UAB MATH-BY-MAIL CONTEST, 2004

ELIGIBILITY. Math-by-Mail competition is an individual contest run by the UAB Department of Mathematics and designed to test logical thinking and depth of understanding of mathematics by high school students. The contest is open to all high school students from participating schools.

PRIZES AND CEREMONY. One first prize ($60 and trophy), one-two second prizes ($40 and a trophy) and some third prizes ($20 and an award diploma) will be handed out (since awards/prizes to individuals are reportable annually by UAB to the IRS, cash prize-winners will have to have a valid U.S. federal tax identification number, will need to fill out W-9/W-8 form prior to receiving cash prizes, and will need to report prizes on their tax returns). All other participants will receive certificates honoring their participation. The school with the best overall performance will be awarded the UAB traveling Math trophy. Prize winners will be invited to an awards ceremony at UAB at a later announced date. Parents and teachers will also be invited. The program will feature awards presentation and refreshments.

RULES AND JUDGING. Math-by-Mail contest is an individual contest. Students must solve problems on their own. All solutions must be self-contained; no references to facts and mathematical results beyond the high school standard curriculum are permitted (nor will they be necessary). The explanations must be detailed and thorough, all answers must be explained and justified! Professors of the Department of Mathematics at UAB will judge the entries and evaluate your reasoning process and method of finding the solution. Criteria will include correctness and elegance of solutions. The decision of the judges is final. The entries will not be returned.

FORMAT AND CONTACTS. The participating schools will receive problems and cover sheets before or on Monday, February 9. On Monday, February 9, the problems and cover sheets should be made available to the students. Every participant must submit a filled copy of the cover sheet; each solution must be written on a separate sheet of paper with the student name and the school name on it. The entries must be postmarked no later than Monday, February 23, and mailed to:

UAB Math-by-Mail Contest
c/o Dr Alexander Blokh
Department of Mathematics, UAB
1300 University Blvd, CH 452
Birmingham AL 35294-1170

For additional information you can contact us at (205)934-2154 or by electronic mail at ablokh@math.uab.edu

UAB MATH-BY-MAIL CONTEST COVER SHEET

(1) YOUR NAME

(2) YOUR SCHOOL

(3) YOUR GRADE

(4) YOUR ADDRESS

(5) YOUR PHONE NUMBER

(6) YOUR E-MAIL ADDRESS (IF ANY)

UAB MATH-BY-MAIL CONTEST 2004, PROBLEMS

Solve the problem(s) and write each solution on a separate sheet of paper with your name and your school's name on it. If you can (and only after you have done as much as you could solving problems), try to see if you can solve more general problems using your methods. Some third prizes will be awarded for particularly good solutions of individual problems, so **you can win a third prize for an excellent job in just one problem!**

Complete your cover sheet (see your teacher to get one) and include it with your submission. Submit your entries (postmarked by **February 23, 2004**) to:

UAB Math-by-Mail Contest
c/o Dr Alexander Blokh
Department of Mathematics, UAB
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PROBLEMS

PROBLEM 1. *Find all two-digit numbers x with digits a, b such that $x^2 = (a+b)^3$.*

PROBLEM 2. *There are n points given on the plane. For any three points A, B, C from this collection the angle ABC is formed. Then among all these angles the least one is chosen. What is the greatest possible value of such an angle (as a function of n)? Note: if three points belong to a line then some angles they form are equal to* 0.

PROBLEM 3. *A square $n \times n$ table is filled with real numbers. For any given row and column the sum of their $2n-1$ numbers taken together is greater than or equal to a real number X. What is the minimal possible value of the sum of all numbers in the table?*

PROBLEM 4. *A closed broken line consists of* 203 *straight links. It is known that no* 2 *links are contained in the same line. What is the greatest number of self-intersections such a broken line may have?*

PROBLEM 5. *A 4×4 table is filled with plus or minus signs. An admissible action is the change of all signs to the opposite in a row or in a column. The minimal number of minuses which can be obtained from a given table after a finite sequence of admissible actions is called the* characteristic *of the table. List all possible characteristics of various 4×4 tables (giving examples of tables generating them) and prove that no other characteristics are possible.*

