omega(x)) \subset \omega(x)$. Observe that this holds in non-compact metric spaces. Now, consider a sequence $f^{n_k-1}(x)$. Since X is compact it has a limit point z - in other words, there exists a subsequence $\{m_j\}$ of $\{n_k - 1\}$ such that $f^{m_j}(x) \to z$. Then $f^{m_j+1} \to f(z)$ and on the other hand since $\{m_j+1\}$ is a subsequence of $\{n_k\}$ we see that $f(z) = y$. This proves that $f(\omega(x)) \supset \omega(x)$.

Now, if $\text{orb}(x)$ is finite then we can find the smallest n for which there exists i such that $f^{n+i}(x) = f^n(x)$ and then pick the smallest such i. Then it is easy to see that f^n is a periodic point of the minimal period i and $\omega(x) = \{f^n(x), \dots, f^{n+i-1}(x)\}$. ∎

Denote the set of all periodic points of f by $\text{Per}(f)$ and the union of all limit set of points of f by $\omega(f)$. Then Proposition 6.4 implies that $\text{Per}(f)] \subset \Omega(f)$ and that $\overline{\text{Per}(f)} \subset \overline{\omega(f)} \subset \Omega(f)$.

Let us classify topological types of limit set for interval maps.

PROPOSITION 6.5. *Let $f : [0,1] \to [0,1]$ be a continuous map. Then for any x either $\omega(x)$ is a nowhere-dense set or $\omega(x)$ is a finite collection of intervals cyclically permuted by f on which a map is surjective.*

Proof. Suppose that $\omega(x)$ is not nowhere dense. Then it has a component I which is a non-degenerate interval. Since $I \subset \omega(x)$ we can choose two integers $m, m+n$ such that $f^m(x) \in I$ and $f^{m+n}(x) \in I$. Hence $f^n(I) \cap I \neq \emptyset$. On the other hand, I is a component of an invariant closed set $\omega(x)$. Therefore in fact $f^n(I) \subset I$. The union of intervals $A = \bigcup_{i=0}^{n-1} f^i(I)$ is then a closed invariant set containing forward iterates of x which implies that $\omega(x) \subset A$. Since we know that $A \subset \omega(x)$ we see that $A = \omega(x)$.

To see now that A is a finite collection of intervals cyclically permuted by f choose the smallest $m > 0$ such that $f^m(I) \cap I \neq \emptyset$. Then since I is a component of $A = \omega(x)$ we have that $f^m(I) \subset I$, and by the minimality of m all smaller iterates of I are pairwise disjoint. Finally, since $f|A$ must be onto, $f^m(I) = I$ as desired. ∎

7 Recurrent Points

An important definition related to that of the limit set is that of a recurrent point. A point x is said to be recurrent if $x \in \omega(x)$. Let us list simple properties of recurrent points.

PROPOSITION 7.1. *A periodic point is recurrent whereas a preperiodic non-periodic point is not recurrent.*

PROPOSITION 7.2. *Any recurrent point x belongs to $\Omega(f)$* (cf Proposition 6.4*).*

PROPOSITION 7.3. *If f is an interval homeomorphism then the only recurrent points of f are its periodic points.*

The notion of a recurrent point is an obvious generalization of that of a periodic point: in the latter case the point comes back *exactly in itself* while in the former case the point comes back *close to itself* with approximation becoming better and better. In some topologically defined cases recurrent points in fact have to be periodic.

PROPOSITION 7.4. *Isolated recurrent points are periodic.*

Proof. If x is isolated and recurrent then for small enough ε the fact that $d(f^n(x), x) < \varepsilon$ implies that $f^n(x) = x$ as desired. ■

It is sometimes useful to consider a set $\overline{\{f^j(x)\}} = \overline{\operatorname{orb} x}$ which is closely related to the limit set of x. Their relationship is described in the following lemma.

LEMMA 7.5. $\overline{\operatorname{orb} x} \supset \omega(x)$*. Moreover,* $\overline{\operatorname{orb} x} = \omega(x)$ *if and only if x is recurrent.*

Proof. If $y \in \omega(x)$ then by the definition in any neighborhood U of y there is a point of $\operatorname{orb} x$ in U. Hence $y \in \overline{\operatorname{orb} x}$. Now, suppose that x is recurrent, that is $x \in \omega(x)$. Then all points of $\operatorname{orb} x$ are in $\omega(x)$ because by Proposition 6.4 $f(\omega(x)) = \omega(x)$. On the other hand any point $z \in \overline{\operatorname{orb} x} \setminus \operatorname{orb} x$ is in $\omega(x)$ because by the definition we will be able to find a sequence of points of $\operatorname{orb} x$ converging to z. So, if x is recurrent then $\overline{\operatorname{orb} x} = \omega(x)$. On the other hand, if x is not recurrent then by the definition $x \in \overline{\operatorname{orb} x} \setminus \omega(x)$ and so $\overline{\operatorname{orb} x} \supset \omega(x), \overline{\operatorname{orb} x} \neq \omega(x)$ which completes the proof. ■

The notion of a recurrent point is important because in fact for every invariant measure μ the set of recurrent points $R(f)$ has full μ-measure. Let us show this using some of the results established in DYNAMICAL SYSTEMS I (generally, in the situation when we need to refer to measure-theoretic results from DYNAMICAL SYSTEMS I we will do this without a proof).

In DYNAMICAL SYSTEMS I, Remark 7.21 it is shown that if $A \subset X$ is a subset such that $\nu(A) = 1$ for every ergodic measure ν then $\mu(A) = 1$ for every measure μ. Hence it is enough to prove that $\nu(R(f)) = 1$ for an ergodic measure ν. To this end let us consider the so-called *support* of the measure ν denoted by $\mathrm{supp}(\nu)$ and defined as follows: for every invariant measure μ the set $\mathrm{supp}(\mu)$ is the set of all points x such that for any neighborhood U of x we have $\mu(U) > 0$.

PROPOSITION 7.6. *The set* $\mathrm{supp}(\mu)$ *is closed, invariant and of full* μ*-measure.*

Proof. Let us show that $A = \mathrm{supp}(\mu)$ is closed. Indeed, if $y \notin A$ then there exists an open set U such that $\mu(U) = 0$. Hence by the definition U is contained in the complement of A which implies that this complement is open and hence A is closed.

Let us show that $\mu(A) = 1$. Indeed, every point $x \notin A$ has a neighborhood U such that $\mu(U) = 0$. Since X is a metric compactum it has a countable basis, so we can now represent the set $X \setminus A$ as a countable union of open sets with zero measure which implies that $\mu(X \setminus A) = 0$ as desired.

Let us show that A is invariant. Indeed, let $x \in A$ while $f(x) \notin A$. Then for some open set U containing $f(x)$ we have that $\mu(U) = 0$. Therefore, $\mu(f^{-1}(U) = 0$ as well while $f^{-1}(U)$ clearly is a neighborhood of x, a contradiction with the assumption that $x \in A$. Hence $f(x) \in A$ and A is invariant. ■

PROPOSITION 7.7. *Let* ν *be an invariant ergodic measure and let* $B = \mathrm{supp}(\nu)$*. Then* ν*-a.e. point* $x \in B$ *is such that* $\omega(x) = B$ *(and so all these points are recurrent).*

Proof. By the construction every point $y \in B$ is such that for every open set U containing y we have $\nu(U) > 0$. Since $\nu(B) = 1$ by Proposition 7.6 then $\nu(Y) = \nu(Y \cap B)$ for any set Y. So, for any open U such that $U \cap B \neq \emptyset$ we have $\nu(U \cap B) > 0$. Therefore by DYNAMICAL SYSTEMS I, Corollary 7.25 we conclude that the orbit of almost every point of B is dense in B as desired. ■

THEOREM 7.8. *For every invariant measure* μ *we have* $\mu(R(f)) = 1$.

Proof. It is enough to prove that if D is a set consisting of non-recurrent points then for every μ we have $\mu(D) = 0$. By Proposition 7.7 this is the case if ν is ergodic. In other words, the ergodic measure of D is always zero. By DYNAMICAL SYSTEMS I, Ergodic Decomposition Theorem (Theorem 7.15) we conclude that then $\mu(D) = 0$ for every measure μ as desired. ■

8 The Center of a Dynamical System and its Depth

The closure $\overline{R(f)}$ of the set $R(f)$ of all recurrent points of a map f is called the *center* of the map f and is denoted by $C(f)$. There is a different way of defining the set $C(f)$. Namely, it can be obtained by taking $\Omega(f)$, then $\Omega(f|_{\Omega(f)})$, etc continuing by means of the so-called *transfinite induction* until the sequence stabilizes. Observe that since on each step the obtained set is closed then the resulting set $C(f)$ is closed as well.

Let us verify that the set $\overline{R(f)}$ is contained in the result of the above described inductive process. If $Y \subset X$ is constructed at some point then on the next step we construct the set $\Omega(f|Y)$ (the initial set is $Y = X$). Indeed, if $x \in Y$ is a recurrent point then there exists a sequence $\{n_k\}$ such that $f^{n_k}(x) \to x$. Therefore for any open U containing x we have that $f^{n_k}(x) \in U$ for big enough k which implies that $x \in \Omega(f|Y)$. Since nothing in the above argument depends on Y we see that $x \in C(f)$ and therefore $R(f) \subset C(f)$. Since $C(f)$ is closed we conclude that indeed $\overline{R(f)} \subset C(f)$.

It was proven by Birkhoff that the resulting set coincides with $\overline{R(f)}$. The number of steps one has to make in order to get the set $C(f)$ is called the *depth* of the center $C(f)$ of the map f. As Example 5.2 shows, $C(f)$ can be smaller than $\Omega(f)$, so the notion of the depth of the center of a map makes sense.

Our main aim now is to study the center of an interval map. A useful tool for us will be the following lemma.

LEMMA 8.1. *Let U be a complementary to $\overline{\mathrm{Per}(f)}$ interval. Then one of the following cases takes place:*

(1) points of U never come back into U;

(2) for any $x \in U, n$ such that $f^n(x) \in U$ we have $x < f^n(x)$;

(3) for any $x \in U, n$ such that $f^n(x) \in U$ we have $x > f^n(x)$.

Proof. Let $x \in U, n$ are such that $f^n(x) \in U$ and $x < f^n(x)$. Then since there are no periodic points of f between x and $f(x)$ we conclude that for any m the direction in which f^m maps x and the direction in which it maps $f^n(x)$ are the same. In other words, either 1) $f^m(x) > x$ and $f^{n+m}(x) > f^n(x)$, or 2) $f^m(x) < x$ and $f^{n+m}(x) < f^n(x)$. Let us apply this to $m = n$; this yields that since f^n maps x to the right, f^n must map $f^n(x)$ to the right as well, and so $x < f^n(x) < f^{2n}(x)$. Hence, f^{2n} maps x to the right, and applying the above claim to f^{2n} we see that since f^{2n} maps x to the right, f^{2n} must map $f^n(x)$ to the right as well and so $x < f^n(x) < f^{3n}(x)$. Clearly, repeating this argument we see that $x < f^{kn}(x)$ for any $k \geq 1$ and therefore for any $z \in U$ we have $z < f^{kn}(z)$.

Now, suppose that there exists another point $y \in U$ and a number r such that $f^r(y) \in U$ and $f^r(y) < y$. Similarly to the previous paragraph we see that then $f^{jr}(y) < y$ for any $j \geq 1$ and therefore for any $z \in U$ we have $f^{jr}(z) < z$.

Now, let us take $s = nr$ and consider $f^{nr}(z)$ and its position with respect to z. By the previous paragraph, $f^{nr}(z) < z$ but by the paragraph before that $f^{nr}(z) > z$, a contradiction which proves that the assumptions of the above two paragraphs cannot coexist and completes the proof. ∎

An informal way to state Lemma 8.1 is to say that points of U return to U to the same side of themselves; this clearly means that there are "right" intervals U and there are "left" intervals U. Let us now prove the following theorem which relies upon Lemma 8.1.

THEOREM 8.2. *For a continuous interval map f we have $C(f) = \overline{\mathrm{Per}(f)}$.*

Proof. Let U be a complementary to $\mathrm{Per}(f)$ interval. Suppose that $x \in U$ is recurrent. Then x has to come back to itself closer and closer under growing to infinity sequence of iterations of f. Now, suppose that the least n with $f^n(x) \in U$ is such that $x < f^n(x)$. Then, as Lemma 8.1 shows, U is the "right" interval and $f^k(z) > z$ for every $z \in U$ and k such that $f^k(z) \in U$. For every $m \neq n$ such that $f^m(x) \in U$ we have by the choice of n that $m > n$ and therefore $f^m(x) > f^n(x)$ which implies that $f^m(x)$ cannot converge to x even along a subsequence. Similar arguments in the case of a "left" interval U finally imply that x cannot be recurrent as desired. ∎

Let us now study the depth of interval maps. To begin with let us give some examples. First, let $f : [0, 1 \to [0, 1]$ be the identity map. Then since $\Omega(f) = [0, 1]$ the depth of the center is 0. Now, consider any orientation preserving homeomorphism f which is not the identity map. Then $\Omega(f) = \mathrm{Per}(f) \neq [0, 1]$ and it is easy to see that $\Omega(f|\Omega(f)) = \Omega(f)$. In other words, the sequence of sets $I, \Omega(f), \Omega(f|\Omega(f)), \ldots$ stabilizes after exactly one step, and therefore the depth of f is 1.

EXERCISE 8.3. Let $f : [-1, 4] \to [-1, 4]$ be defined as follows: $f(-1) = -1, f(0) = 0, f(1) = 3, f(2) = -1, f(3) = 0, f(4) = 1$ and linearly on all the intervals into which $[-1, 4]$ is divided by these points. Show that then $\Omega(f) \neq \Omega(f|\Omega(f))$ by suggesting a point x such that $x \in \Omega(f)$ while on the other hand $x \notin \Omega(f|\Omega(f))$. □

By Exercise 8.3 there are interval maps whose depth is greater than 1. However, as we will show next time it cannot be greater than 2. Thus, the map f from Exercise 8.3 has to be of depth exactly 2. Preparing for the next lecture you are welcome to think about how this can be proven directly, yet this is not a part of your homework.

9 The Depth of a Dynamical System on the Interval

In this lecture we will assume that $f : [0, 1] \to [0, 1]$ is a continuous interval map.

LEMMA 9.1. *Let U be an interval containing no periodic points. Then for any $x \in [0, 1]$ the points of the trajectory $\{x, f(x), \dots\}$ which belong to U, form a strictly monotonic (finite or infinite) sequence.*

Proof. Assume that U is "right", $|\{x, f(x), \dots\} \cap U| > 1$, and $m < n$ are the least with $f^m(x), f^n(x) \in U$. By Lemma 8.1 $f^m(x) < f^n(x)$, and for any $k > n$ with $f^k(x) \in U$ we get $f^n(x) < f^k(x)$. Repeating the argument proves the lemma. ■

COROLLARY 9.2. *Let U be an interval containing no periodic points. If $x \in \Omega(f) \cap U$ then x never comes back into U.*

Proof. Assume that U is a "right" interval. Let $x \in \Omega(f) \cap U$ and contrary to the claim there exists n such that $f^n(x) \in U$. Then $x < f^n(x)$ and we can choose a small neighborhood V of x such that $v < f^n(V), V \cap f^n(V) = \emptyset$. Clearly this implies that for all $m > n$ we have $f^m(V) \cap V = \emptyset$ and x is wandering, a contradiction. ■

THEOREM 9.3. *For an interval map f we have $\Omega(f|\Omega(f)) = \overline{\mathrm{Per}(f)} = C(f)$ and so the depth of an interval map cannot be greater than 2.*

Proof. By Corollary 9.2 if $x \in \Omega(f) \cap U$ where U is complementary to $\overline{\mathrm{Per}(f)}$ then x never comes back into U. Therefore, $U \cap \Omega(f)$ is wandering in $\Omega(f)$ and hence contains no points of $\Omega(f|\Omega(f))$. On the other hand, $\overline{\mathrm{Per}(f)} \subset \Omega(f|\Omega(f))$ as was proved in the previous lecture. Therefore, indeed $\Omega(f|\Omega(f)) = \overline{\mathrm{Per}(f)} = C(f)$ as desired. ■

PROPOSITION 9.4. *Let U be an interval containing no periodic points. Then $U \cap \omega(x)$ consists of at most one point for any x.*

Proof. Follows immediately from Lemma 9.1. ■

Now we show that the set $\Omega(f|\Omega(f)) \cap U$ is at most countable. Let us denote the set $\bigcup_{i=0}^{\infty} f^i(A)$ by $\mathrm{orb}_f(A) = \mathrm{orb}(A)$. The claims in Lemma 9.5 are for students to establish on their own and must be proven in order to understand the lemma.

LEMMA 9.5. *Let U be an interval containing no periodic points, $x \in \Omega(f) \cap U$. Then there exists a wandering semi-neighborhood of x.*

Proof. Without loss of generality assume that $U = (a, b)$ is a "right" interval and show that there exists an $\varepsilon > 0$ such that $I = (x, x + \varepsilon)$ is wandering (i.e., $f^n(I) \cap I = \emptyset$ for any $n > 0$). By way of contradiction assume that this is false.

CLAIM 9.6. Show that under the assumption $x \in \overline{\mathrm{orb}(f(I))}$ for every $\varepsilon > 0$. □

Choose a very small $\varepsilon > 0$ and consider $A = \mathrm{orb}(f(I))$.

CLAIM 9.7. Show that there exists $n > 0$ such that $f^n(f(I)) \cap f(I) \neq \emptyset$. $\square$

The set A is a countable union of intervals. By Exercise 9.7 there exists $n > 0$ such that $f^n(f(I))$ is not disjoint from $f(I)$. Denote $f(I)$ by M and consider $\mathrm{orb}_{f^n}(M) = A'_0$; in this union of intervals two consecutive intervals are non-disjoint, hence A'_0 is an interval of some kind. Then $A'_1 = f(A'_0) = f(\mathrm{orb}_{f^n}(M)) = \mathrm{orb}_{f^n}(f(M))$ is therefore an interval itself, and so are its further images $A'_2 = f^2(A'_0), \dots, f^{n-1}(A'_0) = A'_{n-1}$. It is clear that $f^n(A'_0) = \mathrm{orb}_{f^n}(f^n(M)) \subset A'_0$, and so intervals $A'_0, \dots, A'_{n-1}$ are cyclically permuted by f.

EXERCISE 9.8. Show that connected components of A are cyclically permuted by f. $\square$

Since U is a "right" interval, points of I can be mapped into U only to the right of x. Hence, $(a, x) \cap A = \emptyset$. By Claim 9.6, $x \in \overline{A}$, hence there exists a unique component B of A with $x \in \overline{B}$. By Lemma 8.1 $x \notin A$, hence $x \notin B$. Thus, B is an interval with the left endpoint $x \notin B$. Since by Claim 9.8 components of B are cyclically permuted by f, we can find the least N such that $B, f(B), \dots, f^{N-1}(B)$ are disjoint while $f^N(B) \subset B$.

Let us show that then $x \in f^N(\overline{B})$. Indeed, otherwise there exists a small $\alpha > 0$ such that $L = [x, x+\alpha] \subset I \cap B$ is disjoint from $f^N(\overline{B})$. Since other components of A are disjoint from B we see that L is disjoint from A. On the other hand, $\mathrm{orb}(f(L)) \subset \mathrm{orb}(f(I)) = A$ which implies that L is wandering, a contradiction.

Thus, there exists $y \in \overline{B}$ such that $f^N(y) = x$. Since x is not periodic, $y \neq x$. In fact, y cannot belong to B. Indeed, if so then y belongs to some iterate of I and hence there are points of I which are eventually mapped into x. However this is impossible since U is a "right" interval. So, $y \notin B$ and hence y is the right endpoint of $B = (x, y)$.

We showed that y is the unique f^N-preimage of x in $\bar{B}$. Assume that $y \notin \overline{f^N(B)}$. Then $x \notin \overline{f^{2N}(B)}$. Choose $\alpha' > 0$ so that $L' = [x, x+\alpha'] \subset I \cap B$ is disjoint from $\overline{f^{2N}(B)}$. Then as before L' is wandering, a contradiction. Thus, $y \in \overline{f^N(B)}$. If $y = f^N(x)$ then x is periodic, a contradiction. So, y has preimages inside B which again means that some points of I are eventually mapped into x, a contradiction which completes the proof. ■

THEOREM 9.9. *The set $\Omega(f) \setminus \overline{\mathrm{Per}(f)}$ is at most countable.*

Proof. (suggested by J. Malaugh) Consider a complementary to $\overline{\mathrm{Per}(f)}$ "right" interval U and associate semi-neighborhoods to points $x \in \Omega(f) \cap U$ as in Lemma 9.5. If V and V' are such neighborhoods and $V \cap V' \neq \emptyset$ then one of them contains the left endpoint of the other, a contradiction with the fact that they are wandering as sets while their left endpoints belong to $\Omega(f)$. Hence, all such neighborhoods are pairwise disjoint which implies that there are no more than countably many of them and completes the proof. ■

10 Transitive Maps 1

Under some assumptions we have $\Omega(f) = X$. E.g., define an n-saw map $h : [0,1] \to [0,1]$ so that points $0, 1/n, \dots, 1$ are mapped alternatively into 0 or 1 starting with $h(0) = 0$ while in-between these points the map is defined a linear.

