THE MODULUS OF RENORMALIZATION OF RATIONAL FUNCTIONS: HIGH PERIODS IMPLY SMALL MODULI

ALEXANDER BLOKH, GENADI LEVIN, LEX OVERSTEEGEN, AND VLADLEN TIMORIN

ABSTRACT. The modulus of a polynomial-like (PL) map is an important invariant that controls distortion of the straightening map and hence geometry of the corresponding PL Julia set. Lower bounds on the modulus, so called complex (a priori) bounds, are known in a great variety of contexts. We complement these by upper bounds for moduli of annuli obtained under general geometric assumptions. Then we apply them and prove, in the satellite case, an upper estimate that depends only on the relative period and the degree of the PL map. This rules out a priori bounds in the case when relative periods of renormalizations are unbounded. All results hold for general rational functions of any degree.

1. INTRODUCTION

Since Sullivan’s work on Feigenbaum’s universality [22], complex a priori bounds for polynomial-like renormalizations play a key role in polynomial dynamics. Such bounds, though usually difficult to obtain, have been established and/or crucially used for various classes of quadratic polynomials (see, e.g., [22, 10, 5, 16, 4, 15, 7, 8, 9]). In particular, the bounds imply that the Julia set \( J(f) \) is locally connected and, in a lot of cases, the MLC conjecture at the corresponding points. On the other hand, examples (by Douady and Hubbard) of infinitely (satellite) renormalizable quadratic polynomials with non-locally connected Julia sets show that the bounds do not hold in general (see [19, 21] for qualitative versions and [11, 12, 13] for quantitative ones). Until now, these have been the only known examples of infinitely renormalizable quadratic polynomials without complex bounds.

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In this paper we show that such bounds do not exist in a lot of cases. To this end we make geometric assumptions on cycles of connected polynomial-like Julia sets of rational functions, and estimate from above their moduli of annuli. In Theorem A we verify these assumptions for satellite PL cycles of a given relative period $s$, and show that their moduli are bounded from above by a specific function that depends only on $s$ and on the degree of the PL map. Thus, if relative periods tend to infinity, then moduli of annuli converge to 0 (which makes a priori bounds impossible). E.g., this implies that in the family of infinitely renormalizable polynomials $f_c = z^2 + c$ such that all renormalizations are satellite, for a dense $G_δ$-set of polynomials a priori bounds fail. We also prove Theorem B in which we consider another case when the moduli of annuli of renormalizations of rational functions converge to 0.

2. MAIN RESULTS

Let $\mathbb{P}^1$ be the Riemann sphere, i.e., the complex projective line over $\mathbb{C}$. The space $\mathbb{P}^1$ comes with a conformal structure and a conformal Riemannian metric of constant positive curvature, the spherical metric. Normalize the spherical metric so that the total area of $\mathbb{P}^1$ is $\pi$. With this normalization, $\mathbb{P}^1$ is isometric to a sphere in the Euclidean 3-space of radius $\frac{1}{2}$. For two points $z, w \in \mathbb{P}^1$, we let $\text{dist}(z, w)$ denote the spherical distance between $z$ and $w$. All lengths and areas are also with respect to the spherical metric. For a compact subset $X \subset \mathbb{C}$, let $\text{diam}(X)$ be its (spherical) diameter.

We assume basic knowledge of complex dynamics. Let a rational function $f : \mathbb{P}^1 \to \mathbb{P}^1$ be renormalizable in the sense of Douady and Hubbard, i.e., there is $q > 0$ and Jordan disks $U \subset V$ such that $f^q : U \to V$ is a degree $d^*$ polynomial-like (PL) map with connected Julia set $K^*$. The annulus $A = V \setminus \overline{U}$ is called a fundamental annulus of $K^*$, and the annulus $U \setminus K^*$ is called the root annulus (of $K^*$). The set

$$\mathcal{K} = \{K^*, f(K^*), \ldots, f^{q-1}(K^*)\}$$

is called a PL cycle, and each set $f^i(K^*)$, where $i \geq 0$, is called an element of $\mathcal{K}$ or just a PL set. Each PL set is a connected filled Julia set of some PL map obtained as a suitable restriction of $f^q$. We always denote by $\mathcal{K}$ a PL cycle of period $q$ with first return map $f^q$ of degree $d^*$ on each element of $\mathcal{K}$. For a collection $\mathcal{H}$ of sets, let $\mathcal{H}^+$ be the union of elements of $\mathcal{H}$.

**Definition 2.1** (Primitive and satellite). Let $f$ be a rational function.

1. If all elements of $\mathcal{K}$ are pairwise disjoint (in which case all $f^q$-fixed points in $\mathcal{K}^+$ have exact period $q$) then $\mathcal{K}$ and all its elements are called primitive.
(2) If there is a repelling $f$-cycle $\alpha_0, \ldots, \alpha_{r-1}$ of period $r < q$ in $K^+$ (called the base cycle of $K$) such that $K^+$ has $r$ components $C_0, \ldots, C_{r-1}$ and for each $i$ the set $C_i$ is the union of all elements of $K^+$ containing $\alpha_i$ as a non-separating point and otherwise pairwise disjoint, then $K$ and its elements are called satellite of base period $r$. Setting $q = rs$, the number $s$ is called the relative period of $K$.