UAB MATH-BY-MAIL CONTEST 2003, PROBLEMS AND THEIR SOLUTIONS

PROBLEM 1. *Find all two-digit numbers x with digits a, b such that $x^2 = (a+b)^3$.*

Solution: Let the 2-digit number be $10a + b$ with $a \neq 0$. We need to solve an equation $(10a + b)^2 = (a + b)^3 = N$. Thus, any prime number in the factorization of N is involved in the power which is divisible by both 2 and 3, hence it is divisible by 6. Hence N is in fact the sixth power of some number k. This implies that $a + b$ is the square of some number. On the other hand $a + b \leq 18$. So, $a + b$ may be equal $1, 4, 9, 16$ with the corresponding values of $10a + b$ being $1, 4, 27, 64$. Now, 1 and 4 are not 2-digit numbers while a simple verification shows that 64 is not the answer. On the other hand, $10a + b = 27$ satisfies the conditions.

So, the answer is **27**. □

PROBLEM 2. *There are n points given on the plane. All possible angles with the vertices at these points are made and then the least one is chosen. What is the greatest possible value of such angle? Note: if three points belong to a line then some angles they form are equal to 0.*

Solution: First of all, observe that if there are 3 points which belong to one line, then there are angles equal to 0 formed by them. If however there are no such 3 points then no angle formed by any 3 points equals 0. Hence we may from now on consider only such sets of n points that no 3 point belong to the same straight line.

Now, denote a collection of n points by A. Then we can always choose the so called convex hull of A, i.e. the smallest convex set containing A. Indeed, for any finite ordered collection of points from A we can connect them in this order with the agreement that the last point must be connected to the first one. Some of these broken lines will be actually convex polygons. Of course, not all of such polygons contain all the other points. However, if this is not the case, then there must be a point from A which is outside this polygon. It is easy to see that then we can connect this point and some vertices of the polygon and thus create a new convex polygon with vertices at some of our points which will contain the previous polygon and will not coincide with it. This process cannot last forever, so it will stop with some convex polygon P whose vertices belong to A will contain all points of A. If there is another convex polygon P' with the same properties then it will have to contain P while P will have to contain P' so $P = P'$ and the polygon P (called *convex hull* of A) is well-defined.

Now, suppose that P is a k-gon while there are n points overall. It is known that the sum of angles of P then is $\pi(k-2)$ and hence the least of the angles of P is $\pi(1-\frac{2}{k})$. Suppose that the corresponding vertex of P is a point x. The point x is already connected with two points of A by the sides of P. We can connect x with remaining points of A by straight lines and consider all lines created like this. Clearly there will be $n-1$ such lines. Moreover we can number them moving say in the counterclockwise direction from side of P to the other side of P where both sides come out of x. It is clear that then the least angle of all these angles is going to be at most $\frac{\pi}{n-2}(1-\frac{2}{k})$. Observe that the greater k is the greater this quotient is, so the greatest its value is $\frac{\pi}{n-2}(1-\frac{2}{n})=\pi/n$.

This implies that the greatest possible smallest angle of all the angles formed by the points from a collection of n points is at most π/n. To observe that this level can be achieved consider a right n-gon. Then all its angles are $\frac{\pi(n-2)}{n}$ and if then we connect all its vertices then the minimal angles formed by them are all equal $\frac{\pi}{n}$.

So the answer indeed is $\frac{\pi}{\mathbf{n}}$. □

PROBLEM 3. *A square $n\times n$ table is filled with numbers. For any given a row and a column it is known, that the sum of their numbers is no less than a. What is the minimal possible value of the sum of all numbers in the table?*

Solution: Denote the sum of all entries in the table by N. Choose the row in which the sum of the entries is minimal and denote it by B; denote the sum of all entries of B by b. For the sake of definiteness assume that B is the bottom row of the table. For any column and B the sum of all their entries is at least a. Each such sum can be viewed as the sum of the entries of B and entries of the corresponding column without the bottom entry. Therefore the sum s of numbers for all such pairs (B and a column) is at least na. On the other hand, in this sum s the row B is repeated n times while otherwise we sum up all columns without their bottom entries. Hence in fact $s=N+(n-1)b\geq na$. By the choice of B we have that $b\leq\frac{N}{n}$, so $N+(n-1)\frac{N}{n}\geq s\geq na$ and hence $N\geq a\frac{n^2}{2n-1}$. It remains to give an example of how the table can be filled out so that $N=a\frac{n^2}{2n-1}$, and this can be done by simply putting $\frac{a}{2n-1}$ in each place of the table.

