AN INEQUALITY FOR LAMINATIONS, JULIA SETS AND "GROWING TREES"

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Abstract. For a closed lamination on the unit circle invariant under $z \mapsto z^d$ we prove an inequality relating the number of points in the "gaps" with infinite pairwise disjoint orbits to the degree; in particular, this gives estimates on the cardinality of any such "gap" as well as on the number of distinct grand orbits of such "gaps". As a tool, we introduce and study a dynamically defined growing tree in the quotient space. We also use our techniques to obtain for laminations an analog of Sullivan No Wandering Domain Theorem. Then we apply these results to Julia sets of polynomials.

Introduction

A central object of studying in holomorphic dynamics is the Julia set J of a polynomial P (see e.g. [DH], [F], [J], [Mi1]). J is the boundary of the basin of attraction A_∞ of infinity of P. It is a compact subset of the plane, and, as a rule, a fractal set. Combinatorics, topology of the Julia set and their relations to dynamics are of main interest (see e.g. [BH], [Do1], [DH], [H], [Th]).

Assume, for a moment, that J is a locally connected (and hence connected [Mi1]) set. By Caratheodory theorem [CL], any point $x \in J$ is then accessible from A_∞ by a path, and one can choose it to be a so-called external ray (see [DH] and Sect. 3). Denote by $N(x)$ the number of such external rays tending to x. (Equivalently, $N(x)$ is equal to the number of connected components of $J \setminus \{x\}$.) By the **grand orbit** (of x) we mean the union of all preimages of all iterates of x. Note that the number $N(x')$ is the same for all points x' of the grand orbit of x (if it contains no critical points of P). Also, call a point **preperiodic** (**precritical**) if it is mapped into a periodic (critical) point by $f^k, k \geq 0$. Loosely speaking, we prove that a point $x \in J$ with $N(x) \geq 3$ is preperiodic or precritical provided x is outside of at most $d-2$ grand orbits of P, with $d = \deg(P)$. This is a byproduct of the following inequality.

Theorem A. *Let f be a polynomial whose Julia set is locally connected. Then we have $\sum_{x\in\Gamma}(N(x)-2) \leq k_\sim - k_S - 1 \leq d-2$, where Γ is a non-empty collection of non-preperiodic non-precritical points $x \in J$ from distinct grand orbits, such that $N(x) \geq 3$, $k_\sim$ is the number of distinct grand orbits of non-preperiodic critical points $c \in J(P)$ with*

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$N(P(c)) = 1$ (i.e., $P(c)$ is the landing point of exactly one external ray) and k_S is the number of periodic orbits of the Siegel discs.

An important open problem here is whether the bounds in Theorem A are sharp.

See Theorem 3.1 for more complete results. For a non-locally connected Julia set, we consider Yoccoz puzzle impressions instead of points, and prove a similar inequality (Theorem 3.3).

The inequality in Theorem A follows from Theorem B on laminations and is related to a question of Thurston [Th]. In turn, Theorem A implies the following known fact.

Corollary 1. (cf. [Do1], [Po]). *Let P be a polynomial such that every critical point of P is either attracted by a periodic orbit or preperiodic. Then every $x \in J$ with $N(x) > 2$ is either preperiodic or precritical.*

Corollary 2. (cf. [Ki]). *Under the condition of Theorem A, $N(x) \leq d$ for every not preperiodic not precritical point $x \in J$.*

See [Ki] for more details.

The essence of our proof is to construct and study a growing tree in an appropriate space (which is either a locally connected Julia set or, more generally, a certain quotient space of a lamination). An inspiration for us comes from the theory of Hubbard trees [DH], [Do1] which are introduced for the polynomials whose critical points are preperiodic (in this case our tree is reduced to the Hubbard tree). However, we make no restrictions on the orbits of the critical points. A general growing tree is an increasing sequence of finite trees defined dynamically. Our study could be considered as a development of both the theory of abstract Hubbard trees [DH], [Do1], [Po] and the description of compact sets on the plane by Douady [Do1].

To study Julia sets (including not locally connected ones), we work with laminations in the disc. This object is defined in [Th] to describe basic rules of identification on the unit circle S^1 corresponding to connected polynomial Julia set: $t_1, t_2 \in S^1$ are identified iff external rays of these arguments tend to the same point of the Julia set. Such relations have been studied before ([DH], see also [Mi1]); our definitions and approach are closer to [Do1], see also [McM]. Polynomial Julia sets are the main source of examples of laminations. Yet, the class of laminations we study is more general: there exist closed invariant laminations which do not correspond to the Julia set of any polynomial, see examples in Section 4.

We prove our main inequality (*) for laminations (see Theorem B below) and then deduce the inequalities for Julia sets. To prove (*), we consider the quotient space of a lamination, as in [Do1], and construct the growing tree in the quotient space.

Let us be more precise now. The **lamination** is an equivalence relation $\sim$ on the unit circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ (identified with $S^1 = \{z \in \mathbb{C} : |z| = 1\}$) such that the convex hulls in the unit disc of distinct equivalence classes are disjoint. We assume also that $\sim$ is closed and invariant under the map $\sigma : \mathbb{T} \to \mathbb{T}$, $\sigma(t) = d \cdot t (\mod 1)$ where $d \geq 2$ is a fixed integer (we identify it with the map $z \mapsto z^d$ on S^1). See Section 2 for more details.

Call a class g of $\sim$ **critical** iff the map $\sigma : g \to \sigma(g)$ is not 1-to-1. Let $k_\sim$ be the maximal number of critical classes g from pairwise different orbits such that $\sigma(g)$ is a single point with infinite σ-orbit (i.e., $\sigma(g)$ is an irrational point of $\mathbb{T}$).

Denote by $|A|$ the cardinality of a set A (thus $0 \leq |A| \leq \infty$).

Theorem B. *Let Γ be a non-empty collection of classes of $\sim$, which are neither preperiodic nor precritical, and belong to pairwise disjoint orbits. If $|g| \geq 3$ for every $g \in \Gamma$ then*

$$\sum_{g \in \Gamma} (|g| - 2) \leq k_\sim - 1 \leq d - 2. \qquad (*)$$

Moreover, for any non-preperiodic class g' we have $|g'| \leq 2^d$.

The full version of (*) counts also so-called "Siegel components": see Theorem 2.10.

Theorem B implies the following results.

Corollary 3. ([Th]). *For quadratic laminations (i.e., $d = 2$), every class with at least 3 points is either preperiodic or precritical.*

Corollary 4. *The number of different grand orbits of non-preperiodic non-precritical classes with at least 3 points is at most $d - 2$.*

The question about generalizations of the statement of Corollary 3 to higher degrees is asked in [Th]. The inequality (*) gives an answer. Also, the methods in [Th] and [Ki] differ essentially from ours.

Remark. If J is a connected locally connected Julia set of a polynomial P, then the quotient space $S^1/\sim$ of the lamination $\sim$ corresponding to P is homeomorphic to J. The number $N(x)$, for $x \in J$, equals the cardinality $|g|$ of the corresponding equivalence class $g = p^{-1}(x)$ (where $p : S^1 \to S^1/\sim$ is the factor map). Corollaries 2-4 can be immediately rephrased in this context. For example, $N(x) \leq 2^d$, if x is non-preperiodic. Note that if $x \in J$ is a preperiodic point, the number $N(x)$ is also finite (by so-called Douady's lemma, see e.g.[Mi1]), but may be arbitrary large (for the polynomials of a fixed degree). □

Let $\mathbb{D}$ be the open unit disk, $L_\sim = L$ be the union of $\sim$-hulls, i.e. convex hulls (in the Poincaré metric) of $\sim$-classes. The extension $\simeq$ of $\sim$ onto $\bar{\mathbb{D}}$ mentioned above is defined as follows [Do1]: a $\simeq$-class is a $\sim$-hull or a point of $\bar{\mathbb{D}} \setminus L$. Call a connected component of the complement $\mathbb{D} \setminus L$ a **$\sim$-component**. For an open set Ω in $\mathbb{D}$ let $E(\Omega) = \bar{\Omega} \cap S^1$. Denote by $J_\sim$ the quotient space $\mathbb{T}/\sim$, and by $f : J_\sim \to J_\sim$ the map induced by $\sigma : \mathbb{T} \to \mathbb{T}$. Theorem C says that $\sim$-components and continua in $J_\sim$ are not wandering. Its part (1) is similar to the fundamental Sullivan No Wandering Domain Theorem for the rational maps [Su] (see Section 3). Part (2) generalizes a result of [Le] (cf. [BL1]).

Theorem C. *The following holds for a closed invariant lamination $\sim$.*

1. *Let Ω be a $\sim$-component. Then the set $E(\Omega) \subset \mathbb{T}$ is σ-preperiodic in the following sense: there exist $n \geq 0, m > 0$ with $\sigma^m(E(\Omega)) = \sigma^{m+n}(E(\Omega))$.*
2. *If $M \subset J_\sim$ is a non-degenerate continuum then it is non-wandering (i.e., there exist $n \geq 0, m > 0$ with $f^n(M) \cap f^{n+m}(M) \neq \emptyset$).*

The structure of the paper is the following. In Section 1 we introduce and study growing trees in a metric space. Main result is Theorem 1.3. Then we apply these tools to invariant laminations on the unit circle (Section 2) and prove Theorems B-C stated above. These results find immediate applications in complex dynamics (Section 3). A short last Section 4 is devoted to some examples and generalizations. Notions to be used throughout the paper are written usually in boldface.

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1. Growing trees

A **tree** is a connected compact one-dimensional branched manifold with no subsets homeomorphic to a circle. Let $a \in T$. If $T \setminus \{a\}$ has n connected component, then the **order** of T at a is $\text{ord}_T(a) = n$. The point a is called an **endpoint** (of T) if $\text{ord}_T(a) = 1$, an **inner point** (of T) if $\text{ord}_T(a) = 2$ and a **vertex** (of T) if $\text{ord}_T(a) \geq 3$. Clearly, a tree has finitely many vertices and endpoints. An **arc** (in T) is a subset of T homeomorphic to an interval. An **edge** (of T) is an arc whose endpoints are vertices or endpoints and whose other points are inner points of T. The absence in T of sets homeomorphic to circles makes the arc $[a, b]$ with endpoints $a, b \in T$ well-defined. The number of edges of T is finite. Also, a **germ** of a tree W is a pair (a, S), where $a \in W$ and S is a small semi-neighborhood of a in W containing no vertices/critical points of W inside. Its image is defined as $f(a, S) = (f(a), f(S))$.

Let X be a metric space, $T \subset X$ be a tree, $f : X \to X$ be a continuous map. Denote the sets $\bigcup_{i=0}^{n} f^i(T)$ by T_n and the set $\bigcup_{i=0}^{\infty} f^i(T)$ by T_∞. If (a) $f(T) \cap T \neq \emptyset$, (b) T_n is a tree for any n, and (c) there is a finite set of **critical** points $C_f = \{c_1, \dots, c_k\} \subset T_0$ with $f|T_\infty$ injective in some neighborhood of any $x \in T_\infty \setminus C_f$, then we call the sequence of sets $T_0 \subset T_1 \subset \cdots \subset T_\infty$ (or the set T_∞) a **growing tree**. Also, a point $x \in T_\infty$ is called a **vertex of** T_∞ if x is a vertex of some T_n.

For example, let $T = T_0$ be a letter $E \subset \mathbb{R}^2$ with horizontal segments $[(0, 1), (1, 1)]$, $[(0, 0), (1, 0)]$, $[(0, -1), (1, -1)]$. Let $f(x, y) = (x, 2y)$. Then $T_1 = T \cup f(T)$ consists of 5 horizontal and 1 vertical segments, $T_1 \setminus T_0$ consists of 2 semi-open arcs and, moreover, $T_{n+1} \setminus T_n$ consists of 2 semi-open arcs. This example is illustrated on Figure 1.

Lemma 1.1 shows how trees can grow; the proof of the lemma is left to the reader.

Lemma 1.1. *Let $T \subset T'$ be trees. Then the set $T' \setminus T$ has finitely many components $t_1, \dots, t_l$, all t_i are trees, $\bar{t}_i \cap T = \{x(t_i)\}$ is a point and, moreover, $\mathrm{ord}_{T'}(x(t_i)) \geq \mathrm{ord}_T(x(t_i)) + 1 \geq 2$ for any i.*

In the situation of Lemma 1.1 for a component t of $T' \setminus T$ we call the point $x(t)$ the **basepoint** (of t) and other endpoints of t **outer** endpoints of t (T'). Let the number of outer endpoints of t be $\mathrm{oen}(T, t)$ and the number of all outer endpoints of T' be $\mathrm{oen}(T, T')$. Then $\mathrm{oen}(T, T') = \sum_i \mathrm{oen}(T, t_i)$; e.g., if T' has the shape of the letter H and T is its "plank" then $T' \setminus T$ consists of 4 intervals $\{t_i\}_{i=1}^4$, $\mathrm{oen}(t_i) = 1$ and $\mathrm{oen}(T, T') = 4$. For a growing tree T_∞ Lemma 1.1 implies that $T_{n+1} \setminus T_n = \cup_{j=1}^{k_{n+1}} t_j^{n+1}$ where t_j^{n+1} are components of $T_{n+1} \setminus T_n$ with basepoints x_j^{n+1}.

Consider how the number of outer endpoints changes for a growing tree.

Lemma 1.2. *Let $T_n \subset T_{n+1} \subset T_{n+2}$ come from a growing tree. Then $\mathrm{oen}(T_n, T_{n+1}) \geq \mathrm{oen}(T_{n+1}, T_{n+2})$ and any outer endpoint of T_{n+2} is the image of an outer endpoint of T_{n+1} (and all outer endpoints of any T_n are eventual images of outer endpoints of T_1).*

Proof. If a be an outer endpoint of T_{n+1} then $a = f(b)$ with $b \in T_{n+1} \setminus T_n$. Since f on a component of $T_{n+1} \setminus T_n$ is a homeomorphism then b is an outer endpoint of T_{n+1}. ■

On Figure 1 $\mathrm{oen}(T_0, T_1) = 2$, and actually $\mathrm{oen}(T_n, T_{n+1}) = 2$ for any $n \geq 0$.

By Lemma 1.2 $\mathrm{oen}(T_n, T_{n+1})$ is a non-increasing integer sequence, so $\mathrm{oen}(T_n, T_{n+1}) = \mathrm{oen}(T_\infty)$ for some $\mathrm{oen}(T_\infty)$ and big n (in the above example $\mathrm{oen}(T_\infty) = 2$). We assume that $\mathrm{oen}(T_n, T_{n+1}) = \mathrm{oen}(T_\infty)$. For $x \in T_\infty$ let $r(x)$ be the least number with $x \in f^{r(x)}(T_0)$, defined for all $x \in T_\infty$ (e.g., $r(x) = 0$ for all points $x \in T_0$). Since $f^s(x) \in f^{s+r(x)}(T_0) \subset T_{s+r(x)}$ then $r(f^s(x)) \leq s + r(x)$, but $r(f^s(x)) < s + r(x)$ is possible too. Call $x \in T_\infty$ **slow** if there is $s \geq 1$ with $f^s(x) \in T_{r(x)+s-1}$ and **fast** otherwise.

The **(f)-orbit** of a set $A \subset X$ is $\cup_{n=0}^\infty f^n(A)$, and the **grand (f)-orbit** of A is the set of all points x so that there are $m, n \geq 0$ with $f^m(x) \in f^n(A)$. Let k'' be the **number of pairwise disjoint orbits of fast critical points**. Clearly, a fast point has an infinite forward orbit. Call the grand orbit of a point x **non-cyclic** if it contains no cycles; it is non-cyclic iff the orbit of x is infinite. For a tree $W \subset X$ or a growing tree $T_\infty \subset X$ vertices with infinite orbits and their grand orbits are called **(W-)** or **(T_∞-) exceptional**.

Let T_∞ be a growing tree. A germ (a, S) of some T_m is called **recurrent** if $f^n(a, S) \cap T_0 \neq \emptyset$ for infinitely many n's. For every $x \in T_m$ the **recurrent order** $\mathrm{ord}'_{T_m}(x)$ of T_m at x is the number of recurrent germs of T_m at x. Clearly, $\mathrm{ord}'_{T_m}(x) \leq \mathrm{ord}_{T_m}(x)$ (the equality holds only if all germs of T_m at a are recurrent). A growing tree is **recurrent** if all non-precritical exceptional vertices have recurrent order at least 3.

Call a growing tree **normal** if the images of endpoints of T_0 belong to T_0. Then endpoints of T_0 are slow. Also, call a periodic non-precritical vertex v of T_∞ **irrational** iff a germ of some T_m at v is not periodic. Let p_I be the number of orbits of the irrational vertices of T_∞, and let p_p be the number of preperiodic critical points.

At last, define the local degree of f as follows. Say that $f : T_{m-1} \to T_m$ has **degree** $\deg_{f|T_{m-1}}(c) = q$ at a point c if certain q germs of T_{m-1} at c map into one germ of T_m at $f(c)$ while no $q + 1$ germs do. Let $\max_{m \geq 1}\{\deg_{f|T_{m-1}}(c)\} = \deg_f(c)$; if $\deg_f(c)$ is finite and $\mathrm{ord}_{T_{m-1}}(c)$ is big enough then $f(c)$ remains a vertex of T_m.

Theorem 1.3. *Let T_∞ be a growing tree, $c_1, \dots, c_k$ be critical points of f and we have that $\sum_{i=1}^k \deg_f(c_i) - 1 = D - 1$. Let Γ be a non-empty set of exceptional non-precritical vertices of T_m which belong to pairwise disjoint grand orbits. Then we have that $\sum_{v\in\Gamma}(\mathrm{ord}_{T_m}(v) - 2) \le \mathrm{oen}(T_\infty) - p_I \le k'' - p_I$.*

Moreover, if T_∞ is normal and recurrent then the following inequality can be proven: $\sum_{v\in\Gamma}(\mathrm{ord}'_{T_m}(v) - 2) \le \mathrm{oen}(T_\infty) - p_I - 1 \le k'' - p_I - 1$.