By [6, Proposition 3.4], a PL cycle whose elements do not share critical points is primitive or satellite except that points of the base cycle do not have to be non-separating points of the elements of $K$. From now on we consider only primitive or satellite PL cycles. Observe that sometimes such renormalizations are said to be simple.

Set
$$T(t, D) = \frac{4t - 2}{\sqrt{t - 1}} \tan \frac{D}{2} - 8;$$
if $D$ is fixed and $t$ sufficiently large, then $T(t, D) > 0$, and for any $D_0$ there is a number $t_0(D_0)$ with $T(t, D) > 2D\sqrt{t}$ for $D > D_0$ and $t > t_0(D_0)$.

**Main Theorem.** Let $K$ be a PL cycle for a rational function $f$. Suppose that at least $t$ elements $Z_1, \ldots, Z_t$ of $K$ have diameter at least $D$, the root annuli $U_i \setminus Z_i$ have moduli at least $m$, and the complements $\mathbb{P}^1 \setminus U_i$ have diameters at least $D$. If $T(t, D) > 1$ then
$$m \leq \frac{\pi}{2 \ln T(t, D)}.$$
If $D > D_0$ and $t > t_0(D_0)$ for some $D_0$ then $m < \pi / \ln(4D^2t)$.

The main application is to the satellite case.

**Theorem A.** Let $f$ be a rational function of degree $d \geq 2$ with a satellite PL cycle of arbitrary base period and relative period $s \geq 35$. Then the modulus $m^*$ of any root annulus around a PL set of that cycle satisfies
$$m^* \leq \frac{d^* \pi}{2 \ln \left[\frac{4s-12}{\sqrt{s-2}}(\sqrt{2} - 1) - 8\right]} \leq \frac{d^* \pi}{\ln \frac{s}{34}},$$
so that (1) if $f$ is a rational function, then $m^* \leq \frac{2^{2d-2} \pi}{\ln \frac{s}{34}}$, and (2) if $f$ is a polynomial then $m^* \leq \frac{2^{d-1} \pi}{\ln \frac{s}{34}}$.

Observe that using the spherical metric in this paper, even in dealing with polynomials, is more productive than using the Euclidean metric.

For a sequence of polynomials $f_i$ of degree $d$ with connected $K(f_i)$ and with satellite PL cycles of relative periods $s_i \to \infty$, the moduli of root
annuli tend (uniformly) to zero. As we show in Theorem B, the latter conclusion holds not only in the satellite case.

**Theorem B.** Let \( \{ f_n \} \) be a sequence of degree \( d \geq 2 \) rational functions that converges to a rational function \( f \) of degree \( d \). Assume that for each \( n \) there is a renormalization \( f_n^{q_n} : U_n \to V_n \) of period \( q_n \) with connected PL set \( K_n \), and \( K_n \to K \) in the Hausdorff metric. If (1) \( K \) is non-degenerate, (2) \( q_n \to \infty \), and (3) \( K \) is not contained in a parabolic periodic domain of \( f \) (e.g., if \( f \) has no parabolic points), then an arbitrary sequence of moduli of root annuli of elements of PL cycles generated by \( K_n \) converges to 0.

No conditions in Theorem B can be dropped (Example 5.2).

**Corollary C.** Suppose that a quadratic polynomial \( f(z) = z^2 + c \) admits infinitely many satellite renormalizations with relative periods \( s_i \to \infty \). Then \( \text{mod}(A_i) \leq 2\pi / \ln(s_i/34) \to 0 \) as \( i \to \infty \) for any choice of the fundamental annuli \( A_i \) of these renormalizations.

To state Corollary D we need the following definition.

**Definition 2.2** (Infinitely renormalizable sets). Consider a nested sequence of PL cycles \( K_n \) of \( f \) such that the period of \( K_{n+1} \) is greater than the period of \( K_n \). Under these assumptions, the set \( S = \bigcap_n K_n^+ \) is called an infinitely-renormalizable set for \( f \).

Corollary D follows from Theorem B. It is not new: a proof, due to J. Kahn, in the quadratic case is given in [15, Theorem VI], and can be adjusted to the general situation.

**Corollary D.** If an infinitely renormalizable set \( S \) for a rational function \( f \) is not a Cantor set, then, for any choice of the root annuli of the renormalizations defining the set \( S \), their moduli tend to zero.