LEMMA 10.1. *For any saw map f we have $\Omega(f) = [0,1]$.*

Proof. Assume that f is an m-saw map. Consider any interval I and show that its image is the entire $[0,1]$. Indeed, for any interval J the length of its image $|f(J)| = m|J|$ if J contains no turning points of f. Since this cannot go on forever there is n such that $f^n(I)$ contains a turning point of f and hence f^{n+1} contains 0 or 1. As follows from the definition, 0 is always fixed, and 1 is either fixed or mapped onto 0. Hence, $f^{n+2}(I)$ contains a fixed endpoint of $[0,1]$, and without loss of generality we can assume that $0 = f(0) \in f^{n+2}(I)$. The interval f^{n+2} is then expanded by f until its image covers $1/m$, and then the next image is $[0,1]$ as desired.

This implies that there is a periodic point inside any interval. So periodic points are dense and hence $\Omega(f) = [0,1]$. ∎

A map $f : X \to X$ is *topologically exact* if for any open U there is n such that $f^n(U) = X$.

CLAIM 10.2. *If $f : X \to X$ is topologically exact then $\Omega(f) = X$.*

Proof. The claim follows from the fact that for f there are no wandering open sets. ∎

We say that $f : X \to X$ is *transitive* if there exists $x \in X$ such that $\omega(x) = X$ (in which case we say that x has a dense orbit in X). The existence of a dense in X orbit is an important property. It can be characterized also in a different way. We are going to use in the proof of equivalence of various characterizations the *Baire category method.* Any compact metric space is complete, that is every Cauchy sequence is convergent. In such a space the Baire Theorem holds: the intersection of a countable family of open dense sets is dense.

The intersection of a countable family of open sets is called a *G_δ-set.* The sets containing dense G_δ-sets are in some sense large. They are called *residual* sets (they complements are called the *sets of first category*). Thus, every open dense set is residual, and by Baire Theorem the intersection of a countable number of residual sets is residual. Paradoxically, often it is much easier to prove that a set is residual than just that it is nonempty.

THEOREM 10.3. *Let $f : X \to X$ be a continuous map of a nonempty compact metric space X into itself. Then the following properties are equivalent.*

(a) f is transitive.

(b) The set of points with the dense orbit is residual.

(c) For every open sets $U, V \subset X$ there exists $n \geq 0$ with $f^n(U) \cap V \neq \emptyset$.

(d) For every open sets $U, V \subset X$ and $m \geq 0$ there exists $n \geq m$ with $f^n(U) \cap V \neq \emptyset$.

Proof. Clearly, (b) implies (a) and (d) implies (c). We will show that (a) implies (d), (c) implies (d), and (d) implies (b). Then it will follow that all four conditions are equivalent.

Let us assume (a) and prove (d). Let x be a point with the dense orbit, let $U, V \subset X$ be nonempty open sets, and let $m \geq 0$. There exists k such that $f^k(x) \in U$ and $l \geq k + m$ such that $f^l(x) \in V$. Then $f^n(U) \cap V \neq \emptyset$ for $n = l - k \geq m$.

Now let us assume (c) and prove (d). We start by proving that f is a surjection (that is, maps X *onto* itself). Suppose that $f(X) \neq X$. Take a point $x \in X \setminus f(X)$ and a point $y \in f(X)$. Since $x \neq y$ we can find their neighborhoods $U \ni x$ and $V \ni y$ which are disjoint so that $U \cap V = \emptyset$. Since $f(X)$ is compact, the set $X \setminus f(X)$ is open, and U may be assumed to be contained in $X \setminus f(X)$. Since $f^n(V) \subset f(X)$ for every $n > 0$ we conclude that $f^n(V) \cap U = \emptyset$ for every $n > 0$, and so in fact $f^m(V) \cap U = \emptyset$ for any $m \geq 0$, a contradiction.

Let $U, V \subset X$ be nonempty open sets and let $m \geq 0$. The set $W = f^{-m}(V)$ is open, and since f is a surjection, it is nonempty. By (c), there exists $n \geq 0$ such that $f^n(U) \cap W \neq \emptyset$. Then $f^{n+m}(U) \cap V \neq \emptyset$. This proves (d).

Finally, let us assume (d) and prove (b). The space X has a countable open base (U_i). That is, there exists a sequence of open sets U_i such that for any open set V one can find an open set $U_j \subset V$. Indeed, pick a sequence of finite covers of X by $1/n$-balls and set U_i as the sequence of elements of these covers. If V is open then we can find a ball $B(y, \epsilon)$ inside V. Choose n so that $2/n < \epsilon$ and then a ball U_i of radius $1/n$ containing y. Obviously, $U_i \subset V$.

Let $A_{i,j}$ be the set of those points $x \in X$ for which there exists $n \geq j$ such that $f^n(x) \in U_i$. In other words, $A_{i,j} = \bigcup_{n=j}^{\infty} f^{-n}(U_i)$. Since f is continuous, $A_{i,j}$ is open. We will show that it is dense. Indeed, if $A_{i,j}$ is not dense then there exists an open nonempty set V disjoint from it. That is, $f^n(V) \cap U_i = \emptyset$ for every $n \geq j$. However, this contradicts (d). Therefore $A_{i,j}$ is dense for all i, j. By the Baire Theorem, the set $A = \bigcap_{i,j} A_{i,j}$ is residual. If $x \in A$ then the orbit of x passes through each U_i infinitely often. This means that this orbit is dense. Hence, (b) is proved. ■

From Theorem 10.3 it follows immediately that for a transitive map the set of non-wandering points is the whole space.

11 Transitive Maps 2

The identity on any space consisting of more than one point is a simple example of a map where the set of non-wandering points is the whole space but that is not transitive.

Clearly, by Theorem 10.3 all topologically exact maps are transitive. The opposite is not necessarily true as follows from the following example.

EXAMPLE 11.1. *Consider the map $f : [0, 1] \to [0, 1]$ defined as follows:*

(1) two double-sided sequences $\dots l_{-1} < l_0 < l_1 < \dots$ and $\dots r_{-1} < r_0 < r_1 < \dots$ are chosen so that $\lim_{n\to-\infty} a_n, b_n = 0$ and $\lim n \to \infty = 1$ and the points of the sequences alternate (i.e. $l_i < r_i < l_{i+1}$ for any i);

(2) the map f is defined on $L = \{l_i\}$ as follows: $f(l_i) = l_{i-1}$;

(3) the map f defined on $R = \{r_i\}$ as follows: $f(r_i) = r_{i+1}$;

(4) the map f at $0, 1$ is defined as follows: $f(0) = 0, f(1) = 1$;

(5) the map f is defined linearly on every interval complementary to $R \cup L \cup 0, 1$.

Show that f is transitive but not topologically exact.

Proof. It is easy to see that by the definition $0, 1$ do not have preimages inside $(0, 1)$. Hence the map f cannot be topologically exact.

To show that it is transitive by Theorem 10.3 we need to prove that for any interval I its orbit is dense in $[0, 1]$. Observe that by the definition f is expanding on every interval of monotonicity (has the slope with absolute value greater than 1). Now, let us first prove that there are two images of I, say, $f^n(I)$ and $f^m(I)$ which are not disjoint. To this end we study the length of the iterates of I depending on whether they intersect $L \cup R$ or not.

Consider two cases. First, assume that iterates of I never intersect $L \cup R$. Then since $L \cup R$ are exactly the turning points of f and iterates of I do not cover any of them we see that the length of images of I grows. Thus, we get a sequence of intervals with lengths bounded from below by $|I|$. Clearly, some of them must intersect, and the claim is proven.

Now, suppose that for some k we have $f^k(I) \cap (L \cup R) \neq \emptyset$. Without loss of generality we can assume that $r_0 \in f^k(I)$ and there exists a non-degenerate interval $J \subset I$ with the right endpoint r_0. If we can choose J so that it covers l_0 then its image will intersect itself and hence I will have two non-disjoint images. If J cannot be chosen so that it contains l_0, let us consider its forward images and show that there exists i such that $l_i, r_i \in f^i(I)$. Indeed, otherwise $|I| < |f(I)| < \dots$, so if i is chosen in such a way that $|r_i - l_i| < |I|$ we see that I_i (whose right endpoint is r_i) must stretch from r_i all the way to the left to cover l_i as desired. However in this case $f^{i+1}(I) \cap f^i(I) \neq \emptyset$ and the claim is proven.

Now, if $f^n(I)$ and $f^m(I)$ intersect and $n > m$ then we can denote $f^m(I)$ by J and then we will have that $f^{n-m}(J) \cap J \neq \emptyset$. Clearly, we can now define $A = \operatorname{orb} J$ and apply Exercise 9.7 to this set A. By this exercise, A is a collection of pairwise disjoint intervals cyclically permuted by f. However, if the components of A do not cover points from $L \cup R$ then the length of a component grows as we apply f which makes it impossible to cyclically permute these components, a contradiction. Thus, for the sake of definiteness we may assume that A contains a point of L.

The points of L converge to 0, hence $0 \in \overline{A}$. Since A is a union of intervals, then so is $\overline{A}$, and we conclude that there is a component B of A such that 0 is the left endpoint of $\overline{B}$. Since for some n (which we might call the period of A) we have that $f^n(B) \subset B$ we conclude that $f^n(\overline{B}) \subset \overline{B}$. Let us show that this implies that $\overline{B} = [0, 1]$. Indeed, choose a point $x \in R \cap B$ (clearly such points exists). Then $x < f^n(x) < \dots$ and $\lim_{j\to\infty} f^{nj}(x) = 1$. This implies that $\overline{B} = [0, 1]$ and therefore A - and the orbit of I - is dense in $[0, 1]$. By Theorem 10.3, the map f is topologically exact as desired. ■

In the example below we consider unimodal maps (interval maps with one turning point).

EXAMPLE 11.2. *If $f : [0, 1] \to [0, 1]$ is a unimodal transitive map then either f is topologically exact or there is a fixed point $d \in [0, 1]$ with $f[0, d] = [d, 1], f[d, 1] = [0, d]$.*

Proof. There is a unique turning point $c \in [0, 1]$. We may assume that $f|[0, c]$ is increasing and $f|[c, 1]$ is decreasing. By Proposition 6.4 f is surjective, so $f(c) = 1$. Let us show that there always exists a fixed point $d \in [c, 1]$ in unimodal case. Indeed, since $1 = f(c) \in f([c, 1])$ then either $f([c, 1]) \subset [c, 1]$ or $f([c, 1]) \supset [c, 1]$, and either way a fixed point $d \in [c, 1]$ must exist.

Let us show now that if f is transitive then $f(1) = 0$. Indeed, it is easy to see that if $f(1) > 0$ then $[f(1), 1]$ is invariant. However, this cannot happen because f is transitive, so $f(1) = 0$. Therefore, $f[c, 1] = [0, 1]$. We can repeat this construction pulling $[c, 1]$ back into itself and thus getting smaller and smaller intervals $I_0 \supset I_1 \supset \dots$ around d which are mapped forward and simultaneously "flipped" around d so that $f^(I_k) = I_{k-1}, \dots$. Let us show that this nested sequence of intervals narrows down onto d. Indeed, otherwise the intersection $\cap I_k$ is a non-trivial interval mapped inside itself, a contradiction with transitivity. So, any neighborhood of d eventually covers $[0, 1]$.

EXERCISE 11.3. Finish up the consideration of Example 11.2. □

In fact, in the general interval piecewise-monotone case with finitely many pieces topological transitivity is equivalent to topological exactness, but we will not prove it here. ■

12 Minimal Maps and Transitive Maps

Another example of transitive maps which are not topologically exact is given by circle rotations. Indeed, it is clear that no circle rotation is topologically exact since under any rotation the length of an arc does not change. The full description of transitive circle rotations is given in the following lemma.

LEMMA 12.1. *A circle rotation is transitive if and only if it is an irrational rotation.*

Proof. Indeed, if f is a rational rotation then it is a periodic map. Let us show that in general a periodic map $f : X \to X$ of period p is transitive if and only if X is a periodic orbit. Suppose that X is not a periodic orbit. Then we can find two points x, y such that their (periodic) orbits are disjoint. Moreover, because of continuity we can pick such small balls U, V around x, y respectively that $f^i(U) \cap f^j(V) = \emptyset$ for any $0 \le i, j \le p - 1$. Since $f^p(U) = U, f^p(V) = V$ because of periodicity of f we conclude that in fact $f^n(U) \cap V = \emptyset$ for any n and thus by Theorem 10.3 f is not transitive. In particular, any periodic map of an infinite compact metric space X is not transitive which proves our claim.

Suppose that f is an irrational rotation of the circle and show that for any arc I on the circle there exists n such that $\cup_{i=0}^n f^i(I) = S^1$. Indeed, we may assume that $I = [0, \varepsilon]$. Let us first prove that for some $k \ge 0, l > 0$ we have $f^k(I) \cap f^{k+l}I \ne \emptyset$. Indeed, the length of $f^i(I)$ is the same for all i which proves the claim immediately. Moreover, $f^k(I) \ne f^{k+l}I$ because of the fact that the angle of rotation is irrational. Thus, $f^{k+l}I$ is simply a shift of $f^k(I)$ by a finite distance, say, counterclockwise, which is not disjoint from $f^k(I)$. Clearly, then $f^{k+2l}(I)$ is the shift of $f^{k+l}(I)$ by the same distance in the same direction etc, so after a finite number of steps the union of those shift covers S^1 and we are done with the claim. By Theorem 10.3 then f is transitive which completes the proof. ∎

In fact an orientation preserving circle homeomorphism is transitive iff it is conjugate to an irrational rotation. This is the result of the end of the 19th century which is due to Poincaré and one can safely say that the modern theory of dynamical systems began then.

Another property similar to transitivity but much stronger is *minimality.* A continuous map $f : X \to X$ is *minimal* if every orbit is dense. The terminology can be explained by the following

LEMMA 12.2. *The map $f : X \to X$ is minimal if and only if the only closed invariant subset of X is the space X itself.*

Proof. Let $f : X \to X$ be minimal. If a set $A \subset X$ is invariant then it must contain at least one orbit which by the assumption is dense. If on the other hand A is closed then $A = X$.

On the other hand, assume that the only closed invariant set of f is X. Suppose that still there exists a point x whose orbit is not dense. This implies that $\omega(x) \neq X$. Clearly, $\omega(x)$ is closed and invariant, a contradiction. ∎

Now, if a map $f : X \to X$ has a closed invariant subset A such that the restriction $f|_A$ is minimal then the set A is also called *minimal*. Using a result from the set theory (namely, using Zorn's Lemma) one can prove that any continuous $f : X \to X$ has at least one minimal set. Namely, consider the family $\mathbf{T}$ of all closed invariant sets A of f. Then for any ordered by inclusion subfamily $\mathbf{T}'$ of $\mathbf{T}$ the intersection of all its sets is a non-empty (follows from compactness) closed invariant set. In other words, for any ordered subfamily $\mathbf{T}'$ of T there exists an element of the family $\mathbf{T}$ which is smaller by inclusion than all the elements of $\mathbf{T}'$. Then by Zorn's Lemma it follows that there exists the smallest element of $\mathbf{T}$ which is non-empty.

Clearly, a minimal map is transitive. However this is a much more narrow class of maps. Indeed, a minimal map of an infinite space has no periodic points. Thus standard examples, like the tent map or shifts, are not minimal - in these cases the periodic points are dense in the space. Examples of minimal maps may come from the theory of compact groups: any transitive map of a compact group is minimal. Our examples will be closely related to this.

LEMMA 12.3. *Let $f = R_\alpha$ be an irrational rotation on the circle. Then f is minimal.*

Proof. We use the construction from Lemma 12.1. Indeed, suppose that there exists a point $x \in S^1$ with $\omega(x) \neq S^1$. Then we can find U disjoint from $\omega(x)$. Moreover, choose a smaller V such that $\bar{V} \subset U$. If x enters V infinitely many times then it must have limit points in $\bar{V}$ which is impossible. Hence x enters V finitely many times and there is a subarc W of V such that W is disjoint from $\operatorname{orb} x$. Now, consider the rotation $g = R_{-\alpha}$ by the angle $-\alpha$. Then the finite union of g-images of I covers the entire S^1, a contradiction with the fact that W is disjoint from $\operatorname{orb} x$. ∎

In fact the class of irrational rotations of the circle plays an extremely important role in dynamical systems. To explain main relevant results here we need to go back to Lecture 4 where the notion of semiconjugacy was introduced. Namely, let X and Y be metric spaces and let $f : X \to X$ and $g : Y \to Y$ be continuous maps. If there is an onto map $h : X \to X$ with $h \circ f = g \circ h$ then we say that f and g are *semiconjugate* (by the map h).

It was Poincaré who first proved the following remarkable result.

THEOREM 12.4. *Let $f : S^1 \to S^1$ be a homeomorphism. If f has no periodic points then it is semiconjugate by means of a monotone map h to a unique irrational rotation of the circle by the angle called then* rotation angle of f.

13 Other Properties of Minimal Maps

In what follows we will need the following properties of limit sets which follows immediately from the fact that $\omega(x)$ is closed and invariant.

PROPERTY 13.1. *If $y \in \omega(x)$ then $\omega(y) \subset \omega(x)$.*

We now consider a few examples of minimal maps. We need the following definition: a map $f : X \to X$ is called an *isometry* if $d(f(x), f(y)) = d(x, y)$ for any x, y. Clearly, an isometry is a homeomorphism on the image.

LEMMA 13.2. *Suppose that $f : X \to X$ is a transitive isometry. Then f is minimal.*

Proof. We need to show that for any point y we have $\omega(y) = X$. Indeed, suppose that x is such that $\omega(x) = X$. Then there is a sequence n_k such that $f^{n_k}(x) \to y$. On the other hand there is a sequence m_k such that $f^{m_k}(x) \to x$, and we may assume that $m_k - n_k \to \infty$. Since f is an isometry we conclude that $d(f^{m_k-n_k}(f^{n_k}(x)), f^{m_k-n_k}(y)) = d(f^{m_k}(x), f^{m_k-n_k}y) \to 0$. Since $d(f^{m_k}(x), x)) \to 0$ we see that $f^{m_k-n_k}(y) \to x$ and hence $x \in \omega(y)$. By Property 13.1 this implies that $\omega(y) \supset \omega(x)$. Since $\omega(x) = X$ we conclude that $\omega(y) = X$. ∎

A map $f : X \to X$ is called *uniformly equicontinuous* if for any ε there exists δ such that if $d(x, y) < \delta$ then $d(f^n(x), f^n(y)) < \varepsilon$ for any n. Our main aim is to prove the following lemma which generalizes Lemma 13.2.

LEMMA 13.3. *A transitive uniformly equicontinuous map $f : X \to X$ is minimal.*

Proof. We need to show that for any point y we have $\omega(y) = X$. Indeed, suppose that x is such that $\omega(x) = X$. Then there is a sequence n_k such that $f^{n_k}(x) \to y$. On the other hand there is a sequence m_k such that $f^{m_k}(x) \to x$, and we may assume that $m_k - n_k \to \infty$. So, we have $f^{m_k-n_k}(f^{n_k}(x)) \to x$ and $f^{n_k}(x) \to y$. Since f is uniformly equicontinuous for any $\varepsilon > 0$ we can find δ such that $d(u, v) < \delta$ implies $d(f^m(u), f^m(v)) < \varepsilon$ for any m. Thus from some time on (namely, from the moment when $d(f^{n_k}(x), y) < \delta$) we get $d(f^{m_k-n_k}(f^{n_k}(x)), f^{m_k-n_k}(y)) < \varepsilon$. Since $f^{m_k}(x) \to x$ we see that from some time on (namely, from the time when $d(f^{m_k}(x), x) < \varepsilon$ we have $d(f^{m_k-n_k}(y), x) < 2\varepsilon$. Thus, $f^{m_k-n_k}(y) \to x$. ∎

In fact, we can now study isometries and even uniformly equicontinuous maps without assumptions about their transitivity. In other words, we can fully describe the dynamics of such maps. To begin with, let us consider isometries (uniformly equicontinuous maps can be dealt with the same way).