Also, for an exceptional vertex v, $\mathrm{ord}_{T_m}(v) \le 3 \cdot 2^{D-p_p-1} \le 3 \cdot 2^{D-1}$ (if $p_I = 0$) and $\mathrm{ord}_{T_m}(v) \le (p_I + 2)2^{D-p_p-p_I}$ (if $p_I \ge 1$) while $\mathrm{ord}'_{T_m}(v) \le (p_I + 1)2^{D-p_p-p_I} (\forall p_I)$.

We prove Theorem 1.3 in the rest of Section 1. Denote the *number of vertices (endpoints, edges)* of a tree T by $V(T)$ $(End(T), D(T))$ and the *set of vertices* of T by $\mathcal{V}(T)$.

Lemma 1.4. (1) *For any tree W, $V(W) + 1 = End(W) - 1 - \sum_{v\in\mathcal{V}(W)}(\mathrm{ord}_W(v) - 3)$ (equivalently, $1+\sum_{v\in\mathcal{V}(W)}(\mathrm{ord}_W(v)-2) = End(W)-1$), and so $V(W)+1 \le End(W)-1$.*

(2) $\sum_{j=1}^{k_{n+1}}(V(t_j^{n+1}) + 1) \le \mathrm{oen}(T_\infty) \le \mathrm{oen}(T, T_1)$.

Proof. (1) Induction over the number of edges.

(2) Sum up the inequality from (1) over the components of $T_{n+1} \setminus T_n$. ■

Consider a tree W and its vertices. Call a vertex $v \in W$ **quasi-last** if $f(v)$ is not a vertex of W. Denote the set of quasi-last vertices of W by $QL(W)$. We use this notion in Theorem 1.5 to estimate the number of T_∞-exceptional grand orbits.

Theorem 1.5. *Let $T_\infty = T_0 \subset T_1 \subset \dots$ be a growing tree. Then the following holds.*

(1) *The endpoints of T_m are f^i-images of endpoints of $T_0, 0 \le i \le m$ or f^j-images of critical points, $1 \le j \le m$. In particular, $\mathrm{oen}(T_\infty) \le End(T_0) + k''$, and if all endpoints of T_0 are slow then $\mathrm{oen}(T_\infty) \le k''$.*

(2) *Quasi-last vertices of T_m are critical points which are vertices of T_∞, or vertices of components of $T_m \setminus T_{m-1}$, or their basepoints which are not vertices of T_{m-1}. The images of quasi-last vertices of T_m of the second and third type are either vertices of components of $T_{m+1} \setminus T_m$, or basepoints of such components which are not vertices of T_m.*

(3) $|QL(T_m)| \le k + \mathrm{oen}(T_\infty) \le k + \mathrm{oen}(T_0, T_1)$ *and among points of $QL(T_m)$ there are at most $\mathrm{oen}(T_\infty)$ vertices which are not critical points.*

(4) *The infinite orbit of a vertex x of T_m contains a unique $d_x \in QL(T_m)$ such that $f^i(d_x)$ is not a vertex of T_m for all $i > 0$. The number of T_∞-exceptional grand orbits is at most $k + \mathrm{oen}(T_\infty)$ and the number of them containing no critical points is at most $\mathrm{oen}(T_\infty)$.*

Proof. (1) If $x \in T_m$ is not one of the described images of critical points/endpoints of T_0 then $x = f^s(y) (s \le m), y$ is not an endpoint of T_0, f^s is a local homeomorphism at y and so x is not an endpoint of T_m. If x is a slow critical point/slow endpoint of T_0 then $f^s(x) \in T_{s-1}$ for big s. Hence $f^s(x)$ cannot be an endpoint of T_{m+1} not belonging to T_m. The rest easily follows.

(2) A vertex a of T_{m-1} which is not a critical point is not a quasi-last vertex of T_m because $f(a)$ is a vertex of T_m. So, quasi-last vertices of T_m are either critical points which are vertices of T_∞ or vertices of T_m but not vertices of T_{m-1}, i.e. vertices of components of $T_{m+1} \setminus T_m$ or their basepoints which are not vertices of T_m.

Let v' be a quasi-last vertex of T_m but not a critical point. Consider two cases.

(i) $u = f(v') \notin T_m$. Then u is a vertex of a component of $T_{m+1} \setminus T_m$.

(ii) $u = f(v') \in T_m$. Suppose that u is not a basepoint of a component of $T_{m+1} \setminus T_m$. Then a small neighborhood of v' in T_m maps onto a small neighborhood of u in T_m, and so u is a vertex of T_m, a contradiction.

(3) Immediately follows from (2) and Lemma 1.4(2).

(4) The former part of the claim is obvious; the latter follows from (3) because quasi-last vertices d_x corresponding to vertices from distinct grand orbits are distinct. ∎

Corollary 1.6. *Let T_∞ be a growing tree, A be a finite set, and $M \subset T_\infty$ be a non-degenerate connected set such that eventual preimages of vertices of T_∞ or of points of A are dense in M. Then M is non-wandering.*

Proof. Assume that M is wandering. Then M cannot contain preperiodic points. By our assumptions this implies that points of T_∞-exceptional grand orbits or grand orbits of points of A are dense in M. Hence there are two points $y, z \in M$ from the same grand orbit and thus $f^i(y) = f^j(z)$ for some i, j. If $i \neq j$ then $f^i(M)$ is not disjoint from $f^j(M)$, a contradiction. Thus $i = j$ and so for some $s < j$ the set $f^s(M)$ covers a critical point. Repeating this argument for a subinterval of $f^s(M)$ disjoint from critical points we will find a critical point covered twice by different images of M, a contradiction. ∎

We need a new construction. As we noticed above, for a growing tree T_∞ Lemma 1.1 implies that $T_{n+1} \setminus T_n = \cup_{j=1}^{k_{n+1}} t_j^{n+1}$ where t_j^{n+1} are components of $T_{n+1} \setminus T_n$ with basepoints x_j^{n+1}. We call the germ of $\bar{t}_j^{n+1}$ at x_j^{n+1} the **base germ** (of $\bar{t}_j^{n+1}$). Also, germs at vertices of T_∞ are called **v-germs**.

Let $u \in T_{m+1}$ be a non-preperiodic point for which there are points $v \in T_m$ and numbers k_v with $v, f(v), \dots, f^{k(v)}(v)$ vertices of T_m but not critical points and $f^{k(v)+1}(v) = u$ not a vertex of T_m. Thus, v gets mapped through vertices of T_m which are not critical points until $f^{k(v)}(v)$ is a quasi-last vertex of T_m (and so $f^{k(v)+1}(v) = u$ is a vertex of T_{m+1}). Let the set of all such v be $O_m(u)$ and the set of all points u with non-empty $O_m(u)$ be Π_m. The union O_m of sets $O_m(u)$ over all $u \in \Pi_m$ consists of all non-preperiodic points v such that $v, f(v), \dots, f^{k(v)}(v)$ are vertices of T_m but not critical points of f while $f^{k(v)+1}(v)$ is not a vertex of T_m.

Define the set $OG_m(u)$ of germs as follows: if $x \in O_m(u), (x, A)$ is a germ of T_m at $x, i \leq k(x) + 1$ and $f^i(x, A)$ is a germ of T_{m+1} but not a germ of T_m then we include $f^i(x, A)$ in $OG_m(u)$. The union of sets $OG_m(u)$ over all $u \in \Pi_m$ is denoted by OG_m. Also, for a germ (v, A) of T_m at $v \in O_m(u)$ we count how many times along the way to u its images will be germs of T_{m+1} but not germs of T_m and denote this number by $\psi(v, A)$. In other words, $\psi(v, A)$ counts the number of times $f^i(v, A)$ is in $T_{m+1} \setminus T_m$ for $i = 1, dots, k(v) + 1$. Then we sum up $\psi(v, A)$ over all germs of T_m at v and denote this sum by $\varphi(v)$. Define $\xi(u)$ for $u \in \Pi_m$ as 2 if $u \notin T_m$ and 0 if $u \in T_m$.

Lemma 1.7. (1) $|\Pi_m| \leq \mathrm{oen}(T_\infty)$ *and every point $u \in \Pi_m$ is either a vertex of a component of $T_{m+1} \setminus T_m$, or a basepoint u of such a component with* $\mathrm{ord}_{T_m}(u) \leq 2$.

(2) *For $u \neq u' \in \Pi_m$ the sets $OG_m(u), OG_m(u')$ are disjoint.*

(3) *A germ of OG_m is either a v-germ or a base germ of a component of $T_{m+1} \setminus T_m$.*

(4) *Let $u \in \Pi_m, v \in O_m(u)$. If u is a vertex of a component of $T_{m+1} \setminus T_m$ then $\psi(v, A) \geq 1$ for any germ (v, A) of T_m at v while if u is a basepoint of a component*

of $T_{m+1} \setminus T_m$ *then* $\psi(v, A) \geq 1$ *for all germs* (v, A) *of* T_m *at* v *except for at most* $\mathrm{ord}_{T_m}(u) \leq 2$ *germs whose all images under* $f, \dots, f^{k(v)+1}$ *are germs of* T_m*.*

(5) $|OG_m(u)| - \xi(u) \geq \varphi(v) - \xi(u) \geq \mathrm{ord}_{T_m}(v) - 2$ *where* $u \in \Pi_m$ *and* $v \in O_m(u)$*.*

(6) *If in the above situation* $\psi(v, A) \geq 2$ *for some germ at* v*, or if* $\psi(v, A) \geq 1$ *for some germ at* v *mapped by* $f^{k(v)+1}$ *into a germ of* T_m *at* u*, or* $\mathrm{ord}_{T_m}(u) = 1$ *(the last two conditions may take place only if* $u \in T_m$ *is a basepoint) then* $|OG_m(u)| - \xi(u) \geq \varphi(v) - \xi(u) \geq \mathrm{ord}_{T_m}(v) - 1$*.*

Proof. (1) Follows from Theorem 1.5(2) and Theorem 1.5(3).

(2) Let $u \neq u' \in \Pi_m$ but (v, A) is a common germ of $OG_m(u)$ and $OG_m(u')$. Then u, u' are not vertices of T_m. Since $f^{k(v)+1}$ is the first power of f mapping v into a point which is not a vertex of T_m then $u = u' = f^{k(v)+1}(v)$, a contradiction.

(3) Let $u \in \Pi_m, v \in O_m(u)$. By the definition points $v, f(v), \dots, f^{k(v)}$ are vertices of T_m. Germs of T_{m+1} but not of T_m at these points are base germs of components of $T_{m+1} \setminus T_m$. Now, by (1) the point $u = f^{k(v)+1}(v)$ is a vertex/basepoint of a component of $T_{m+1} \setminus T_m$, so a germ of T_{m+1} but not of T_m at u is a v-germ/a base germ of T_{m+1}.

(4) Let $u \in \Pi_m$ be a vertex of a component of $T_{m+1} \setminus T_m, v \in O_m(u)$, and (v, A) be a germ of T_m at v. Then $f^{k(v)+1}(v, A)$ is not a germ of T_m, thus there is the minimal power $f^i, i \leq k(v) + 1$ of f such that $f^i(v, A)$ is not a germ of T_m but a germ of T_{m+1}, which has to be counted in $\psi(v, A)$. Hence, $\psi(v, A) \geq 1$. The second claim follows similarly.

(5) The function $\psi(v, A)$ counts the number of images of the germ (v, A) under $f^i, 1 \leq i \leq k(v) + 1$ which are germs of T_{m+1} but not of T_m. Since points $f^i(v), 0 \leq i \leq k(v)$ are not critical then the sets of germs initiated by distinct germs at v are disjoint. By the definition these germs are germs from $OG_m(u)$. The former inequality $|OG_m(u)| - \xi(u) \geq \varphi(v) - \xi(u)$ follows. To prove the latter consider two cases. If $u \in \Pi_m$ is a vertex of a component of $T_{m+1} \setminus T_m$ then by (4) $\psi(v, A) \geq 1$ for any germ (v, A) of T_m at v, hence $\varphi(v) \geq \mathrm{ord}_{T_m}(v)$ and so $\varphi(v) - \xi(u) \geq \mathrm{ord}_{T_m}(v) - 2$. If $u \in \Pi_m$ is the basepoint of a component of $T_{m+1} \setminus T_m$ then by (4) $\psi(v, A) \geq 1$ for all germs (v, A) of T_m at v except for at most $\mathrm{ord}_{T_m}(u) \leq 2$. Thus $\varphi(v) - \xi(u) \geq \mathrm{ord}_{T_m}(v) - 2$.

(6) Conditions of the claim imply $\varphi(v) \geq \mathrm{ord}_{T_m}(v) + 1$ (if $u \notin T_m$) or $\varphi(v) \geq \mathrm{ord}_{T_m}(v) - 1$ (if $u \in T_m$). So, they imply $\varphi(v) - \xi(u) \geq \mathrm{ord}_{T_m}(v) - 1$. By (5) we get the required inequality. ∎

Theorem 1.8 relies upon Lemma 1.7 and Lemma 1.4(1).

Theorem 1.8. *Let* Γ *be a non-empty set of exceptional vertices of* T_m *which are non-precritical and belong to pairwise disjoint grand orbits. Then the following holds.*

(1) $\sum_{v \in \Gamma} (\mathrm{ord}_{T_m}(v) - 2) \leq \sum_{u \in \Pi_m} (|OG_m(u)| - \xi(u)) \leq \mathrm{oen}(T_\infty)$.

(2) $\mathrm{ord}_{T_m}(v) \leq \mathrm{oen}(T_\infty) + 2$ *for any exceptional vertex* v *of* T_n *and, moreover, if there are* s *different grand orbits of such vertices then* $\mathrm{ord}_{T_m}(v) \leq \mathrm{oen}(T_\infty) - s + 3$*.*

(3) *If* $\sum_{v \in \Gamma} (\mathrm{ord}_{T_m}(v) - 2) = \mathrm{oen}(T_\infty)$ *then:*

(a) *the set* OG_m *coincides with the set of all v-germs/base germs of components of* $T_{m+1} \setminus T_m$ *and for any germ* (v, A) *we have* $\psi(v, A) \leq 1$*;*

(b) *the set* Π_m *is exactly the set of all vertices of components of* $T_{m+1} \setminus T_m$ *united with the set of all basepoints of such components which are not vertices of* T_m*;*

(c) *all points of* Π_m *belong to distinct grand orbits, a vertex/basepoint* v *of a component of* $T_{m+1} \setminus T_m$ *may belong to the grand orbit of* $u \in \Pi_m, u \neq v$ *only if* v *is a basepoint* *which is a vertex of T_m with $f^{k(v)+1}(v) = u$, and no point of Π_m is ever mapped into a vertex of T_{m+1} by a positive iterate of f;*

(d) *no condition from Lemma 1.7(6) takes place (in particular no basepoint of a component of $T_{m+1} \setminus T_m$ is an endpoint of T_m);*

(e) *no germ of OG_m maps into a v-germ of T_{m+1} by a positive iterate of f.*

Proof. (1) By the definition in the sum $U = \sum_{u \in \Pi_m} (|OG_m(u)| - \xi(u))$ every v-germ/base germ of a component of $T_{m+1} \setminus T_m$ is counted at most once. Assume that $T_{m+1} \setminus T_m = \cup_{j=1}^{k_{m+1}} t_j^{m+1}$ where t_j^{m+1} are components of $T_{m+1} \setminus T_m$. Then moving from U to the sum $V = \sum_{j=1}^{k_{m+1}} (1 + \sum_{u \in \mathcal{V}(t_j^{m+1})} (\text{ord}_{t_j^{m+1}}(u) - 2))$ we in fact add some v-germs/base germs of some components of $T_{m+1} \setminus T_m$ but also subtract numbers 2 taken over all vertices of components of $T_{m+1} \setminus T_m$ which do not belong to Π_m. Now, if $x \notin \Pi_m$ is a vertex of a component of $T_{m+1} \setminus T_m$ then germs of T_{m+1} at x do not belong to OG_m and therefore are added when we move from U to V. This proves that $U \le V$, and by Lemma 1.4(1) we get $U \le V = \sum_{j=1}^{k_{m+1}} \text{oen}(t_j^{m+1}) = \text{oen}(T_\infty)$. It remains to notice that since all $v \in \Gamma$ belong to disjoint grand orbits then points $f^{k(v)+1}(v), v \in \Gamma$ are all distinct which together with the estimate from Lemma 1.7(5) implies $\sum_{v \in \Gamma} (\text{ord}_{T_m}(v) - 2) \le \sum_{u \in \Pi_m} (|OG_m(u)| - \xi(u))$ and completes the proof of the claim.

(2) Follows from (1).

(3) (a) If $\sum_{v \in \Gamma} (\text{ord}_{T_m}(v) - 2) = \text{oen}(T_\infty)$ then all inequalities from above become equalities. In particular this means that the set of germs OG_m is the set of **all** v-germs/base germs of components of $T_{m+1} \setminus T_m$. Also, by Lemma 1.7(6) if $\psi(v, A) \ge 2$ for a germ (v, A) then for the point $u = f^{k(v)+1}(v) \in \Pi_m$ we have $\text{ord}_{T_m}(v) - 2 \le |OG_m(u)| - \xi(u) - 1$ and by the above inequalities $\sum_{v \in \Gamma} (\text{ord}_{T_m}(v) - 2) \le \text{oen}(T_\infty) - 1$, a contradiction. This proves the claim (a).

(b) If a vertex of a component of $T_{m+1} \setminus T_m$ or a basepoint of such component which is not a vertex of T_m does not belong to Π_m then the germs at this point do not belong to OG_m because by the definition such point cannot appear in any orbit segment $v, \dots, f^{k(v)+1}(v)$ for $v \in O_m$, a contradiction with (a).