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3. **Packing modulus**

Recall that, in our normalization, the spherical metric is given by

\[
ds = \frac{|dz|}{1 + |z|^2}
\]
with respect to an affine coordinate $z$ given by the stereographic projection of the sphere. In terms of the $z$-coordinate, the sphere is identified with $\mathbb{C} \cup \{\infty\}$, where $\mathbb{C}$ is the $z$-plane, and $\infty$ is the only point on the sphere that is not represented by a finite value of $z$. In particular, a circle of (Euclidean) radius $\rho$ centered at 0 in the $z$-plane has radius
\[
\int_0^\rho \frac{dx}{(1 + x^2)} = \arctan(\rho)
\]
in spherical metric.

3.1. The Teichmüller extremal problem. Recall the Teichmüller extremal problem: find the maximal possible value of $\text{mod}(U \setminus Z)$, where an open topological disk $U \subset \mathbb{C}$ and a full continuum $Z$ are such that $0, -1 \in Z$ while $U$ does not contain the disk $\{z \in \mathbb{C} \mid |z| \leq \varepsilon\}$. A solution to the Teichmüller extremal problem is given, e.g., in [1], paragraphs 4.11 – 4.12. It reads that the maximal $m = \text{mod}(U \setminus Z)$ is attained when $Z = [-1, 0]$ while $U = \mathbb{C} \setminus [\varepsilon, \infty)$. It follows that $m$ satisfies the inequality
\[
m \leq \frac{\pi}{2 \ln(16/\varepsilon)},
\]
which is asymptotically sharp for small $\varepsilon$. (The precise maximal value of $m$ is given by a certain ratio of elliptic integrals.) The inequality displayed above is equivalent to $\varepsilon \geq 16e^{-\pi/2m}$. If the value $m = \text{mod}(U \setminus Z)$ is fixed, then one concludes from the solution of the Teichmüller problem that the disk $\{z \in \mathbb{C} \mid |z| < 16e^{-\pi/2m}\}$ is contained in $U$.

3.2. Modulus vs. cross-ratio. Recall that the cross-ratio of four points $z, z_1, z_0, z_\infty \in \mathbb{C} \cup \{\infty\}$ is defined as
\[
[z, z_1; z_0, z_\infty] = \frac{(z_0 - z)(z_\infty - z_1)}{(z_0 - z_1)(z_\infty - z)}.
\]
Cross-ratio is invariant under complex projective (Möbius) transformations. If a projective transformation takes $z_1, z_0, z_\infty$, respectively, to $1, 0, \infty$, then the image of $z$ is $[z, z_1; z_0, z_\infty]$. We now reformulate the solution of the Teichmüller extremal problem in projectively invariant terms.

**Lemma 3.1.** Suppose that an annulus $A \subset \mathbb{P}^1$ of modulus $m$ separates points $z_0, z_1$ from points $z, z_\infty$. Then
\[
m \leq \frac{-\pi}{2 \ln(16/\|[z, z_1; z_0, z_\infty]\|)}
\]

**Proof.** Let $[z, z_1; z_0, z_\infty] = \sigma$. Applying a suitable projective transformation, we may assume that $z_0 = 0, z_1 = -1, \text{ and } z_\infty = \infty$. Then $z = -\sigma$. By the solution of the Teichmüller problem, $m \leq \frac{-\pi}{2 \ln(|\sigma|/16)}$. \qed
The following is an immediate corollary of Lemma 3.1.

**Corollary 3.2.** Let $Z_1, Z_\infty \subset \mathbb{P}^1$ be disjoint full continua such that $z_0, z_1 \in Z_1$ and $z_\infty \in Z_\infty$. Set $A = (\mathbb{P}^1 \setminus Z_\infty) \setminus Z_1$ and $\text{mod}(A) = m$. Then the open disk given in terms of the $z$-coordinate by the inequality $|[z, z_1; z_0, z_\infty]| < 16e^{-\pi/2m}$ contains the point $z_0$ and is disjoint from $Z_\infty$.

**Proof.** Set $W = \{z \in \mathbb{C} \mid |[z, z_1; z_0, z_\infty]| < 16e^{-\pi/2m}\}$. Clearly, $z_0 \in W$. Let $z \in Z_\infty$ be such that $|[z, z_1; z_0, z_\infty]| < 16e^{-\pi/2m}$. Since $A$ separates the points $z_0, z_1$ from the points $z, z_\infty$ then by Lemma 3.1

$$m \leq \frac{-\pi}{2 \ln \left(\frac{1}{16} |[z, z_1; z_0, z_\infty]| \right)}.$$ 

Since $m > 0$, the above inequality implies that $\ln \left(\frac{1}{16} |[z, z_1; z_0, z_\infty]| \right) < 0$; hence it implies that $|[z, z_1; z_0, z_\infty]| \geq 16e^{-\pi/2m}$, a contradiction. \hfill $\Box$

Let us now estimate the spherical radius of the disk around $z_0$ contained in the disk $|[z, z_1; z_0, z_\infty]| \leq R$.

**Lemma 3.3.** Assume that $\text{dist}(z_0, z_1) = \delta_1$ and $\text{dist}(z_0, z_\infty) = \delta_\infty$. Then the closed spherical disk around $z_0$ of spherical radius

$$\rho_0 = \arctan \left( \frac{R}{\cot \delta_1 + (1 + R) \cot \delta_\infty} \right)$$

is contained in the disk $|[z, z_1; z_0, z_\infty]| \leq R$.