LEMMA 13.4. *Consider an isometry $f : X \to X$ of a metric compact space X into itself. Then all points of X are recurrent (i.e., $x \in \omega(x)$ for any $x \in X$).*

Proof. Consider $x \in \omega(x)$ and assume that $x \notin \omega(x)$. We can define the distance $d(x, \omega(x))$ as the infimum of all distances between x and points of $\omega(x)$; since all sets are compact there exists a point $y \in \omega(x)$ such that for any $z \in \omega(x)$ we have $d(x, z) \geq d(x, y) = d(x, \omega(x))$. Now, it is easy to show that the fact that f is an isometry implies that $d(x, \omega(x)) = d(f(x), \omega(x))$. Indeed, since $d(x, y) = d(f(x), f(y))$ then $d(x, \omega(x)) \geq d(f(x), \omega(x))$. On the other hand, for every $z \in \omega(x))$ there exists $z' \in \omega(x)$ such that $f(z') = z$. If z is such that $d(f(x), \omega(x)) = d(f(x), z)$ then picking the appropriate z' in this case we get that $d(f(x), \omega(x)) = d(x, z') \geq d(x, \omega(x))$. Thus, $d(x, f(x)) = d(f(x), \omega(x)$. Since $d(f^n(x), \omega(x)) \to 0$ we see that $d(x, \omega(x)) = 0$ as desired. ■

These lemmas deal with minimal maps in a specific situation when the map is an isometry or is uniformly equicontinuous. There are more general properties of minimal maps which we will discuss now. In fact these are more traditional properties associated with minimal maps. They will help us describe isometries of compact spaces in full.

LEMMA 13.5. *Two minimal sets of $f : X \to X$ are either disjoint or coincide.*

Proof. Indeed, if $A \cap B \neq \emptyset$ then for any $x \in A \cap B$ we have $\omega(x) = A = B$. ■

Lemma 13.5 implies that if f is isometry then the entire space is partitioned into pairwise disjoint minimal sets which actually can be considered as a full description of dynamics of f.

LEMMA 13.6. *A space with isolated points which admits transitive maps must be finite with all such maps being cyclic permutations.*

Proof. If x is a point and $\{x\}$ is an open set then x must have a dense orbit and must also be mapped onto itself by some power of the map as desired. ■

LEMMA 13.7. *If $f : [0, 1] \to [0, 1]$ is continuous and A is an infinite minimal set of f then A is a Cantor set.*

Proof. By Lemma 13.6 A has no isolated points. Suppose it has a non-degenerate component I. Then by minimality for some n we have $f^n(I) \subset I$ which implies that there is a periodic point $x \in I$, contradiction with the minimality of A. ■

Later on we will give a simple example of a minimal map of a classic 'middle-third' Cantor set C_0 into itself which can be transformed into a minimal map of any Cantor set $C \subset [0, 1]$ and then extended to a continuous map of $[0, 1]$ linearly on complementary intervals.

14 Strongly recurrent points

A point is recurrent if it comes back arbitrarily close to itself. However in our definition we do not specify how often it should be coming close to itself, nor do we say how the moments when the point comes, say, ε-close to itself, relate to the moments when the point comes $\varepsilon/2$-close to itself etc. It turns out that the frequency with which a recurrent point comes close to itself is related to the fact that the limit set of the point is minimal or not.

A point $x \in X$ is said to be *strongly recurrent* if for every open set U containing x there exists a positive integer $N = N(U)$ such that if $f^m(x) \in U$ with $m \geq 0$ then $f^{m+k}(x) \in U$ for some $0 < k \leq N$. Thus a strongly recurrent point is one which is recurrent with 'bounded return times'. The close connection between strong recurrence and minimal sets was established by Birkhoff in 1912. The following result is essentially due to him.

Theorem 14.1. *If M is a minimal set, then any point $x \in M$ is strongly recurrent. Conversely, if x is strongly recurrent then its limit set is minimal.*

Proof. Let M be a minimal set and assume that $x \in M$ is not strongly recurrent. Then there exists an open neighborhood U of x such that, for some increasing sequence (n_k) of positive integers, $f^{n_k})x_{\in}U$ but $f^n(x) \notin U$ for $n = n_k+1, n_k+2, \ldots, n_k+k$. Moreover, we may suppose that $f^{n_k}(x) \to y$ where $y \in \overline{U} \cap M$. Since $f^m(y) \in U$ for some m (remember that M is minimal and U contains points of M, e.g. x) then we can find a neighborhood V of y such that $f^m(V) \subset U$. In other words, all points close to y must come back into U under f^m. However points $f^{n_k})x)$ stay away of U for longer and longer periods of time. Together with the fact that $f^{n_k}(x) \to y$ this yields a contradiction. More precisely, choose $k > m$ so that $f^{n_k}(x) \in V$, then $f^{n_k+m}(x)$ both belongs to U and does not belong to U, a contradiction.

In fact, we can prove a bit more, namely we can prove that if for any open U there exists a number $N = N(U)$ such that in any N consecutive iterations of any point of M at least one belongs to U. Let us give an alternative proof of this fact. Indeed, since M is minimal, all points of M enter U at some moment. Choose for every $y \in M$ its neighborhood V such that for some m we have $f^m(V) \subset U$. Then we will get an open cover of M and we can choose its finite subcover. Let $N = N(U)$ be the maximum of the powers m involved in this construction. Then any point of M belong to one of those neighborhoods V, hence any point of M enters U under some f^i with $i \leq N$ as desired.

Let us now prove the converse statement: if x is strongly recurrent then $\omega(x) = A$ is minimal. Indeed, suppose not. Then there exists a point $y \in A$ such that $\omega(y) \neq A$. In particular, $x \notin \omega(y)$. Since $y \in A$ we can find a point $f^n(x)$ which is very close to y

and therefore a very long segment of the orbit of x which follows $f^n(x)$ will stay close to the appropriate images of y and hence far away from x. Clearly, this contradicts the assumption about strong recurrence of x and completes the proof. ∎

COROLLARY 14.2. *A point x is strongly recurrent if and only if for any $y \in \omega(x)$ we have that $x \in \omega(y)$.*

Let us go back to Theorem 14.1, namely to its second part where we prove that if a point is strongly recurrent then its limit set is minimal. It turns out that this claim holds for non-compact metric spaces as well. Indeed, if x is strongly recurrent but there exists $y \in \omega(x)$ such that $x \notin \omega(y)$ then we can find arbitrarily long segments of the orbit of x which stay away of a sufficiently small neighborhood of x, a contradiction. It turns out that indeed the property of minimality and strong recurrence show up in other fields in Math where we do not necessarily deal with compact metric spaces, and here is one of them.

A set $E \subset Z$ is said to be *relatively dense* if for any $\varepsilon > 0$ there exists a number $l > 0$ such that in any interval $[a, a+l]$ there exists at least one element of the set E. Given a function $Z \to X$ where X is a complete metric space, a number t is said to be an ε-almost period if $\sup d(f(n+t), f(n)) \leq \varepsilon$. If for every ε the set of the ε-almost periods of f is relatively dense then the function f is said to be *almost periodic.*

CLAIM 14.3. *A function is almost periodic if and only if it is a strongly recurrent point of the shift map acting on the metric space of all functions with* sup-*metric.*

Proof. If f is strongly recurrent in the above introduced space then for a given ε it comes back to its ε-neighborhood with bounded time gaps. However, the fact that some shift of the function is ε-apart from it means exactly that $\sup d(f(n+t), f(t)) \leq \varepsilon$. Thus, the set of ε-almost periods is relatively dense. The opposite direction can be proven just as easily. ∎

It turns out however that even though compactness is not directly involved in the definition it is related to almost periodic functions in an essential way.

CLAIM 14.4. *If f is almost periodic then the set $A = \overline{f(Z)}$ is compact.*

Proof. Since A is closed it is enough to show that it has finite ε-nets for any $\varepsilon > 0$. Choose big enough N which corresponds to the "frequency" of ε-almost periods of f and consider all points $f(i)$ with $-N \leq i \leq N$. Then given j we can always approximate $f(j)$ by $f(i), -N \leq j \leq N$ because the numbers t such that $d(f(n), f(n-t)) \leq \varepsilon\, \forall n$ can be found in any interval of the length N, in particular inside $[i-N, i+N]$. For such t we have $d(f(i), f(i-t)) \leq \varepsilon$ while $-N \leq i-t \leq N$. ∎

15 Minimal Sets and Groups 1

As we saw before, minimal sets can occur in some non-compact cases. However we also saw that there is some connection to compactness in the case of almost periodic functions. It turns out that this connection can be easily established under more general circumstances, namely for isometries of complete metric spaces X.

Consider isometries in more detail. Now we do not make an assumption about the compact properties of the complete metric space X. Still we can define the limit sets of points and study them. Unlike in the compact case, the limit set of a point x now can be empty. Then there is nothing to study, so we will now consider non-empty limit sets.

THEOREM 15.1. *Let $f : X \to X$ be an isometry. If x is such that $\omega(x) \neq \emptyset$ then x is recurrent and $\omega(x)$ is minimal.*

Proof. Let $y \in \omega(x)$ and $f^{n_k}(x) \to y$. We may assume that the sequence n_k is chosen in such a way that $n_{k+1} - n_k \to \infty$. Now, $d(f^{n_{k+1}}(x), f^{n_k}(x)) \to 0$. Then since f is an isometry we conclude that $d(f^{n_{k+1}-n_k}(x), x) \to 0$ and so x is recurrent.

To show that the set $\omega(x)$ is minimal we need to show that it contains no smaller invariant closed subsets; equivalently, we need to show that for any $y \in \omega(x)$ we have that $x \in \omega(y)$. Indeed, we can simply mimic the arguments from Lemma 13.2 here. Suppose that $f^{n_k}(x) \to y$ while $f^{m_k}(x) \to x$ for another sequence m_k. We can always assume that $m_k - n_k \to \infty$. Together with the fact that f is an isometry this implies that $f^{m_k-n_k}(y) \to x$ because $d(f^{m_k-n_k}(y), x) \leq d(f^{m_k-n_k}(y), f^{m_k-n_k}(f^{n_k}(x)) + d(f^{m_k}(x), x)$. ∎

The fact that the set $\omega(x)$ is minimal in this non-compact case does not imply that x is strongly recurrent. If we go over the proof of Theorem 14.1 we see that the initial points of longer and longer time segments within which the point x stays away from a neighborhood of x may have no accumulation points which is exactly why x may be recurrent with minimal limit set but not strongly recurrent.

CLAIM 15.2. *If x is strongly recurrent then $\omega(x)$ is compact.*

Proof. As before in Claim 14.4 we need to show that $\omega(x)$ has finite ε-nets for any ε. Choose N such that for any k there exists $k-i$ with $i \leq N$ such that $d(x, f^{k-i}(x)) \leq \varepsilon$. Let us show that the point $x, f(x), \ldots, f^N(x)$ form an ε-net in $\omega(x)$. Indeed, for every k let us find $k-i$ such that $d(x, f^{k-i}(x)) \leq \varepsilon, i \leq N$. Apply f^i to this inequality; then by the properties of isometries we have that $d(f^i(x), f^k(x)) \leq \varepsilon$. Thus, every point of the orbit of x is ε-approximated by the points $x, \ldots, f^N(x)$ which implies that these points form an ε-net for $\omega(x)$. ∎

From now on we study transitive isometries of compact metric spaces (or, equivalently by Lemma 13.2, minimal isometries of compact metric spaces - or, equivalently by Theorem 14.2 and Claim 15.2, limit sets of strongly recurrent points in complete metric spaces).

Our next step in studying dynamics of isometries $f : X \to X$ is to introduce a nice additional structure on X. To do so we need a simple observation concerning the distances between points in an f-orbit. Indeed, fix a number i; then $d(x, f^i(x)) = d(f^s(x), f^{s+i}(x))$ for any s. In other words, inside one orbit the distance between points depends only on the difference in powers of f defining those points. In particular, if $d(x, f^i(x))$ is very small then for every s the distance between $f^s(x)$ and $f^{i+s}(x)$ is very small. By continuity this implies that the distance between z and $f^i(z)$ is very small for all $z \in \omega(x)$.

However simple, this observation gives us a nice tool which we will apply to introduce a new structure on the limit sets of points of X. Namely, fix a recurrent point x of an isometry f. By Lemma 13.2 $\omega(x)$ is minimal (observe that in the proof of Lemma 13.2 we do not use the fact that X is compact). Consider two points $y, z \in \omega(x)$. They are the limits of sequences $f^{n_k}(x), f^{m_k}(x)$. Let us consider a new sequence of iterates of f defined as $f^{m_k+n_k}(x)$ and called the *resulting* sequence.

Let us show that the resulting sequence converges. Since the space X is complete it is enough to show that the resulting sequence is Cauchy. So we need to estimate the distance between $f^{n_k+m_k}(x)$ and $f^{n_l+m_l}(x)$ if $k < l$ are very big. We have the following:

$$f^{n_l+m_l}(x) = f^{n_l-n_k}(f^{m_l-m_k}(f^{n_k+m_k}(x))).$$

In other words, the point $f^{n_k+m_k}(x)$ is mapped into the point $f^{n_l+m_l}(x)$ by applying first $f^{m_l-m_k}$ and then $f^{n_l-n_k}$. Since we want to prove that $f^{m_k+n_k}(x)$ and $f^{m_l+n_l}(x)$ are very close it is enough to prove that $f^{m_l-m_k}$ and $f^{n_l-n_k}$ shift points in $\omega(x)$ only a tiny bit. But as follows from the observation made above, all the points of $\omega(x)$ are shifted by the same distance if we apply the same power of f. So we need to find points from $\omega(x)$ which are shifted only a little by $f^{m_l-m_k}$ and by $f^{n_l-n_k}$. This is not very difficult. Indeed, points $f^{m_k}(x)$ and $f^{m_l}(x)$ are very close to y, therefore we get the claim that $f^{m_l-m_k}$ shifts points of $\omega(x)$ only a little. The points $f^{n_k}(x)$ and $f^{n_l}(x)$ are very close to z, therefore we get the claim that $f^{n_l-n_k}$ shifts points of $\omega(x)$ only a little. This implies that indeed $f^{m_k+m_l}(x)$ and $f^{n_k+n_l}(x)$ are very close and so the resulting sequence is indeed a Cauchy sequence.

We would like to introduce the *product* of y and z as follows: $y \cdot z = \lim f^{m_k+n_k}(x)$. Let us check that this is a correct definition. Indeed, any resulting sequence converges by the above. Suppose that two resulting sequences converge to different limits. Then we can mix them and obtain a resulting sequence which does not converge, a contradiction. Denote the limit point of a resulting sequence by ζ; declare that the product of y and z denoted by $y \cdot z$ equals ζ. We study its properties in the next lecture.

16 Minimal Sets and Groups 2

As we showed in the previous lecture the compact $X = \omega(x)$ (where x is a strongly recurrent point) is endowed with the product structure and thus becomes an algebraic object with certain metric properties. In fact it becomes a *compact metric commutative group*. Let us check algebraic properties of X and the operation $\cdot$.

PROPERTY 16.1. $y \cdot z = z \cdot y$.

Proof. Follows from the fact that $f^{n_k+m_k} = f^{m_k+n_k}$ with the appropriate choice of sequences. ∎

PROPERTY 16.2. $(y \cdot z) \cdot \zeta = y \cdot (z \cdot \zeta)$.

Proof. Follows from $f^{(n_k+m_k)+l_k} = f^{n_k+(m_k+l_k)}$ with the appropriate choice of sequences. ∎

PROPERTY 16.3. $x \cdot y = y \cdot x = y$ *for any* $y \in X$.

Proof. Choose $n_k = 0$ as a sequence such that $f^{n_k}(x) \to x$ and apply the definition. ∎

PROPERTY 16.4. *Denote* $f(x)$ *by* z*. Then for any* y *we have* $f(y) = z \cdot y$.

Proof. Suppose that $f^{n_k}(x) \to y$. Then $f(y) = \lim f^{n_k+1}(x)$ which by the definition of the product $\cdot$ equals $z \cdot y$ (we use the fact that if $m_k = 1$ then $f^{m_k}(x) \to z$). ∎

PROPERTY 16.5. *For any* $y \in X$ *there exists a unique* $z \in X$ *such that* $y \cdot z = z \cdot y = x$.

Proof. Let us show that if z exists then it is unique. Indeed, if $y \cdot z = y \cdot z' = x$ then $z' \cdot y \cdot z = z' = z$.

To find z consider sequences $n_k < m_k$ such that $f^{n_k}(x) \to y$, $f^{m_k}(x) \to x$. For $m_k - n_k = l_k$ we have $d(f^{l_k}(y), f^{l_k}(f^{n_k}(x))) = d(y, f^{n_k}) \to 0$ as $k \to \infty$. Since $f^{l_k}(f^{n_k}(x)) = f^{m_k}(x) \to x$ we see that $f^{l_k}(y) \to x$. As we saw before, $d(f^r(x), f^{r+s}(x)) = d(\zeta, f^s(\zeta))$ for every $\zeta \in \omega(x)$. This implies that $d(f^{l_i}(y), f^{l_k}(y)) = d(f^{l_i}(x), f^{l_k}(x))$. This implies that $f^{l_k}(x)$ is a Cauchy sequence and therefore it converges to a point $z \in \omega(x)$. Hence, by the definition $z \cdot y = \lim f^{l_k+n_k}(x) = \lim f^{m_k}(x) = x$ as desired. ∎

This shows that in fact all transitive isometries of compact metric spaces are *transitive group translations*. Indeed, the role of the *neutral* element (sometimes called the *unity* or the *unit* element) is played by x because multiplying by x does not change any element y by Property 16.3. The action of f on X is simply multiplying all elements of X by $y = f(x)$ as follows from Property 16.3. For any z the inverse z^{-1} of z in the sense of the group action exists by Property 16.5. Moreover, the group in question is commutative by Property 16.1.

PROPERTY 16.6. *The group action is isometric, i.e. $d(y, z) = d(a \cdot y, a \cdot z)$ for any elements y, z, a of the group.*

Proof. By the definition we need to pick sequences k_i, l_i, m_i such that $f^{k_i} \to y, f^{l_i} \to z, f^{m_i} \to a$. Then we have that $a \cdot y = \lim f^{k_i+m_i}(x)$ and $a \cdot z = \lim f^{l_i+m_i}(x)$. Since f is an isometry then $d(f^{k_i+m_i}(x), f^{l_i+m_i}(x)) = d(f^{k_i}(x), f^{l_i}(x))$. Therefore taking the limits we conclude that $d(a \cdot y, a \cdot z) = d(y, z)$ as desired. ■

For example, the circle is a compact metric group whose action is isometric: $\alpha \cdot \beta = \alpha + \beta \bmod 2\pi$. Another example is two-dimensional torus which is the product of two circles: $\mathbf{T}^2 = S^1 \times S^1$ with the addition of coordinates as a group action. Moreover, this can be generalized onto n-dimensional tori.

This allows us to look at the map f from a different point of view. Namely, in the group X every point $z \in X$ defines the operation which is the multiplication by z. In fact, this operation is an isometry of X into itself. Thus, we define a map Φ from the space X into the space of all isometries of X or more generally into the space $C(X, X)$ of all continuous maps of X into itself. The products of elements of X are then mapped by φ into the compositions of the corresponding maps. Moreover, the inverse element to z is mapped into the inverse map of $\varphi(z)$.

Let us show that φ keeps the distance. To do so we need to define the distance between maps from $C(X, X)$; as usual, let us do this using the sup-metric. More precisely, for every two maps $F, G \in C(X, X)$ let $d_C(F, G) = \sup_{z \in X}(d(F(z), G(z))$ (since X is compact, d_C is always assumed at some point z').

Now, let us compute the distance between $\varphi(a)$ (which is simply the multiplication by a as a map of X into itself) and $\varphi(b)$ (which has similar meaning). For any z we have that $d(a \cdot z, b \cdot z) = d(a, b)$ because as we have seen in Property 16.6, multiplication by z itself is an isometry. Clearly this implies the desired, and we can say that φ is an isometric isomorphism between X and the appropriate subgroup of $C(X, X)$.

Our next aim is to see if in some cases even with weaker assumptions on the map $f : X \to X$ of a compact metric space X into itself we can make similar conclusions as in the case considered above. Judging from Lemmas 1.2 and 13.3 it is natural to consider transitive uniformly equicontinuous map for this purposes. This is done in Exercise 16.7 below.

EXERCISE 16.7. Let $f : X \to X$ be a transitive uniformly equicontinuous map. Introduce a new metric in X defined as follows: $d'(x, y) = \sup_{n \geq 0}(d(f^n(x), f^n(y))$. Verify that this indeed a metric. Prove that this metric defines the same topology on X as d. Prove that f becomes a transitive isometry in the new metric and that therefore all the above results apply to f. □

17 Recurrent maps of Cantor sets 1

There are several ways in which the previous results can be generalized. As we saw, transitive (or, equivalently, minimal) isometries are simply transitive isometric translations in compact commutative groups. If y is the element of the group X by which we multiply all its elements then since the map is by assumption transitive there is a sequence n_k such that $y^{n_k} \to e$ where e is the unit element of X. Since X can be considered as a family of isometries of X onto itself, this in other words means that $\varphi_y^{n_k} \to id_X$ where id_X is the identity map of X into itself.