(c) By (a), **all** points $u \in \Pi_m$ appear as $f^{k(v)+1}(v)$ for various $v \in \Gamma$. Since points of Γ belong to distinct grand orbits we see that so do all points of Π_m. This proves the first part of the claim (c). Let us show that a point $u \in \Pi_m$ and a vertex/basepoint $v \ne u$ of a component of $T_{m+1} \setminus T_m$ may belong to the same grand orbit only in one way: $u = f^{k(v)+1}(v)$. Indeed, if $v \in \Pi_m$ then we get two points $u \ne v$ in Π_m which belong to the same grand orbit, a contradiction with the first statement of (c). So, $v \notin \Pi_m$.

By (b) this means that v is a basepoint of a component of $T_{m+1} \setminus T_m$ which is also a vertex of T_m. Then $f^{k(v)+1}(v) = u' \in \Pi_m$. If $u' \ne u$ then u' and u will be points of Π_m which belong to the same grand orbit, a contradiction. Hence $u' = u$ as desired. Finally, assume that for a point $u \in \Pi_m$ there exists $n > 0$ such that $f^n(u)$ is a vertex of T_{m+1}. By what we have already proven in (c), $f^n(u)$ cannot be a vertex/basepoint of a component of $T_{m+1} \setminus T_m$ which is not a vertex of T_m. Thus it is a vertex of T_m. On the other hand, if $f^n(u)$ is a vertex of T_m then since its orbit avoids critical points and is infinite we can find the first quasi-last vertex of T_m in the orbit of $f^n(u)$ which must be a vertex/basepoint of a component of $T_{m+1} \setminus T_m$, a contradiction.

(d) If a condition from Lemma 1.7(6) takes place then $|OG_m(u)| - \xi(u) \geq \varphi(v) - \xi(u) \geq \mathrm{ord}_{T_m}(v) - 1$ for appropriate points $v \in O_m(u)$. By (1) this would imply that $\sum_{v \in \Gamma}(\mathrm{ord}_{T_m}(v) - 2) \leq \mathrm{oen}(T_\infty) - 1$, a contradiction with the standing assumption.

(e) Let $(x, A) \in OG_m$. If $x \in \Pi_m$ then by (c) its orbit avoids vertices of T_{m+1} and we are done. Let $x \notin \Pi_m$. By (a) and (b) then (x, A) is the base germ of a component of $T_{m+1} \setminus T_m$ at a vertex x of T_m. So, x maps into a point $u \in \Pi_m$ by $f^{k(x)+1}$ and for all $i, 0 \leq i \leq k(x)$ the points $f^i(x)$ are vertices of T_m. Hence, $f^i(x, A), 0 \leq i \leq k(x)$ cannot be a v-germ of a component of $T_{m+1} \setminus T_m$. However, $f^{k(x)+1}(x, A)$ cannot be a germ of $T_{m+1} \setminus T_m$ since then $\psi(x, A)$ would be greater than 1 while by (a) we have $\psi(x, A) = 1$. Finally, if $j > k(x) + 1$ then $f^j(x)$ is not a vertex of T_{m+1} by (c). Thus, $f^i(x, A)$ is not a v-germ of T_{m+1} for any i which completes the proof. ∎

Let k_f be the number of fast critical points. Then $k'' \leq k_f$ (the definitions of k'' and other constants can be found before the statement of Theorem 1.3).

Corollary 1.9. *Let T_∞ be a growing tree, $c_1, \dots, c_k$ be critical points of f, $\deg_f(c_i) = d_i, 1 \leq i \leq k$ and x be an exceptional vertex of T_∞. Then the following holds.*

(1) $\mathrm{ord}_{T_m}(x) \leq (\mathrm{oen}(T_\infty) + 2) \prod_{i=1}^k d_i$ *for any m.*

(2) *If $\sum_{i=1}^k (d_i - 1) = D - 1$ and all endpoints of T_0 are slow then $k'' = 0$ implies $\mathrm{ord}_{T_m}(x) \leq 2^{D - p_p}$, $k'' \geq 1$ implies $\mathrm{ord}_{T_m} \leq 2^{D-p_p-k_f}(k''+2) \leq 2^{D-p_p-k_f}(k_f+2) \leq 2^{D-p_p-k''}(k''+2)$, so always $\mathrm{ord}_{T_m}(x) \leq 3 \cdot 2^{D-p_p-1} \leq 3 \cdot 2^{D-1}$.*

Proof. (1) We may assume that x is a vertex of T_m with big m. Since the orbit of x is infinite there exists a number l such that the orbit of $f^l(x)$ is disjoint from the set of critical points of f. Then $\mathrm{ord}_{T_{m+l}}(x) \leq \mathrm{oen}(T_\infty) + 2$ by Theorem 1.8(2) applied to $f^l(x)$. By the definition of the degree this implies that $\mathrm{ord}_{T_m}(x) \leq (\mathrm{oen}(T_\infty) + 2) \prod_{i=1}^n d_i$.

(2) For our vertex $x \in T_m$ there exists a finite collection of numbers $i_1 < i_2 < \dots < i_l$ with $f^{i_j}(x), 1 \leq j \leq l$ being all critical points of f in the orbit of x. If $k'' = 0$ then the maximal value of l is k and by Theorem 1.8(2) we have $\mathrm{ord}_{T_m}(f^{i_l+1}(x)) \leq \mathrm{oen}(T_\infty) + 2 \leq 2$. Applying the arguments from (1) to the map f^{i_l+1} at x we get then that $\mathrm{ord}_{T_m}(x) \leq 2 \prod_{j=1}^{k-p_p} d_{i_j}$ where the product is taken over all non-preperiodic critical points. Since every preperiodic critical point has degree at least 2 we see that $\sum_{j=1}^{k-p_p}(d_{i_j} - 1) \leq D - p_p - 1$. Hence $\prod_{j=1}^{k-p_p} d_{i_j} \leq 2^{D-p_p-1}$ and $\mathrm{ord}_{T_m}(x) \leq 2 \prod_{j=1}^{k-p_p} d_{i_j} \leq 2^{D-p_p} \leq 2^D$.

Assume that $k'' \geq 1$. Then points $f^{i_1}(x), \dots, f^{i_{l-1}}(x)$ are slow, so fast critical points come from the set of other critical points from which p_p preperiodic critical points must be excluded. Hence $k_f \leq k - p_p - l + 1$ and so $l \leq k - p_p - k_f + 1$. Since by the assumption all endpoints of T_0 are slow, by Lemma 1.5(1) $\mathrm{oen}(T_\infty) \leq k''$ and by Theorem 1.8(2) $\mathrm{ord}_{T_m}(f^{i_l+1}(x)) \leq \mathrm{oen}(T_\infty) + 2 \leq k'' + 2$. Applying the arguments from (1) to the map f^{i_l+1} at x we get that $\mathrm{ord}_{T_m}(x) \leq (k''+2) \prod_{j=1}^l d_{i_j}$, and the maximal value of the expression on the right is achieved when l is maximal, i.e. for $l = k - p_p - k_f + 1$. Observe that in the sum $\sum_{j=1}^l (d_{i_j} - 1)$ at least $k_f + p_p - 1$ critical points are not involved, hence this sum cannot exceed $D - 1 - (p_p + k_f - 1) = D - p_p - k_f$ and we get that $\prod_{j=1}^l d_{i_j} \leq 2^{D-p_p-k_f}$. Thus $\mathrm{ord}_{T_m}(x) \leq (k''+2)2^{D-p_p-k_f}$ and the rest easily follows. ∎

It is useful to consider the "letter E" example described right before Lemma 1.1 and illustrated on Figure 1 in terms of Theorem 1.8. Indeed, for the growing tree T_0 from that example $\text{oen}(T_\infty) = 2$. There are two grand orbits of exceptional vertices of T_∞, namely the grand orbit of $(0, 1)$ and the grand orbit of $(0, -1)$. A non-empty set Γ of exceptional vertices of T_m which are non-precritical and belong to pairwise disjoint grand orbits can at most consist of two points v_1, v_2 from these orbits. Then $\text{ord}_{T_m}(v_1) = \text{ord}_{T_m}(v_2) = 3$, and we have $\sum_{v\in\Gamma}(\text{ord}_{T_m}(v)-2) = (\text{ord}_{T_m}(v_1)-2)+(\text{ord}_{T_m}(v_2)-2) = 1+1 = 2 = \text{oen}(T_\infty)$ (compare with Theorem 1.8(1)). Moreover, since there are $s = 2$ distinct grand orbits by Theorem 1.8(2) we get $\text{ord}_{T_m}(v_i) \le \text{oen}(T_\infty) - s + 3 = 3, i = 1, 2$ as indeed is the case. The reader can easily verify the rest of Theorem 1.8 in this case.

The “letter E” example is naturally non-recurrent. We strengthen our estimates by making mild recurrent assumptions. Let (a, A) be a germ of a tree W and $W(a, A)$ be the component $W(a, A)$ of $W \setminus \{a\}$ containing (a, A) : a is an endpoint of $W(a, A)$ and (a, A) is the unique germ of $W(a, A)$ at a. Call $W(a, A)$ the **grape** of W generated by (a, A). If T_∞ is a growing tree and (a, A) is a germ of T_m for $m \ge 0$, the set $\cup_{m\ge0}T_m(a, A) = T_\infty(a, A)$ is called the **grape** of T_∞ generated by (a, A). A germ (a, A) and the grape $T_\infty(a, A)$ are **recurrent** if $f^n(T_\infty(a, A)) \cap T_0 \ne \emptyset$ for infinitely many n. For any $x \in T_m$ the **recurrent order** $\text{ord}'_{T_m}(x)$ of T_m at x is the number of recurrent germs of T_m at x. Then $\text{ord}'_{T_m}(x) \le \text{ord}_{T_m}(x)$ (the equality holds only if all germs of T_m at a are recurrent). Lemma 1.10 classifies behaviors of base germs and their grapes.

Lemma 1.10. *Let t be a component of $T_{m+1} \setminus T_m$, (x, A) be its base germ and $D = T_\infty(x, A)$. Then the following are possible behaviors of D.*

(1) *For all $n \ge 0$ we have $f^n(x) \in T_m$ and the set $f^n(D)$ is disjoint from T_m.*

(2) *There exists a number j such that $x, f(x), \dots, f^j(x) \in T_m$, j is the minimal such number that $f^j(D) \cap T_m \ne \emptyset$ and then $f^j(x, A)$ is a germ of T_m.*

(3) *There exists the minimal number $s < \infty$ such that $f^s(x) \notin T_m$ and all $D, f(D)$ $\dots, f^{s-1}(D)$ are disjoint from T_m. Denote by τ the component of $T_{m+1} \setminus T_m$ such that $f^s(x) \in \tau$ and let (y', B) be the base germ of τ. Then $f^s(D) \subset T_\infty(y', B)$.*

Proof. If (1) does not hold consider the orbit of D until one of the following takes place: either $f^j(x) \in T_m$ and $f^j(D) \cap T_m \ne \emptyset$ or $f^j(x) \notin T_m$. The first possibility corresponds to the case (2) of the lemma, and it remains to prove then that $f^j(x, A)$ is a germ of T_m. Indeed, by the minimality of j the sets $D, f(D), \dots, f^{j-1}(D)$ are disjoint from T_m and thus contain no critical points inside. Let $y \in D$ be such that $f^j(y) \in T_m$; by the definition of the grape the germ of $[x, y]$ at x is (x, A). Then $[x, y] \subset D, f[x, y] \subset f(D), \dots, f^{j-1}[x, y] \subset f^{j-1}(D)$ contain no critical points of f. Hence $f^j[x, y] = [f^j(x), f^j(y)] \subset T_m$ is a homeomorphic image of $[x, y]$, and so the f^j-image of the germ (x, A) of $[x, y]$ at x is a germ of T_m at $f^j(x)$ as desired.

Suppose now that $x, f(x), \dots, f^{s-1}(x) \in T_m, f^s(x) \notin T_m$ for some $s < \infty$ and all $D, f(D), \dots, f^{s-1}(D)$ are disjoint from T_m (case (3)). We show that then $f^s(D)$ cannot contain y'. Consider the maximal component $K \subset T_m$ of the f^{-1}-preimage of τ containing $f^{s-1}x$ and show that $y' \in f(\bar{K})$. Indeed, if not then we extend K further until y' is reached by the image of $\bar{K}$, a contradiction. Moreover, critical points do not belong to K because otherwise their images would belong to $T_{m+1} \setminus T_m$ while they all belong to T_1 and m is greater than 1. Hence $f|\bar{K}$ is a homeomorphism, just like $f|\overline{f^{s-1}(D)}$ (recall that all $f^i(D)$ are disjoint from T_m for $0 \le i < s$ and hence do not contain critical points of f).

Choose $z \in \bar{K}$ with $f(z) = y'$. Since $f|\bar{K}$ is a homeomorphism then $f[f^{s-1}(x), z] = [f^s(x), y]$. Now, if $y \in f^s(D)$ then similarly we can find a point $\zeta \in f^{s-1}(D)$ such that $f[f^{s-1}(x), \zeta] = [f^s(x), y]$. However this implies that $f^{s-1}(x)$ is a critical point and $f(f^{s-1}(x)) \in T_{m+1} \setminus T_m$ which is impossible because m is big. The contradiction implies that $y \notin f^s(D)$. Clearly, this in turn implies that $f^s(D) \subset T_\infty(y, B)$ as desired. ∎

Theorem 1.11 strengthens Theorem 1.8 for the recurrent order. A growing tree is **recurrent** if all non-precritical exceptional vertices have recurrent order at least 3.

Theorem 1.11. *Let Γ be a non-empty set of exceptional vertices of T_m which are non-precritical and belong to pairwise disjoint grand orbits. Then $\sum_{v \in \Gamma}(\text{ord}'_{T_m}(v) - 2) \leq \text{oen}(T_\infty) - 1$. In particular, if T_∞ is recurrent then an exceptional vertex is either precritical or belongs to one of at most $\text{oen}(T_\infty) - 1$ grand orbits, and overall there are at most $k + \text{oen}(T_\infty) - 1$ exceptional grand orbits.*

Proof. If the inequality in question does not hold then since $\text{ord}'_{T_m}(x) \leq \text{ord}_{T_m}(x)$ we get by Theorem 1.8(1) that $\sum_{v \in \Gamma}(\text{ord}'_{T_m}(v) - 2) = \sum_{v \in \Gamma}(\text{ord}_{T_m}(v) - 2) = \text{oen}(T_\infty)$ and all germs in OG_m (by Theorem 1.8 these are all germs of components of $T_{m+1} \setminus T_m$ at their vertices/basepoint) are recurrent. Also, all vertices of components of $T_{m+1} \setminus T_m$ and their basepoints belong to various orbits of points of Γ, thus they all are non-precritical. Our aim is to draw a contradiction from these facts (cf. [Le]).

We consider the cases from Lemma 1.10 which may realize for the base germ (x, A) of a component t of $T_{m+1} \setminus T_m$. Let $T_\infty(x, A) = D$. Clearly, case (1) does not realize since (x, A) is recurrent and hence $f^n(D)$ cannot be disjoint from T_m for all n. If case (2) realizes then for some $j > 0$ we have $x, f(x), \ldots, f^j(x) \in T_m$, j is the minimal such number that $f^j(D) \cap T_m \neq \emptyset$ and $f^j(x, A)$ is a germ of T_m. Now, by Theorem 1.8(3)(d) the conditions from Lemma 1.7(6) cannot take place, so in particular basepoints of components of $T_{m+1} \setminus T_m$ are not endpoints of T_m and hence are vertices of T_{m+1}.

Consider the segment $x, f(x), \ldots, f^j(x)$ of the orbit of x. Observe that by Theorem 1.8(3)(e) (x, A) is never mapped into a v-germ of T_{m+1}, hence $f^j(x)$ is not a vertex of T_{m+1}, in particular $f^j(x)$ is not a basepoint of a component of $T_{m+1} \setminus T_m$. Let i be the maximal number no greater than j such that $f^i(x)$ is the basepoint of a component of $T_{m+1} \setminus T_m$; by what we have just proven $i < j$ and all points $f^{i+1}(x), \ldots, f^j(x)$ are not basepoints of components of $T_{m+1} \setminus T_m$. Therefore the germs $(f^i(x), U)$ and $(f^i(x), V)$ of T_m at $f^i(x)$ (there are at least two of them since such basepoints are not endpoints of T_m) are mapped by $f, f^2, \ldots, f^{j-i}$ into germs of T_m. In particular, $f^{j-i}(f^i(x), U)$ and $f^{j-i}(f^i(x), V)$ are germs of T_m. If two of the germs $f^j(x, A), f^j(f^i(x), U)$ and $f^j(f^i(x), V)$ coincide then there must be a critical point among points $f^i(x), \ldots, f^j(x)$. However, this is impossible as we saw in the first paragraph of the proof. Hence $f^j(x)$ is a vertex of T_{m+1} and so $f^j(x, A)$ is a v-germ of T_{m+1} which is impossible by Theorem 1.8(3)(e). Thus the case (2) of Lemma 1.10 does not take place.

So, the remaining possibility for a base germ of a component $T_{m+1} \setminus T_m$ is (3) from Lemma 1.13. Let $T_{m+1} \setminus T_m = \cup_{j=1}^{k_{m+1}} t_j^{m+1}$ where t_j^{m+1} are components of $T_{m+1} \setminus T_m$. Denote by D_j the grape of T_∞ generated by the base germ of t_j^{m+1}. Then by what we have proven for any j there exists s and i such that $D_j, f(D_j), \ldots, f^{s-1}(D_j)$ are disjoint from T_m and $f^s(D_j) \subset D_i$. This implies that for some r and l we have $D_r, f(D_r), \ldots, f^{l-1}(D_r)$

are disjoint from T_m while $f^l(D_r) \subset D_r$. Clearly, this implies, that the base germ (and actually all germs) of D_r are not recurrent, a contradiction which proves the claim. ■

Corollary 1.12 is similar to Corollary 1.9 but deals with the recurrent order at vertices.