**Proof.** By a suitable rotation of the sphere, one can arrange that $z_0 = 0$ and $z_1 = \tan \delta_1$. Set $\tau_1 = \tan \delta_1$ and $\tau_\infty = \tan \delta_\infty$. If $|z| \leq \tan \rho_0$, then we see that $|z| \leq \frac{R \tau_1 \tau_\infty}{\tau_1 + \tau_\infty + R \tau_1}$ and, therefore, $\frac{|z| \tau_1 + \tau_\infty}{\tau_1 \tau_\infty - |z|} \leq R$.

On the other hand, it is easy to see that $\frac{|z| z_\infty - \tau_1}{\tau_1 z_\infty - z} \leq \frac{|z| \tau_1 + \tau_\infty}{\tau_1 \tau_\infty - |z|}$, which in the end implies that $\frac{|z| z_\infty - \tau_1}{\tau_1 z_\infty - z} = |[z, z_1; z_0, z_\infty]| \leq R$. \hfill $\Box$

### 3.3. Packing modulus

A pointed continuum is defined as a pair $(z, Z)$ consisting of a continuum $Z$ and a point $z \in Z$. Suppose now that we are given $t$ full pointed continua $(z_i, Z_i), Z_i \subset \mathbb{P}^1$ where $i = 1, \ldots, t$. Fix $D > 0$. The $D$-packing modulus of this collection of full pointed continua is defined as the supremum of positive real numbers $m$ with the following properties: there are Jordan domains $U_i \supset Z_i, i = 1, \ldots, t$ such that

$$\text{mod}(U_i \setminus Z_i) \geq m, \quad z_i \notin U_j \ (j \neq i), \quad \text{diam}(\mathbb{P}^1 \setminus U_i) \geq D.$$ 

The function $T(t, D)$ is defined in the Introduction. Recall that if $D$ is fixed and $t$ sufficiently large, then $T(t, D) > 0$, and for any $D_0$ there is a number $t_0(D_0)$ with $T(t, D) > 2D\sqrt{t}$ for $D > D_0$ and $t > t_0(D_0)$. 

Theorem 3.4. The $D$-packing modulus $\tilde{m}$ of $t \geq 2$ full pointed subcontinua of $\mathbb{P}_1$ of diameter at least $D$ is such that, if $T(t, D) > 1$, then

$$\tilde{m} \leq \frac{\pi}{2 \ln\left[\frac{D - 2}{\sqrt{t} - 1} \tan \frac{D}{2} - 8\right]} = \frac{\pi}{2 \ln T(t, D)}.$$ 

If $D > D_0$ and $t > t_0(D_0)$ for some $D_0$ then $\tilde{m} < \frac{\pi}{\ln(4D^2t)}$.

First we prove Lemma 3.5 by combining Corollary 3.2 with Lemma 3.3.

Lemma 3.5. Let $U \subset \mathbb{P}_1$ be a topological disk, and $Z \subset U$ a full continuum with $\text{mod}(U \setminus Z) = m$. If the diameters of both $Z$ and $\mathbb{P}_1 \setminus U$ are at least $D$, then the round disk of spherical radius

$$\rho = \arctan\left(\frac{8e^{-\pi/2m}}{(1 + 8e^{-\pi/2m}) \cot \frac{D}{2}}\right)$$

around any point of $Z$ is contained in $U$.

Proof. Fix a point $z_0 \in Z$; we need to prove that all points of $\mathbb{P}_1 \setminus U$ are at least at distance $\rho$ from $z_0$. Let $\Delta$ be the distance from $z_0$ to the farthest point $z_\infty$ of $\mathbb{P}_1 \setminus U$. Since there are points $x, y \in \mathbb{P}_1 \setminus U$ with $\text{dist}(x, y) \geq D$ then $\text{dist}(z_0, x) \geq D/2$ or $\text{dist}(z_0, y) \geq D/2$. Thus, $\Delta \geq D/2$. Considering the function $\psi(t) = \text{dist}(t, z_0)$ on $Z$ we see that $\psi(Z)$ is a closed interval that stretches from 0 to at least $D/2$; hence there exists a point $z_1 \in Z$ such that $\text{dist}(z_0, z_1) = D/2$. Set $\delta_1 = D/2$ and $\delta_\infty = \Delta$.

