UAB MATH-BY-MAIL COMPETITION

ELIGIBILITY

Math-by-Mail competition is an individual contest presented by the UAB Department of Mathematics and designed to test logical thinking and depth of understanding of mathematics by high school students. The contest is open to all high school students from participating schools starting at the freshman level.

PRIZES AND CEREMONY

One first prize ($60), one-two second prizes ($40) and 5 third prizes ($20) will be handed out together with award diplomas. All other participants will receive certificates honoring their participation. The school with the best overall performance will be awarded the UAB traveling Math trophy. Prize winners will be invited to an awards ceremony at UAB on Friday May 11th. Parents and teachers will also be invited. The program will feature awards presentation and refreshments.

RULES AND JUDGING.

Math-by-Mail contest is an individual contest. Students must solve problems on their own. All solutions must be self-contained; no references to facts and mathematical results beyond the high school standard curriculum are permitted (nor will they be necessary).

The explanations must be detailed and thorough. Professors of the Department of Mathematics at UAB will judge the entries. Your reasoning process and method of finding the solution will be evaluated. Criteria will include correctness and elegance of solutions. The decision of the judges is final. The entries will not be returned.

FORMAT AND CONTACT INFORMATION

Problems and a cover sheet will be faxed to participating schools on Thursday April 26th. Every participant must include a copy of the cover sheet; each solution must be written on a separate sheet of paper with the student name and the school name on it. Submit your entry by Tuesday, May 1st to:

UAB Math-by-Mail Contest
Department of Mathematics
UAB, UAB Station
Birmingham AL 35294-1170

For additional information you can contact us at (205)934-8549 or by electronic mail at ablokh@math.uab.edu

UAB MATH-BY-MAIL CONTEST COVER SHEET

(1) YOUR NAME

(2) YOUR SCHOOL

(3) YOUR GRADE

(4) YOUR ADDRESS

(5) YOUR PHONE

(6) YOUR E-MAIL ADDRESS (IF ANY)

UAB MATH-BY-MAIL CONTEST 2001, PROBLEMS

Solve the problem(s) and write each solution on a separate sheet of paper with your name and your school's name on it. You may choose to solve only one problem, several problems, or all problems. If you can (and only after you have done as much as you could solving problems), try to see if you can solve more general problems using your methods. Some third prizes will be awarded for particularly good solutions of individual problems, so **you can win a third prize for an excellent job in just one problem!**

Complete your cover sheet (see your teacher to get one) and include it with your submission. Submit your entries (postmarked by May 1st) to:

UAB Math-by-Mail Contest
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UAB, UAB Station
Birmingham AL 35294-1170

PROBLEMS

PROBLEM 1. *Find all integers n with initial digit 6 which have the following property: if the initial digit is deleted, the resulting integer m is equal to $\frac{n}{25}$.*

PROBLEM 2. *We are given 27 coins of the same denomination; we know that one of them is counterfeit and that it is lighter then the others.*

1. *Locate the counterfeit coin by using three weighings on a pan balance.*
2. *How many weighings are necessary to solve the same problem with 243 coins and how would you do this?*

PROBLEM 3. *Point O is given in the plane containing a square $ABCD$. Prove that the distance from O to one of the vertices of the square is not greater than the sum of the distances from O to the other three vertices.*

PROBLEM 4. *Every living person has shaken hands with a certain number of other persons. Prove that the number of people who have shaken their hands an odd number of times is even.*

PROBLEM 5. *Given 5 positive integers, it is known that any four of them can be divided into two groups, two in each group, so that the sum in each group is the same.*

1. *Show that all these integers are equal.*
2. *Solve the same problem with 13 integers for which it is known that any 12 of them can be divided in two groups, six in each group, so that the sum in each group is the same.*

UAB MATH-BY-MAIL CONTEST 2001, PROBLEMS AND THEIR SOLUTIONS

PROBLEM 1. *Find all integers n with initial digit 6 which have the following property: if the initial digit is deleted, the resulting integer m is equal to $\frac{n}{25}$.*

Solution Since no number with one or two digits starting with 6 is divisible by 25 we may assume that n has at least 3 digits and therefore can be written as $6 \cdot 10^{k+2} + m$ where $0 \leq m \leq 10^k - 1$ and $k \geq 0$. By the assumptions $25m = n$, so $24m = 6 \cdot 10^{k+2} = 600 \cdot 10^{k-2}$ which implies that $m = 25 \cdot 10^k$. We see that $n = 6 \cdot 10^{k+2} + 25 \cdot 10^k = 625 \cdot 10^k, k \geq 0$. □

PROBLEM 2. *We are given 27 coins of the same denomination; we know that one of them is counterfeit and that it is lighter then the others.*

1. *Locate the counterfeit coin by using three weighings on a pan balance.*
2. *How many weighings are necessary to solve the same problem with 243 coins and how would you do this?*

Solution First show that if we have $3m$ coins which are all of the same weight except for exactly one which is lighter then we can get a pile of m coins containing a light coin. Indeed, divide the coins into three piles, m in each. Put any two of these three on the balance. Then if one of these two piles is lighter it must contain the light coin, and if they have the same weight then the pile which is not on the balance contains the light coin.