So, the answer is $\mathbf{N}=\mathbf{a}\frac{\mathbf{n^2}}{\mathbf{2n-1}}$ □

PROBLEM 4. *A closed broken line consists of 203 links. It is known that no 2 links are contain in the same line. What is the greatest number of self-intersections such broken line may have?*

Solution: Observe that a link in the broken line can intersect with no more than 200 other links at points which are not its endpoints. Thus, the overall number of intersections counted for each link giving rise to it is at most 40600, and so the overall number of self-intersections is at most 20300 (we need to divide 40600 by 2 because each self intersection is counted twice). To get this number of self-intersections, take the right 203-gon constructed inside a unit circle and connect its vertices in the order described below. Number the vertices from 1 through 203 counterclockwise. Then take each vertex A, count 101 vertices from it and get the next vertex B. Connect A with B and do this for each vertex A. For example, 1 is the connected with 102, 102 is connected with 203, 203 is connected with 101 and so on.

It is easy to see that then we do get a closed broken line with 203 links. Each link in this line intersects all other links except for the ones adjacent to it because by the construction all other links begin and end on the distinct sides of the chosen link. It remains to show that no 3 links intersect at one point. Indeed, each link connects two points on the circle such that the arc between these two points is $2\pi\dfrac{101}{203}$. Let us show that given an angle $\alpha < \pi$, 3 distinct chords in the circle which are bounded by the arcs of the radian measure α cannot intersect at one point. Indeed, let us associate to any chord the segment connecting the center of the circle and the midpoint of the chord and call this segment a characteristic segment of the chord. Clearly, the characteristic segment of a chord is perpendicular to it.

The fact that our chords correspond to the arcs of the same radian measure means that the length of their characteristic segments is the same and equals, say, $x < 1$ (recall that the radius of the circle is 1). Fix a chord A and see at what points other chords from our collection of chords can intersect A. To do so we measure the distance d between the midpoint of A and the point of intersection. Then if the angle between the characteristic segment of A and that of some other chord is β then we see that the distance d is $x\tan(\beta/2)$. Moreover, depending on the side from which the other chord is located with respect to A the point of intersection may be on either side of the midpoint of A. This implies the desired and completes the solution.

So, the answer is **20300** □

PROBLEM 5. *A* 4×4 *table is filled with pluses or minuses. An admissible action is the change of all signs to the opposite in a row or in a column. The minimal number of minuses which can be obtained from a given table after a few admissible actions is called the characteristic of the table. List* *all possible characteristics of various 4×4 tables (giving examples of tables generating them) and prove that no other characteristics are possible.*

Solution: Any table with pluses and minuses which cannot be changed (in one or more steps) so that the overall number of minuses decreases is called *minimal*. Then any such minimal table cannot be changed to decrease the number of minuses in one step either, hence the number of minuses in any row or column is no more than 2 and the overall number of minuses in such a table is no more than 8.

Let T be a table. Denote by $c(T)$ its characteristic. Suppose that M is a minimal table and show that $c(M) \leq 4$. Consider a few cases assuming by way of contradiction that $c(M) > 4$. First of all suppose that there are 4 cells in M which contain minuses and form a horizontal-vertical rectangle (in other words, these cells are places of intersection of two rows with two columns). Without loss of generality we may assume that these cells are actually the corner-cells of the table. Then one more minus must stand somewhere in the table. It cannot stand in a boundary cell of the table since otherwise there will be three minuses in one row or column. However if it stands inside a small 2-subtable in the middle of the given table then we can change signs in the top and the bottom rows and get 3 minuses in the same column, i.e. the column of the 5-th minus. Clearly this contradicts the minimality of M.