By Corollary 3.2, the topological disk $W$ that consists of points $z$ with $[z, z_1; z_0, z_\infty] < 16e^{-\pi/2m} = R$ is contained in $U$. By Lemma 3.3, the disk $W$ contains a round disk of radius $\rho_0(R, \delta_1, \delta_\infty)$ centered at $z_0$ where $\rho_0$ is the function defined in Lemma 3.3. Thus, the round disk of radius $\rho_0(R, \delta_1, \delta_\infty)$ centered at $z_0$ is contained in $U$. The explicit formula for the radius is

$$r = \rho_0\left(16e^{-\pi/2m}, \frac{D}{2}, \Delta\right) = \arctan\left(\frac{16e^{-\pi/2m}}{\cot \frac{D}{2} + (1 + 16e^{-\pi/2m}) \cot \Delta}\right)$$

and since this function is increasing in $\Delta$, then $U$ contains the round disk centered at $z_0$ of radius

$$r = \rho_0\left(16e^{-\pi/2m}, \frac{D}{2}, \frac{D}{2}\right) = \arctan\left(\frac{8e^{-\pi/2m}}{(1 + 8e^{-\pi/2m}) \cot \frac{D}{2}}\right),$$

as desired. \qed

Proof of Theorem 3.4 Consider a collection of $t$ pointed continua $(z_i, Z_i)$ of diameter at least $D$. Suppose that $m$ is less that the $D$-packing modulus of these continua. Let $\rho$ be as in Lemma 3.5. Write $B_i$ for the disk of spherical radius $\rho/2$ centered at $z_i$. Then, by Lemma 3.5, the disks $B_i$ are...
pairwise disjoint. The spherical area of each of these disks is $\pi(1 - \cos \rho)$. On the other hand, $t$ such pairwise disjoint disks fit into the sphere $\mathbb{P}^1$ of total area $\pi^2$. It follows that

$$t\frac{\pi}{2}(1 - \cos \rho) \leq \pi \implies \cos \rho \geq 1 - \frac{2}{t} = \frac{t - 2}{t}. $$

Therefore $1/(\tan^2 \rho + 1) = \cos^2 \rho \geq (t - 2)/t^2$ which implies that

$$\tan \rho = \frac{8e^{-\pi/2m}}{(1 + 8e^{-\pi/2m}) \cot \frac{D}{2}} \leq \frac{2\sqrt{t - 1}}{t - 2}. $$

The above inequality holds for every $m$ less than the packing modulus. Passing to the limit, we see that it also holds for the packing modulus $\tilde{m}$. It is easy to verify that this is equivalent to the claim of the theorem. □

4. DYNAMICAL IMPLICATIONS

In this section we obtain dynamical corollaries of Theorem 3.4.

**Definition 4.1.** Suppose that $K$ is a PL cycle and $K^* \in K$. Suppose that $x \in K^*$ is a point such that for any root annulus $U \setminus \tilde{K}^*$ around an element $\tilde{K}^* \neq K^*$ of $K$ we have $x \notin \tilde{U}$. Then $x$ is called a proper point of $K^*$.

Let us show that the concept of a proper point is not vacuous.

**Lemma 4.2.** Let $K$ be a PL cycle and $K^* \in K$; then $K^*$ contains at least two proper points.

**Proof.** Let the period of $K^*$ be $q$. Clearly, there are points of period $q$ in $K^*$ (it is immediate in the primitive case and easy in the satellite case; we leave details to the reader). The definition now implies that any such point of $K^*$ is proper. Another option is to consider a critical point of $f^q|_{K^*}$; it is easy to see that all such points are proper points of $K^*$, too. Finally, points of $K^*$ that map to $\alpha$ under the first return map are proper (here $\alpha$ is a point of the base cycle that belongs to $K^*$). The lemma follows. □

**Proof of the Main Theorem.** By Lemma 4.2 all elements of $K$ contain proper points. Now the desired upper bound on $m$ follows from Theorem 3.4. □

Next we consider the satellite PL cycle of base period 1 and relative period $s$. Assume that $f$ is a degree $d$ rational function with PL cycle $K$ of period $q$. Fix an element $K^*$ of a PL cycle $K$. Consider the sets $\tilde{K}^*$, $f(K^*)$, $\ldots$, $f^{q-1}(K^*)$. Let $U \setminus \tilde{U}$ be a fundamental annulus of $K^*$. Define $U_i$, $V_i$ as appropriate iterated pullbacks of $U$ and $V$ containing $K_i^* = f^i(K^*)$. Namely, $U_i$ is the component of $f^{i-q}(U)$ containing $K_i^*$. Similarly, $V_i$ is the component of $f^{i-q}(V)$ containing $K_i^*$. Clearly, $V_0 = U$. Observe that $U_i \setminus K_i^*$ is a root annulus of $K_i^*$. Also, the map $f^q : U_i \to V_i$ is a PL map.
with PL set $K_i^*$, where $i = 0, \ldots, q - 1$. Thus, the sets $K_i^*$ are also PL sets of $f$. Let $d^*$ be the degree of the PL map $f^* = f^t : U \to V$.

**Lemma 4.3.** Let $K^*$ be a $q$-periodic PL set for a degree $d$ rational function $f$. Then $d^* \leq 2^{2d-2}$, and for each $i = 0, \ldots, q - 1$,

$$\frac{\mod(U \setminus K^*)}{d^*} \leq \mod(U_i \setminus K_i^*) \leq \mod(U \setminus K^*).$$

If $f$ is a polynomial of degree $d$ then $d^* \leq 2^{d-1}$.