Corollary 1.12. *Let T_∞ be a growing tree, $c_1, \dots, c_k$ be critical points of f, $\deg_f(c_i) = d_i, 1 \le i \le k$ and x be an exceptional vertex of T_∞. Then the following holds.*

(1) $\mathrm{ord}'_{T_m}(y) \le (\mathrm{oen}(T_\infty) + 1) \prod_{i=1}^n d_i$ *for any m.*

(2) *If $\sum_{i=1}^k (d_i - 1) = D - 1$ and all endpoints of T_0 are slow then we have that* $\mathrm{ord}_{T_m} \le 2^{D-p_p-k_f}(k''+1) \le 2^{D-p_p-k_f}(k_f+1) \le 2^{D-p_p-k''}(k''+1) \le 2^{D-p_p} \le 2^D$.

Proof. The proof almost literally repeats that of Corollary 1.9. The only exception is that instead of the estimate $\mathrm{ord}_{T_m}(v) \le \mathrm{oen}(T_\infty) + 2$ taken from Theorem 1.8(2) we now rely upon the estimate $\mathrm{ord}'_{T_m}(v) \le \mathrm{oen}(T_\infty) + 1$ from Theorem 1.11. ■

Proof of Theorem 1.3. It suffices to prove the theorem for m large enough. Let v be an irrational vertex of period p. Fix T_m with big m so that a non-periodic germ at v has already appeared. If points $v, \dots, f^{p-1}(v)$ are not basepoints of a component of $T_{m+1} \setminus T_m$ then the germs of T_m at them are mapped in each other and form an invariant set of germs. By the assumption points $v, \dots, f^{p-1}(v)$ are not critical, so the set of germs in question is the union of several periodic orbits of germs, a contradiction.

Hence, for any periodic orbit of an irrational vertex from some time on there is a germ at one of its points v' which is a base germ of a component t' of $T_{m+1} \setminus T_m$, and there are at least p_I such germs. By the definitions, a point $u \in \Pi_m$ is not the base point of t', and a germ of OG_m is not the base germ of t'. Thus in the estimate from Theorem 1.8(1) in the sum $\sum_{j=1}^{k_{m+1}} (1 + \sum_{u \in \mathcal{V}(t_j^{m+1})} (\mathrm{ord}_{t_j^{m+1}}(u) - 2)) = \sum_{j=1}^{k_{m+1}} \mathrm{oen}(t_j^{m+1})$ the above mentioned p_I germs must be excluded, and we obtain

$$\sum_{v \in \Gamma} (\mathrm{ord}_{T_m}(v) - 2) \le \sum_{u \in \Pi_m} (|OG_m(u)| - \xi(u)) \le \mathrm{oen}(T_\infty) - p_I.$$

If we suppose that the second inequality from the theorem does not hold, then

$$\sum_{v \in \Gamma} (\mathrm{ord}'_{T_m}(v) - 2) = \sum_{v \in \Gamma} (\mathrm{ord}_{T_m}(v) - 2) = \mathrm{oen}(T_\infty) - p_I$$

and all germs in OG_m are recurrent. Also, all vertices of components of $T_{m+1} \setminus T_m$ and their basepoints which are not irrational belong to orbits of points of Γ, so they are not precritical. Repeating arguments from Theorem 1.11 we get a contradiction.

To prove the remaining estimates for $\mathrm{ord}_{T_m}(v)$ and $\mathrm{ord}'_{T_m}(v)$ observe that by the first part of the theorem we have $k'' \ge p_I$ and then apply Corollaries 1.9 and 1.12. ■

Call a growing tree T_∞ **strongly recurrent** if for any $x \in T_m$ which is not an endpoint of T_m there exists k with $f^k(x) \in T_0$. Strongly recurrent trees are recurrent; for them the order and the recurrent order at points of T_∞ are the same and all recurrent order estimates apply. In fact, all growing trees arising below in our study of laminations and polynomials are strongly recurrent. The following Proposition 1.13 will be used later on.

Proposition 1.13. *Let T_∞ be a strongly recurrent normal growing tree with s critical points of degree 2. Then the following two cases may take place:*

1. *if T_∞ is not a finite tree then the number of all grand orbits of exceptional vertices of T_∞ is at most $2k - s - 1$;*
2. *if T_∞ is a finite tree then all exceptional vertices of T_∞ are precritical and the number of their grand orbits is at most k.*

In particular, if all turning points are of degree 2 (that is, $s = k$) then the number of all exceptional orbits of vertices of T_∞ is at most $k - 1$ if T_∞ is infinite and k if T_∞ is finite, and in the latter case exceptional vertices are all precritical.

Proof. (1) Assume that T_∞ is not finite. Then by the definition there are fast critical points. Consider now a few cases.

Assume that there exist exceptional non-precritical vertices of T_∞ with non-cyclic grand orbits. Denote the set of all turning points of degree 2 by S_2. Suppose that there are r points $c \in S_2$ which become vertices of T_m for big m, and $s - r$ points $c \in S_2$ which never become vertices of T_m. Let us show that $\text{oen}(T_\infty) \le k - r$. Indeed, if $c \in S_2$ is a vertex of T_m for some m then $f(c)$ cannot be an endpoint of T_{m+1} because $\text{ord}_{T_{m+1}}(f(c)) \ge 2$. Since T_∞ is strongly recurrent we see that $f(c)$ is mapped back into T_0 by some power of f, hence c is slow. So, the number of fast turning points of T_0 is at most $k - r$. By Lemma 1.5(1) this implies that $\text{oen}(T_\infty) \le k - r$ and so by Theorem 1.3 the number of all non-cyclic grand orbits of non-precritical vertices is at most $k - r - p_I - 1 \le k - r - 1$.

On the other hand, a turning point which never becomes a vertex of T_m cannot contribute to the list of non-cyclic grand orbits of vertices of T_∞. Since there are $s - r$ such critical points of degree 2 we conclude that the number of non-cyclic grand orbits of critical points is at most $k - (s - r)$. Therefore the overall number of non-cyclic grand orbits of vertices of T_∞ is at most $k - r - 1 + k - (s - r) = 2k - s - 1$.

Now, assume that there are no exceptional non-precritical vertices of T_∞. In this case the only source of exceptional vertices are critical points, and so the number of their grand orbits is at most k. Hence if $s < k$ we are done. If $k = s$ (i.e., all critical points are of degree 2) then the fact that there exist fast critical points implies that at least one of them is not a vertex of T_∞ (vertices of a strongly recurrent growing tree are slow), and so the estimate from above is $k - 1 = k - s - 1$.

(2) If T_∞ is finite then the infinite orbit of a vertex must contain a critical point, so the number of all their grand orbits is at most k. ∎

2. Laminations

Here we construct a growing tree in the quotient space of a closed invariant lamination motivated by some ideas of holomorphic dynamics [DH], [Do1], [Le]. Then we apply results of Section 1 and prove Theorem B and the full version of Theorem A of Introduction (see Theorem 2.10).

Let us start with precise definitions. Consider an equivalence relation $\sim$ on the unit circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ (identified with $S^1 = \{z \in \mathbb{C} : |z| = 1\}$) with the following properties ([Do1], [McM], cf. [Th]):

(E1) $\sim$ is **closed**: the graph of $\sim$ is a closed set in $\mathbb{T} \times \mathbb{T}$;

(E2) $\sim$ defines a **lamination**, i.e. it is **unlinked**: if $t_1 \sim t_2 \in S^1$ and $t_3 \sim t_4 \in S^1$, but $t_2 \not\sim t_3$, then the open intervals in $\mathbb{C}$ with the endpoints t_1, t_2 and t_3, t_4 are disjoint;

(E3) each class of equivalence $\sim$ is totally disconnected.

Call $\sim$ a **closed lamination**. We always assume that it is **non-degenerate**, i.e. has a class of more than one point. Equivalence classes of $\sim$ are called **($\sim$-)classes**; for $x \in S^1$ let $\mathrm{Cl}(x)$ be its class. A $\sim$-class that consists of exactly two (2) points is called a **leaf** while a $\sim$-class that consists of at least three (3) points is called a **gap** (cf. [Th]). Note that laminations in [Th] do not always arise from an equivalence relation on $\mathbb{T}$. Also, a gap in [Th] is defined as a component of $\mathbb{D} \setminus \{$*the union of convex hulls of leaves*$\}$. Our definitions are closer to [Do1], [McM].

Fix an integer $d > 1$, denote by $\sigma_d = \sigma : \mathbb{T} \to \mathbb{T}$ the map $\sigma(t) = d \cdot t(\mod 1)$ and identify it with the map $z \mapsto z^d$ on S^1. Say that a subset of S^1 is *split* into classes if it contains a class of each its element. The relation $\sim$ is called **(σ-)invariant** iff:

(D1) $\sim$ is **forward invariant**: for a class g, the set $\sigma(g)$ is a class too

which implies that

(D2) $\sim$ is **backward invariant**: for a class g, its preimage $\sigma^{-1}(g) = \{x \in \mathbb{T} : \sigma(x) \in g\}$ is split into classes;

(D3) for any gap g, the map $\sigma : g \to \sigma(g)$ is a covering map with positive orientation.

Call a class g **critical** iff the map $\sigma : g \to \sigma(g)$ is not 1-to-1. Let $k_\sim$ be the maximal number of critical classes g such that $\sigma(g)$ is a single point with the infinite σ-orbit (i.e., $\sigma(g)$ is an irrational point of $\mathbb{T}$) and the orbits of g are pairwise disjoint.

Let $\mathbb{D}$ be the open disk bounded by S^1, $L_\sim = L$ be the union of **$\sim$-hulls**, i.e. convex hulls (in the Poincaré metric) of $\sim$-classes; by the definition $\sim$-hulls are contained in $\bar{\mathbb{D}}$ but not in $\mathbb{D}$. Define an extension $\simeq$ of $\sim$ onto $\bar{\mathbb{D}}$ as follows [Do1]: a $\simeq$-class is a $\sim$-hull or a point of $\bar{\mathbb{D}} \setminus L$. Extend $\simeq$ onto $\mathbb{C}$ by declaring that a point in $\mathbb{C} \setminus \bar{\mathbb{D}}$ is equivalent only to itself. Call a connected component of the complement $\mathbb{D} \setminus L$ a **($\sim$-)component**. Given an open set Ω in $\mathbb{D}$, denote by $E(\Omega)$ the set $\bar{\Omega} \cap S^1$. Below Ω is usually bounded by geodesics intersecting each other only at their endpoints on the circle, and then $\bar{\Omega}$ is the convex hull of the set $E(\Omega)$.

We construct a growing tree in the quotient space of $\simeq$ and apply results of Section 1. First we need new definitions. Let (x, y) be the Poincaré geodesic in $\mathbb{D}$ joining $x, y \in S^1$. Call (x, y) a **($\sim$-)geodesic** if $x \sim y$. We identify the $\sim$-geodesic (x, y) with the pair of points $\{x, y\}$ and speak of these two objects interchangeably. If (x, y) is a $\sim$-geodesic we say that σ **maps** (x, y) onto (x', y') if $\sigma(x) = x'$ and $\sigma(y) = y'$. By $< x, y >$ we mean one of two arcs in S^1 with endpoints x, y.

Lemma 2.1. *Let Ω be a $\sim$-component. Then $E(\Omega) = E$ is a Cantor set and $\sigma(E) = E(\Omega')$, where Ω' is a $\sim$-component. Moreover, let $x_1, x_2 \in E$ be the endpoints of a component I of $S^1 \setminus E = E'$. Then $x_1 \sim x_2, \mathrm{Cl}(x_1) \subset \bar{I}$, and if $x'_1 \in E$ is such that $\sigma(x'_1) = \sigma(x_1)$ then one of the following cases holds:*

1. *$\sigma(x_1) \neq \sigma(x_2)$ and there is $x'_2 \in E$ such that $\sigma(x'_2) = \sigma(x_2)$ and x'_1, x'_2 are endpoints of another component of E';*
2. *$\sigma(x_1) = \sigma(x_2)$ and there is $x'_2 \in E$ such that $\sigma(x'_2) = \sigma(x_1)$ and x'_1, x'_2 are endpoints of a component of E';*
3. *$\sigma(x_1) = \sigma(x_2)$ and there is no $x'_2 \in E$ such that x'_1, x'_2 are endpoints of a component of E'.*

Proof. For the sake of definiteness we assume that a point which runs within I from x_1 to x_2 has to run counterclockwise.

First we show that $x_1 \sim x_2$ and $\mathrm{Cl}(x_1) \subset \bar{I}$. Let l be a component of $\partial\Omega \setminus \{x_1, x_2\}$, which is disjoint with S^1. Any point $x \in l$ is then the limit of a sequence of points x_n so that each x_n lies in a boundary of a $\sim$ hull. Hence, $x_n \in l_n$ where l_n are pairwise disjoint $\sim$-geodesics. Consider two possibilities.

(i) The sequence $\{l_n\}$ is finite. Then x belongs to one of them, $l(x)$.

(ii) The sequence $\{l_n\}$ is infinite. Then x belongs to a geodesic $l(x)$ which is the limit of l_n.

Since the geodesics $l(x)$ for different $x \in l$ are either disjoint or coincide, we see that $l(x) = (x_1, x_2)$ for every $x \in l$. Thus $l = (x_1, x_2)$. Moreover, the endpoints of l_n are $\sim$ equivalent and the lamination is closed, therefore, $x_1 \sim x_2$. Also, Ω is disjoint with the $\simeq$ classes, therefore $\mathrm{Cl}(x_1) \subset \bar{I}$. Denote $\mathrm{Cl}(x_1)$ by K.

Let us show that E is a Cantor set. The fact that $\mathrm{Cl}(x_1) \subset \bar{I}$ implies that x_1 is not an isolated point in E. Indeed, otherwise there is another complementary to E arc $< z, x_1 >$ and by the above proven $z \sim x_1$, a contradiction to $\mathrm{Cl}(x_1) \subset \bar{I}$. Clearly, this means that there are no isolated points in E at all. To prove that E is a Cantor set it remains to prove that E contains no subintervals. This follows from the fact that some σ-iterate of any interval covers S^1.

Let $I' = S^1 \setminus I$. Let J' be the arc running clockwise from $\sigma(x_1)$ to $\sigma(x_2)$ and $J = S^1 \setminus J'$. Then J' contains σ-images of small semi-neighborhoods of x_1, x_2 non-disjoint from E. We show that J is disjoint from $\sigma(E)$. It is clear if $\sigma(x_1) = \sigma(x_2)$, so we assume that $\sigma(x_1) \neq \sigma(x_2)$. By (D3) for every class-preimage of $\sigma(K)$ we can find two points x_1', x_2' with $\sigma(x_1') = \sigma(x_1), \sigma(x_2') = \sigma(x_2)$ such that the closure of the arc T running counterclockwise from x_1' to x_2' contains $\mathrm{Cl}(x_1')$. Moreover, T is disjoint from E because there are points of E in a small counterclockwise semi-neighborhood of, say, x_1' and the geodesic (x_1', x_2') separates T from those points. Thus, the union A of all such arcs T is disjoint from E too. On the other hand by the construction A covers all preimages of J. Therefore, $\sigma(E)$ is disjoint from J as claimed which implies that $\sigma(E) \subset \bar{J}'$.

Since $\sim$ is a lamination (i.e., by (E2)) any two arcs T from the previous paragraph are either disjoint or contain one another. Choose a maximal by inclusion arc $T = < x_1', x_2' >$ from the family of arcs chosen in the previous paragraph; we may assume that it is running counterclockwise from x_1' to x_2'. Let us show that it is complementary to E. Indeed, otherwise there is another complementary to E arc $S \supset T$ such that the endpoints of S and T do not intersect. Applying the arguments from above to S instead of I we can find an open arc containing I which is disjoint from E, a contradiction with $x_1 \in E, x_2 \in E$. Thus, all maximal arcs from the family of arcs chosen in the previous paragraph are complementary to E. Denote them $T_1, \dots, T_r$.

Let us show that $\sigma(K) \subset \overline{J}$. Let x_1'' be the counterclockwise closest to x_2 point such that $\sigma(x_1) = \sigma(x_1'')$ (i.e., x_2'' is of the form $x_1 + j/d$ for some j). Let R be the arc running counterclockwise from x_2 to x_1''. If $\sigma(K) \not\subset \overline{J}$ then inside R there must be points of a class K' such that $\sigma(K') = \sigma(K)$ which is impossible because A contains K' and is on the other hand disjoint from R.

Let us show that the alternative (1)-(3) follows. Assume that $x_1' \in E$ is such a point that $\sigma(x_1') = \sigma(x_1)$; let $\mathrm{Cl}(x_1') = K'$. If $\sigma(x_1) \neq \sigma(x_2)$ then by the proven above x_1'

is an endpoint of a maximal arc $< x_1', x_2' >= A_j$ which is complementary to E. If $\sigma(x_1) = \sigma(x_2)$ then by (D3) we see that $u = \sigma(K)$ is a one-point set. Hence K' consists of a few points from $\sigma^{-1}(u)$. If $K' = \{x_1'\}$ then the case (3) holds. Otherwise by the above analysis the case (2) holds.

This completes the proof. ■

Let us introduce some maps and spaces. First, $K = \bar{\mathbb{D}}/\simeq$ is the quotient space, called the **pinched disc defined by** $\sim$ ([Do1]). Denote the interior of K by F. The factor space $\mathbb{C}/\simeq$ is called the **pinched plane**; K is imbedded in $\mathbb{C}/\simeq$. Let $p : \mathbb{C} \to \mathbb{C}/\simeq$ be the factor map. Then $p : \mathbb{C} \setminus \bar{\mathbb{D}} \to (\mathbb{C}/\simeq) \setminus K$ and $p : \mathbb{D} \setminus L \to F$ are homeomorphisms. The set $J = p(S^1) = p(L)$ is the boundary of K in $\mathbb{C}/\simeq$. Also, call $A_\infty = (\mathbb{C}/\simeq) \setminus K$ the **basin of infinity** of a map f defined as follows. Since the map $\sigma(z) = z^d$ acts on S^1 and on $\mathbb{C} \setminus \bar{\mathbb{D}}$ and the relation $\sim$ is σ-invariant, we can introduce a map $f : J \to J$; also, since $p : \mathbb{C} \setminus \bar{\mathbb{D}} \to A_\infty$ is a homeomorphism, $f : J \to J$ extends to the map $f : A_\infty \to A_\infty$ as $f = p \circ \sigma \circ p^{-1}$. Observe, that K, J are compact, connected and locally connected because $p : \bar{\mathbb{D}} \to K$ is continuous. Finally, J and A_∞ are completely f-invariant, and $f|_{J \cup A_\infty}$ is continuous. We fix a metric on $\mathbb{C}/\simeq$ compatible with the topology which makes $\mathbb{C}/\simeq$ a Hausdorff metric space.