We call a continuous map $f : X \to X$ *recurrent* if there exists a sequence n_k such that $f^{n_k} \to id_X$. Clearly, transitive isometries are recurrent. To what extent can we generalize the properties of transitive isometries onto all transitive recurrent maps? Are there any compact sets for which these two classes coincide? We will study these questions below. First we establish a simple property of recurrent maps.

LEMMA 17.1. *Recurrent maps are homeomorphisms.*

Proof. Let a map f be such that for some sequence $\{n_k\}$ we have $f^{n_k} \to \mathrm{id}_X$. Let us show now that must be a homeomorphism. Indeed, if f is not onto then for $z \notin f(X)$ the convergence $f^{n_k}(z) \to z$ is impossible, a contradiction. On the other hand, if $f(y) = f(z) = u$ then $f^{n_k-1}(u)$ must converge to both y and z, which is impossible. So f is a continuous bijection and therefore any recurrent map is a homeomorphism. ∎

In what follows we will need a construction which can be useful in other cases as well. Namely, let $g : X \to X$ be a continuous map of a metric compact space into itself and $x \in X$. Fix m and consider m sets defined as $A_0 = \omega_{f^m}(x), A_1(x) = \omega_{f^m}(f(x)), \dots, A_{m-1} = \omega_{f^m}(f^{m-1}(x))$.

CLAIM 17.2. *In the above situation* $\omega(x) = \cup_{i=0}^{m-1} A_i$ *and* $f(A_i) = A_{i+1} (0 \le i \le m-1), f(A_{m-1}) = A_0$.

Proof. Clearly, $\omega(x) \supset \cup_{i=0}^{m-1} A_i$. On the other hand, if $y \in \omega(x)$ then we can always choose a subsequence $f^{n_k}(x) \to y$ so that all n_k have the same remainder modulo m. If this remainder is r then we can write that $f^{ms_k+r}(x) \to y$ for some sequence s_k. By the definition it follows that $y \in A_r$.

Now, if $y \in A_i$ then $f^{ms_k+i}(x) \to y$ for some sequence s_k. Then $f^{ms_k+s+1}(x) \to f(y)$ and so $f(y) \in A_{i+1}$. Thus, $f(A_i) \subset A_{i+1}$; the rest is left to the reader. ∎

The construction can be extended. Consider the set of *vertices* $V_0, \dots, V_{m-1}$ which correspond to sets $A_0, \dots, A_{m-1}$. If two sets intersect, we connect the corresponding vertices with a non-oriented edge. This produces a *graph* G. Now, call a subset of G connected if there is a path from any vertex of any vertex of this subset. Clearly, the graph G can be divided into maximal connected subsets denoted by $B_0, \dots, B_{k-1}$. This allows us to define the unions of sets $A_0, \dots, A_{m-1}$ corresponding to vertices from $B_0, \dots, B_{k-1}$. These pairwise unions of sets $A_0, \dots, A_{m-1}$ are denoted $C_0, \dots, C_{k-1}$ respectively.

CLAIM 17.3. *Prove that for any i there exists $j = \varphi(i)$ with $f(C_i) = C_j$ (this defines a map $\varphi : \{0, \dots, k-1\} \to \{0, \dots, k-1\}$). Moreover, φ is a cyclic permutation of $\{0, \dots, m-1\}$; i.e., the orbit of 0 under φ is the entire set $\{0, \dots, m-1\}$.*

Proof. Essentially, this was done before for interval maps. The general arguments are similar. First, we need to show that φ exists. To this end observe that if A_k, A_l are contained in the same C_i then there exists a chain of sets A_r connecting them and such that each next set is non-disjoint from the previous one. Therefore the chain of images of A_r will connect $f(A_k)$ and $f(A_l)$ and so $f(A_k)$ and $f(A_l)$ are contained in the same C-set, say, in C_j. Since this can be done for every similar pair of k, l we see that there exists $j = \varphi(i)$ such that $f(C_i) = C_j$. The fact that it is a cyclic permutation of $\{0, \dots, m-1\}$ follows from the fact that f cyclically permutes $A_0, \dots, A_{m-1}$. ■

EXERCISE 17.4. Consider the above construction for an irrational rotation of the circle and determine what are the sets A_i, C_i in that case. □

Now, we apply this construction in the particular case of a recurrent transitive map of a Cantor set. It turns out that one can describe all such maps of a Cantor set which happens to be equivalent to the description of all transitive uniformly equicontinuous maps of Cantor sets. By a Cantor set we understand a metric space X which is homeomorphic to the standard Cantor set $C \subset \mathbf{R}$ but may have a different metric.

Observe that for any ε there is a cover of X by pairwise disjoint clopen (closed and open) sets of diameter less that ε. Consider such cover $\{U_i\}_{i=1}^n$. Then there exists $\delta > 0$ such that for any $i, j, x \in U_i, y \in U_j$ we have $d(x, y) > \delta$.

CLAIM 17.5. *Let m be such that $d_C(f^m, \mathrm{id}_X) < \delta$. Show that then $f^m(U_i) = U_i$.*

Proof. Indeed, first of all choose any $y \in U_i$. Then $d(f^m(y), y) < \delta$. Thus by the choice of δ we get $f^m(y) \in U_i$. Since y was chosen arbitrarily, $f^m(U_i) \subset U_i$ for every i. On the other hand, f and therefore all its powers are homeomorphisms by Claim 17.1. Hence, $f^m(U_i) = U_i$ for any i. ■

Claim 17.5 implies that for any $z \in U_i$ we have $\omega_{f^m}(z) \subset U_i$. This will be used later when we construct models for transitive recurrent maps of Cantor sets. Also, in our investigation we will rely upon Claims 17.2 and 17.3.

18 Recurrent maps of Cantor sets 2

We continue our study using the same notation as before. The map $f : X \to X$ of a Cantor set X is now assumed to be recurrent and transitive. Let $x \in U_1$ have a dense orbit. Set $\omega_{f^m}(x) = A_0, \dots, \omega_{f^m}(f^{m-1}(x)) = A_{m-1}$ where m is chosen in Claim 17.5. For every $i, 0 \leq i \leq m-1$ choose $j_i = j$ so that $f^i(x) \in U_j$. Clearly, j is well defined. Now, the fact that $f^m(U_s) = U_s$ for every s implies that the entire f^m-orbit of $f^i(x)$ is contained in U_j. Therefore $A_i \subset U_j$ where A_j are constructed as before.

Now, construct sets $C_0, \dots, C_{k-1}$ exactly like in Claims 17.2 and 17.3. Then any set C_i can be contained in only one set U_j, and so $k \geq m$. Also, the diameter of any set C_i is at most ε. By Claim 17.3 these sets are cyclically permuted.

This gives us the first step in the construction which can be continued similarly. Namely, we can consider now the restriction of f^k onto C_0. Then $f^k : C_0 \to C_0$. Moreover, as we know there are iterations of f arbitrarily close to id_X. It is easy to see then that their k-th powers will also converge to id_X. Indeed, the following claim holds (the proof is simple and left to the reader).

CLAIM 18.1. *Let k be fixed and consider a map $\Phi : C(X,X) \to C(X,X)$ defined as follows: $\Phi(f) = f^k$. Then Φ is continuous. In particular, if we have $f^{m_i} \to \mathrm{id}_X$ then $f^{km_i} \to \mathrm{id}_X$.*

To establish the fact that our previous arguments apply to $f^k : C_0 \to C_0$ we need to check if this map is transitive.

CLAIM 18.2. *The map $f^k : C_0 \to C_0$ is transitive.*

Proof. Indeed, since C_0 is open then there exists a point $x \in C_0$ with a dense orbit. All powers of f which are not divisible by k map x into C_i with $i \neq 0$. Hence for every point $y \in C_0$ and a sequence $f^{m_i}(x) \to y$ we conclude that from some time on $m_i = kn_i$. Therefore, $(f^k)^{n_i}(x) \to y$ which implies that $y \in \omega_{f^k}(x)$ and hence $C_0 \subset \omega_{f^k}(x)$. Since the f^k-orbit of x is contained in C_0 and so $C_0 \subset \omega_{f^k}(x)$, this completes the proof. ∎

This implies that $f^k : C_0 \to C_0$ has the same properties as the original map and the arguments can be repeated for $f^k : C_0 \to C_0$. Since the choice of ε is ours we can use a much smaller than before number and thus get a set $D_0 \subset C_0$ such that for some l the sets $f^k(D_0), \dots, f^{k(l-1)}(D_0)$ are disjoint while $f^{kl}(D_0) = D_0$. Moreover, $f^{jk+i}(D_0) \subset C_i$ for any j and $0 \leq i \leq m-1$. In fact, all sets $f^i(D_0), 0 \leq i \leq kl-1$ are disjoint and contained in the appropriate set $C_{i \bmod k}$. This finishes up the next step in the construction which then can be continued.

So, as we have already seen, there exists a nested sequence of compact sets $X^1 \supset X^2 \supset \ldots$ with the following properties.

- For every i there exists the integer n_i such that the sets $X^i = X^i_0, f(X^i_0) = X^i_1, \ldots, f^{n_i-1}(X^i_0) = X^i_{n_i-1}$ are pairwise disjoint while $f^{n_i}(X^i_0) = X^i_0$; moreover, the union of all these sets is the entire space X.
- $\max\{\operatorname{diam} X^i_k : 0 \le k \le n_i - 1\}$ strictly monotonically decreases to 0 as $i \to \infty$.

It follows that for any i we have $n_{i+1} > n_i$ and n_{i+1} is a multiple of n_i.

Now, consider the following construction. Let $y \in X$. Then for any i there exists the unique $k_i(y) = k$ such that $y \in X^i_k$. This defines the sequence $\varphi(y)$ where $\varphi(y)_i = k_i(y)$. It is easy to see that $0 \le \varphi(y)_i \le n_i - 1$, and the property that singles out the number $\varphi(y)_i$ for us is that $\varphi(y)_i$ is the unique number between 0 and $n_i - 1$ such that $y \in f^{\varphi(y)_i}(X^i_0)$.

To continue describing the properties of $\varphi(y)$ we need to introduce some notions from a very elementary number theory. Suppose that a number n is given. By $a \equiv b (\operatorname{mod} n)$ one means that n divides $b - a$; in this case one says that a is *congruent* to b modulo n. If $0 \le b < n$ then one refers to b as the *residue* of $a (\operatorname{mod} n)$.

CLAIM 18.3. *The congruence relation is an equivalence relation. If* $a \equiv b (\operatorname{mod} n)$ *and* $a' \equiv b' (\operatorname{mod} n)$ *then* $a + b \equiv a' + b' (\operatorname{mod} n), a - b \equiv a' - b' (\operatorname{mod} n)$ *and* $ab \equiv a'b' (\operatorname{mod} n)$.

The equivalence classes of the congruence relation are called *residue classes* or *congruence classes*. We can introduce addition, subtraction and multiplication on the set of all residues $\operatorname{mod} n$. The set of all residues $\operatorname{mod} n$ is called the *ring* of residues modulo n and is denoted by M_n, and when we speak of addition, subtraction and multiplication modulo n we mean the algebraic operations defined above.

In the next lecture we go back to the problem of description of a recurrent transitive Cantor set map $f : X \to X$. As we have seen, with any point $y \in X$ we can associate a sequence $\varphi(y)$ in a certain way which provides for a kind of symbolic dynamics in this setting. We want to describe the properties of such sequence. This would allow us to then describe the set of all sequences with these properties and construct a number-theoretic model for our dynamical system.

19 Recurrent maps of Cantor sets 3

In the previous lectures we developed the tools which will allow us now to construct the number-theoretic model for recurrent maps of Cantor set. To apply these tools we need the following lemma in which we use the notation from Lecture 17.

Lemma 19.1. *$X^i_k \supset X^{i+1}_l$ if and only if $l \equiv k (\mathrm{mod}\, n_i)$. This implies that for any y the sequence $\varphi(y)$ has the property that*

$$\varphi(y)_i \equiv \varphi(y)_{i+1} (\mathrm{mod}\, n_i).$$

Moreover, for any sequence m_i with this property there exists the unique point $y \in X$ such that $\varphi(y) = \{m_i\}$.

Proof. As one can see from the statement of Lemma 19.1, we need to study what sets of the next level are contained in the sets of the previous level. Since $X^i_0 \supset X^{i+1}_0$ it is clear that $f^k(X^i_0) = X^i_k \supset f^k(X^{i+1}_0) = X^{i+1}_k$. Suppose that $X^i_k \supset X^{i+1}_l = f^{l-k}(X^{i+1}_k)$. Since the set X^{i+1}_k is contained in X^i_k we see that it is mapped back into the same X^i_k by f^{l-k}. This is possible only if n_i divides $l-k$, that is only if $l \equiv k (\mathrm{mod}\, n_i)$.

This line of arguments can be reversed: if $l \equiv k (\mathrm{mod}\, n_i)$ then the set X^{i+1}_k which is contained in X^i_k, is mapped back into the same X^i_k by f^{l-k} and thus $X^i_k \supset X^{i+1}_l$ as desired. In other words, the next level sets are contained in the previous level sets if and only if their subscripts are congruent modulo n_i.

Now, let $\varphi(y) = \{m_i\}$. Then by the definition $y \in f^{m_i}(X^i_0)$ and $y \in f^{m_{i+1}}(X^{i+1}_0)$. This implies that $f^{m_i}(X^i_0) \supset f^{m_{i+1}}(X^{i+1}_0)$ and therefore by the previous paragraph n_i divides $m_{i+1} - m_i$ as desired.

On the other hand, if a sequence $\{m_i\}$ has the property that $m_i \equiv m_{i+1} (\mathrm{mod}\, n_i)$ then again by the results of the first paragraph of this proof $X^i_{m_i} \supset X^{i+1}_{m_{i+1}}$ and so we get a nested sequence of compact sets $X^i_{m_i}$. In addition their diameters decrease to 0. Therefore the intersection of these sets is a point, say, z, and it follows immediately from the construction that $\varphi(z) = \{m_i\}$. ∎

Denote the set of all sequences $\{m_i\}$ such that $m_i \equiv m_{i+1} (\mathrm{mod}\, n_i)$ for any i by $S(\{n_i\})$. We conclude that $\varphi : X \to S(\{n_i\})$ is a 1-to-1 map. Let us study $S(\{n_i\})$ from the set-theoretic and number-theoretic point of view. It is easy to see that $S(\{n_i\})$ is a subset of a bigger space, namely the product space $M = \prod_i M_{n_i}$. Also, the definition of $S(\{n_k\})$ implies that if $\{m_k\} \in S(\{n_k\})$ then for any $i < j$ we have $m_j \equiv m_i (\mathrm{mod}\, n_i)$. Thus, if the number m_j is known this defines all $m_i, i \leq j$ as the residues from M_{n_i} which are congruent to m_j modulo n_i.

To finish this series of observations concerning the structure of the set $S(\{m_i\})$ without studying the dynamics on it let us observe that M can be endowed with a metric as follows: $\rho(\bar{x}, \bar{y}) = 2^{-k}$ where k is the first place where the two sequences $\bar{x}, \bar{y}$ have distinct entries (this is our usual definition). Clearly, this is a metric. Moreover, we will need a particular case of a well-known topological result which is known as the Tychonoff Theorem stated here as follows.

PROPOSITION 19.2. (M, ρ) *is a compact metric space.*

On the other hand it is easy to check that $S(\{n_i\})$ is a closed subset of M which is the purpose of the next exercise.

PROPOSITION 19.3. $S(\{n_i\})$ is a closed subset of M and is therefore a compact metric space. □

Proof. It is enough to observe that if a sequence of elements of $S(\{n_i\})$ converges in topology on M then from some time on in this sequence the coordinates stabilize. ■

Let us now consider some examples. The best known example is that of the so-called 2-*adic adding machine*. Usually it is considered differently, but our approach dictates the interpretation. So, consider $S(\{2^i\}) = G$. This is a set of all sequences $\{m_i\}$ such that $m_{i+1} \equiv m_i (\operatorname{mod} 2^i)$ where all m_i come from the ring M_i of all residues $\operatorname{mod} 2^i$. For example, $(0, 0, \dots) \in G$ and also $(1, 1, \dots) \in G$.

Let us look at a sequence $\{m_k\}$ in terms of the changes which m_k can experience as we move from m_k to m_{k+1}. Then since $m_{k+1} \equiv m_k (\operatorname{mod} 2^i)$ and $m_{k+1} < 2^{k+1}$ we see that in fact either $m_{k+1} = m_k$ or $m_{k+1} = m_k + 2^k$. This actually suggests another more traditional interpretation of G where every point is characterized by the sequence of zeros or ones depending on whether the move from k to $k+1$ involves adding 2^k to the number or not. We will not discuss this connection in much detail.

Another example is when the sequence $\{n_i\}$ stabilizes from some time on (this does not really correspond to what we study, i.e. to recurrent maps of Cantor sets, but is related as we shall see later). If for some j and all $i \geq j$ we have $m_i = m_j$ then the set $S(\{n_i\})$ is finite because every possible value of m_j (and there are n_j such values) determines all m_i with $i < j$ (this follows from the definition) as well as m_i with $i \geq j$ (this follows from the assumptions). Thus n this case we can say that $S(\{n_i\})$ is simply a periodic orbit of period n_j. In fact once we introduce the algebraic structure on M and $S(\{n_i\}$ we will see that $S(\{n_i\})$ is isomorphic to M_j in the sense of this structure too.

Clearly, M has an algebraic structure - two elements of M can be summed up coordinatewise. In fact, M is an infinite compact group with respect to this operation. We will now study the connection between this structure and the dynamics.

20 Recurrent maps of Cantor sets 4

We continue by studying the properties of the map φ.

LEMMA 20.1. *$\varphi : X \to S(\{n_i\})$ is a homeomorphism.*

Proof. Since for any i all the sets $X^i_k, 0 \le k \le n-1$ are pairwise disjoint we see that for any i there exists a positive number δ_i which is less than the distance between any of those sets. Hence if $d(x,y) < \delta_i$ then both points x, y belong to the same set X^i_k and so $\varphi(x)_i = \varphi(y)_i = k$ which implies that in fact the first place where $\varphi(x)$ and $\varphi(y)$ may differ is $i+1$ and thus the distance between $\varphi(x)$ and $\varphi(y)$ is at most 2^{-i-1}. In other words, if $d(x,y) < \delta_i$ then $\rho(\varphi(x), \varphi(y)) \le 2^{-i-1}$. Since i can be chosen arbitrarily large we see that $\varphi : X \to S(\{m_i\})$ is continuous. Since φ is 1-to-1, the lemma is proven. ∎

We can replace the old metric on X by the new one, and in general think of X as $S = S(\{n_i\})$. The question which we have to answer now is about the map into which our original map f is transformed. That is, the existence of the homeomorphism $\varphi : X \to S$ allows us to consider the map $g = \varphi f \varphi^{-1} : S \to S$, and we now need to describe this map explicitly. To do this consider the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ \big\downarrow{\scriptstyle\varphi} & & \big\downarrow{\scriptstyle\varphi} \\ S & \xrightarrow{g} & S \end{array}$$

It follows from the construction that for any $y \in X$ we have $y = \cap_{i=0}^{\infty} f^{\varphi(y)_i}(X^i_0)$. As we apply f to this we get that $f(y) = \cap_{i=0}^{\infty} f^{\varphi(y)_i+1}(X^i_0)$. In other words the sequence which corresponds to $f(y)$ is $\{\varphi(y)_i + 1\}$. This implies that in fact the map g in question acts on a sequence $\{m_i\} \in S(\{n_i\})$ as follows: $g(\{m_k\}) = \{m_k + 1\}$ where for every k the addition is understood modulo n_k (so that $(n_k - 1) + 1 = 0$). In other words, g shifts all sequences by $(1, 1, \ldots)$ (notice, that $(1, 1, \ldots) \in S(\{n_k\})$).

Clearly, the ρ-distance between points does not change under g (the sequences which differ for the first time at a certain place will differ at the same place when shifted by $(1, 1, \ldots)$). The following exercise is also useful here.

EXERCISE 20.2. Prove directly that the shift g is transitive. □

Therefore the map g is a transitive isometry and all the machinery developed so far applies. Moreover, the description given (through sequences and addition of $(1, 1, \ldots)$) is the explicit description we have been looking for. The maps like g are called *generalized* *adding machines* and appear mostly in studying one-dimensional and complex dynamics. Their existence is closely related to a very deep phenomenon recently discovered for such low-dimensional dynamical systems. This phenomenon is called *renormalization* and we are now ready to discuss it in the case of unimodal interval maps.

Let $f : [0,1] \to [0,1]$ be a unimodal interval map. Say that an interval $J \subset [0,1]$ is *periodic* if $J, f(J), \dots, f^{n-1}(J)$ are pairwise disjoint while $f^n(J) = J$. Suppose that there is a periodic interval J which contains the critical point c of f. Then let us consider the map $f^n : J \to J$. This map is well-defined; moreover, the following can be proven.