**Proof.** Indeed, $d^*$ is the product of $d_i^* = \deg(f^i(K^*) \to f^{i+1}(K^*))$ over $i = 0, \ldots, q - 1$. The function $f$ has $2d - 2$ critical points, counting multiplicities. Positive integers $d_i^*$ satisfy the inequality

$$(d_0^* - 1) + \cdots + (d_{s-1}^* - 1) \leq 2d - 2,$$

and the maximal possible value of the product of $d_i^*$ subject to this inequality is attained when $2d-2$ different numbers $d_i^*$ are equal to 2, and all remaining $d_i^*$ are equal to 1. In this case, the product is $2^{2d-2}$, as claimed. The case of a polynomial is similar. \qed

**Proof of Theorem A.** First let us prove the claim for satellite PL cycles of base period 1; the general case will follow by passing to a suitable iterate of $f$. The idea is to conjugate $f$ to a suitable rational function by a projective transformation of $\mathbb{P}^1$. Let $Z_1, \ldots, Z_s$ be all elements of the given PL cycle $K$. Choose proper points $z_1 \in Z_1, \ldots, z_s \in Z_s$. Consider root annuli $U_i \setminus Z_i$ around all elements $Z_i \in K$ such that their moduli are at least $m$.

Let $\alpha$ be the base point of $K$, and $z_0$ be a fixed point of $f$ different from $\alpha$. We claim that $z_0$ does not belong to $K^+$. Otherwise all $Z_i \in K$ contain both $\alpha$ and $z_0$, a contradiction with definitions.

By a suitable projective transformation of $\mathbb{P}^1$, map $\alpha$ to 0 and $z_0$ to infinity. It follows that $U_1, \ldots, U_s$ do not contain $\infty$ (observe that, by definition, the $f^*$-image of the boundary of $U_i$ is disjoint from itself for any $i$). Assume that $z_1$ is a point of $\{z_1, \ldots, z_s\}$ closest to 0. Applying a multiplication by an appropriate positive number one can arrange that $\dist(0, z_1) = \pi/4$. If follows that, for any $i$ with $2 \leq i \leq s$, one has $\diam(Z_i) \geq \pi/4$ and $\diam(\mathbb{P}^1 \setminus U_i) \geq \pi/4$ (the latter claim follows from the fact that $z_1, \infty \notin U_i$).

Apply the Main Theorem to the map $f$ with $D = \pi/4$ and $t = s - 1$. Since $\tan(\pi/8) = \sqrt{2} - 1$, then $T(t, \pi/4) = 4^{\frac{t}{\sqrt{2}-1}}(\sqrt{2} - 1) - 8$, and by the Main Theorem we see that

$$m \leq \frac{\pi}{2 \ln T(t, \pi/4)}, \quad \forall t \geq 33.$$
By Lemma 4.3, the modulus of any root annulus for a satellite PL cycle of period $s \geq 34$ satisfies

$$m \leq d^* \frac{\pi}{2 \ln T(s - 1, \pi/4)} = \frac{d^* \pi}{2 \ln \left[ \frac{4s-12}{\sqrt{s-2}}(\sqrt{2} - 1) - 8 \right]},$$

as desired. \hfill \Box

5. Theorem B and related comments

Proof of Theorem B. In view of Lemma 4.3, it is enough to prove that we have $\text{mod}(V_n \setminus K_n) \to 0$. By way of contradiction, assume that $\text{mod}(V_n \setminus K_n)$ stay away from 0 for an infinite subsequence of numbers $n$. Pulling $V_n$ and $U_n$ back under $f_n^n$, we may assume that there are also Jordan disks $W_n$ with $f_n^n : V_n \to W_n$ being PL-maps. By Lemma 3.5, a neighborhood of $K$ is contained in all sets $U_n$. Passing to a subsequence, assume that there is a domain $U$ such that for all $n$ we have $K_n \subset U \subset U_n$, and $\text{mod}(U \setminus K_n) \geq m$ for some $m > 0$.

Lemma 5.1. The set $U$ is contained in the Fatou set of $f$.

Proof. The exceptional set of $f$ is the maximal finite subset $E_f \subset \mathbb{P}^1$ with the property $f^{-1}(E) \subset E$. By [18, Lemma 4.9], $E_f$ consists of one or two points. Moreover, if $E_f$ is nonempty, then $f^2$ is Möbius conjugate to a polynomial. Assume, by way of contradiction, that $U \cap J(f) \neq \emptyset$. Then, by [18] (see Corollary 14.2 and the following remark), the complement of $f^j(U)$ is a subset of an arbitrarily small neighborhood of $E_f$, for all sufficiently large $j$. Since $q_n \to \infty$, we may assume that $q_1$ is already sufficiently large, so that $f^{q_1}(U) \cup O(E_f) = \mathbb{P}^1$ for a small neighborhood $O(E_f)$ of $E_f$ (if $E_f$ is empty, then $f^{q_1}(U) = \mathbb{P}^1$). We may also assume that $q_1 < q_2 < \cdots < q_n < \ldots$. Consider two cases.