According to a theorem of Moore [Mo], the pinched plane $\mathbb{C}/\simeq$ is homeomorphic to the plane.

Proposition 2.2. *Let U be a connected component of the interior F of K. Then its closure is a topological disc. In particular, the boundary ∂U is a Jordan curve.*

Proof. $\bar{U}$ is the quotient of the closure of a $\sim$-component Ω by a closed equivalence relation on $\partial\Omega$ whose classes are points of S^1 and closed arcs in $\bar{\mathbb{D}}$ with the endpoints in S^1. Therefore, it is homeomorphic to $\bar{\mathbb{D}}$. ■

Our next aim is to extend f to $F \neq \emptyset$ (no extension is necessary if $F = \emptyset$).

Lemma 2.3. *If U is a component of F, then $f(\partial U)$ is a boundary of some component U' of F, the map $f : \partial U \to \partial U'$ preserves orientation and is an unbranched degree l covering map with $l \geq 1$ finite.*

Proof. Follows from Lemma 2.1. ■

Lemma 2.4. *If U is a component of F, such that $f^p(\partial U) = \partial U$, for some $p \geq 1$, then the map $f^p : \partial U \to \partial U$ is topologically conjugate either to*

(S) an irrational rotation on S^1, or to

(A) the map $z \mapsto z^l$, for some $l \geq 2$.

Moreover, if $x \in \partial U$ is such that the $\sim$-class $g = p^{-1}(x)$ is not a point, then g is either eventually mapped into a point (and thus precritical) or preperiodic; so if Ω is a $\sim$-component such that $p(\Omega) = U$ then $\sigma^p|\partial\Omega$ is not injective.

Proof. Denote by g the map $f^p : \partial U \to \partial U$. It is enough to show that g has no wandering intervals (i.e., non-trivial arcs $I \subset \partial U$ with $g^k(I) \cap g^n(I) = \emptyset, k \neq n$). Indeed, if g has no wandering intervals then by Lemma 2.3, $g : \partial U \to \partial U$ is conjugate to the rotation (if $l = 1$) or the map $z \mapsto z^l$ (if $l > 1$), see e.g. [MS]. Moreover, the rotation has to be irrational, because the map σ has finitely many periodic orbits of each period.

To prove that there are no wandering intervals we find a finite non-empty set $A \subset \partial U$ and a dense set $S \subset \partial U$ such that any point $x \in S$ eventually hits A (i.e., there exists $k \geq 0$ s.t. $g^k(x) \in A$). Let Ω be a $\sim$-component such that $p(\Omega) = U, S = \{p(l)\}$ where l runs over the geodesics in $\partial\Omega$. Also, let $A = \{p(l_b)\}$ where l_b runs over the family A' of geodesics in $\partial\Omega$ with the radial length (the length of the shortest arc of $S^1 \setminus l_b$) at least $1/(2d^p)$. By Lemma 2.1, the geodesics l are dense in $\partial\Omega$, hence S is dense in ∂U. Also, A is finite because the number of the geodesics l_b as above in the boundary of the same component Ω is at most $2d^p$. Finally, A is non-empty because any geodesic l on $\partial\Omega$ will be eventually mapped by σ^p onto a geodesic of radial length at least $1/(2d^p)$.

Note that if the case (A) holds then $\sigma^p|\Omega$ is not injective because $z \mapsto z^l, l \geq 2$ is not. Suppose that the case (S) holds. Then some geodesics in A' have to map into points since otherwise by the previous paragraph they will all be preperiodic, a contradiction with the case (S). So again $\sigma^p|\Omega$ is not injective which completes the proof. ∎

We call a $\sim$-component U for which the condition of the lemma holds **periodic Siegel** iff (S) holds and **periodic attractive** iff (A) holds (cf. with rational maps [Mi1]).

Proposition 2.5. *The following properties hold.*

1. *Let $g \subset S^1$ be a $\sim$-class or the set $E(\Omega)$ for some $\sim$-component Ω. Then the number of such sets g with the additional property that $\sigma : g \to S^1$ is not injective, is finite. In particular, the number of components U of F such that $f : \partial U \to \partial U$ is an unbranched degree l covering map, $l \geq 2$, is finite.*
2. *The number of all periodic components of F (Siegel and attractive) is finite.*

Proof. (1) Every g satisfying the assumptions, contains two points $x, y \in S^1$ with $\sigma(x) = \sigma(y)$, and so the radial distance between x, y equals to j/d for some $j = 0, 1, ..., [d/2]$. The geodesic (x, y) lies in the convex hull of g and these convex hulls are pairwise disjoint, thus these geodesics are pairwise disjoint too. However there may be only finitely many pairwise disjoint geodesics (x, y) such that the radial distance between x, y equals to j/d for some $j = 0, 1, ..., [d/2]$, hence there are finitely many sets g.

(2) Follows from (1) and the last claim of Lemma 2.4. ∎

We strengthen Proposition 2.5 later in Proposition 2.8.

To extend f from $J = \partial K$ to components of F choose a component U of F and consider the grand orbit of U (the components U^n with the boundaries contained in $f^n(\partial U)$, $n = 0, \pm1, \pm2, \dots$).

Case A. ∂U is invariant under f^p for some p, and $f^p : \partial U \to \partial U$ is an unbranched degree l covering with $l \geq 1$. Let U^i be the component with the boundary $f^i(\partial U)$, $i = 0, 1, ..., p-1$. Keeping the dynamics on $f^i(\partial U)$ we extend it on all U^i in two steps.

(1) Extend $f^p : \partial U \to \partial U$ to $f_{p,U} : \bar{U} \to \bar{U}$ as follows. Using Lemma 2.4, consider a homeomorphism $H : \partial U \to S^1$ conjugating f^p to g_l (g_l is an irrational rotation if $l = 1$, and $g_l(z) = z^l$ otherwise): $g_l \circ H = H \circ f^p$ on ∂U. The map g_l is defined on $\bar{\mathbb{D}}$ and fixes zero. Extend H to a homeomorphism $\bar{H} : \bar{U} \to \bar{\mathbb{D}}$ and let $a_U = \bar{H}^{-1}(0) \in U$. The desired extension of f^p on $\bar{U}$ is $f_{p,U} = \bar{H}^{-1} \circ g_l \circ \bar{H}$. Note that $f_{p,U}|_{\partial U} = f^p$.

Define the set $\mathcal{G}_U = \mathcal{G}_0 = \{\Gamma_z\}_{z \in \partial U}$ of curves in $\bar{U}$ as $\Gamma_z = \bar{H}^{-1}(r_x)$ where $x = H(z) \in S^1$ and r_x is the radius in $\bar{\mathbb{D}}$ between 0 and $x \in S^1$. Then the system of curves $\mathcal{G}_U$ is invariant under $f_{p,U}$, each Γ_z joins $z \in \partial U$ with $a_U = a_0 = \bar{H}^{-1}(0)$, the curves of $\mathcal{G}_U$ form a **foliation** of $\bar{U} \setminus a_U$ (i.e., fill in this set and are pairwise disjoint), and $f_{p,U}(a_U) = a_U$.

(2) We set $U^0 = U^p = U, \mathcal{G}_p = \mathcal{G}_0, a_p = a_0$ and define maps $f_i : \bar{U}^i \to \bar{U}^{i+1}$ ($i = 1, ..., p-1$) so that $f_{p,U} = f_{p-1} \circ f_{p-2} \circ \cdots \circ f_0$. Simultaneously we define points $a_{U^i} = a_i \in U^i$ and foliations $\mathcal{G}_{U^i} = \mathcal{G}_i = \{\Gamma_z\}_{z \in \partial U^i}$ of $\bar{U}^i \setminus a_i$. We begin by defining maps $f_i, i = 1, \ldots, p-1$ as follows:

(a) f_i is a continuous extension of $f : \partial U^i \to \partial U^{i+1}$;

(b) f_i is an unbranched degree l_i covering map with a unique branched point a_i such that $a_{i+1} = f_i(a_i)$ (here l_i is the degree of the map $f : \partial U^i \to \partial U^{i+1}$);

(c) $a_{i+1} = f_i(a_i)$ and $\mathcal{G}_{i+1} = f_i(\mathcal{G}_i)$ (that is, the foliation $\mathcal{G}_i$ is obtained as a pull-back of $\mathcal{G}_{i+1}$ under the map f_i which is possible because $f_i(a_i) = a_{i+1}$).

To begin with the foliation $\mathcal{G}_p = \mathcal{G}_0$ and the point a_p are defined. Let $f_{p-1} \circ \cdots \circ f_1 = h$ and $r = \prod_{i=1}^{p-1} l_i$. Then $h : U^1 \to U$ is of degree r. Define a map $f_0 = h^{-1} \circ f_{p,U}$ first along a curve $\Gamma_{z_0} \in \mathcal{G}_U$. As the point z moves along ∂U^0, extend the germ of f_0 over the curves Γ_z from the map $f : \partial U^0 \to \partial U^1$ to a well-defined map $f_0 : \bar{U}^0 \to \bar{U}^1$ so that $f_{p,U} = f_{p-1} \circ f_{p-2} \circ \cdots \circ f_0$ and properties (a)-(c) above are satisfied for $f_i, 1 \le i \le p-1$.

By the construction, the union of curves of families $\mathcal{G}_i, i = 0, ..., p-1$, is invariant under the map $\bar{f} : \cup_{i=0}^{p-1} U^i \to \cup_{i=0}^{p-1} U^i$ defined as $\bar{f}|U^i = f_i$. Each curve $\Gamma_z \in \mathcal{G}_i$ joins the point $z \in \partial U^i$ and the marked point a_i, and the curves of $\mathcal{G}_i$ form a foliation of $\bar{U} \setminus a_i$.

Case B. U is a preimage W^{-m} of a periodic component W, i.e. $f^m(\partial W^{-m}) = \partial W$. Consider all preimages $W^{-n}, n \ge 1$, other than iterates of W and introduce the dynamics on all W^{-n} inductively (first on all W^{-1}, then all W^{-2}, etc) as follows. We have done it on each periodic W. Assume we have already defined the map $f_{V'} : V' \to f_{V'}(V')$ on every component V' which is not an iterate of W such that $f^i(\partial V') = \partial W$ for some $0 \le i \le n-1$. If now $f^n(\partial V) = \partial W$ and $f : \partial V \to f(\partial V)$ is an l-cover ($l \ge 1$) we define f_V on $\bar{V}$ in such a way, that $f_V|\partial V = f$, $f_V : \bar{V} \to f(\bar{V})$ is a covering map with a chosen point a_V (which is a unique branch point if $l > 1$) such that $f_{f^{n-1}(V)}(a_V) \circ \cdots \circ f_{f(V)} \circ f_V = a_W$. Preimages of the curves of $\mathcal{G}_W$ inside components V form families of curves $\mathcal{G}_V$ which are in fact foliations of sets $V \setminus \{a_V\}$.

Case C. If U is a wandering domain ($f^k(\partial U) \cap f^r(\partial U) = \emptyset$, $k \ne r$), fix a high forward iterate V of U, so that maps $f^n : \partial V \to f^n(\partial V), n > 0$, are isomorphisms. Mark a point $a_V \in V$ and choose a foliation $\mathcal{G}_V = \{\Gamma_x\}_{z \in \partial V}$ of $V \setminus \{a_V\}$, where Γ_z is a curve joining a_V and $z \in \partial V$. Define f on all images of V so that it becomes a homeomorphic extension of f defined on their boundaries; for any such image $U = f^n(V)$ also define the point $a_U = f^n(a_V)$. Now define f on all preimages of all images of V as in Case B.

We get a continuous map $\bar{f} : \mathbb{C}/\simeq \to \mathbb{C}/\simeq$ of the pinched plane as follows (here we define some new notions mimicking [DH], [Do1]). First, $\bar{f}$ coincides with f on $(\mathbb{C}/\simeq) \setminus F$ and with f_U on all components of F. Every component U of F has the marked point a_U called the **center** of U, and $a_{f(U)} = \bar{f}(a_U)$. Every set $\bar{U} \setminus a_U$ is foliated by the curves Γ_x joining a_U with points $x \in \partial U$; these curves, called **internal rays**, form the family $\mathcal{G}_U$. The union $\mathcal{G}(K)$ of $\mathcal{G}_U$ over all components U of F is $\bar{f}$-invariant. An arc l in K is called **legal** if for any component U of the interior F of K, the set $l \cap \bar{U}$ is contained in the union of two internal rays. Talking of an arc defined by a map $\gamma : [0,1] \to K$ we often denote this arc (i.e. the set $\gamma([0,1])$) by γ. Also, by a **loop** in K we mean a continuous map $\gamma : [0,1] \to K$ such that $\gamma(t) \ne \gamma(\tau)$, for all $0 \le t < \tau \le 1$, except if $\gamma(0) = \gamma(1)$.

It is easy to see that the map $\bar{f}$ is a local homeomorphism at any point x of the pinched plane except for a finitely many (by Proposition 2.5) **critical** points $c_1, ..., c_m$ of the form: either $c_i = p(g) \in J$, where g is a critical $\sim$-class, or $c_i = a_U$, where a_U is the center of a component U of F and $f : \partial U \to f(\partial U)$ is an l-cover, $l \geq 2$ (note that each critical point of the latter type is preperiodic whenever U is preperiodic).

Indeed, sets J, F, and A_∞ are completely invariant under the map $\bar{f}$. Moreover, by the construction, for every point $x \in J$ there is a neighbourhood U such that $\bar{f}$ is one-to-one on every component of $U \setminus J$. Therefore, it is enough to check that $\bar{f}|J = f$ is a local homeomorphism at any non-critical point. Let us check that f is actually an open map everywhere; we do this by way of contradiction. If f is not open at x then there is its neighborhood U and a sequence of classes x_n such that $f(x_n) \to f(x)$ while no class $f(x_n)$ has preimages in U. We can assume that $x_n \to y$ and then $f(y) = f(x)$. Then we can choose points $x'_n \in x_n$ which converge to a point $x' \in y$ so that $\sigma(x') \in f(y) = f(x)$. By the properties of laminations we can find a point $z' \in x$ such that $\sigma(z') = \sigma(x')$ which implies that there exists a sequence of points $z'_n \to z'$ such that $\sigma(z'_n) = \sigma(x'_n)$. Choosing a subsequence, we may assume that classes z_n of points z'_n converge in J, and then they can only converge to the class x. On the other hand, classes z_n from some time on belong to U which proves that classes $f(z_n) = f(x_n)$ belong to $f(U)$, contrary to our assumption. The verification of the fact that f is 1-to-1 at a non-critical point is just as elementary as is left to the reader as a useful exercise.

The **external ray** R_t of argument $t \in \mathbb{T}$ is the curve $p(\{r \exp(2\pi i t) : r > 1\})$, the external rays $R_t, t \in \mathbb{T}$ foliate the basin of infinity A_∞. If $r \to 1$ then the point $p(r \exp(2\pi i t))$ of R_t tends to the point $x = p(\exp(2\pi i t))$ in J (R_t **lands** at x) and vice versa, every point $x = p(\exp(2\pi i t)) \in J$ is a landing point of the external ray R_t.

Lemma 2.6. *The set K is arcwise connected and has the following properties:*

1. *there is no loop γ in K which is the union of finitely many legal arcs;*
2. *given points $x, y \in K$, there exists a unique legal arc in K with endpoints at x, y;*
3. *if γ is a legal arc, then $\bar{f}(\gamma)$ is a finite union of the legal arcs containing no loops.*

Proof. K is arcwise connected because it is the image of $\bar{\mathbb{D}}$ under a continuous map p.

(1) If γ lies in a component U of F, the statement clearly holds. Otherwise fix points $a \neq b \in \gamma$ who split γ into two closed arcs γ_1, γ_2, so that $\gamma_1 \cap \gamma_2 = \{a, b\}$. Consider subsets $\tilde{\gamma} = p^{-1}(\gamma), \tilde{\gamma}_i = p^{-1}(\gamma_i), i = 1, 2$ of $\bar{\mathbb{D}}$. Since $p^{-1}(x)$ is a connected closed subset of the plane for any $x \in \mathbb{C}/\simeq$, the sets $\tilde{\gamma}_1, \tilde{\gamma}_2$ are compact connected subsets of $\bar{\mathbb{D}}$ while $\tilde{\gamma}_1 \cap \tilde{\gamma}_2 = p^{-1}(a) \cup p^{-1}(b)$ is not connected. Hence ([Ku]) $\tilde{\gamma}_1 \cup \tilde{\gamma}_2 = \tilde{\gamma}$ separates the plane. Let $\tilde{A}$ be a bounded component of $\mathbb{C} \setminus \tilde{\gamma}$. Since $\tilde{\gamma}$ consists of $\simeq$ classes, $\tilde{A}$ consists of $\simeq$ classes as well. Also, $\tilde{A}$ is open. Then $\tilde{A} \subset \mathbb{D}$ because $\tilde{\gamma} \subset \bar{\mathbb{D}}$ and so if $\tilde{A}$ hits $\mathbb{C} \setminus \mathbb{D}$ it must be unbounded. Hence, $\tilde{A}$ is disjoint from any $\sim$-class because otherwise it would contain points of $\bar{\mathbb{D}}$ (every $\sim$-class contains points of $\bar{\mathbb{D}}$ by definition), and so $\tilde{A}$ contains an interior point $\tilde{x}$ of a $\sim$-component Ω. Thus, the point $x = p(\tilde{x})$ lies in the component $U = p(\Omega)$ of F.