CLAIM 20.3. *$f^n|J$ is a unimodal map.*

Proof. Clearly, the only critical point contained in $\cup_{i=0}^{n-1} f^i(J) = \operatorname{orb} J$ is c. The map f^n is then the composition of n maps each of which is the restriction of f onto images of J and therefore has a unique critical point. ■

If a periodic interval J of period $n > 1$ exists then the map f is said to be *renormalizable* while the map $f^n|J$ is said to be a *renormalization* of f. A variety of deep and very hard questions are related to this are considered in the dynamical systems theory. Some problems arise if one considers the possibility for $f^n|J$ to be renormalizable itself, and then repeats this for several, possibly infinitely many, times thus giving rise to the so-called *infinitely renormalizable* unimodal maps. Let us consider what happens in this case in more detail.

Let f be an interval map (not necessarily unimodal). Define periodic intervals of f exactly as above. Suppose that there exists a sequence $I_0 \supset I_1 \supset \dots$ of periodic intervals with periods $n_0 < n_1 < \dots$. Then we can consider the set $S = \cap_{j=0}^{\infty} \operatorname{orb} I_j$. This set is similar to the sets $S(\{n_j\})$ with one important difference. Indeed, the intervals from the orbits of intervals I_k can play the role of sets X_k^i cyclically permuted by the map. However we know that the maximal diameter of a set X_k^i goes to 0 as $i \to \infty$. To establish similar facts for intervals is often hard. In a related Exercise 20.4 below we use the notation introduced before; also, call a set A *roaming* if all sets $A, f(A), \dots, f^n(A), \dots$ are pairwise disjoint.

EXERCISE 20.4. Suppose that an interval map f has no roaming intervals. Prove that then $f|S$ is conjugate to the shift by $(1, 1, \dots)$ in $S(\{n_j\})$. □

Summarizing we can say that sets $S(\{n_i\})$ with the map g are faithful extensions of the notion of a periodic orbit on infinite sets. Abusing the language, we can say that the set $S(\{n_i\}) = S$ is like a periodic orbit whose period is a sequence $\{n_i\}$ (here by extension we mean that $g|S$ can be approximated by periodic orbits of periods n_j in a natural sense).

21 Intrinsic properties of limit sets 1

In the first half of this lecture we plan to consider some additional properties of limit sets. In general to be able to say that a given set is the limit set of a point we need to find such a point. Sometimes however it might be difficult to do. Still we may need to be able to distinguish limit sets from the rest of compact invariant sets. To this end we need to establish certain intrinsic properties of limit sets which would enable us to verify these properties and to sort out limit sets from non-limit sets, at least in some cases.

As an introduction let us consider the following question: is it always so that $f|\omega(x)$ is transitive? It may seem the case since the orbit of x if in a way dense in $\omega(x)$. However if x is not recurrent and therefore $x \notin \omega(x)$ then $f|\omega(x)$ might be non-transitive.

Still, since the orbit of x approaches $\omega(x)$ we may hope for some weak version of transitivity taking place for $f|\omega(x)$. To start our investigation let us first prove a technical lemma which establishes a formal connection between the orbit of x and $\omega(x)$.

Lemma 21.1. *Given x for any $\varepsilon > 0$ there exists $N(\varepsilon) = N$ such that for any $n > N$ we have $d(f^n(x), \omega(x)) < \varepsilon$.*

Proof. Suppose that for some $\varepsilon > 0$ such a number N does not exist. Then there exists a sequence n_k of integers such that $d(f^{n_k}(x), \omega(x)) \geq \varepsilon$. Refining this sequence we may assume that $f^{n_k}(x)$ converges to a point y (here we use the fact that X is a compact). Then on the one hand y must belong to $\omega(x)$, on the other hand the fact that $d(f^{n_k}(x), \omega(x)) \geq \varepsilon$ implies that $d(y, \omega(x)) \geq \varepsilon$, a contradiction. ∎

As the first step in our investigation of a version of transitivity for limit sets let us see what kinds of finite sets can be limit sets. Since the map on any limit set is surjective by Proposition 6.4, the fact that $\omega(x)$ is finite implies that it breaks into a finite collection of cycles. The most natural assumption about an analog of transitivity which must hold then is that in fact if $\omega(x)$ is finite then it must be one cycle.

LEMMA 21.2. *If $\omega(x)$ is finite then it is a cycle.*

Proof. We begin by suggesting a sketch of the proof which will be made formal later on. Suppose that $\omega(x)$ is the union of several (more than one) cycles. Then we can represent $\omega(x)$ as the union of two invariant disjoint sets A and B. Since x gets closer and closer to $\omega(x)$ we can find arbitrarily far down the road orbit segments in which the point x moves from being very close to A to being very close to B.

On the other hand, the set A is invariant, therefore and because of the continuity the distance between the points in the orbit segment of x and the set A cannot grow immediately. In other words, as i grows the point $f^i(x)$ moves gradually farther and farther away from A until it reaches a small neighborhood of B. Because of continuity, at some moment along the way the point will have to be located far away from A but not yet too close to B. Since this argument is universal, there will be infinitely many moments of time when the image of x is located "between" the sets A and B. Hence there must be a point of $\omega(x)$ "between" A and B, that is not belonging to $A \cup B$. However, this is impossible because $\omega(x) = A \cup B$.

Let us now suggest a formal (ε, δ)-version of this argument. Since we consider a compact metric space, the distance between A and B is positive. Denote it by ε and choose by the uniform continuity of f some $\delta < \varepsilon/3$ such that if $d(z, z') < \delta$ then $d(f(z), f(z')) < \varepsilon/3$. Then by Lemma 21.1 there exists a number N such that for any $n > N$ we have $d(f^n(x), \omega(x)) < \delta$.

Fix $n > N$ and consider the point $f^n(x)$. Then by the same Lemma 21.1 there is a point $y_n \in \omega(x)$ such that $d(f^n(x), y_n) < \delta$. The point y_n belongs to either A or B, and for the sake of definiteness we can assume that $y_n \in A$. Now, since $B \subset \omega(x)$ then there must exist arbitrarily large numbers m such that $d(f^m(x), B) \leq \delta$. Choose such $m > n$ that this holds and consider the segment of the orbit of x from $f^n(x)$ through $f^m(x)$.

Now, because of the choice of N and the assumption that $n > N$ we see that for every $i \geq n$ either $d(f^i(x), A) < \delta$ or $d(f^i(x), B) < \delta$. Since we know that $d(f^m(x), B) < \delta$ we see that $d(f^m(x), A) > \delta$. Therefore there exists the minimal $j > n$ such that $d(f^j(x), A) > \delta$. Let us see how far can $f^j(x)$ be from B. By the choice of j we have that $d(f^{j-1}(x), A) \leq \delta$. Therefore by the choice of δ we have $d(f^j(x), A) \leq \varepsilon/3$, and so by the choice of ε we see that $d(f^j)x), B) \geq 2\varepsilon/3 > \delta$. In other words, the distance between $f^j(x)$ and both A and B is greater than δ. Since $\omega(x) = A \cup B$ this implies that $d(f^j(x), \omega(x)) > \delta$, a contradiction with the choice of N. ∎

The following lemma will only be started in class; you will have to finish its proof using the ideas from Lemma 21.2.

LEMMA 21.3. *If $a \in \omega(x)$ is a periodic point then either $\omega(x)$ coincides with the periodic orbit of a or a is not isolated in $\omega(x)$ (so that in particular $\omega(x)$ is infinite).*

Proof. Suppose that a has the minimal period m. By Lemma 21.2 if $\omega(x) \neq \text{orb}(a)$ then $\omega(x)$ is infinite. Consider sets $A_i = \omega_{f^m}(f^i(x)), 0 \leq i \leq m-1$. Then by Claim 17.2 we see that none of the sets A_i is finite. Assume for the sake of definiteness that $a \in A_0$ and prove that a is not isolated in A_0. From now on assume that $f^m = g$; then $A_0 = \omega_g(x)$ and $a \in \omega_g(x)$ is a g-fixed point isolated in $\omega_g(x)$. ∎

EXERCISE 21.4. Finish the proof of Lemma 21.3 using ideas from Lemma 21.2. □

22 Intrinsic properties of limit sets 2

The main result in this direction is the following theorem.

THEOREM 22.1. *Suppose that $L = \omega(x)$ and $G \subset L$ is non-empty and open in L in the subspace metric. If $G \neq L$ then $f(\bar{G}) \not\subset G$.*

Proof. By way of contradiction suppose that $G \neq L$ and contrary to what we want to prove $f(\bar{G}) \subset G$. This implies that $\bar{G} \neq L$ either. Indeed, if $\bar{G} = L$ then by the properties of limit sets we have $f(\bar{G}) = L$. On the other hand we know that $f(\bar{G}) \subset G \neq L$, a contradiction.

Now, consider two non-empty compact sets, $f(\bar{G})$ and $F = X \setminus G$. By the assumptions they are disjoint (in fact, $f(\bar{G}) \subset G$ is equivalent to the fact that they are disjoint). Hence there is a positive ε which is less than the distance between any two points from $f(\bar{G})$ and F respectively. Consider the union G' of all $\varepsilon/2$-balls centered at points from $f(\bar{G})$ and prove that from some time on the orbit of x stays in G'.

First of all observe that by the choice of ε for any $y \in G'$ we have $d(y, F) > \varepsilon/2$. Indeed, otherwise there exist points $z \in F$ and $z' \in f(\bar{G})$ such that $d(y, z) \leq \varepsilon/2$ and $d(y, z') < \varepsilon/2$. This implies (by the triangle inequality) that $d(z, z') < \varepsilon$, a contradiction with the choice of ε.

Now, by continuity of f we can choose the number $\delta < \varepsilon/2$ so that $d(y, z) < \delta$ implies $d(f(x), f(y)) < \varepsilon/2$. Also, by Lemma 21.1 we can choose $N = N(\delta)$ such that for any $n > N$ we have $d(f^n(x), \omega(x)) < \delta$. Finally, since $G \neq \emptyset$ we can find a point $z \in G$ and a number $M > N$ such that $d(f^M(x), z) < \delta$.

Let us show by induction that for $i \geq M + 1$ we have $f^i(x) \in G'$. Indeed, for $i = M + 1$ we have $d(f^i(x), f(z)) < \varepsilon/2$, so since $f(z) \in f(\bar{G})$ we see that $f^i(x) \in G'$. Now, suppose that for some $i \geq M + 1$ we have $f^i(x) \in G'$. As we have seen, this implies that for any point $z \in F$ we have $d(f^i(x), F) > \varepsilon/2$. On the other hand for some $z' \in \omega(x)$ we must have $d(f^i(x), z') < \delta$ because $i > M \geq N$. It follows that $z' \in G$. Now, $d(f^{i+1}(x), f(z')) < \varepsilon/2$ and $f(z') \in f(\bar{G})$. Hence $f^{i+1}(x) \in G'$ and the induction step is made.

We get that x stays in G' from some time on. This implies that $\omega(x) \subset \bar{G}'$. However since the distance between any point of G' and F is at least $\varepsilon/2$ we conclude that F and $\bar{G}'$ are disjoint. Clearly, this contradicts the fact that $F \neq \emptyset$ is a subset of $\omega(x)$ and proves the theorem. ∎

It is easy to see that Lemma 21.2 and Lemma 21.3 in fact follow from Theorem 22.1. Also, Theorem 22.1 establishes a sort of dynamical connectivity of the limit sets. It provides an approach to the following problem of embedding a given dynamical system in another system as a limit set.

PROBLEM 22.2. *Suppose that $f : X \to X$ is a continuous map of a metric compact X into itself. What are the conditions on f which are necessary and sufficient for the existence of another compact metric space $Y \supset X$ and a map $F : Y \to Y$ such that $F|X = f$ and there exists a point $y \in Y$ such that $\omega_F(y) = X$.*

The conditions of Theorem 22.1 are necessary for the existence of Y and F. It turns out that they are also sufficient as was discovered in the 60-s. We will not prove it here, rather we will give an example which has the flavor of the general proof.

Consider the identity map of the circle $S^1 = S$ onto itself. Let us suggest an embedding of this dynamical system as a limit set as follows. First, consider an annulus A with S being its inner circle and another circle B of radius R being its outer circle. We will consider polar coordinates for points of A.

To begin with let us define a map on this annulus so that it rotates every circle concentric with S and B by the same angle depending only on the radius of the circle. We could choose these angles continuously and in such a way that they converge to zero as the circle converge to S and then the forthcoming construction would still be possible. However for the sake of definiteness let us assume that the angle by which the circle of radius r is rotated is simply $r - 1$.

Clearly, this map $F : A \to A$ is well defined and continuous. However, it does not solve the problem because the annulus here is foliated into invariant circles and all limit sets of points outside S will stay in their circles and thus cannot have the limit set S. On the other hand, of course for every point $x \in S$ we have $\omega(x) = \{x\}$. Still, the closer points are to S the less they move which suggests the idea of the limit set of some of these points being equal to S^1 if they are pushed to S.

Let us realize the idea of pushing points closer to S. We begin by choosing a map g of the interval $I = [1, R]$ onto itself such that $g(1) = 1, g(R) = R$ and $g(x) < x$ for any $1 < x < R$. Thus, g pushes inner points of I to the left. Observe that we do not specify the properties of g yet. Define a map $G : A \to A$ as follows: $G(\rho, \theta) = (g(\rho), \theta + \rho - 1)$. In other words, we keep the same rotations as before but now push points closer and closer to S with the rate defined by the local behavior of g at 1, more specifically by the speed with which 1 attracts points to the right of it under the map g.

EXERCISE 22.3. Suggest a map g so that there exists $y \in A$ with $\omega(y) = S$. Describe the limit sets of points for G under extra-assumption that $g'(1)$ exists and $g'(1) < 1$. $\square$

23 The Sharkovskii Theorem 1

In the theory of discrete dynamical systems, periodic orbits (called also *cycles*) play a very important role. The problem of coexistence of various types of cycles for a given map admits particularly nice answers in dimension one. However, one has to decide what to understand by a "type" of a cycle. For interval maps, the most widely adopted choice for this is to look only at the period of a cycle. Then the results are very strong. For instance, the following Sharkovskiĭ Theorem holds. To state it let us first introduce the *Sharkovskiĭ ordering* for the set $\mathbf{N}$ of positive integers:

$$3 \succ 5 \succ 7 \succ \ldots \succ 2 \cdot 3 \succ 2 \cdot 5 \succ 2 \cdot 7 \succ \ldots 2^2 \cdot 3 \succ 2^2 \cdot 5 \succ 2^2 \cdot 7 \succ \ldots \succ 8 \succ 4 \succ 2 \succ 1$$

Denote by $Sh(k)$ the set of all positive integers m such that $k \succ m$, together with k, and by $Sh(2^\infty)$ the set $\{1, 2, 4, 8, \ldots\}$. Denote also by $Per(f)$ the set of periods of cycles of a map f (by a period we mean the least period).

Theorem 23.1 (The Sharkovskĭ Theorem). *If $f : [0,1] \to [0,1]$ is a continuous map, $m \succ n$ and $m \in Per(f)$, then $n \in Per(f)$. Therefore there exists $k \in \mathbf{N} \cup \{2^\infty\}$ such that $Per(f) = Sh(k)$. Conversely, if $k \in \mathbf{N} \cup \{2^\infty\}$ then there exists a continuous map $f : [0,1] \to [0,1]$ such that $Per(f) = Sh(k)$.*

We shall call a cycle *divergent* if it has points $x < y$ such that $f(x) < x$ and $f(y) > y$. A cycle that is not divergent will be of course called *convergent*.

We shall use in the proofs the standard technique of loops of intervals. An interval J *f-covers* an interval K (we write then $J \to K$) if $K \subset f(J)$. If we have a *loop* of intervals $J_0 \to J_1 \to \ldots \to J_{n-1} \to J_0$ then there is a periodic point x such that $f^i(x) \in J_i$ for $i = 0, 1, \ldots, n-1$ and $f^n(x) = x$. We will say that the orbit of x is *associated* to the loop. Any piece $J_i \to \ldots \to J_j$ (or $J_j \to \ldots \to J_0 \to \ldots \to J_i$) of the loop will be called a *block*.

LEMMA 23.2. *If f has a divergent cycle then it has points of all periods.*

Proof. We prove first that if a cycle P is divergent then there are points $x < y < z$ of P such that $f(x) < x$, $f(y) \geq z$ and $f(z) \leq x$. Indeed, first of all let us choose two points u and x such that $x < u$ are adjacent in P and $f(x) < x, f(u) > u$. This is possible because P is divergent. Now, consider the cycle P starting at u. At some moment there will be the first time when the point is mapped to the left of u. Denote the corresponding image of u by z. Thus we have that $u, f(u), \ldots, f^k(u) = z$ are all points to the right of u while $f(z) = f^{n+1}(u) \leq x$. Observe that by the properties of u we have $u < z$ and $n > 1$.

Consider now the moment before n when u is mapped to the right of z or at z for the first time. That is, consider the least $i \geq 0$ such that $f^i(u) < z \leq f^{i+1}(u)$. Then $0 < i < n$.

Set $y = f^i(u)$ and check that the triple of points x, y, z satisfies the conditions. Indeed, $x < y < z$ by the choice of these points. On the other hand, $f(x) < x, f(y) \geq z$ and $f(z) \leq x$ as well. This proves our claim.

Then there are fixed points a, b of f such that $x < a < y < b < z$ and $f(t) > t$ for every $t \in (a, b)$. Indeed, there are some fixed points in (x, y) and in (y, z). Simply choose the closest to y fixed points a, b on either side and observe that the direction in which points are mapped inside (a, b) must be to the right since otherwise there would be other fixed points in (a, b), a contradiction. Moreover, there is a point $c \in (a, y)$ such that $f(c) = b$. Set $J = [a, c]$, $K_1 = [c, y]$, and $L = [b, z]$. Then the interval J f-covers J, K_1, the interval K_1 f-covers L, and the interval L f-covers J, K_1.

To find a periodic point of period q where $q > 2$ is a given number we take a periodic orbit Q associated to the loop $J \to J \to \ldots \to J \to K_1 \to L \to J$ with $q-3$ J's followed by a block $J \to K_1$, then a block $K_1 \to L$ and then a block $L \to J$. This loop passes only once through K_1. The only points of K_1 that belong to other intervals of the loop are c and y, and they clearly do not belong to Q. Therefore the period of Q is equal to the length of the loop, that is q.

It remains to find a point of period 2. To this end consider the loop $K_1 \to L \to K_1$. Any periodic orbit associated with it will have to have the period 2 because K_1 and L are disjoint. So, the lemma is proved. ∎

If a certain combinatorial behavior of points on the interval guarantees another type of behavior (perhaps for some other points) then we say that the former type of behavior **forces** the latter type of behavior. Thus we can restate our lemma as follows: divergent cycles force cycles of all periods. We can also restate a part of the Sharkovskĭtheorem as follows: if $m \succ n$ then any cycle of period m forces a cycle of period n. This point of view is very important in one- and two-dimensional dynamics.

By Lemma 23.2 it is enough to consider the case when f has no divergent cycles. So the situation to be considered is as follows: P is an f-cycle which is convergent and of period m, we are given that $m \succ n$, prove that then f has a cycle of period n. The main part of the argument deals with cycles of odd periods. Indeed, it is easy to see that to establish the result it is enough to prove that a) a point of any odd period $2k+1$ forces points of all even periods and also points of all odd periods greater than $2k+1$, and b) a point of period 4 forces a point of period 2. In the next lectures we will prove these two claims and then we will see how they imply the Sharkovskĭtheorem. In fact this implication is a general one (is not related to the fact that the maps are one-dimensional or other specifics), follows from the definition of the period and can be proven for any dynamical system.

24 The Sharkovskiĭ Theorem 2

We will give a new proof of the Sharkovskiĭ theorem which relies upon some new ideas. These new ideas allow one to prove a stronger version of the Sharkovskiĭ result but these stronger results will only be stated, not proved.

However to begin with we need to justify the techniques used in finding points associated with loops of intervals. That is, we will prove the following lemma.

LEMMA 24.1. *If $J_0 \to \dots J_{n-1} \to J_0$ is a loop of intervals then there exists a point x associated with it. Moreover, if these intervals are disjoint then x must have the period n, and if these intervals are not disjoint but their interiors are disjoint then the only way the period of x may be less than n is when the entire orbit of x consists of some of the endpoints of the intervals $J_1, \dots, J_{n-1}$.*

Proof. By the elementary reasons the fact that J_{n-1} F-covers J_0 implies that there is a subinterval $J_{n-1} \subset J_{n-1}$ such that $f(J'_{N-1}) = J_0$. Then J_{n-2} f-covers J'_{n-1} which implies that we can find $J'_{n-2} \subset J_{n-2}$ such that $f(J'_{n-2}) = J'_{n-1}$. Repeating this argument we can find an interval $J'_0 \subset J_0$ such that $f(J'_0) \subset J_1, f^2(J'_0) \subset J_2, \dots, f^n(J'_0) = J_0$. By the elementary reasons there exists a point $x \in J'_0$ such that $f^N(x) = x$, and clearly this point x is associated with the loop in question as desired.