(1) $E_f = \emptyset$; then $f^{q_1}(U) = \mathbb{P}^1$, and since $f_n^n \to f^{q_1}$ as $n \to \infty$, by compactness of $\mathbb{P}^1$ it follows that $f_n^{q_n}(U) = \mathbb{P}^1$ for large $n$, a contradiction with $f_n^{q_n}(U) \subset V_n$.

(2) $E_f \neq \emptyset$; then we may assume that $f$ is a polynomial, possibly replacing $f$ with $f^2$ and $f_n$ with $f_n^2$. In this case $f^{q_1}(U)$ is the whole of $\mathbb{C}$ except, perhaps, for a small neighborhood of $\infty$ and $V_n \supset f_n^{q_n}(U) \supset J(f_n)$ for large $n$. This is a contradiction with Lemma 4.2 applied to $K_n$ and $V_n$. \hfill \Box

It follows that $U \subset \Omega$ where $\Omega$ is a Fatou component of $f$. The set $\Omega$ cannot be a component of the basin of an attracting cycle of $f$ as otherwise, for large $n$, the PL set $K_n \subset U$ would be in the basin of an attracting cycle of $f_n$, a contradiction. By way of contradiction assume that $\Omega$ is not in the basin of a parabolic cycle of $f$. Thus $\Omega$ is eventually mapped to a periodic
rotation domain (Siegel disk or Herman ring) under $f$. It is safe to assume that $\Omega$ is itself a periodic rotation domain of period $p$.

As $\mod(U \setminus K_n) \geq m$, there is $\delta > 0$ such that $\dist(\partial U, K_n) > \delta$ for all $n$. Now, since $f^p : \Omega \to \Omega$ is conjugate to an irrational rotation, one can fix $s > 0$ such that, for each $z \in \overline{U}$ we have $\dist(z, f^{sp}(z)) < \delta/2$. As $f_n \to f$, it follows that for every $n$ large enough, $f^{sp}_n(K_n) \subset U \subset U_n$ and $f^{sp}_n(K_n) \neq K_n$. The sets $K_n$ and $K'_n = f^{sp}_n(K_n)$ are in the same PL cycle. Hence the proper point of $K'_n$ that exists by Lemma 4.2 cannot lie in $U_n$. This contradiction concludes the proof.

Observe that the last part of the proof of Theorem B can be deduced from the Main Theorem. Indeed, the fact that the limit set $K$ of the sequence $\{K_n\}$ is contained in a rotation domain implies that for some $k > 0$ the number of components of the $f_n$-orbit of $K_n$ of diameter greater than $k$ goes to infinity with $n$, which contradicts the Main Theorem.

Corollary D almost immediately follows from Theorem B where we let $f_n = f$ for all $n$ because no point of an infinitely-renormalizable set can belong to a parabolic domain. More precisely, let $m_n$ be the modulus of a root annulus of some element of $K_n$, for each $n$. By the condition, there is a non-degenerate component $K$ of $S$. Then $K = \cap_{n>0} K_n$ for some $K_n \in \mathcal{K}_n$. By Theorem B, Corollary D follows.

All three conditions (1) – (3) are essential for the conclusion of Theorem B. Namely, it is clear that the conclusion breaks down without condition (2). As for conditions (1) and (3), counterexamples can be found in the real unimodal family $f_c(z) = z^d + c$, with $c \in \mathbb{R}$, for every even positive $d$. Indeed, in that case it is known that there is a universal complex bound for any renormalization of $f_c$ of period $s$ whenever $f^{2s}_c$ has no attracting or neutral fixed point, see Theorem A of [10] (complemented by the following paragraph), see also [5, 16]. Thus it is enough to find corresponding examples of real $f_c$ with periodic intervals. It follows that the conclusion breaks down without condition (1).

As for (3), the following counterexample shows that this assumption is also necessary.

**Example 5.2.** We claim that there exists a real sequence $(a_n)$ such that:

(i) the sequence $a_n$ is decreasing, and $a_n \searrow a$;
(ii) the map $f_a$ has a parabolic 3-cycle of multiplier 1;
(iii) there is a symmetric w.r.t. 0 and $f_{a_n}$-periodic interval $L_n$ of period $q_n$ such that $f^{q_n}_{a_n} : L_n \to L_n$ is a unimodal map;
(iv) maps $f^{2q_n}_{a_n}$ have no attracting/parabolic fixed points;
(v) periods $q_n$ tend to infinity;
(vi) diameters of $L_n$ stay away from 0.
As mentioned, by applying the above universal complex bounds, for the sequence \( f_{a_n} \) and a special choice of renormalizations \( f_{a_n}^q : U_n \to V_n \) where \( L_n \subset U_n \Subset V_n \), there exists \( \delta > 0 \) that depends exclusively on the degree \( d \) such that \( \text{mod}(V_n \setminus U_n) > \delta \), i.e., the conclusion of Theorem B breaks down. On the other hand, conditions (1) – (2) still hold.