Now, $\tilde{A} \subset \mathbb{D}$ implies $\Omega \cap \tilde{\gamma} \neq \emptyset$. Hence $\gamma \cap U \neq \emptyset$ too. By the definition of a loop, $\gamma \cap U$ is a finite union of internal rays. Moreover, since γ has no points of self-intersection, $x \in \gamma \cap \bar{U} = \Gamma_{x_1} \cup \Gamma_{x_2}$ where $x_1, x_2 \in \partial U$ and $\Gamma_{x_1}, \Gamma_{x_2}$ are the corresponding internal rays. Let $A = p(\tilde{A})$; clearly, A is an open and connected subset of a pinched disk. Then one of two open arcs of $\partial U \setminus \{x_1, x_2\}$ lies in A, a contradiction with $A \subset U$.

(2) Let $\gamma : [0,1] \to K$ be a curve, connecting $x = \gamma(0)$ with $y = \gamma(1)$. Then the set $\gamma \cap \bar{U}$ is closed for any component U of F. Let α_U, β_U be the least and the greatest numbers with $\gamma(\alpha) \in \bar{U}, \gamma(\beta) \in \bar{U}$. For every U, we can redefine γ on $[\alpha_U, \beta_U]$ so that γ maps the interval $[\alpha_U, \beta_U]$ onto $\Gamma_{\gamma(\alpha_U)} \cup \Gamma_{\gamma(\beta_U)}$. We proceed this way, applying the construction on every step to the current map γ. It is easy to see that the sequence of maps γ (and corresponding curves) converges to a legal arc with endpoints x, y as desired. By (1) this arc is unique.

(3) Follows immediately from (1). ∎

Given $x, y \in K$, denote by $[x, y]$ a unique well-defined by Lemma 2.6 legal arc in K with ends at x, y. Now we step by step define a growing tree $T_0 \subset T_1 \subset \dots$ in K for the map $\bar{f}$. Let $\beta = p(0)$ ($0 \in S^1$ is a fixed point of the map $\sigma(z) = z^d$ of S^1). Then β is also a fixed point of $\bar{f}$. By (D2) any $\sim$-class in $\sigma^{-1}(\mathrm{Cl}(0))$ contains at least one point of $\sigma^{-1}(0)$. Hence there are no more than d preimages of β; denote them by $\{\gamma_i\}$ and then define the **initial tree** $T_0 = \cup_i[\gamma_i, \beta]$.

Let $T_n = \cup_{i=0}^n \bar{f}^i(T_0)$. By Lemma 2.6 all T_n are trees. Given $x \in J$, denote by $N(x)$ the number of the external rays landing at x. In the next proposition we study the trees T_n and the orbits of the points $x \in J$ with $N(x) \geq 2$. We say that two external rays R_{t_1} and R_{t_2} are **separated (by the tree T_0)** if t_1 and t_2 lie in different components of $\mathbb{T} \setminus \{0, 1/d, 2/d, ..., 1 - 1/d\}$. Recall, that $k_\sim$ is the maximal number of critical $\sim$-classes g with infinite and pairwise disjoint orbits, and such that $\sigma(g)$ is a point.

Proposition 2.7. *The following properties hold.*

(1) *If separated external rays R_{t_1}, R_{t_2} land at the same point x then $x \in T_0$.*
(2) *All critical points of $\bar{f}$ belong to T_0 and $T_0 \subset T_1 \subset \dots$ is a growing tree.*
(3) *If $x \in J$ then $N(x) = |p^{-1}(x)|$.*
(4) *If $M \subset J$ is a continuum or $M = \{x\}$ with $N(x) \geq 2$ then there exists i with $f^i(M) \cap T_0 \neq \emptyset$. Moreover, the following holds:*
(a) *in the case of continuum there are infinitely many i such that $\bar{f}^i(M) \cap T_0 \neq \emptyset$ and the set of points eventually mapped into T_0 is dense in M.*
(b) *if $x \in J$ is not an $\bar{f}$-preimage of a critical point or of β then $N(x) \geq 2$ if and only if there are infinitely many i such that $\bar{f}^i(x)$ is an inner point of the tree T_0.*
(c) *if $N(x) \geq 3$, then, for every finite $n \leq N(x)$, and for some i, m the point $\bar{f}^i(x)$ is a vertex of T_m with $\mathrm{ord}_{T_m}(x) \geq n$.*
(5) *For every component U of F, some iterate $V = \bar{f}^i(U)$ intersects T_0; moreover, the center a_V of V lies in T_0, and $\bar{V} \cap T_0$ is homeomorphic to the n-od with the branching point at a_V.*
(6) $\mathrm{oen}(T_\infty) \leq k_\sim$.

Proof. (1) Since β and all its f-preimages γ_i belong to T_0 we see that $\sim$-classes corresponding to γ_i include points $0, 1/d, \dots, (d-1)/d$. Since $g = p^{-1}(x)$ is a $\sim$-class containing t_1, t_2 then the convex hull of g intersects $p^{-1}(T_0)$ and hence $x \in T_0$.

(2) If g contains i/d for some i then $p(g) \in T_0$ by the definition of T_0. Otherwise suppose that $p(g) \notin T_0$. Then by (1) all rays with arguments from g are not separated.

On the other hand, since g is critical there must be two points x and $x+j/d, 1 \le j \le d-1$ in g, a contradiction. Similarly, if c is a critical point which is the center of a Fatou domain U, then there are external rays landing on the boundary of U which are mapped into one ray. These rays are separated, hence there are two angles of the form $i/d, j/d$ such that the legal arc connecting i/d with j/d must cross U and thus must pass through c showing that in this case $c \in T_0$ too. Together with Lemma 2.6 this implies that indeed $T_0 \subset T_1 \subset \dots$ is a growing tree.

(3) Follows from the definition of the external rays.

(4) In our situation we can find two external rays R_{t_1}, R_{t_2} landing at points of M so that for some i either one of the rays $\bar{f}^i(R_{t_1})$, $\bar{f}^i(R_{t_2})$ has the argument j/d (and lands at a point of T_0) or the rays $\bar{f}^i(R_{t_1})$, $\bar{f}^i(R_{t_2})$ are separated in which case by connectivity the continuum $\bar{f}^i(M)$ must intersect T_0. This proves the main claim of (4).

To prove (a) observe that subcontinua of arbitrarily small diameters are dense in M and that the image of a continuum under a power of f is a continuum itself.

To prove (b) observe that under the assumptions of the proposition the first claim can applied to x infinitely many times, so $\bar{f}^i(x) \in T_0$ for infinitely many i. Since x is not a preimage of β then $\bar{f}^i(x)$ is not an endpoint of T_0. If on the other hand $f^k(x)$ is not an endpoint of T_0 then there are at least two external rays landing at $f^k(x)$ and so there are at least two external rays landing at x implying $N(x) \ge 2$.

Consider claim (c). First we show that any non-critical $x \in J$ has a neighborhood U such that for any $y \in J \cap U$ the cyclic order on the set $p^{-1}(x) \cup p^{-1}(y)$ is preserved by σ. Indeed, $\simeq$ is a closed equivalence relation on the plane such that every equivalence class is closed, connected and nonseparating. Hence, there is an arbitrarily small neighborhood $\tilde{U}$ of $p^{-1}(x)$ such that $\tilde{U}$ consists of $\simeq$-classes. We can set $U = p(\tilde{U})$, and by (D2) the property is satisfied.

Let $N(x) \ge 3$. Fix $n, 3 \le n \le N(x)$. Let $R_{t_i}, i = 0, \dots, n-1$ be external rays landing at x in the cyclic order of their arguments $t_0, ..., t_{n-1}$. By (D2) this order will not change under iterations of σ. For each $i = 0, ..., n-1$, find the minimal $r_i = r > 0$ so that $\bar{f}^r(R_{t_i}), \bar{f}^r(R_{t_{i+1}})$ are separated. By (D2) the arc $I_i =< \sigma^r(t_i), \sigma^r(t_{i+1}) >$ containing no σ^r-images of other t_j is well-defined. Also, by the first part of claim (4) $\bar{f}^r(x)$ is an inner point of T_0, so there are points $y \in T_0$ arbitrarily close to x such that $p^{-1}(y) \subset I_i$.

Repeating this we find numbers $r_i, i = 0, \dots, n-1$. Let R be their maximum. Pick a small neighborhood U of x so that for all $y \in U$ all the iterates $\sigma^j, 0 \le j \le R$ preserve the cyclic order on $p^{-1}(x) \cup p^{-1}(y)$. Then choose points $y_i \in U$ so that $\sigma^{r_i}(p^{-1}(y_i)) \subset I_i$ and $\bar{f}^{r_i}(y_i) \in T_0$. Since σ^R preserves the cyclic order on $p^{-1}(x)$, the cyclic order of points $\{\sigma^R(t_i)\}$ is the same as that of points $\{t_i\}$. Thus the pairwise disjoint arcs $<\sigma^R(t_i), \sigma^R(t_{i+1})>$ are well-defined. By the choice of U each set $\sigma^R(p^{-1}(y_i))$ is contained in $<\sigma^R(t_i), \sigma^R(t_{i+1})>$. Since $\bar{f}^R(y_i) = \bar{f}^{R-r_i}(\bar{f}^{r_i}(y_i)) \subset T_{R-r_i}$ we get the desired estimate $\text{ord}_{T_m}(x) \ge n$ for a sufficiently big m.

(5) Let Ω be the corresponding to U component of $\bar{\mathbb{D}} \setminus L$. Take any two $t_1, t_2 \in E(\Omega)$, which are non-precritical, non-preperiodic, and whose σ-images are not $\sim$-equivalent (it is possible since $E(\Omega)$ is a Cantor set). Then $\bar{f}^i$-iterates of external rays R_{t_1} and R_{t_2} land at distinct points of $\partial f^i(U)$ and are separated for some i. Let Ω' be the $\sim$-component with $E(\Omega') = \sigma^i(E(\Omega))$. Then $p^{-1}(T_0)$ intersects $\partial\Omega'$ at least at two points. Since T_0 consists of legal arcs and by Lemma 2.6 we conclude that $\bar{f}^i(\bar{U})$ is of the desired form.

(6) By Lemma 1.4(5), $\mathrm{oen}(T_\infty) \le k''$ where k'' is the number of distinct grand orbits of fast critical points of $\bar f : T_\infty \to T_\infty$. We estimate the number of fast critical points. Each critical point which maps into T_0 by a positive iterate of $\bar f$ is slow. So, by (5) critical points of $\bar f$ in the interior F of K are slow. Also, each preperiodic critical point is slow. Let $c \in J$ be a non-preperiodic critical point whose forward iterates do not hit other critical points. Let $g = p^{-1}(c)$ be the corresponding critical class. If $N(\bar f(c)) = |\sigma(g)| \ge 2$, then, by (4), some iterate of c is a point of T_0, i.e. c is slow. Therefore, $k'' \le k_\sim$. ■

Proof of Theorem C. (1). Assume to the contrary that for a component Ω, the set $E(\Omega)$ is not preperiodic under σ. Replacing Ω by its sufficiently high iterate we may assume that no iterate of $U = p(\Omega)$ contains a critical point of $\bar f$.

As in the proof of Proposition 2.7(4c), we show that for any s we can find high iterate of U whose boundary intersects T_m for some m at least at s points.

Now, the only vertex of T_m which an iterate of U may contain must be the center of this iterate. Thus, by the above proven we can find high iterates of U which contain a vertex v of T_m with $\mathrm{ord}_{T_m}(v) = s$ for any s. By Theorem 1.10(1) this implies that the vertex in question is preperiodic and thus U is also preperiodic.

(2) Suppose that a continuum M is wandering and so all its iterates are pairwise disjoint. We may assume that all iterates of M avoid critical points. First we show that $M \subset T_\infty$ is impossible. To this end let us prove that preimages of the endpoints of T_0 and of vertices of T_∞ are dense in M. Choose a point $x \in M$ which is not an endpoint of T_0, a small interval $I \subset M$ containing x and a component V of $S^1 \setminus \{j/d\}_{j=0}^{d-1}$ such that for any point $y \in I$ there exists an external rays landing at y with the argument from V. Denote the set of all such rays by A. Then there exists the minimal $k > 0$ such that two rays from the set $f^k(A)$ have the arguments which belong to different components of $S^1 \setminus \{j/d\}_{j=0}^{d-1}$. Since M contains no preperiodic points this can only happen if $f^k(I)$ contains either a vertex of T_∞ or an endpoint of T_0 and the claim about the density of preimages of the endpoints of T_0 and of vertices of T_∞ in M is proven. By Corollary 1.6 this implies that M is non-wandering, a contradiction.

Let us now consider the general case when $M \subset J$. By (1) and Lemma 2.4 M cannot contain a subcontinuum of the boundary ∂U of a Fatou component. Let us use this to prove that if $x \ne y \in M$ then the legal arc I connecting x, y must be contained in M. Indeed, otherwise we may assume that $I \cap M = \{x, y\}$ and consider closed connected sets $p^{-1}(M)$ and $p^{-1}(I)$. Since their intersection $p^{-1}(x) \cup p^{-1}(y)$ is not connected we conclude by [Ku] (cf. Lemma 2.6(1)) that the set $A = p^{-1}(M) \cup p^{-1}(I)$ separates the plane. Denote by B a bounded component of $\mathbb{C} \setminus A$; then as in the proof of Lemma 2.6(1) we can show that B consists of $\simeq$-classes, is open and contained in $\mathbb{D}$. Therefore $p(B)$ contains no points of J (otherwise B would contain points of S^1) and so $p(B)$ must be a subset of a Fatou component. This implies that M contains a connected subset of the boundary of this Fatou component, a contradiction.

So, the legal arc I is a subset of M. Since I never covers a critical point then all the images of I are all legal arcs. Moreover, by Proposition 2.7(4)(a) we can find two points $x \ne y \in I$ eventually mapped into T_0, hence we can find a subarc $L \subset I$ which is mapped into T_m for big enough m. However, by the first paragraph of the proof this image of L is non-wandering, a contradiction. ■

Example. Consider the filled-in Julia set $K(f)$ (see e.g. [Mi1] and next Section) of a polynomial $f(z) = z^2 + \exp(2\pi i t)z$, where $t = (5^{1/2} - 1)/2$. Then 0 is the Siegel fixed point, which is the "center" of the Siegel disc U. By [Pe1], the Julia set J of f is locally connected. Because the critical point c of f lies in J, we do not need to change the dynamics, so that $\bar{f} = f$. The initial tree T_0 is the arc $[\gamma, \beta]$, where β is the repelling fixed point of f, and $f(\gamma) = \beta$, $\gamma \neq \beta$. Since $c \in \partial U$ (see [He]), the tree T_n consists of T_0 and n internal rays in U, $f^i(\Gamma), i = 1, ..., n$, where Γ is one of the two internal rays of $U \cap T_0$, which joins the fixed point 0 and the critical point c. Note that all iterates $f^i(\Gamma)$, $i = 0, 1, 2, ...$, are pairwise disjoint (except at 0) and form a dense subset in $\bar{U}$. Thus, T_n has the only vertex at 0, and $\mathrm{ord}_{T_n}(0) = n + 2$, in particular, tends to ∞. Note that in this example no germ of the (periodic) vertex at 0 is recurrent. □

Denote by k_c the total number of all critical classes of $\sim$ and by k_A the number of different orbits of all components of $\sim$ on whose boundaries f is not 1-to-1 (clearly, then k_A is less than or equal to the number of the periodic orbits of attracting components of $\sim$). Obviously, $k_\sim \le k_c$. Now, let g be a $\sim$-critical class. Define degree $d(g)$ of g as the degree of the cover $\sigma : g \to \sigma(g)$; if $|g|$ is finite, then $d(g) = |g|/|\sigma(g)|$ (cf. [Th]). Let Ω be a component of $\sim$. Then $f : \partial U \to f(\partial U)$ is a covering map, where $U = p(\Omega)$ is the corresponding component of F. Define degree $d(\Omega)$ as the degree of this map and call Ω **critical** iff $d(\Omega) \ge 2$.

Proposition 2.8 (cf. [Th]). $\sum(d(g) - 1) + \sum(d(\Omega) - 1) = d - 1$ *where g and Ω run over the critical classes and components respectively. In particular, $k_\sim \le d - 1$, and, moreover, $k_\sim + k_A \le k_c + k_A \le d - 1$.*

Proof. We follow an idea of [Th, Proposition: total critical degree of a lamination]. Actually, we prove the following more general statement. Let $\mathbb{D}_0$ and $\mathbb{D}_1$ be two copies of the unit disc, and $\sim_0$, $\sim_1$ be closed laminations on their boundaries $\partial\mathbb{D}_0, \partial\mathbb{D}_1$ respectively. Let τ be any orientation preserving endomorphism of degree $d \ge 1$ (if $d = 1$, τ is a homeomorphism), such that τ satisfies (D1)-(D3) (with obvious changes of notations). The definition of $\sim_0$ critical class g and its degree (to be the degree of the map $\tau : g \to \tau(g)$) is unchanged. To define the degree of a $\sim_0$ component, let us consider the natural extensions $\simeq_0$, $\simeq_1$, and the quotient spaces $K_0 = \overline{\mathbb{D}}_0 / \simeq_0$, $J_0 = \partial\mathbb{D}_0 / \sim_0$ and $K_1 = \overline{\mathbb{D}}_1 / \simeq_1$, $J_1 = \partial\mathbb{D}_1 / \sim_1$ together with projections p_0, p_1. Let $f_\tau : J_0 \to J_1$ be the induced map. Observe that Lemmas 2.1, Proposition 2.2, and Lemma 2.3 still hold (with obvious changes of notations) since we don't use iterates of the map τ in the proofs. It allows us to define the degree $d(\Omega)$ of a $\sim_0$ component Ω as the degree of the map $f_\tau : \partial U \to \partial U'$ where $U = p_0(\Omega)$ (and U' defined similarly). It is convenient to set $d(g) = d(\Omega) = 1$ for any not critical $\sim_0$ class g and component Ω. Now we prove the formula

$$\Sigma(\sim_0; \sim_1) = d - 1 \qquad (**)$$

where $\Sigma(\sim_0; \sim_1) = \sum(d(g) - 1) + \sum(d(\Omega) - 1)$ and g and Ω run over ALL $\sim_0$ classes and components respectively. It is clear that (**) implies the statement. We prove it by induction over the degree d of τ. If $d = 1$, (**) holds (both sides of (**) vanish). Assume that (**) holds for the degrees $1, ..., d - 1$, and prove it for $d \ge 2$. First, assume that each critical $\sim_0$ class g as well as its $\sim_1$ image $\tau(g)$ are gaps. Notice that then if

(x_1, x_2) is a $\sim_0$ geodesic in a boundary of a gap g, then $\tau((x_1, x_2))$ is a $\sim_1$ geodesic in the boundary of the gap $\tau(g)$. Hence, in Lemma 2.1 only alternative (1) holds. In particular, $\tau : E(\Omega) \to \tau(E(\Omega))$ is a covering map (and its degree is $d(\Omega)$). Therefore, one can extend $\tau : \partial\mathbb{D}_0 \to \partial\mathbb{D}_1$ to a covering map Ψ of $\mathbb{D}_0$ onto $\mathbb{D}_1$ in such a way, that it preserves the relations $\simeq_0, \simeq_1$. In this case, the equality (**) expresses the Riemann-Hurwitz formula for the cover Ψ.