Consider now the case when all intervals J_i are disjoint. In this case the point x cannot have the period less than n since if the period of x is $k < n$ then $f^k(x) = x \in J_0$ while on the other hand $f^k(x) \in J_k$ and $J_k \cap J_0 = \emptyset$, a contradiction.

Finally, suppose that some intervals are non-disjoint and x has the period $k < n$. Let us prove that then the entire orbit of x consists of endpoints of the intervals J_i. Indeed, otherwise let y be a point from the orbit of x which belongs to the interior of some interval J_i. Then on the one hand we have $f^k(y) = y \in J_i$ while on the other hand $f^k(y) \in J_{i+k \bmod n}$ and since $k < n$, the latter interval is disjoint from the interior of J_i, a contradiction. ∎

LEMMA 24.2. *If an interval map f has a point of period 4 then it must have a point of period 2.*

Proof. Consider all possible cyclic permutations which represent all the possible orders in a cycle of period 4; for each such order we will suggest a loop of two disjoint intervals which would force the existence of a cycle of period 2 as desired. Below we consider cycles of period 4 which consist of points $a < b < c < d$; also keep in mind that by Lemma 23.2 we may assume that the cycle in question is convergent.

1. $f(a) = b, f(b) = c, f(c) = d, f(d) = a$. Then the loop of intervals could be constructed as follows. First choose a fixed point $x \in (c, d)$. Then consider the loop of intervals $[b, c] \to [x, d] \to [b, c]$.

Let us show that this is the only possible case when 3 points of the orbit are mapped to the right and one point is mapped to the right. Indeed, since we assume that our cycle is convergent we see that a, b, c must be mapped to the right and d must be mapped to the left. Now, the point a has to have the preimage in the cycle, and this preimage must be mapped to the left, so we necessarily have $f(d) = a$. On the other hand, c maps to the right, so we must have $f(c) = d$. Now, b maps to the right as well, and by this time the only point available for being the image of b is c, thus $f(b) = c$. It is clear now that $f(a) = b$ and we are done.

Clearly, if the situation is symmetric to this one - that is, if there are three points mapped to the left and one point mapped top the right - then the arguments can be repeated. Therefore, it remains to consider the case when there are two points mapped to the right and two points mapped to the left in our orbit.

Consider first the case when the set of points a, b maps onto the set of points c, d. Then since we deal with a periodic orbit, the set of points c, d maps back onto the set of points a, b. Therefore $[a, b] \to [c, d] \to [a, b]$ is a loop of intervals and since they are disjoint we have the desired.

Suppose that the set $\{a, b\}$ does not map onto $\{c, d\}$. Then since $f(b) \in \{c, d\}$ we conclude that $f(a) = b$. On the other hand the point c maps to the left, and since the point b is already taken by $f(a)$ the only way it can happen is when $f(c) = a$. Now, the point b cannot be mapped onto c since then we would have a cycle of period 3, not 4. Thus, $f(b) = d$ and $f(d) = c$. It remains to consider the same loop of intervals as before: $[a, b] \to [c, d] \to [a, b]$ which completes the proof. ∎

Let us now describe the plan for the next lecture. We will study odd-periodic convergent cycles. Suppose that f has a convergent cycle of period $2l + 1 \geq 3$. Then of course f has a cycle of the smallest possible odd period $2m + 1$ greater than 1; chose one of such cycles and denote it by P. The fact that $2m + 1$ is the smallest possible period of such a cycle very strongly influences the structure of P (in fact this structure can be fully described up to the orientation). In fact the whole idea of studying a dynamical object which minimizes of maximizes some parameter is very useful because in this case one can speak more definitely about the properties of the dynamical object in question.

The analysis of the cycle of minimal odd period greater than 1 yields that f must have periodic points of all odd periods greater than or equal to $2m + 1$ and also points of all even periods which proves the 'odd' part of the Sharkovkǐ theorem.

25 The Sharkovskiĭ Theorem 3

We will concentrate now upon odd-periodic convergent cycles. A good idea here would be to consider cycles of period 3 and prove for them that the Sharkovskiĭ theorem holds. Also, it is clear that our arguments need to be done up to the orientation.

LEMMA 25.1. *If f has a cycle of period 3 then f has cycles of all even periods.*

Proof. Without loss of generality and up to the orientation the cycle of period 3 can be described as the set $x < y < z$ such that $f(x) = y, f(y) = z, f(z) = x$. Let us prove that then f must have cycles of all even periods. Indeed, choose a fixed point $a \in (y, z)$ and then a point $b \in (x, y)$ so that $f(b) = a$. Now, take a point $c \in (a, z)$ such that $f(c) = b$. Finally let $e \in (y, a)$ be such that $f(e) = c$.

Set $I = [b, y], J = [e, a]$ and $K = [a, c]$. First prove that there exists a point of period 2. To this end consider a loop of intervals $I \to K \to I$; by Lemma 24.1 there exists a point of period 2 in I as desired. Now, let $s > 1$. To find a point of period $2s$ consider a loop of length $2s$ defined as follows:

$$I \to K \to I \to K \to \ldots \to J \to K \to I.$$

Then by Lemma 24.1 there exists a point x such that

$$x \in I, f(x) \in K, f^2(x) \in I, f^3(x) \in K, \ldots, f^{2s-2}(x) \in J, f^{2s-1}(x) \in K, f^{2s}(x) = x.$$

Let us show that x has the period $2s$. Indeed, if it has the period $k < 2s$ then $f^k(x) = x \in I$ and s must be a factor of $2s$. Hence $k \leq s < 2s - 2$ which implies that the entire orbit of x is contained in the union $I \cup K$. On the other hand there must be a point of the orbit of x which belongs to J. Therefore there is a point of the orbit of x which belongs to $(I \cup K) \cap J = \{a\}$ which is impossible since a is a fixed point and its orbit is disjoint from I. This completes the proof of the lemma. ■

LEMMA 25.2. *If f has a cycle of period 3 then f has cycles of all odd periods.*

Proof. Let us use the notation from Lemma 25.1. Set $I' = [x, b], J' = [y, a]$ and $K' = [a, z]$ and consider loops of intervals using I', J' and K'. Let $2s + 1, s \geq 2$ be given. Then consider the loop of length $2s + 1$ defined as follows:

$$J' \to K' \to J' \to K' \to \ldots J' \to K' \to I' \to J'$$

By Lemma 24.1 there is a point x such that

$$x \in J', f(x) \in K', f^2(x) \in J', f^3(x) \in K', \ldots, f^{2s-1}(x) \in K', f^{2s}(x) \in I', f^{2s+1}(x) = x.$$

If x has the period $k < 2s+1$ then $f^k(x) = x$ and $k \leq s < 2s$. Therefore the entire orbit of x is contained in the union $J' \cup K'$ which is disjoint from I', a contradiction to $f^{2s}(x) \in I'$. This shows that the period of x is $2s+1$ and completes the proof. ∎

Let us use these ideas to prove the Sharkovskiĭ theorem for odd periods in general.

LEMMA 25.3. *If f has a cycle of an odd period greater than 1 then it has a cycle of all greater odd periods.*

Proof. First, let us consider all cycles which f has and choose a cycle of the smallest possible odd period among them. Clearly, it is enough to prove the lemma for these cycles. Moreover, by Lemma 23.2 it is enough to consider convergent cycles. So, suppose that f has a convergent cycle P of period $2m+1, m \geq 1$ and that periods of all other f-cycles of odd periods are greater than or equal to $2m+1$. Let us denote the point of the cycle $x_1 < x_2 < \ldots x_{2m+1}$. Let us assume that there are l points of the orbit mapped by f to the right, and without loss of generality let us assume that $l > 2k+1-l$. Then all points $x_1 \ldots, x_l$ are mapped to the right, and all points $x_{l+1}, \ldots, x_{2m+1}$ are mapped to the left. Hence there is a fixed point $a \in (x_l, x_{l+1})$.

Moreover, since $l > 2m+1-l$ we see that not all the points of P to the left of a are mapped to the right of a. Choose the greatest j such that $f(x_j) < a$. Then since $f(x_{j+1} > a$ there must be a point $b \in (x_j, x_{j+1})$ such that $f(b) = a$. For the sake of convenience in the proof of Lemma 25.3 we will from now on use the notation $[z, a]$ for the interval with the endpoints at a and z regardless of whether $z < a$ or $z > a$. This allows us to introduce a useful notation: set $I_{x_r} = [x_r, a]$ for $1 \leq r \leq 2m+1$ except for $r = j$ for which we set $I_{x_j} = [x_j, b]$.

Now, suppose that $2s+1 > 2m+1$ is given and construct a loop of intervals of length $2s+1$ defined as follows:

$$I_{x_l} \to I_{x_{l+1}} \to I_{x_l} \to I_{x_{l+1}} \to \ldots I_{x_l} \ldots I_{f(x_l)} \to \ldots I_{f^{2m}}(x_l) \to I_{x_l}.$$

Then by Lemma 24.1 there exists a point y associated with this loop. Suppose that its period is less than $2s+1$ show (by way of contradiction) that this is impossible. Indeed, if so then y has the period $k < 2s+1$ which has to be odd. Then by the choice of $2m+1$ we see that $k = 1$. However this is impossible because a point from the orbit of y must belong to I_j, yet another point of the orbit of y must belong to I_{l+1} and $I_j \cap I_{l+1} = \emptyset$. ∎

In the forthcoming Lemma 26.1 we will show that odd periods force all even periods and also will show how the general claim of the Sharkovskiĭ theorem follows from Lemma 25.3 and some general arguments. In fact let me encourage those of students who are interested in these techniques to try to do all this on your own. If you succeed you will get a chance to present your proof in class next time and will get extra-credit for that.

26 The Sharkovskiĭ Theorem 3

LEMMA 26.1. *If f has a cycle of odd period greater than 1 than it has cycles of all even periods.*

Proof. We will use the notation and agreements from Lemma 25.3, however we will need several new constructions. Namely, consider $f|\{x_{j+1}, x_{j+2}, \ldots, x_l\}$ and choose the point x_k at which the above restriction assumes its maximum (it may happen that $k = j + 1$ or $k = l$ but in any case $j + 1 \leq k \leq l$). Clearly, $f(x_k) \geq f(x_l) > a$.

Consider now all points of the cycle P located between x_{l+1} and $f(x_k)$. Let us show that the minimum of f restricted upon these points cannot be greater than x_j. Indeed, otherwise it is easy to see that $B = \{x_{j+1}, \ldots, f(x_k)\}$ is an invariant subset of P (because points of B cannot escape from B by the choices and assumptions we made). However, this subset does not contain x_j, a contradiction. Therefore, there exists a point $c \in (a, f(x_k))$ such that $f(c) = b$.

Choose a point $e \in (x_k, a)$ so that $f(e) = c$. Denote the interval $[b, x_k]$ by I, the interval $[e, a]$ by J, and the interval $[a, c]$ by K. If we compare the picture with the picture from Lemma 25.1 we will see that it is absolutely identical. Thus, literally repeating the arguments from the proof if Lemma 25.1 we see that f has points of all even periods as desired. ∎

There is an important information which we would like to add at this point to what we have already learned about the cycles of of odd periods greater than 1. To do so we need the following definition. Suppose that for an interval map f there exist k disjoint closed intervals $I_1, I_2, \ldots, I_k$ such that $f(I_j) \supset \cup_{r=1}^{k} I_r$ for every $1 \leq r \leq k$. Then we say that f has a k**-horseshoe**. If f has a k-horseshoe for some $k > 1$ then we simply say that f has a **horseshoe**. As it turns out the existence of horseshoes is very important for interval maps and for low-dimensional dynamical systems in general.

We rely upon Lemma 26.1 in order to prove the following claim.

LEMMA 26.2. *If f has a point of an odd period then f^2 has a horseshoe.*

Proof. Let use the notation from Lemma 26.1 and consider disjoint intervals $I = [b, x_k]$ and $J = [e, a]$. Then $f^2(I) \supset I \cup J$ and $f^2(J) \supset I \cup J$. Therefore f^2 has a horseshoe as desired. ∎

We are ready now to prove the Sharkovkiĭ theorem. In fact, by Lemmas 25.3 and 26.1 we see that it holds for odd periods. Let us prove that it holds for any period $n = 2^m(2k+1)$ case by case.

LEMMA 26.3. *If f has a point of period 2^m then it has points of all periods 2^j with $j \leq m$.*

Proof. It is enough to show that if a map f has a point of period 2^i then it has a point of period 2^{i-1} and apply this repeatedly. To this end consider $g = f^{2^{i-2}}$. Then g has a point of period 4. Therefore by Lemma 24.2 it has a point of period 2. Thus, there exists a point x such that $g^2(x) = x$ while $g(x) \neq x$. Then $f^{2^{i-1}}(x) = x$ while $f^{2^{i-2}}(x) \neq x$. Suppose that x is of f-period less than 2^{i-1}. Then its period has to be 2^j with $j < i-1$ which would imply that $f^{2^{j-1}}(x) = x$, a contradiction. So, the period of x is really 2^{i-1} as desired. ∎

The next lemma completes the proof of the Sharkovskiĭ theorem.

LEMMA 26.5. *If f has a cycle of period $n = 2^m(2k+1)$ with $k \geq 1$ then f has cycle of all periods s such that $n \succ s$.*

Proof. Without loss of generality we may assume that f has no cycles of periods $r \succ n$ (otherwise we could simply consider the Sharkovskiĭ-strongest period of cycle of f which will be of the same form, prove Lemma 26.4 for this period, and this would imply the desired results for n). In other words, we may assume that n is the Sharkovskiĭ-strongest period of a cycle of f (sometimes this number is called the Sharkovskiĭ **type** of f).

Under these assumptions consider the map $g = f^{2^m}$. Then g has a point of period $2k+1$, therefore g has points of all even periods and of all odd periods greater than or equal to $2k+1$. Let us deduce from it that f has points of all periods of the form $2^l(2s+1)$ such that $n \succ 2^l(2s+1)$.

First let us show that f has points of all periods which are powers of 2 (i.e., consider the case when $s = 0$). Indeed, for every l the map g has a point x of period 2^l by Lemma 26.1. Then $f^{2^{m+l}}(x) = x$ while $g^i(x) \neq x$ for $1 \leq i < 2^l$. This implies that the minimal f-period of x is 2^{m+l} which by Lemma 26.3 implies that f has points of all periods $2^j, j \leq m+l$. Since l here can be chosen arbitrarily, the claim is verified.

Now, let $s \geq 1$. Consider a point x of g-period $2^{l-m}(2s+1)$ with either $s \geq k$ or $l > m$; such a point must exist by Lemma 25.3. Then $f^{2^l(2s+1)}(x) = x$. The minimal f-period of x has to be equal to $2^u(2v+1)$ where $u \leq m$ and $2v+1$ is a factor of $2s+1$. Let us show that in fact $u = l$ and $v = s$. Consider several cases. If $u < m$ then since $2^m(2s+1)$ is the Sharkovskiĭ type of f we conclude that $v = 0$. However this implies that the g-period of x is 1, a contradiction. If $n \geq m$ then the g-period of x is $2^{u-m}(2v+1)$ and since it has to be equal to $2^{l-m}(2s+1)$ we conclude that $l = u$ and $v = s$ as desired. So, we showed that for any $n = 2^m(2k+1), k \geq 1$ the existence of a point of period n forces the existence of points of all periods weaker than n. This completes the proof of the lemma and also of the Sharkovskiĭ theorem. ∎

27 No division and horseshoes 1

The following lemma follows immediately from Lemma 26.2.

LEMMA 27.1. *If an interval map f has points whose period is not a power of 2 then some power of f has a horseshoe.*

In fact this statement can be easily reversed. To do so we need to recall the following definition: the Sharkosvkiĭ **type** of a map is the strongest period of its cycle (if it exists) and 2^∞ in the only case when the strongest period does not exist (which is when the periods of cycles of f are $\{1, 2, 4, \dots, 2^i, \dots\}$. Then the following lemma holds.

LEMMA 27.2. *An interval map f has the Sharkovskiĭ type $2^n, 0 \le n \le \infty$ if and only if no power of f has horseshoes.*

Proof. By Lemma 27.1 it is enough to show that if a map f is such that for some m the map f^m has a horseshoe then the Sharkovskiĭ type of f cannot be 2^n. Indeed, by way of contradiction let us assume that f^m has a horseshoe while the Sharkovskiĭ type of f is 2^n. By the assumption there are two intervals I, J such that $I \cap J = \emptyset$ and $f^m(I) \cap f^m(J) \supset I \cup J$. Choose a prime number p which is coprime with $2m$, and consider a f^m-loop of intervals of length p defined as follows:

$$I \to_{f^m} I \to_{f^m} I \to \dots \to_{f^m} J \to I.$$

Consider a point y associated with this loop. What can the exact f-period k of y be equal to? By the assumption about the map, it must be a power of 2. Therefore it cannot be a factor of p (since p is coprime with $2m$). Now, k must be a factor of pm because $f^{pm}(y) = y$. We conclude that k is a factor of m which implies that $f^m(y) = y$. However it contradicts the fact that $y \in I$ while $f^{m(p-1)}(y) \in J$ because I and J are disjoint. ∎

We will continue our study of one-dimensional topological dynamics by studying in great detail the maps of the interval whose sets of periods consist of powers of 2 only. Our aim now is to describe possible limit sets of points under the assumption that the Sharkovskiĭ type of a map f is $2^n, 0 \le n \le \infty$. In fact we will start by studying the periodic points of such maps.

Let us say that a periodic orbit P of an interval map f has **division** if it can be divided into two subsets P' and P'' such that $P' < P'', f(P') = P'', f(P'') = P'$ (so that in particular P is even-periodic).

LEMMA 27.3. *Let f be an interval map whose all cycles have even periods. Then all its cycles have division.*

Proof. By way of contradiction assume that P is a cycle with no division. One case can be considered immediately - if P is divergent then it forces cycles of all periods, a contradiction. So we may assume that P is convergent. Then as always we can consider the subset P' of P such that for $x \in P$ we have $f(x) > x$ if and only if $x \in P'$ as well as the set $P'' = P \setminus P'$; since P is convergent, we have $P' < P''$. Denote a fixed point of f located between P' and P'' by a.

Since P has no division we can without loss of generality assume that there are points of P' mapped back into P'. Let x be the least point of P and consider the orbit of P starting it at x. Then if $z \in P'$ and $f(z) \in P'$ (and by the assumption such point z must exist) then for at least one of these two points (denote it by y) we know that the power of f which maps x onto this point is an odd number. Since by the assumption the period $2m$ of x is even, we see that there exists an odd number $2i + 1 < 2m$ such that $f^{2m-2i-1}(y) = x < y$. On the other hand, $f^{2m-2i-1}(x) > x$ because x is the leftmost point of P. Hence, there is a point $u \in (x, y)$ such that $f^{2i+1}(u) = u$.

Now, by the assumption there are no points of odd periods greater than 1. Therefore, u is a fixed point: $f(u) = u$. Consider now the same orbit P starting at the closest to u from the right point of P denoted by v. Let us repeat the arguments from the proof of Lemma 23.2 in order to show that f has points of all possible periods.

Similar to Lemma 23.2 we prove first that there are points $u < y' < z'$ of P such that $f(y') \geq z'$ and $f(z') \leq u$. Indeed, consider the cycle P starting at v. At some moment there will be the first time when the point is mapped to the left of u. Denote the preceding image of v by z'. Thus we have that $v, f(v), \dots, f^n(v) = z'$ are all points to the right of u while $f(z') = f^{n+1}(u) \leq u$. Observe that $u < z'$ and $n > 1$.

Consider now the moment before n when v is mapped to the right of z' or at z' for the first time. That is, consider the least $i \geq 0$ such that $f^i(v) < z' \leq f^{i+1}(v)$. Then $0 < i < n$. Set $y' = f^i(v)$ and check that points y', z' satisfy the conditions. Indeed, $u < y' < z'$ by the choice of these points. On the other hand, $f(y') \geq z'$ and $f(z') < u$ as well. This proves our claim.

From this point on there is literally no difference between the end of the proof of Lemma 23.2 and the end of the proof of this lemma. This implies that f has cycles of all periods, a contradiction. So we conclude that if f does not have points of odd periods greater than 1 then every cycle of f has division. ∎

In the next lecture we will consider interval maps of type $2^n, n \leq \infty$ and apply Lemma 27.3 to study their cycles.

28 No division and horseshoes 2

Consider a cycle P of f of period 2^n. Then P is said to have **maximal division** if the following holds: 1) for every $i < n$ divide P into 2^i segments of equal length; 2) then the map permutes these segments. Suppose that P has division in the above sense up until the number of segments into which P is divided is 2^i. This means not just that P can be divided into 2^i segments permuted by the map but also that on the first, second etc step in the above described process the division takes place.