Now, since there are at least 5 minuses in M then there exists a row with 2 minuses. Again without loss of generality we may assume that they stand in the bottom corners. If the 2nd or the 3rd column has 2 minuses then we can change signs in the bottom row and get 3 signs in the same column, a contradiction. Hence either 1st or 4th column must contain a minus. Again without loss of generality we may assume that the left top corner contains minus. Consider the possible locations of the guaranteed remaining 2 minuses. Observe that they cannot stand in the 1st column or in the bottom row; the right top corner cannot contain minus either.

Suppose that there is one more minus in the top row; for the sake of definiteness assume that it stands in the 2nd column. Then the 5th minus must stand in the 3rd column or in the 4th column. If it stands in the 3rd column we can change signs in the top and bottom rows and get 3 minuses in the 3rd column; if it stands in the 4th column we can change signs in the top row and get 3 minuses in the 4th column. Either way we get a contradiction, so there are no more minuses in the top row. By the symmetry there are no minuses in the 4th column. Hence the 2 minuses must stand in the 2×2-subtable in the center. Changing signs in the top and bottom rows we get a column in the middle with 3 minuses, a contradiction. So, M cannot contain more than 4 minuses.

Let us show now that $c(M)$ may be equal to $0, 1, 2, 3, 4$. Indeed, suppose the only minuses in M are k minuses standing on the main diagonal of the table and show that then $c(M) = k$. Observe, that then to begin with there are k columns with the following property: no two columns from this collection coincide or are reflections (in the obvious sense) of each other. Let us prove that then the same property will hold for the same columns after several changes of signs. Indeed, if 2 columns are not equal/reflections of each other, then neither are they after one change regardless of whether the change is a column change or a row change. Thus, after any number of changes there will be k columns which are neither pairwise equal nor pairwise reflections of each other. This implies that even if one of them has no minuses after several changes then al others must have at least one minus, and so there are at least $k - 1$ minuses in any table obtained from M. Since the parity of the number of minuses is obviously kept after all the changes this implies that in fact the number of minuses in any table obtained from M is at least k as desired.

Another way to construct examples of tables M such that $c(M) = 0, 1, 2, 4$ was suggested by Dr N. Chernov. Observe that if 4 small squares in the table stand at the intersections of two rows and two columns (in other words, form a rectangle with horizontal and vertical sides) then the product of their entries does not change after any number of admissible actions (because any such action replaces two of the entries by their opposites). So, we can proceed as follows: a) divide the table into 4 subtables each of which is formed by 4 small squares in the table located in the upper left, upper right, lower left and lower right part of the table). Now, choose, say, 3 of those subtables and out in them exactly one minus while filling out the rest of the entire table with pluses. As we saw, admissible actions do not lower the number of minuses in any of those 4 subtables into which the table is divided. Hence, for this table M we have $c(M) = 3$. Similarly we can suggest tables M with $c(M)$ equal to $0, 1, 2, 4$.

So, the answer is that $c(M)$ can be $\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}$ only. $\square$

UAB MATH-BY-MAIL 2004 AWARD CEREMONY

1. Tea
2. Speech: Dr Mayer (overall 83 students participated in the contest!).
3. Handing out awards.

a) Schools' Honorable Mentions:

ASFA, 1 participant

Auburn High, 2 participants

Buckhorn High, 1 participant

Grissom High, 3 participants

Hoover High School, 21 participants

Minor High, 1 participant

Vestavia High, 54 participants

b) Traveling Trophy: Vestavia High School

c) Individual Trophies:

Honorable Mentions:

Ellie Barr, Vestavia

Mingwei Gu, Hoover

Misha Kotov, Vestavia

Mark Mennemeyer, Vestavia

Vedran Oruc, Vestavia

Christopher Turndrop, Vestavia

Third prize:

David Harris, Vestavia

Second prize:

Xun Liu, Auburn

First Prize:

Fan Young, ASFA