If there is a critical class g so that $\tau(g)$ is either a point or a leaf, then we use the induction hypothesis as follows.

(1) $\tau(g)$ is a point. Note that $|g| \le d$. Cut the disc $\mathbb{D}_0$ along all geodesics of the boundary of the convex hull of g, and then glue the endpoints of the resulting arcs to form $|g|$ new circles $\partial\mathbb{D}_{0,j}, 1 \le j \le |g|$. The original lamination $\sim_0$ and the map τ are then split into closed laminations $\sim_{0,j}$ on $\partial\mathbb{D}_{0,j}$ and the corresponding maps $\tau_j : \partial\mathbb{D}_{0,j} \to \partial\mathbb{D}_1$, $1 \le j \le |g|$, which preserve the corresponding relations. If d_j is the degree of τ_j, then $d_j < d$ and $\sum_j d_j = d$. By the induction hypothesis, $\Sigma(\sim_{0,j}; \sim_1) = d_j - 1$. Then $\Sigma(\sim_0; \sim_1) = \sum_j \Sigma(\sim_{0,j}; \sim_1) + (d(g) - 1) = \sum_j d_j - |g| + |g| - 1 = d - 1$.

(2) $\tau(g)$ is a leaf $A = \{a, b\}$ of $\sim_1$. Note that $|g| \le 2d < \infty$. Let $x_1, x_2, x_3 \in S^1$ be 3 adjacent points of the $\sim_0$ class g. Then $\tau(x_1) = \tau(x_3) = a$ and $\tau(x_2) = b$. Cut the disc $\mathbb{D}_0$ along the geodesic $l = (x_1, x_3)$ inside the convex hull of g, and glue the endpoints of the two resulting arcs. Then we get two circles $\partial\mathbb{D}_{0,j}$ with the corresponding laminations $\sim_{0,j}$ and two maps $\tau_j : \partial\mathbb{D}_{0,j} \to \partial\mathbb{D}_1$, $j = 1, 2$. Then g induces classes g_1 and g_2: g_1 is a leaf in $\sim_{0,1}$ obtained from the triangle with vertices x_1, x_2, x_3 by gluing the edges (x_1, x_2) and (x_3, x_2); g_1 is not critical (its image by the map τ_1 is the leaf A) while g_2 is either a not critical leaf of $\sim_{0,2}$ (if $|g| = 4$) or a critical gap with less points which is mapped to the leaf A by the second new map τ_2. Moreover, $d(g_2) = d(g) - 1$. If d_j is the degree of τ_j, $j = 1, 2$, then $d_1 + d_2 = d$ and, by the induction hypothesis, we have $\Sigma(\sim_{0,j}; \sim_1) = d_j - 1, j = 1, 2$. But $\Sigma(\sim_{0,1}; \sim_1) + \Sigma(\sim_{0,2}; \sim_1) = \Sigma - (d(g) - 1) + (d(g_2) - 1) = \Sigma - 1$ where $\Sigma = \Sigma(\sim_0; \sim_1)$ is the sum for the initial lamination. Thus $\Sigma - 1 = (d_1 - 1) + (d_2 - 1) = d - 2$, and we are done. ■

In Lemma 2.9 we classify the growing tree T_∞ using results from Section 1.

Lemma 2.9. *The growing tree T_∞ is normal and recurrent; moreover, for any exceptional non-precritical vertex v of T_m all germs of T_m at v are recurrent.*

Proof. The fact that T_∞ is normal follows from the definition of T_0. To prove the rest it is enough to check whether for any exceptional non-precritical vertex v of T_m all germs of T_m at v are recurrent. Observe that the intersection of any Fatou component with any T_m is the union of some legal arcs, thus the only vertices of T_∞ which belong to F are centers of Fatou components. By Theorem 2.8 all such points are preperiodic and thus they are not exceptional vertices of T_∞. Thus we may consider an exceptional non-precritical vertex v of T_m such that $v \in J$. First observe that v cannot belong to the closures of two Fatou components because any Fatou component is preperiodic and v is not. Since v is a vertex of T_m then among sides of T_m at v there are at least two such that the corresponding semineighborhoods of v are not contained in internal rays. Hence there are at least two external rays landing at v. We use it in order to prove that every germ A of T_m at v is recurrent. Consider two cases.

(a) A is a germ of an internal ray of some preperiodic component U. By Proposition 2.7(5), one can assume $\bar{U} \cap T_0$ is not empty and $a_U \in T_0$. Moreover, passing to an iterate, we assume that $\bar{f}(U) = U$ and $\bar{f}(a_U) = a_U$. Thus A belongs to an internal ray l of U, which joins a_U and $v \in \partial U$. Since at least 2 external rays land at v then by Proposition 2.7(4) there are infinitely many $k > 0$ such that $\bar{f}^k(v) \in T_0$. For any such k, $\bar{f}^k(A)$ is a germ of the internal ray $\bar{f}^k(l)$, which joins points a_U and $\bar{f}^k(v)$ of T_0. Thus, A is recurrent.

(b) If $A = (v, S)$ (where S is a side of T_m at v) is not a germ of an internal ray, then there is a sequence $x_n \to v$ such that $x_n \in A' \cap J$ where $A' \in S$ is a semi-neighborhood of v from the side S. Then x_n as a landing point of at least two external rays of $\bar{f}$ returns to T_0 under infinitely many iterates and hence A is recurrent which completes the proof. ■

Now we prove the full version of Theorem B of Introduction. Let k_S be the number of periodic orbits of Siegel components and let $k_p \le k_S + k_A$ be the number of all periodic orbits of Fatou components of a lamination $\sim$.

Theorem 2.10. *Let Γ be a non-empty collection of classes of $\sim$, such that:*

(a) *any $g \in \Gamma$ is non-preperiodic under the map σ (i.e., each $t \in g$ is irrational);*
(b) *the orbits of $g \in \Gamma$ are pairwise disjoint;*
(c) *$|g| \ge 3$ for every $g \in \Gamma$ (i.e., g is a gap);*
(d) *σ^n is injective on g for every $n = 1, 2, \ldots$ and every $g \in \Gamma$.*

Then $\sum_{g\in\Gamma}(|g| - 2) \le k_\sim - k_S - 1 \le d - 2 - k_p \le d - 2$ so that the number of classes in Γ is at most $k_\sim - k_S - 1 \le d - 2 - k_p \le d - 2$ and for every $g \in \Gamma$ we have $|g| \le k_\sim - k_S + 1 \le d - k_p \le d$. Furthermore, for any non-preperiodic (not necessarily non-precritical) $\sim$-class g' we have $|g'| \le (k_S + 1)2^{d-k_A-k_S}$ which implies both $|g'| \le 2^{d-k_A}$ and $|g'| \le (k_S + 1)2^{d-k_p}$ so that in any case $|g'| \le 2^d$.

Proof. First we estimate the number of orbits of irrational vertices of T_m. The forward orbit of each periodic component of F (and of its center) intersects T_0 (Proposition 2.7(5)), the number of periodic components is finite (Proposition 2.5(2)), so there exists a finite $m_0 > 0$, such that every Siegel component intersects T_{m_0}, and its center lies in T_{m_0}. Now, the periodic orbit of a Siegel component contains a component U non-disjoint from T_0 such that $U \cap T_0$ contains its center $a_U \in T_0$ and two internal rays l_1, l_2. Note that $\bar{f}^p(a_U) = a_U$, where p is the period of U. Since $\bar{f}^p|U$ is conjugate to an irrational rotation, the germs of l_1, l_2 at a_U are not periodic and a_U is an irrational vertex of T_∞. So, the number p_I of orbits of irrational vertices of $T_m, m \ge m_0$ is at least k_S (i.e., $p_I \ge k_s$).

Now, let us pass to the proof of the theorem. We prove the first series of inequalities of the theorem; obviously, they imply the rest. Choose classes $g_1, \ldots, g_r$ in Γ and denote by $z_1 = p(g_1), ..., z_r = p(g_r) \in J$ their projections. By (d) $z_1, ..., z_r$ are non-precritical, by (a)-(b) they belong to distinct grand orbits of non-preperiodic points, and by (c) $|g_i| = N(z_i) \ge 3, 1 \le i \le r$. Let us prove that $\sum_{i=1}^r (n_i - 2) \le k_\sim - 1$ for any finite $3 \le n_i \le N(z_i), i = 1, ..., r$. By Proposition 2.7(4), for a large enough l the points $v_i = f^l(z_i)$ are vertices of some T_m with $\mathrm{ord}_{T_m}(v_i) \ge n_i$. By Lemma 2.9 T_∞ is a normal recurrent growing tree and moreover $\mathrm{ord}'_{T_m}(v_i) = \mathrm{ord}_{T_m}(v_i)$. Thus by Theorem 1.3 we have $\sum_{i=1}^r (n_i - 2) \le \sum_{v\in\bar{f}^l(\Gamma)}(\mathrm{ord}'_{T_m}(v) - 2) \le \mathrm{oen}(T_\infty) - k_S - 1$. Since by

Proposition 2.7(6) $\text{oen}_{T_\infty)} \le k_\sim$ we see that $\sum_{i=1}^r (n_i - 2) \le k_\sim - k_S - 1$ and therefore $\sum_{g\in\Gamma}(|g|-2) \le k_\sim - k_S - 1$. It remains to notice that by Proposition 2.8 $k_\sim \le d-1-k_A$, so $k_\sim - k_S - 1 \le d-1-k_A-k_S-1 = d-2-k_p$. To complete the proof observe that the number p_p of preperiodic critical points is at least k_A, so by Theorem 1.3 we get $|g| \le (p_I+1)2^{d-p_p-p_I} \le (k_S+1)2^{d-k_A-k_S}$ as desired. ■

Finally we state Proposition 2.11 which deals with the number of **all** non-cyclic grand orbits of gaps (i.e., without the assumption of their non-precriticality).

Proposition 2.11. *Suppose that there are k critical classes and s critical classes of degree 2. Then the following two cases may take place:*

(1) *if T_∞ is not a finite tree then the number of all grand orbits of non-preperiodic gaps of $\sim$ is at most $2k - s - 1 \le d - 2$;*
(2) *if T_∞ is a finite tree then non-preperiodic gaps of $\sim$ are precritical and the number of their grand orbits is at most $k \le d - 1$.*

Proof. Follows from Proposition 1.13. The only inequality we need to check is $2k-s-1 \le d-2$. To this end denote the number of critical classes of degree greater than 2 by x. Then by Proposition 2.8 $s+2x \le d-1$. This implies that $x \le (d-1-s)/2$ and therefore $k = s+x \le (d-1+s)/2$. We conclude that $2k-s-1 \le d-1+s-s-1 = d-2$ as desired. ■

3. Connected Julia sets of polynomials

Let K be a connected and full (i.e., $\mathbb{C} \setminus K$ is connected) compact set in $\mathbb{C}$. By the Riemann mapping theorem, one can find a unique analytic isomorphism $\varphi_K : \mathbb{C} \setminus K \to \mathbb{C} \setminus \bar{\mathbb{D}}$ such that $\varphi_K(z)/z$ tends to a positive constant as $z \to \infty$ (we assume that K is not a single point). The **external ray** $R_t(K)$ of argument $t \in \mathbb{T}$ is the set $\varphi_K^{-1}(\{r\exp(2\pi i t) : r > 1\})$; it **lands** at a point $x \in \partial K$ iff $R_t(K)$ has the only limit point in K at x. Then t is called an **external argument** of the point x. By the Caratheodory theorem mentioned in the Introduction, K is locally connected if and only if the map φ_K^{-1} extends continuously to a map $\varphi_K^{-1} : S^1 \to \partial K$ which induces a **closed lamination** on $\mathbb{T} : t_1 \sim_K t_2$ if and only if $\varphi_K^{-1}(t_1) = \varphi_K^{-1}(t_2)$ (clearly, $\sim$ satisfies the conditions (E1)-(E3) from Section 2). Then K is homeomorphic to the space $\bar{\mathbb{D}}/\simeq$ defined in Section 2, and so ∂K is homeomorphic to the boundary $\partial(\bar{\mathbb{D}}/\simeq)$ of $\bar{\mathbb{D}}/\simeq$ in $\mathbb{C}/\simeq$ [Do1].

Below K is the **filled-in Julia set** $K(P)$ of a polynomial P of degree $d \ge 2 : K(P) = \{z : P^n(z) \not\to \infty, n \to \infty\}$, see e.g. [Mi1]. The classical **Julia set** ([F], [J]) $J(P)$ of P is then $\partial K(P)$. The compact $K(P)$ is full since it contains all bounded components of $\mathbb{C} \setminus J(P)$ (maximal principle). Its complement $A_\infty(P) = \{z : P^n(z) \to \infty, n \to \infty\}$ is the **basin of infinity** of P. If $K(P)$ is connected then the Riemann map $\varphi_{K(P)} : \mathbb{C} \setminus K(P) \to \mathbb{C} \setminus \bar{\mathbb{D}}$ and the external rays of $K(P)$ (or P) are defined. A connection with the dynamics is then expressed by the following fact (it follows from the Schwarz reflection principle): the Riemann map $\varphi_{K(P)}$ coincides with the so-called **Bottcher function**, a map at infinity, which conjugates P to the map $\sigma(z) = z^d$, i.e., $\varphi_{K(P)} \circ P \circ \varphi_{K(P)}^{-1} = \sigma$ on $\mathbb{C} \setminus \bar{\mathbb{D}}$. (Without loss of generality, one can assume that the polynomial P is monic: $P(z) = z^d + \dots$. Then $\varphi_{K(P)} \sim z$ at infinity.)

Assume first that $K(P)$ (or, equivalently, $J(P)$) is locally connected. Then the closed lamination $\sim_{K(P)}$ on $\mathbb{T}$ is defined. Also, the map $\varphi^{-1}_{K(P)}$ and last equation extend to S^1, and we obtain a fundamental fact that $\sim_{K(P)}$ is invariant under σ, i.e. satisfies also the conditions (D1), (D2), (D3) from Section 2. Then the classical Julia set $J(P)$ is homeomorphic to $\partial(\bar{\mathbb{D}}/\simeq_{K(P)}) = J_{\simeq_{K(P)}}$. If we now define the map $f : J_{\simeq_{K(P)}} \to J_{\simeq_{K(P)}}$ as in Section 2 then the map $P : J(P) \to J(P)$ is conjugate to the map $f : J_{\simeq_{K(P)}} \to J_{\simeq_{K(P)}}$ and we can apply results of Section 2 to deduce dynamical properties of the polynomial P.

Let $F(P)$ be the interior of $K(P)$. It is called **Fatou set** of P. If its component U is periodic, i.e. $f^p(U) = U, p \geq 1$, then by the Wolff-Denjoy theorem U is **attracting** or **Siegel** (both cases are possible) according to whether the P^p-iterates of points of U converge to a P^p-fixed point in $\bar{U}$, or $P^p|U$ is conjugate to an irrational rotation in $\mathbb{D}$ (our definition of an attracting domain includes also what is usually called parabolic one).

Let $N(x)$ be the number of external arguments of the point $x \in J(P)$ (i.e., the number of points in the closed non-empty set $\{t \in S^1 : \varphi^{-1}_{K(P)}(t) = x\}$). If n is big enough then P is a local homeomorphism at $P^n(x)$, hence the number $N(P^n(x))$ does not depend on n. Denote this number by $N_\infty(x)$. By [Ki], $N(x)$ is always finite and $N_\infty(x) \leq d$ (in [Ki] this is proven for polynomials with connected Julia set). More information is contained in Theorem 3.1, which follows from Theorem 2.10; in Theorem 3.1 the numbers $k_\sim, k_S, k_A$ and k_p are defined as in Section 2 for the lamination associated with the polynomial P. In other words, $k_\sim$ is the number of pairwise disjoint orbits of non preperiodic critical points $c \in J(P)$ with $N(P(c)) = 1$ (i.e., $P(c)$ is the landing point of exactly one external ray), k_S is the number of all Siegel periodic orbits, k_A is the number of distinct orbits of all Fatou components containing critical points and k_p is the number of all periodic orbits of the components of the Fatou set $F(P)$.