So, it may happen that the map has division in the above sense up until 2^i and then when P is divided into 2^{i+1} segments the division fails. Then P is said to have division **of depth** 2^i. Our main aim is to show that the previous results imply that all cycles of maps of type $2^n, n \leq \infty$ have cycles with maximal division. To this end we prove the following lemma in which the usual notation concerning convergent cycles is used.

LEMMA 28.1. *Given a convergent cycle P let $u = \max P', v = \min P''$. Then there exists a number n such that $f^n([u, v]) \supset P$.*

Proof. Denote by a a fixed point between u and v. Also, set $I = [u, v]$. Then $f(I) \supset I$, and so $f^{n+1}(I) \supset f^n(I)$ for any n. Suppose that for some k the sets $A = f^k(I) \cap P$ and $B = f^{k+1}(I) \cap P$ are the same. Then $A = B$ is an invariant subset of P which is possible only if $A = B = P$ as desired. ∎

THEOREM 28.2. *If f is of type $2^n, n \leq \infty$ then all cycles of f have maximal division.*

Proof. We will start by considering the case when P has division of depth 2 but does not have division of depth 4. The main ideas of the general proof are present in this particular case, still the notation is easier to follow, so to consider first this case and then the general case is appropriate.

Let P be a cycle of period 2^n. By Lemma 27.3 P has division. So, if we divide P into 2 segments $P_1 < P_2$ then they are permuted by f. Suppose that P does not have division of depth 4. This implies that if we divide P into 4 segments $P'_1 < P'_2 < P'_3 < P'_4, P'_1 \cup P'_2 = P_1, P'_3 \cup P'_4 = P_2$ then they are not cyclically permuted by f. Observe that by Lemma 27.3 P'_1, P'_2 are permuted by f^2, and P'_3, P'_4 are permuted by f^2.

These assumptions imply that $f(P'_1)$ cannot be equal to P'_3 or P'_4 because otherwise f will have division of depth 4. Therefore $f(P'_1)$ intersects both P'_3 and P'_4. Set $u = \max P'_3, v = \min P'_4$. Then there are points $x < y$ in P'_1 such that the f-images of these two points lie to distinct sides of $[u, v]$. Now, by Lemma 28.1 applied to $f^2|P_2$ we see that for some power f^{2i} we have $f^{2i}([u, v]) \supset P_2$. Together with the facts that $f([x, y]) \supset [u, v]$ and

$f(P_2) = P_1$ this implies that $f^{2i+2}([x, y]) \supset P_1$.

Similarly we see that there are points $x' < y'$ in P_2' mapped to different sides of $[u, v]$ which implies that for some j we have $f^{2j}([x', y']) \supset P_1$. Hence if we choose $N = \max(2i+2, 2j)$ we will have two disjoint intervals I, J such that $f^N(I) \supset I \cup J, f^N(I) \supset I \cup J$, that is a horseshoe, a contradiction to the assumptions of the theorem and Lemma 27.2. This proves that f must have at least division of depth 4.

The arguments in the general case are similar. Let P be a cycle of period $2^n, n \geq 2$. By Lemma 27.3 P has division. If P does not have maximal division then it has division of depth $2^i, i \geq 1$ but does not have division of depth 2^{i+1}. This implies that if we divide P into 2^i segments $P_1, \dots, P_{2^i}$ then they are cyclically permuted by f whereas if we divide each of these segments into two new segments of equal length then the new segments are not permuted by f. Observe however that if P_j is divided into two segments P_j', P_j'' then by Lemma 27.3 we have that $f^{2^i}(P_j') = P_j''$ and $f^{2^i}(P_j'') = P_j'$.

The fact that P has no division of depth 2^{i+1} implies the following (we use the notation from the previous paragraph): 1) there exist two segments A, B from the collection of 2^i segments such that $f(A) = B$; 2) if we divide each segment A, B into segments of equal cardinality $A' < A''$ and $B' < B''$ then $f(A')$ does not coincide with either B' or B''.

Repeating the arguments from the case of depth 4 choose points $u = \max B'$ and $v = \min B''$. Also, choose $x, y \in A'$ such that $f(x)$ and $f(y)$ lie to different sides of $[u, v]$. We can also find $x', y' \in A''$ such that $f(x')$ and $f(y')$ lie to different sides of $[u, v]$ too. Then as before there exists m such that $f^m([u, v]) \supset A$ which implies that there exists l such that $f^l([x, y]) \supset A, f^l([x', y'])$. Therefore f^l has a horseshoe, a contradiction with Lemma 27.2 according to which no power of f has a horseshoe. ∎

The next step in our investigation is to consider the limit sets of all points for maps of type $2^n, n \leq \infty$. It turns out that some arguments applicable so far only to periodic orbits can be extended onto infinite limit sets. Let us start by introducing the following definition: a limit set A is said to be **divergent** if there are points $z < y$ in it such that $f(z) < z$ and $f(y) > y$ (observe that the definition applies to periodic orbits which are a particular case of limit sets). A non-divergent limit set is said to be convergent. The following lemma follows from the last part of the proof of Lemma 23.2 and is given here without a proof. To state it we need the following definition: say that f **admits** L**-scheme** if, up to the orientation there are point a, y, z such that $f(z) \leq a = f(a) < y < z < f(z)$.

LEMMA 28.3. *If f admits L-scheme then f has points of all periods.*

29 No division and horseshoes 3

We will now study **all** limit sets of maps of type 2^n. The next lemma is an important tool.

LEMMA 29.1. *Suppose that f does not admit L-scheme (e.g., this is so if the set of periods of cycles of f is not the set of all positive integers). Consider the set $\omega(t)$ which is not a fixed point. Then there are fixed points $a' < b' \le c' < d'$ and a number N such that for $i \ge N$ we have $f^i(t) \in (a', b') \cup (c', d')$, the map f maps all points of (a', b') to the right of themselves and all points of (c', d') to the left of themselves.*

Proof. Consider the set A of all fixed points a of f such that there exists n with $a < f^n(t) < f^{n+1}(t)$. Then there exists a point $a' = \max A$. We claim that there exists a number N such that $f^n(t) > a'$ for any $n > N$ and prove it step by step. First we show that if $a < f^n(t) < f^{n+1}(t)$ then $f^i(t) > a$ for any $i \ge n$. Changing the notation we may assume that $a < t < f(t)$, and will have to prove that $a < f^i(t)$ for any i.
Indeed, otherwise $t, f(t), \dots, f^n(t) = z$ are all points to the right of a while $f(z) = f^{n+1}(t) < a$. Consider now the moment before n when t is mapped to the right of z or at z for the first time. That is, consider the least $i \ge 0$ such that $f^i(t) < z \le f^{i+1}(t)$. Then $0 < i < n$. Set $y = f^i(t)$ and check that the points a, y, z form an L-scheme. Indeed, $a < y < z$ by the choice of these points. On the other hand, $f(a) = a, f(y) \ge z$ and $f(z) \le a$ as well. This shows that f admits L-scheme, a contradiction.

Now, let $a' = \sup A$. If there exists N such that $a' < f^N(x) < f^{N+1}(x)$ then the claim follows by the proven above. Assume now by way of contradiction that whenever we have $f^i(t) > a'$ we have $f^{N+1}(t) < f^N(t)$. This implies that if $a'' \in A$ then $\omega(t) \subset [a'', a'] \cup f([a'', a'])$. Indeed, by the proven above $\omega(t) \ge a''$. On the other hand, when the point t gets to the right of a', it can only get there from $[a'', a']$ after which by our assumption it must be mapped to the left until it enters $[a'', a']$ again, and then this process repeats. Clearly this implies that $\omega(t) \subset [a'', a'] \cup f([a'', a'])$.

Now, by the assumption $a' = \sup A \notin A$, and we conclude that a' is approached from the left by the points of A. Hence, the point $a'' < a', a'' \in A$ can be chosen arbitrarily close to a'. Together with the proven in the previous paragraph it implies that $\omega(x) = a'$, a contradiction. So, indeed there exists N such that $a' < f^N(x) < f^{N+1}(x)$, for $i \ge N$ we have $f^i(t) > a'$, and a' is the greatest fixed point with such property.

Choose the fixed point b' so that the interval (a', b') is complementary to the set of all fixed points of f. Then all points of (a', b') are mapped to the right by f and on the other hand all points of the orbit of t to the right of b' are mapped to the left of themselves. Repeating the arguments now for the left direction we see that there exists a similar interval (c', d'). Clearly, $b' \le c'$ which completes the proof. ∎

The next lemmas deal with maps of type 2^n. We now introduce the notation which will be used in these lemmas. Suppose that the type of f is $2^n, n \leq \infty$. Let x be a point with infinite limit set $\omega(x)$. Choose (by Lemma 29.1) fixed points $a' < b' \leq c' < d'$ and replace x if necessary by its forward image so that $f^i(x) \in (a', b') \cup (c', d')$ for all $i \geq 0$, the map f maps all points of (a', b') to the right of themselves and all points of (c', d') to the left of themselves. Set $y = \min \omega(x)$.

LEMMA 29.2. *In the above notation if n is such that $y < f^n(x) < b'$ then $f^{n+1}(x) > b'$.*

Proof. By way of contradiction let us assume that for some n we have $y < f^n(x) < b'$ and $f^{n+1}(x) < b'$. Changing notation we may without loss of generality assume that $n = 0$. Set $I = [x, f(x)]$ and consider the sequence $x, f(x), \ldots$. We can always choose the least number $i \geq 0$ such that $f^{i+1}(x) > b'$. Then we have $x < f^i(x) < b' < f^{i+1}(x)$. Clearly, $J = [f^i(x), b'] \subset f^i(I)$ and also the intervals J and I have at most one points in common (namely, if $i = 1$ then $f(x)$ is this common point). Now, since $y \in \omega(x), y < x$ then there exists a number j such that $f^{i+j}(x)$ is so close to y that $f^{i+j}(x) < x$. This implies that $f^{i+j}(I) \supset I \cup J$ and $f^j(J) \supset I \cup J$. As follows from Exercise 29.3 (see below) this is impossible for maps of type 2^n, a contradiction. ∎

EXERCISE 29.3. *If g is a map of type $2^n, n \leq \infty$ then it is impossible to have two closed intervals I, J with at most one point in common and two powers k, l such that $f^k(I) \supset I \cup J$ and $f^l(J) \supset I \cup J$.*

LEMMA 29.4. *In the above notation $f(y) > b'$.*

Proof. First observe that by Lemma 29.2 $y \neq a'$. Indeed, otherwise we can choose n so that $f^n(x) > a'$ is close to a' and hence $f^{n+1}(x) > f^n(x)$ by Lemma 29.1 while on the other hand $f^{n+1}(x) < b'$ by continuity. This contradicts Lemma 29.2.

Hence, $y \neq a'$. Suppose that $f(y) \leq b'$. Since $y \in \omega(x)$ there exists $z \in \omega(x)$ such that $f(z) = y$. Since $y \neq a'$ we see that $z \neq y$ and hence $z > b'$. Now, let us show that there exists a point $z' \in [y, b']$ such that $f(z') = z$. Indeed, let $\max f|[y, b'] = s < z$. Then it is easy to see that $\omega(x) \subset [y, s]$, a contradiction with $z \in \omega(x)$. So, $z' \in [y, b']$ such that $f(z') = z$ exists. Clearly, $z' \in (y, b')$ which implies that $[y, z']$ and $[z', b']$ form a horseshoe in the non-strict sense of Exercise 29.2, a contradiction. ∎

Clearly, Lemma 29.2 and Lemma 29.4 imply the following corollary.

COROLLARY 29.5. *Replacing x by its forward image we may assume that $f^{2n}(x) \in (a', b')$ while $f^{2n+1}(x) \in (c', d')$ for any non-negative integer n. Thus, if $B = \omega(x) \cap (\infty, b']$ and $C = \omega(x) \cap [c', \infty)$ then $f(B) \subset C, f(C) \subset B$.*

30 No division and horseshoes 4

LEMMA 30.1. *Consider an interval map f of type $2^n, n \le \infty$. Then for every point x either $\omega(x)$ is a periodic orbit, or $\omega(x)$ is infinite and contains no periodic points at all.*

Before we start the actual proof it is worth pointing out that in general the claim of the lemma does not hold. Indeed, consider the full tent map f. Then, as we know, f is transitive and therefore the entire interval $[0,1]$ is a limit set. On the other hand, there are of course periodic points in $[0,1]$ (in fact, we have showed before that they are dense). Hence, for some maps there are infinite limit sets which contain periodic points. In fact, this example can be easily generalized: any transitive interval map has the entire interval as its limit set which contains some periodic points. The aim of Lemma 30.1 is to show that for maps of type $2^n, n \le \infty$ this is impossible.

Proof. It is enough to prove that if $\omega(x)$ is infinite then it cannot contain a fixed point. Indeed, if this is proven, we can simply consider powers of the map and thus prove that periodic points of f cannot belong to $\omega(x)$ either.

By way of contradiction assume that $a \in \omega(x)$ where a is a fixed point of f. Consider $\omega_{f^2}(x) = A$ and $\omega_{f^2}(f(x)) = B$. Then $\omega_f(x) = A \cup B$ and also $f(A) = B, f(B) = A$. This implies that $a \in A \cap B$. By Corollary 29.5 we may assume that $a = b' = c', A \subset (a', b']$ and $B \subset [c', d)$. Hence A is the infinite limit set of a point under the map f^2 such that the greatest point of A is f^2-fixed. By Corollary 29.5 this is impossible. ∎

COROLLARY 30.2. *Let $\omega(x)$ be the limit set of a map f of type 2^n which is not a fixed point. Let $a' < b' \le c' < d'$ are the fixed points whose existence is proven in Lemma 29.1. Then there exist points $z, y, u, v \in \omega(x)$ such that $a' < z \le y < b' \le c' < u \le v < d'$ such that $\omega(x) \subset [z, y] \cup [u, v]$ and the following holds: $f(\omega(x) \cap [z, y]) = \omega(x) \cap [u, v], f(\omega(x) \cap [u, v]) = \omega(x) \cap [z, y]$.*

Proof. By Lemma 29.1 there are fixed points $a' < b' \le c' < d'$ such that the orbit of x is contained in the union $(a', b') \cup (c', d')$ and all points in (a', b') (resp. (c', d')) are mapped to the right (resp. left). Set $z = \min \omega(x), y = \max \omega(x) \cap [a', b'], u = \min \omega(x) \cap [c', d']$ and $v = \max \omega(x)$. Then by Lemma 30.1 all these points are not fixed and hence $y < u$. The rest follows from Corollary 29.5. ∎

THEOREM 30.3. *Suppose that f is a map of type $2^n, n < \infty$. Then every limit set of f is a periodic orbit.*

Proof. Replacing the map by its power we see that it is enough to prove the result assuming that the map f is of type 1. In other words, assume that the only periodic points of f are its fixed points, and prove that then $\omega(x)$ is a fixed point for any x.

BWOC assume that $\omega(x)$ is not a fixed point. Then by Corollary 30.2 there are fixed points $a' < b' \le c' < d'$ and points $z, y, u, v \in \omega(x)$ such that $a' < z \le y < b' \le c' < u \le v < d'$, $\omega(x) \subset [z, y] \cup [u, v]$ and the following holds: $f(\omega(x) \cap [z, y]) = \omega(x) \cap [u, v], f(\omega(x) \cap [u, v]) = \omega(x) \cap [z, y]$. This implies that $f([z, y]) \supset [u, v]$ while $f([u, v]) \supset [z, y]$. Therefore there exists a 2-periodic point inside $[z, y]$, a contradiction with the assumption that the only periodic points of f are fixed points. ∎

Consider a map f of type 2^∞ and describe its infinite limit sets. The first step is made in Lemma 30.2. We now need to continue dividing the intersections of the limit set $\omega(x)$ with the segments of the real line into smaller intersections with segments. The proof is close to the analogous part of the proof of Theorem 28.2 in which we prove that for a map of type 2^n the periodic orbits must have maximal division. However first we prove an analog of Lemma 28.1 for infinite limit sets.

LEMMA 30.4. *Suppose that $\omega(x)$ is such that the following holds:*

1. *there exists a fixed point $a \notin \omega(x)$ such that $z = \min \omega(x) < a < \max \omega(x) = y$;*
2. *every point of $\omega(x) \cap (-\infty, a]$ is mapped strictly to the right while every point of $[a, \infty) \cap \omega(x)$ is mapped strictly to the left;*
3. *neither points z, y nor points $u = \max \omega(x) \cap (\infty, a], v = \min \omega(x) \cap [a, \infty)$ form a 2-periodic orbit.*

Then there exists a number n such that $f^n([u, v]) \supset [z, y] \supset \omega(x)$.

Proof. By Lemma 28.1 it is enough to consider the case of an infinite set $\omega(x)$. Set $I = [u, v]$. Then $f(I) \supset I$ because $f(u) \ge v, f(v) \le u$, and so $f^{n+1}(I) \supset f^n(I)$ for any n. The union $J = \bigcup_{k=0}^{\infty} f^k(I)$ is an invariant interval. Clearly, u or v belongs to the interior of $f(I)$ since u, v do not form a 2-periodic orbit (e.g., if $f(v) < u$ then u belongs to the interior of $f(I)$).

Let us show that $\overline{J} \supset [z, y]$. Otherwise we may assume that $z \notin \overline{J}$. Since there are points of $\omega(x)$ inside the interior of J we may assume that $x \in J$. Since $\overline{J}$ is an invariant interval this implies that $\omega(x) \subset \overline{J}$, a contradiction to $z \in \omega(x)$.

Now, neither z nor y is a fixed point, hence these points have preimages in $\omega(x)$ distinct from them. Thus, there are points $z', y' \in \omega(x)$ with $y' \le u, f(y') = y$ while $z' \ge v$ and $f(z') = z$. By the assumptions y, z do not form a periodic orbit of period 2, hence we may assume that $y' \in J$ and so $y \in J$. This implies that $z \in J$ and so $J \supset [z, y]$. Therefore there exists n such that $f^n(I) \supset [z, y]$ as desired. ∎

31 No division and horseshoes 5

We continue our study of maps of type 2^∞. Writing $A < B$ about two sets we mean that all points of A are to the left of B and $A \cap B = \emptyset$. Also, for every set A let $[A]$ be its **convex hull**, that is the smallest interval of some kind containing A.

LEMMA 31.1. *Let $\omega(x)$ be infinite. Then there are sets $D_1 < D_2 < D_3 < D_4$ such that $\bigcup_{j=1}^4 D_j = \omega(x)$ and exactly one of the following holds.*

1. $f(D_1) = D_3, f(D_3) = D_2, f(D_2) = D_4, f(D_4) = D_1$;

2. $f(D_1) = D_4, f(D_4) = D_2, f(D_2) = D_3, f(D_3) = D_1$.

Proof. We use the notation from Corollary 30.2. Set $A = [z, y] \cap \omega(x)$ and $B = [u, v] \cap \omega(x)$. Then $A < B$ and it follows that $f(A) = B, f(B) = A$. Now, replacing if necessary x by its forward image we can assume that $A = \omega_{f^2}(x)$ and $B = \omega_{f^2}(f(x))$. Then we can apply Corollary 30.2 to A, B and f^2. This implies that there are sets $D_1 < D_2 < D_3 < D_4$ such that $D_1 \cup D_2 = A, D_3 \cup D_4 = B$ and $f^2(D_1) = D_2, f^2(D_2) = D_1, f^2(D_3) = D_4, f^2(D_4) = D_3$.

Let us show that either $f(D_1) = D_3$ or $f(D_1) = D_4$. By way of contradiction let us assume that $f(D_1) \cap D_3 \neq \emptyset$ and $f(D_1) \cap D_4 \neq \emptyset$. This implies that $f([D_1])$ contains the entire closed interval I which stretches from the maximum of D_3 through the minimum of D_4. Now, by Lemma 30.4 we see that there exists a number n such that $f^n(I) \supset [D_3 \cup D_4]$ and therefore $f^{n+1} \supset [D_1 \cup D_2]$. On the other hand, $f^2([D_2]) \supset [D_1]$ which implies that $f^{n+3}([D_2]) \supset [D_1 \cup D_3]$. So, some power of f has a horseshoe, a contradiction.