Now, if we apply this procedure to any collection of 3^k coins step by step we will get one coin after k weighings which solves both parts of the problem. What if we had n coins such that $3^k < n \leq 3^{k+1}$ coins? Also, can this method be improved? That is, is there an algorithm such that for $n = 3^k$ it takes less that k weighings no matter what the results of individual weighings are? □

PROBLEM 3. *Point O is given in the plane containing a square $ABCD$. Prove that the distance from O to one of the vertices of the square is not greater than the sum of the distances from O to the other three vertices.*

Solution Compare AO with $DO + BO$. Draw perpendiculars from O to the lines AD (denote it OE) and AB (denote it OF). Then $OA < OF + FA = OF + OE$ (triangle inequality) while $OF < OB$ (right triangle OFB) and $OE < OD$ (right triangle OED). Therefore $OA < OB + OE$ as desired.

Observe that we proved more than required: the distance from O to any vertex of the square is not greater than the sum of the distances from O to certain two vertices.

What if initial quadrilateral were not a square but, say, a rectangle? a rhombus? a parallelogram? an arbitrary quadrilateral? Can you establish inequalities similar to the one proven above in these cases? □

PROBLEM 4. *Every living person has shaken hands with a certain number of other persons. Prove that the number of people who have shaken their hands an odd number of times is even.*

Solution Let us denote numbers of handshakes made by different people by $a_1, a_2, \ldots$. The sum of all numbers a_i equals the overall number of handshakes multiplied by 2, so it is even. The sum of all even a_i's is even. If there is an odd number of odd numbers a_i then their sum is odd (every time you add an odd number you change the parity to a different one, so if you do this odd number of times you will get an odd sum), and the overall sum will be odd which is impossible. Therefore there must be an even number of odd a_i's as desired.

Do you know of another solution to this problem? Could you suggest other problems which could be solved by similar methods? □

PROBLEM 5. *Given 5 positive integers, it is known that any four of them can be divided into two groups, two in each group, so that the sum in each group is the same.*

(1) *Show that all these integers are equal.*
(2) *Solve the same problem with* 13 *integers for which it is known that any* 12 *of them can be divided in two groups, six in each group, so that the sum in each group is the same.*

Solution Suppose that $2n+1$ integers $a_1, a_2, \ldots, a_{2n+1}$ are such that any group of $2n$ of them can be divided into two groups of n numbers with equal sums. Let us prove that then they are all equal.

First show that they have the same parity. Set $a_1 + a_2 + \cdots + a_{2n+1} = S$. Then for any i the number a_i equals the difference between S and the sum of the remaining $2n$ integers; since by the assumption the latter sum is even (can be divided into two groups of n numbers with equal sums) we see that the parity of a_i and the parity of S are the same.

Now, let a be the minimal value of all a_i's, and consider new numbers $b_i = a_i - a$. Then b_i have the same property as the original collection of numbers because for any collection of $2n$ numbers b_i we can consider the corresponding collection of numbers a_i, break it into two groups of n elements in each with equal sums (denoted by s), and then observe that the same partition of the corresponding group of numbers b_i will also give two groups of n elements with equal sums (equal to $s - na$). Hence the positive integers $b_1, \ldots, b_{2n+1}$ have the same property as the original collection.

Since one of b_i is zero then by the proven above they all have to be even, so we can divide them all by 2 and get a collection of integers which will have the same properties as the original collection. Clearly this can be repeated infinitely many times (nothing prevents us from continuing) which in the case of integer numbers is only possible if the integers are from some time on equal to 0. However, this means that they were equal to zero from the very beginning, and so they were all equal to a as desired.

This problem in fact can be generalized onto collections of real numbers having the same property as the one in the claim of the problem; in fact, one can show that if one can prove that all integer solutions are those with all numbers equal to each other then this implies that all real solutions are like that. The methods used however are non-elementary, and it would be very interesting if we could obtain the same deduction by purely elementary methods. $\square$

UAB MATH-BY-MAIL 2001 AWARD CEREMONY

1. Tea

2. Speech: Dr Mayer (overall 52 students participated).

3. Handing out awards.

a) Schools Honorable Mentions with Certificates of Participation for their students:

i) Alabama School of Fine Arts - 12 participants;

ii) Vestavia High School - 4 participants;

iii) Altamont High School - 12 participants;

iv) Hoover High School - 24 participants.

b) Traveling Trophy: Hoover School

c) Individual Trophies:

Third prizes:

i) Moneer Helu, ASFA;

ii) David Hall, Altamont;

iii) Irving Ye, Hoover;

iv) Katie Baldwin, Hoover;

v) Adam Roth, Hoover.

Second Prizes:

i) Weichen Zhu, Hoover;

ii) Evan Roseman, Homewood.

First Prize:

Will Shanks, Altamont.