Theorem 3.1. *Suppose that P is a polynomial with the locally connected Julia set. Let Γ be a collection of points $z \in J(P)$, such that the points of Γ belong to pairwise disjoint grand orbits, every $z \in \Gamma$ is non-preperiodic and $N_\infty(z) > 2$. Then $\sum_{z\in\Gamma}(N_\infty(z) - 2) \leq k_\sim - k_S - 1 \leq d - 2 - k_p$. In particular, $N_\infty(z) \leq k_\sim - k_S - 1 \leq d - k_p \leq d$ for every $z \in J(P)$ which is non-preperiodic, and the number of all grand orbits of such points is $r \leq k_\sim - k_S - 1 \leq d - 2 - k_p \leq d - 2$. Also, for any non-preperiodic (not necessarily non-precritical) point y we have $N(y) \leq (k_S + 1)2^{d-k_A-k_S}$ which implies both $N(y) \leq 2^{d-k_A}$ and $N(y) \leq (k_S + 1)2^{d-k_p}$ and in any case $N(y) \leq 2^d$.*

Remark. Assume that every critical point of P is either preperiodic or is attracted to an attracting orbit. Then $J(P)$ is locally connected [DH], and we apply the theorem with $k_\sim = 0$. Therefore, $N_\infty(z) \leq 2$ for any point $z \in J(P)$ with an infinite orbit. For such polynomials our tree coincides with the Hubbard tree [DH] and the result can be extracted from [Do1], [Po]. □

The Sullivan No Wandering Domain Theorem [Su] states that every component of normality of a rational function is preperiodic. In particular, every component U of the Fatou set $F(P)$ is preperiodic by P. In the framework of connected locally connected Julia sets of polynomials our approach obviously yields an alternative proof of this result which turns out to be an immediate corollary of Theorem C. Moreover, we have

Theorem 3.2. *Suppose that P is a polynomial with the locally connected Julia set. Then every connected component of the interior of $K(P)$ is preperiodic and any subcontinuum of $J(P)$ is non-wandering.*

We apply our results to the case of polynomials P whose (filled-in) Julia sets are connected but not necessarily locally connected. One of the main problems here is to actually construct the lamination. This can be done if we assume that all cycles of P are repelling. Then the interior of $K(P)$ is empty and $J(P) = \partial K(P) = K(P)$. Also, the Bottcher function $\varphi_{K(P)} : A_\infty \to \mathbb{C} \setminus \bar{\mathbb{D}}$ and the external rays of P are defined, yet *a priori* the equivalence relation on $\mathbb{T}$ is not since the map $\varphi^{-1}_{K(P)}$ does not extend continuously to the unit circle. Still, a closed invariant lamination on $\mathbb{T}$ related to P can be constructed. We briefly describe the construction as in [Ki].

By a Douady's theorem (see e.g. [Mi1], [Pe2]), for a repelling periodic point a of P there are $1 \le k < \infty$ external rays landing at a. Their arguments form periodic orbit(s) of the map σ (so, they are rational numbers). Vice versa, every external ray R_t with a rational argument t lands, and the landing point is preperiodic. Define a rational lamination $\sim_{K(P),\mathbb{Q}}$ on $\mathbb{Q}/\mathbb{Z} \subset \mathbb{T}$ as follows: $t \sim_{K(P),\mathbb{Q}} t'$ iff $t, t' \in \mathbb{Q}$ and the external rays $R_t, R_{t'}$ land at the same point; every $\sim_{K(P),\mathbb{Q}}$ class is finite and preperiodic under $\sigma|_{\mathbb{Q}/\mathbb{Z}\to\mathbb{Q}/\mathbb{Z}}$. Then $\sim_{K(P),\mathbb{Q}}$ is an non-degenerate invariant lamination on $\mathbb{Q}/\mathbb{Z}$; it satisfies the conditions (E2), (E3), (D1), (D2), (D3); moreover, $\sim_{K(P),\mathbb{Q}}$ contains infinitely many periodic classes with at least two points (to see this, compare the number of cycles of $\sigma|_{S^1}$ and $P|_{\mathbb{C}}$ and use the fact that all P-cycles are repelling).

Extend $\sim_{K(P),\mathbb{Q}} |_{\mathbb{Q}/\mathbb{Z}}$ to an equivalence relation $\sim_{K(P)} |_{\mathbb{T}}$ as follows: *a class of $\sim_{K(P)}$ is either a class of $\sim_{K(P),\mathbb{Q}}$ or a maximal subset of $\mathbb{T}$ unlinked with any class of $\sim_{K(P),\mathbb{Q}}$*. To study $\sim_{K(P),\mathbb{Q}}$ and $\sim_{K(P)}$, we use the following construction of Yoccoz **puzzle structure** (cf. [Le], [Mi2], [Ki]).

Denote by $\beta_1, \dots, \beta_\ell$, $\ell \le d-1$, the fixed points of P which are the landing points of $d-1$ external rays of P of arguments $0, \frac{1}{d-1}, \frac{2}{d-1}, \dots, \frac{d-2}{d-1}$ (these rays are fixed by P).

Since P has d distinct repelling fixed points, there exists at least one repelling fixed point which is the landing point of finitely many external rays with non-zero rational combinatorial rotation number. Let $R_1, R_2, \dots, R_q, q \ge 2$, be all external rays landing at all such fixed points. Fix an equipotential curve Γ, and let W_0 be a bounded component of $\mathbb{C} \setminus \Gamma$. The components of $W_0 \setminus \cup_{i=1}^q R_i$ are called the (open) Yoccoz puzzle pieces of depth zero denoted $Y_0^{(i)}, 1 \le i \le q'$ where $q' \ge 2$. All components of the preimages $P^{-k}(Y_0^{(i)}), 1 \le i \le q'$, are said to be the (open) pieces of depth $k \ge 0$. Let $Y_0 \supset Y_1 \supset Y_2 \supset \dots \supset Y_n \supset \dots$ be a sequence of nested pieces. Denote

$$K = \bigcap_{n=1}^{\infty} \overline{Y}_n$$

a non-empty intersection of their closures. K is either a point, or a continuum. Now we distinguish two cases.

1. Every K obtained as above is either a point or a wandering continuum. Then *the final Yoccoz structure is the union of the pieces of all depths constructed above.*

2. For some continuum K as above, and for some positive integers n, m ($n \ne m$), $P^n(K) \cap P^m(K) \ne \emptyset$.

Then for any such continuum K we associate critical point(s) of P and a renormalization of P as follows. By the construction, either $P^n(K) = P^m(K)$ or $P^n(K)$ intersects $P^m(K)$ at a fixed point a of P with a non-zero rotation number. Because the combinatorial rotation number of a is rational, in the latter case again $P^{n'}(K) = P^{m'}(K)$ (with other $n' \neq m'$). In either case, for the image J_1 of K under an iterate of P, one holds $P^{N_1}(J_1) = J_1$, for some *minimal* $N_1 \geq 1$.

Let Y_{n_1} be a piece of the depth n_1, so that $J_1 \subset \overline{Y}_{n_1}$. If n_1 is large enough, the boundary of Y_{n_1} does not contain a critical point of P^{N_1} because otherwise there would be a periodic critical point of P. The map $P^{N_1} : Y_{n_1} \to Y_{n_1 - N_1}$ is a covering map. Moreover, since the intersection of the (closed) pieces contained J_1 is equal to J_1, for n_1 large, there are no critical points of P^{N_1} in $\overline{Y}_{n_1} \setminus J_1$. Thickening the pieces $Y_{n_1}, Y_{n_1 - N_1}$ if necessary (i.e., if J_1 intersects ∂Y_{n_1}, see [Mi2]) we obtain a covering map $P^{N_1} : \hat{Y}_{n_1} \to \hat{Y}_{n_1 - N_1}$ so that $\hat{Y}_{n_1}$ is compactly contained in $\hat{Y}_{n_1 - N_1}$. Therefore, J_1 must contain a critical point of P^{N_1} (otherwise J_1 would be a point, by a contraction principle), and $P^{N_1} : \hat{Y}_{n_1} \to \hat{Y}_{n_1 - N_1}$ is a polynomial-like mapping with the Julia set J_1 [DH1]. We associate to K (and to J_1) all critical points of P contained in J_1 and in its forward iterates. In particular, the number of pairwise disjoint orbits of continua as above is bounded by the number of different critical points of P. We also have $J_1 = \{x \in \overline{Y}_{n_1} : P^{iN_1}(x) \in \overline{Y}_{n_1}, \quad i = 0, 1, \dots\}$ (a renormalization of P). Call N_1 the period of this renormalization. By the theory of polynomial-like maps [DH1] and by the beginning of the construction, J_1 contains at least one fixed point of P^{N_1} with a non-zero rotation number (w.r.t. the map $P^{N_1} : J_1 \to J_1$). Let a_1 be such a point. Looking at the uniformization plane of $P^{N_1} : J_1 \to J_1$ we see that the rotation number of a_1 as a periodic point of P is also non-zero. Observe that this implies that the period of the renormalization $N_1 \geq 2$ (otherwise a_1 would be a fixed point of P with a non-zero rotation number which is impossible since all of them are in the boundary of the pieces of zero depth).

Now repeat the procedure. Namely, let J_1^j, $1 \leq j \leq r$, for some $r \leq d$, be all (first) renormalizations (obtained as above) from pairwise disjoint orbits. Let $N_1^j \geq 2$ be the period of J_1^j. Choose the same depth n_1 for all J_1^j, so that, for $1 \leq j \leq r$, $J_1^j = \{x \in \overline{Y}^j_{n_1} : P^{iN_1^j}(x) \in \overline{Y}^j_{n_1}, \quad i = 0, 1, \dots\}$.

Define now the *Yoccoz structure of the first renormalization* as the union of the previous pieces up to the depth $n_1 - 1$. For each J_1^j, consider all external rays of P which land at the fixed points of $P^{N_1^j} : J_1^j \to J_1^j$ with non-zero rotation number (like a_1 above). The forward images of these rays under iterates of P divide the pieces $Y^j_{n_1}$ and all other pieces of the depth n_1 of the previous (first) renormalizations, which are met by the forward trajectories of J_1^j, into finitely many components (since the rays are periodic). Unite these components with all remaining pieces of the depth n_1 and call them *the zero depth pieces of the structure of the second renormalization.* Taking all the components of their preimages by $P^k, k = 1, 2, ...$, we either finish the construction (like in the case 1), or come to next renormalizations. Note that each next renormalization J_2 is contained in one of the previous renormalizations J_1 chosen among J_1^j (the one marked by the same critical point of P as J_2), and the period N_2 of J_2 is divisible by the period N_1 of J_1. Moreover, $N_2/N_1 \geq 2$ by the same reason as $N_1 \geq 2$. Then we proceed as before, constructing *the Yoccoz structure of the third renormalization*, and so on. Note that on each step the zero depth pieces are bounded by a fixed equipotential and by a forward invariant and a finite system of external rays of P. Therefore, on each step the Yoccoz puzzle structure is well defined.

Define the *height* of a piece Y as the unique $h \geq 0$ such that the boundary of Y contains arcs of equipotential $P^{-h}(\Gamma)$. By the construction, for every point $x \in J(P)$, the intersection $Y(x)$ of a nested sequence $\overline{Y^h(x)}$ of the closures of pieces containing x, where h is the height of the piece Y^h, is either a point, or a wandering continuum as $h \to \infty$ (we have proved this, if the number of renormalizations is finite; if this number is infinite, the intersection is wandering because the periods of the renormalizations tend to infinity growing every time by the factor at least 2). The set $Y(x)$ is called the **puzzle impression** of x.

For a puzzle impression Y denote by $\Lambda(Y)$ the set of arguments of the external rays R of P, such that every limit point $y \in J(P)$ of R lies in Y. Then $\Lambda(Y)$ is a $\sim_{K(P)}$-class. Conversely, every $\sim_{K(P)}$ class is $\Lambda(Y)$ for some puzzle impression Y. Call a puzzle impression Y_c **critical** if it contains a critical point of P. Since $P : Y_n \to P(Y_n)$ is a covering map, for every piece Y_n, $\sigma : g \to \sigma(g)$ is a cover for every $\sim_{K(P)}$ class g.

The construction implies (see, e.g., [Ki]) that if $g = \{t_1, t_2, ..., t_p\}$ is a $\sim_{K(P),\mathbb{Q}}$ class (in cyclic order), then, for every $\varepsilon > 0$, and every adjacent pair t_i, t_{i+1}, there exists another $\sim_{K(P),\mathbb{Q}}$ class g', such that g' contains points both in $< t_i, t_i + \varepsilon >$ and in $< t_{i+1} - \varepsilon, t_{i+1} >$ where the arcs are understood in the natural way. Using this fact, one can show that $\sim_{K(P)}$ **satisfies** (E1), (E2), (E3), (D1), (D2), (D3) and Theorem B applies. To state its version applicable in our case we need notation. Given a puzzle impression Y let $N(Y)$ be the number of points in the set $\Lambda(Y)$ (when we worked with polynomials with locally connected Julia sets puzzle impressions were points so our current notation extends the previous one in a natural way; on the other hand now we cannot write $N(x)$ for all points, only for those which are puzzle impressions). It is easy to see that $N(Y)$ is finite: if Y is preperiodic, then Y is a point (as we explain above, puzzle impressions are either wandering or points) and $N(Y)$ is finite by the above quoted theorem of Douady, otherwise $N(Y)$ is finite by [Ki] or by Theorem B. In the case of a non-preperiodic puzzle impression Y for all $n > 0$ big enough the number $N(P^n(Y))$ is a constant denoted by $N_\infty(Y)$. As always denote by $k_\sim$ the number of different grand orbits of non preperiodic critical puzzle impressions Y_c of P with $N(P(Y_c)) = 1$.

We use also the standard notions of prime end impressions and prime ends of a continuum [CL].

Theorem 3.3. *Let P be a polynomial with all periodic points repelling and connected Julia set $J(P)$. Then the following holds.*

1. *Let Γ be a collection of puzzle impressions Y contained in pairwise disjoint non preperiodic grand orbits, and $N_\infty(Y) \geq 3$ for $Y \in \Gamma$. Then $\sum_{Y \in \Gamma}(N_\infty(Y) - 2) \leq k_\sim - 1 \leq d - 2$.*
2. *Let $z \in J(P)$ have an infinite orbit. Then z lies in at most 2^d prime end impressions of $J(P)$ while for $n > 0$ large enough $P^n(z)$ lies in at most $k_\sim + 1 \leq d$ prime end impressions of $J(P)$ (cf. [Ki]) and there are at most $3(k_\sim - 1) \leq 3(d-2)$ prime ends such that if the point z lies outside the grand orbits of the impressions of these prime ends, then $P^n(z)$ belongs to at most 2 prime end impressions.*

Proof. (1) Clearly, $N_\infty(Y) = |\sigma^n(\Lambda(Y))|$ if $n > 0$ is big enough, and the inequality follows from Theorem B.

(2) Let I_t be the impression of a prime end corresponding to the external ray R_t of argument t [CL]. Then I_t is a subset of some puzzle impression Y. Hence, $N(Y)$ is greater than or equal to the number of prime end impressions contained in Y which is greater than or equal to the number of the prime end impressions containing a point $x \in Y$. So, the first claim follows from Theorem A while from (1) $N_\infty(Y) \le k_\sim + 1$ as desired. The last inequality of the theorem follows from the fact that $\sum_{Y\in\Gamma} N_\infty(Y) \le 3(k_\sim - 1)$ which in turn follows easily from the inequality of (1). ∎

4. Concluding remarks

1. First we give a rather brief description of examples of closed invariant laminations which are not topological models of the Julia set of any polynomial.

The first example, in degree 2, is constructed as follows[1]. Consider the Julia set J ("airplane") of a real quadratic polynomial $z^2 - 1.75488...$; then the critical point 0 is 3-periodic and J is connected locally connected. Take a closed invariant lamination corresponding to J. If U is a component of the basin of attraction to the superattractive cycle containing 0, declare that the external arguments of the points of ∂U form an equivalence class of the new lamination $\sim_2$. Other classes are not changed. It gives the closed invariant lamination $\sim_2$ with an infinite periodic critical class (note that it contains a dense subset of rational points). Therefore, there is no Julia set of a polynomial that corresponds to $\sim_2$.

Another (simple) example exists in degree 3. Take $\sigma(z) = z^3$, divide S^1 into arcs $I_1 = [0, 1/3], I_2 = [1/3, 2/3], I_3 = [2/3, 1]$, then pick the fixed point $a = 1/2$ of σ in I_2 and also its preimage $b = 1/6$ in I_1, and consider $J = [0, b]$ and $K = [1/3, a]$. Let A be the set of all points staying forever in $J \cup K$ under the forward iterates of σ (clearly, $\sigma|A$ is homeomorphic to the 2-shift). Declare A and all its preimages classes of the lamination we are constructing (a subset B of S^1 is said to be a **preimage** of A iff there exists $n \ge 0$ such that if l is a minimal closed arc in S^1 containing B then $\sigma^n : l \to [0, 1/2]$ is a one-to-one map, and $\sigma^n(B) = A$). Also, any point of S^1 outside all preimages of A is a class of the lamination. Since the first preimage of A other than A itself is disjoint with A and the length of the minimal arc containing n-preimage of A tends to zero as $n \to \infty$, we obtain a closed σ-invariant lamination $\sim_3$. Now, A is an infinite critical class of $\sim_3$, such that $\sigma(A) = A$. Rational points are dense in A.

2. In a very recent preprint [Ki1] the rational laminations that correspond to polynomials with connected Julia set are characterized. This opens a way to generalize some results of Section 3 to all polynomials with connected Julia set. Namely, given such polynomial P, consider corresponding to P rational lamination and take its closure as in [Ki1]. Then we obtain a closed invariant lamination to whom our results can be applied. Note however that the topological Julia set (the quotient space of such lamination) is not always homeomorphic to the original Julia set even if the original one is locally connected. For instance, the rational lamination of the quadratic polynomial with Siegel fixed point (see Example of Section 2) is trivial (each rational point of $\mathbb{T}$ is a class).

[1] The example appeared during a discussion of the first author with Jan Kiwi

Hence, its closure is also trivial, i.e. the corresponding topological Julia set is a circle. See [Ki1] for a general statement.

3. For Cantor Julia sets, see [BL1].

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