It remains to observe that if $f(D_1) = D_3$ then $f(D_3) = f^2(D_1) = D_2$, $f(D_2) = f^2(D_3) = D_4$ and $f(D_4) = f^2(D_2) = D_1$. On the other hand, if $f(D_1) = D_4$ then $f(D_4) = f^2(D_1) = D_2$, $f(D_2) = f^2(D_4) = D_3$ and $f(D_3) = f^2(D_2) = D_1$. This completes the proof of Lemma 31.1. ∎

LEMMA 31.2. *Let $\omega(x)$ be infinite. Then for every n there is a collection of sets $D_1^n < D_2^n < \ldots < D_{2^n}^n$ such that $\bigcup D_j = \omega(x)$ and the following holds.*

1. *For every i we have $D_{2i-1}^n \cup D_{2i}^n = D_i^{n-1}$;*

2. *The sets $D_j^n, j = 1, \ldots, 2^n$ are cyclically permuted by the map f.*

Proof. We prove Lemma 31.2 by induction. In fact the base of induction has already been established in Lemma 31.1. To make the step of induction let us assume that the claim holds for sets D_j^{n-1}. Then D_j^{n-1} is actually a $f^{2^{n-1}}$-limit set to which Corollary 30.2 is applicable. Hence we can divide D_j^{n-1} into sets $D_{2j-1}^n < D_{2j}^n$ such that

$f^{2^{n-1}}(D^n_{2j-1}) = D^n_{2j}$ and $f^{2^{n-1}}(D^n_{2j}) = D^n_{2j-1}$ for every j.

Let us fix j, divide the set D^{n-1}_j by Corollary 30.2 as described and then do the same with its image $f(D^{n-1}_j)$ which is by the inductive hypothesis one of the sets from the collection of sets on the $(n-1)$-st level. Assume that $f(D^{n-1}_j) = D^{n-1}_k$ and divide D^{n-1}_k into two subsets D^n_{2k-1} and D^n_{2k} as prescribed by Corollary 30.2. We are now in the situation which is almost literally the same as that of Lemma 31.1 with the only difference that the maps applied to our sets are other powers of f.

Thus we can slightly amend the arguments fro Lemma 31.1 and make them applicable to the situation of Lemma 31.2. This will yield that there are powers s, t of f such that $f^s([D^n_{2j-1}]) \supset [D^{n-1}_j]$ and $f^t([D^n_{2j}]) \supset [D^{n-1}_j]$ which implies that some power of f has a horseshoe, a contradiction. This completes the proof of Lemma 31.2. ■

The numbering of the sets of the n-th level used in Lemma 31.2 was spatial. We would like to change it now to a temporal numbering. That is, let us always denote the leftmost of the sets of the n-th level by $D^n_0 = E^j_0$ and then denote the sets $f^j(E^n_0)$ by E^n_j. In other words, the sets E^n_j are the sets D^n_j in a different, temporal and not spatial, order. It follows that $\omega(x) = \cup_{j=0}^{2^n-1} E^n_j$. Therefore for every point $y \in \omega(x)$ there exists a unique number $j_n \in \{0, 1, \ldots, 2^n - 1\}$ such that $y \in E^n_{j_n}$. This defines a map $\varphi : \omega(x) \to M = \prod M_{2^n}$ where M_{2^n} is the set of all residues $\bmod 2^n$.

LEMMA 31.3. *The map φ is continuous and maps $\omega(x)$ onto the subset of M defined before as $S(1, 2, 4, \ldots, 2^n, \ldots)$.*

Proof. The fact that φ is continuous follows from the fact that the sets D^n_j are disjoint and closed. To prove that the image is exactly $S(1, 2, \ldots)$ one needs to show that if $\varphi(y) = (j_0, j_1, \ldots)$ then $j_{k+1} \equiv j_k \bmod 2^k$. Indeed, it follows from the definition that $y \in E^k_{j_k} = f^{j_k}(E^k_0)$. Now, $E^{k+1}_0 \subset E^k_0$. Hence $f^{j_k}(E^{k+1}_0) = E^{k+1}_{j_k} \subset E^k_{j_k}$ and

$$f^{j_k+2^k}(E^{k+1}_0) = E^{k+1}_{j_k+2^k} \subset f^{2^k}(E^k_{j_k}) = E^k_{j_k}.$$

Since there are two sets of level $k+1$ whose union is $E^k_{j_k}$ we see that

$$E^{k+1}_{j_k} \cup E^{k+1}_{j_k+2^k} = E^k_{j_k}.$$

Hence the set of level $k+1$ which contains y is either $E^{k+1}_{j_k}$ or $E^{k+1}_{j_k+2^k}$, and so by the definition $j_{k+1} = j_k$ or $j_{k+1} = j_k + 2^k$.

By the previous paragraph for every j and k we have $E^k_j = E^{k+1}_j \cup E^{k+1}_{j+2^k}$. Therefore for every sequence $(j_0, j_1, \ldots) \in S(1, 2, \ldots)$ the sets $E^k_{j_k}$ form a nested sequence and have a non-empty intersection. Any point of such intersection then is projected onto $(j_0, j_1, \ldots)$ which proves that φ is onto and completes the proof of the lemma. ■

32 No division and horseshoes 6

Set $S(1,2,\dots) = H$ and set the shift by $(1,1,\dots)$ in H be denoted by τ. Consider an infinite limit set $\omega(x)$ of a map f of type 2^∞ and the map φ defined in the previous lecture.

LEMMA 32.1. *The map φ semiconjugates $f|\omega(x)$ with τ.*

Proof. Indeed, let $y \in \omega(x)$ and $\varphi(y) = (j_0, j_1, \dots)$. Then $y \in f^{j_k}(E_0^k)$ for every k. Hence $f(y) \in f^{j_k+1}(E_0^k)$. Then $\varphi(f(y)) = (j_0+1, j_1+1, \dots) = \tau(\varphi(y))$ as desired. ∎

Lemma 32.1 provides a model for any infinite limit set of a map of type 2^n. The remaining question is as follows: how non-injective is the map φ? To answer this question observe that H is an uncountable set. Indeed, every point in H is a sequence $(j_0, j_1, \dots)$ where $j_{k+1} \equiv j_k \bmod 2^k$. Now, let us associate with this sequence the sequence $(i_0, \dots)$ of zeros and ones defined as follows: $i_k = 0$ if $j_k = j_{k+1}$ and $i_k = 1$ if $j_{k+1} = j_k + 2^k$. Clearly, this is a 1-to-1 correspondence between H and all possible sequences of zeros and ones. Therefore, H is uncountable.

LEMMA 32.2. *There are no more than countably many elements* $\mathbf{i}$ *of* H *such that* $\varphi^{-1}(\mathbf{i})$ *is not a point.*

Proof. By the definition for every $(j_0, j_1, \dots) \in H$ we have that $\varphi^{-1}(j_0, \dots) \subset \bigcap [E_{j_k}^k] = R(j_0, \dots)$ where $R(j_0, \dots)$ is a closed (perhaps degenerate) interval; let us call such intervals R-intervals. Since sets $E_i^k, 0 \le i \le 2^k - 1$ are ordered on the line we conclude that R-intervals determined by different sequences from H are disjoint. It follows from the construction that on every step the endpoints of $E_{j_k}^k$ belong to the $\omega(x)$ which implies that the endpoints of any R-interval belong to the $\omega(x)$ (we need this remark later).

Now, since the φ-preimages of sequences in H are contained in their R-intervals we see that if an R-interval is degenerate then the corresponding φ-preimage is a point too. Finally, since there are no more than countably many non-degenerate R-intervals then there are no more than countably many points of H whose φ-preimages are non-degenerate. ∎

We want to study now the orbit of an R-interval and prove that such interval is roaming (recall that a set A is said to be **roaming** if all the images of this set are pairwise disjoint). We will do this in a sequence of steps.

LEMMA 32.3. *An R-interval contains no periodic points.*

Proof. It is enough to consider a non-degenerate R-interval $J = [a, b]$. Since points a, b belong to $\omega(x)$ they are not periodic. Suppose that $c \in (a, b)$ is a periodic point. Then it can only be of period 2^k for some k. Consider the interval of the form $[E^k_j]$ containing J. Then J is contained in one interval of next level of the form $[E^{k+1}_i]$ where $a, b \in E^{k+1}_i$. By our description of the limit set $\omega(x)$ we see that the entire set E^{k+1}_i is mapped by f^{2^k} either to the right of itself or to the left of itself. For the sake of definiteness let us assume that $f^{2^k}(E^{k+1}_i) > E^{k+1}_i$.

As follows from our description of the limit set $\omega(x)$, the entire set E^k_j is actually the ω-limit set of some iteration of x under f^{2^k}. This iteration y of x can always be chosen so that y is very close to b, hence $c < y$ and also $f^{2^k}(y) > y$. On the other hand, $a \in \omega_{f^{2^k}}(y)$ which means that for some high power f^{r2^k} of f^{2^k} we will have $f^{r2^k}(y) < c$. Denoting the map f^{2^k} by g we have the following picture: $g^r(y) < c = g(c) < y < g(y)$. As we have seen several times before, this picture implies that g admits L-scheme, a contradiction which proves the lemma. ■

LEMMA 32.4. *The orbit of an R-interval contains no periodic points.*

Proof. Suppose otherwise. Then we may assume that an R-interval $I = [a, b]$ is non-degenerate and that some point $c \in (a, b)$ is mapped into a periodic point by some power of f. The periodic orbit in question must be of period 2^m for some m; iterating f more we can always assume that the power of f which maps c onto a 2^m-periodic point is itself a multiple of 2^m, that is that for some r the point $f^{r2^m}(c) = v$ is of period 2^m. For the sake of definiteness let us assume that $v < a$ and consider the orbit of the interval $[v, c]$ under the map $g = f^{2^m}$.

We claim that the entire orbit of $[v, c]$ under g is non-strictly to the left of c. Indeed, otherwise there is a point $d \in [v, c]$ and a number l such that $g^l(d) > c$. Then since v is a g-fixed point we see that there is a point $d' \in (v, d)$ such that $g^l(d') = c$. In other words, we have $v = g(v) = g^m(c) < d' < g^l(d') = c$. Clearly, this type of behavior is impossible for maps of type 2^n, a contradiction. Therefore the closure of orbit of $[v, c]$ under g is an non-degenerate closed interval $J = [u, c]$.

Choose $k > m$ and the set E^k_i so that $I \subset [E^k_i]$. Then $a, b \in E^k_i$. Observe that as before we can always choose an iterate y of x very close to a and in such a way that $\omega_{f^{2^k}}(y) = E^k_i$. However in this case $y \in J$ and therefore $\omega_{f^{2^k}}(y) \subset \omega_g(y) \subset J$, a contradiction to $b \in \omega_g(y)$. Thus the assumption that I contains a preimage of a periodic point fails. ■

We are ready now to prove that all R-intervals are roaming which will be done next time. In the proof we will need the following claim established in Lecture 9.

CLAIM 32.5. *Let J be a closed non-roaming interval. Then there exist numbers n, m such that $A = \mathrm{orb}(f^n(I))$ is a cycle of pairwise disjoint intervals $I_0, \dots, I_{m-1}$ cyclically permuted by the map.*

33 No division and horseshoes 7

LEMMA 33.1. *An R-interval is roaming.*

Proof. Let $J = [a, b]$ be an R-interval and assume by way of contradiction that it is not roaming. Since by Lemma 30.1 $\omega(x)$ contains no periodic points we may assume that J is non-degenerate. Also, by the remark in the proof of Lemma 32.2 we have $a, b \in \omega(x)$.

By Claim 32.5 we can choose the numbers m, n and intervals $I_0, \dots, I_{m-1}$ as in this claim. Let us show that I_0 must contain an f^m-fixed point. Indeed, otherwise consider $f^m|\overline{I}_0$; this map has no fixed points inside I_0 and therefore all points of I_0 converge to one of the endpoints of I_0, say, u which must be a periodic point. On the other hand, the orbits of a and b are from some time on contained in $\mathrm{orb}(f^n(J))$ and therefore the limit sets of a and b must contain u. Since $a, b \in \omega(x)$ this is a contradiction to the fact that there are no periodic points in $\omega(x)$ (Lemma 30.1). Thus, there exists a point $c \in J$ and a number k such that $f^k(c)$ is periodic, contrary to proven in Lemma 32.4. ∎

LEMMA 33.2. *The semiconjugacy φ is at most* 2*-to-*1*.*

Proof. Consider any R-interval $I = [a, b]$. Then since it is roaming there are no points of $\omega(x)$ inside (a, b). Indeed, otherwise there is a point $c \in \omega(x) \cap (a, b)$, hence x will enter (a, b) infinitely many times and so I is not roaming, a contradiction. This proves the claim. ∎

We can now summarize all we know about the limit sets of maps of type 2^n.

THEOREM 33.3. *Let f be a map of type 2^n. Then the following holds.*

1. *Any periodic orbit of f has maximal division.*
2. *If $n < \infty$ then any limit set of f is a periodic orbit.*
3. *If $n = \infty$ then f may have infinite limit sets. If $\omega(x)$ is an infinite limit set of f then $f|\omega(x)$ is semiconjugate to the shift by $(1, 1, \dots)$ in $S(1, 2, 4, \dots)$, this semiconjugacy is at most* 2*-to-*1 *and is not $1 - 1$ at no more than a countable set of points.*

We can actually extend our understanding of the dynamics of maps of type 2^∞ a little bit. Let us go back to the construction and consider the intervals $[E_i^k]$. As follows from the construction, these intervals have f^{2^k} which contain themselves. Therefore given an interval $[E_i^k]$ we see that its orbit is a cycle of intervals $I_0, \dots, I_{m_k-1}$. The thing is, we do not know what exactly is the period m of this cycle of intervals, and conceivably it can be less than 2^k. In fact, it is indeed possible to have $m_K < 2^k$. However the following lemma still takes place.

LEMMA 33.4. *In the situation described above the periods m_k grow to infinity.*

We will not prove Lemma 33.4 but suggest it as a very good exercise which can be done similarly to how we prove Lemmas 32.3 and 32.4. This lemma in fact allows us to consider the dynamics of maps of type 2^∞ and their infinite limit sets in a very easy way. Namely, any infinite limit set of f is contained in a nested sequence of cycles of intervals of periods which are powers of 2. This brings us to the situation which is similar to the one above but is not confined by a specific structure of the periods of cycles of intervals.

Namely, let $A_0 \supset A_1 \supset \ldots$ be a sequence of cycles of intervals $I_0 \supset I_1 \ldots$ respectively such that their periods $m_0 < m_1 < \ldots$ grow. Let $A = \bigcup_{i=0}^{\infty} A_i$. Then it follows that every m_{n+1} is a multiple of m_n, and A is a closed invariant set. It turns out that we can define a map $\varphi : A \to S(m_0, m_1, \ldots)$ the same way this was done before for infinite limit sets of a map of type 2^∞. Namely, if $y \in A$ then for any j there is a well-defined number k_j such that $y \in f^{k_j}(A_0)$, and define $\varphi(y)$ as the sequence $(k_0, k_1, \ldots)$. It turns out that the map φ in fact has the usual properties which are listed below in Lemma 33.5.

LEMMA 33.5. *Using the terminology introduced above we can say that the map φ semiconjugates $f|A$ with $\tau : S(m_0, \ldots) \to S(m_0, \ldots)$ where τ is defined as the shift by $(1, 1, \ldots)$ in $S(m_0, \ldots)$. Moreover, φ is monotone in the following sense: the preimage of any point is an interval (perhaps degenerate).*

Since Lemma 33.5 can be proven by familiar methods we skip its proof. Observe that A cannot contain a periodic point. Indeed, if it does then the periodic orbit of this point is entirely contained in A which means that cycles of intervals involved in the definition of A cannot have periods greater than the period of that periodic point, a contradiction. However, even more can be proven.

LEMMA 33.6. *Suppose that there exists a limit set $\omega(x)$ such that $\omega(x) \cap A \neq \emptyset$. Then in fact $\omega(x) \subset A$.*

Proof. Indeed, if $y \in A \cap \omega(x)$ then since y is non-preperiodic by Lemma 33.5 we have that for every i there exists a power n_i which maps y *inside* the interior of A_i. The fact that $f^{n_i}(y) \in \omega(x)$ implies than that some image of x is inside A_i, and hence the entire orbit of x is inA_i from that time on. We conclude that $\omega(x) \subset A_i$, and since this holds for any i we see that $\omega(x) \subset A$ as desired. ∎

In the next lecture, last lecture of the course, we describe without proof a big picture of dynamics for interval maps; Theorem 33.3 is only a particular case of this picture.

34 Decomposition of the union of limit sets of an interval map

We use the notation from the previous section. By Lemma 33.6, the limit sets non-disjoint from A are contained in A.

THEOREM 34.1. *Given the set A consider the union B of all limit sets non-disjoint from A (and hence contained in A). Then B is the limit set itself. Any other limit set are either disjoint from B or contained in B.*

B is a **maximal by inclusion limit set which contains no periodic points**. In fact **all** maximal by inclusion limit set which contain no periodic points are like that. The limit sets maximal by inclusion among all limit sets play an important role below; we will simply call them **maximal limit sets**. Those of them which are considered in Theorem 34.1 (i.e. the ones which do not contain cycles) are called **adding machines**. The remaining maximal by inclusion limit sets of an interval map f can be of two types. First of all these are maximal by inclusion among all limit sets periodic orbits (cycles).

LEMMA 34.2. *Attractive in the usual sense periodic orbits are maximal limit sets.*

Some other cycles are also maximal limit sets (e.g. fixed points of the identity map). Call such cycles **maximal cycles**. The remaining are infinite limit sets maximal by inclusion and containing periodic orbits. It turns out that these are the most interesting in terms of dynamics limit sets. They are called **basic** limit sets and can be described as follows.

THEOREM 34.3. *A basic limit set is a set C for which there exists a cycle of intervals D such that the following holds.*

1. *$f|D$ is monotonically semiconjugate to a transitive map g on a cycle of intervals;*
2. *for any $x \in B$ and any neighborhood U of x in D the orbit of U is dense in D;*
3. *the semiconjugacy φ collapses all intervals complementary to C in D and is therefore at most* 2*-to-*1 *on C;*
4. *C is a Cantor set.*

In fact, like in the case of adding machines one can see that if there is a limit set B with properties (1)-(3) from Theorem 34.3 then B has to be maximal. Indeed, suppose that $I \supset B = \omega(x)$ is an invariant interval and all properties listed in (1)-(3) in Theorem 34.3 are satisfied. Assume that there is another limit set $\omega(y)$ such that $B \cap \omega(y) \neq \emptyset$. Clearly, there are two cases then: either y enters I at some point or y y does not do so.

Consider the first case. If $f^n(y) \in I$ then without loss of generality we may assume that $y \in I$. If $y \in B$ then of course $\omega(y) \subset B$ as desired. If $y \notin B$ then y belongs to a complementary to I interval $J = (a, b)$. Now, if J projects by φ onto a non-preperiodic point of g then the fact that φ semiconjugates f and g implies that J is a roaming interval. This in turn implies that its length converges to zero and therefore that every point in $\overline{J}$ has the same limit set. In particular $\omega(y) = \omega(a) \subset B$ where the latter containment follows from the fact that the endpoints of J belong to B. So in this case $\omega(y) \subset B$. Observe that then $\omega(y)$ cannot be an adding machine. Indeed, otherwise it is contained in cycle of intervals of greater and greater periods while as we know by Theorem 34.3 any neighborhood of any point of $omega(y) \subset B$ is dense in I, a contradiction.

If J projects onto a g-preperiodic point then we may assume that J is a periodic interval, and since $y \in J$ then $\omega(y) \subset \operatorname{orb} J$. Since $\omega(y)$ is non-disjoint from B then the set $\omega(y) \cap B$ is finite and contains periodic points. Observe that in this case $\omega(y)$ cannot be an adding machine because adding machines contain no periodic points.

If y does not enter I but $\omega(y)$ has common points with B then these common points must come from the endpoints of I, so $\omega(y) \cap B$ is finite and contains periodic points (and hence $\omega(y)$ cannot be an adding machine). This analysis implies the following result.

LEMMA 34.4. *Suppose that B is a basic set and $\omega(y) \cap B \not\emptyset$. Then $\omega(y)$ is not an adding machine and nor is it a maximal cycle. Moreover, either $\omega(y) \subset B$ or $\omega(y) \cap B$ is finite and contains periodic points.*

The idea of decomposition of the union $\omega(f)$ of limit sets for interval maps is as a follows. Given $A = \omega(y)$ consider all possible cycles of intervals containing A. If their periods grow to infinity, we have an adding machine. If not, we pick the smallest by inclusion cycle of intervals containing A, and then prove that in fact A is contained in the basic set corresponding to this cycle of intervals. Thus we get the following Decomposition Theorem (similar to Lemma 34.4 methods show that there are at most countably many basic sets).

THEOREM 34.5. *The set $\omega(f)$ can be decomposed into countably many basic sets, some adding machines and some maximal cycles. In this collection intersections are possible only between basic sets and is always finite.*

The main tool now becomes Theorem 34.3 which tells us that to study basic sets it is enough to study transitive interval maps which serve a model for basic sets. E.g., it turns out that transitive interval maps have so-called **specification** property which implies that measures concentrated on periodic orbits are dense in all invariant measures.

THEOREM 34.6. *Every ergodic measure of an interval map f can be approximated arbitrarily well by a measure concentrated on a periodic orbit.*