The sequence \((a_n)\) can be defined using Lavaurs maps. The construction is motivated by [14]. Start with a map \( f_a \) satisfying \((ii)\). Let \( F = f_a^3 \) and \( x_0, x_0 < 0 \), be the point of the parabolic 3-cycle of \( f \) that contains 0 in its immediate basin of attraction. Locally, \( F(z) = z + A(z - x_0)^2 + B(z - x_0)^3 + \ldots \) where \( A < 0 \). Let \([c_{-2}, 0]\) be the maximal interval containing \( x_0 \) on which \( F \) is increasing. Here \( c_{-2} < 0 \), \( f_a^3(c_{-2}) = 0 \) and \( F(c_{-2}) = c_1 := f_a(0) \) where \( c_1 < c_{-2} \). The interval \((c_1, c_{-2})\) contains a point \( c_{-1} \) such that \( f_a(c_{-1}) = 0 \) so that \( f_a^2 \) is decreasing on \([c_1, c_{-1}]\) and increasing on \([c_{-1}, 0]\). Consider the fundamental interval \( I_- = [F(0), 0] \) of the immediate attracting basin of \( x_0 \). This means that the sets \( F^n(I_-) \) for \( n \geq 0 \) have disjoint interiors, and their union covers the interval \((x_0, 0]\) all of whose points converge to \( x_0 \) under forward iterations of \( F \). Also, consider the interval \( I_+ = [c_1, c_{-2}] \). This is a fundamental interval for the backward iteration of \( F^{-1} : [c_1, x_0] \to [c_{-2}, x_0] \) which, from now on, denote the inverse branch of strictly increasing \( F : [c_{-2}, x_0] \to [c_1, x_0] \).

Now, consider attracting \( \varphi_- \) and repelling \( \varphi_+ \) Fatou coordinates of \( F \) near \( x_0 \), see e.g., [18, 20]. Recall briefly the definition. Let \( \delta > 0 \) be small and \( D_-, D_+ \) be the disks centered at \( x_0 + \delta, x_0 - \delta \) respectively with radius \( \delta \). Then \( F(D_-) \subset D_- \) while \( D_+ \subset F(D_+) \). This defines, after identification of \( z \) with \( F(z) \) at the boundary, two cylinders \( U_- = D \setminus F(D_-) \) and \( U_+ = F(D_+) \setminus D_+ \). By the Riemann mapping theorem, \( U_\pm \) can be uniformized by “straight” cylinders: there exist \( \varphi_\pm \) mapping the cylinders \( U_\pm \) conformally onto vertical strips of width 1 conjugating \( F \) to \( T_1 \) where \( T_\sigma : z \mapsto z + \sigma \) denotes the translation by \( \sigma \). That is,

\[
(5.1) \quad \varphi_-(F(z)) = T_1(\varphi_-(z))
\]

for \( z \in U_- \) and, correspondingly,

\[
(5.2) \quad \varphi_+(F(z)) = T_1(\varphi_+(z))
\]

for \( z \in U_+ \). By symmetry, we may assume that \( \varphi_\pm(z) = \varphi_\pm(-z) \).

Then \( \varphi_- \) extends by (5.1) to an orientation reversing homeomorphism \( \varphi_- : (x_0, 0] \to [\varphi_-(0), +\infty) \) and \( \varphi_+ \) extends by (5.2) to an orientation reversing homeomorphism \( \varphi_+ : [c_1, x_0) \to (-\infty, \varphi_+(c_1)] \). In particular, \( \varphi_-(I_-) = [\varphi_-(0), \varphi_-(F(0))] \) where \( \varphi_-(F(0)) = \varphi_-(0) + 1 \) and \( \varphi_+(I_+) = [\varphi_+(c_{-2}), \varphi_+(c_1)] \) where \( \varphi_+(c_1)) = \varphi_+(c_{-2}) + 1 \). Notice that the Fatou coordinates \( \varphi_\pm \) are unique up to post-composition by a real translation. This allows us to fix the choice of \( \varphi_\pm \) in such a way that \( X := \varphi_-(0) = \varphi_+(c_{-2}) \).
which means that \( \varphi_-([x_0, 0]) = [X, +\infty) \) while \( \varphi_+([c_1, x_0]) = (-\infty, X + 1] \). Hence, \( \varphi_-(I_-) = \varphi_+(I_+) = [X, X + 1] \) and the following map is a well-defined orientation preserving homeomorphism: \( g_0 := \varphi_+^{-1} \circ \varphi_- : I_- \to I_+ \).

More generally, let

\[
g_\sigma = \varphi_+^{-1} \circ T_\sigma \circ \varphi_-
\]

be the Lavaurs map. If \( \sigma \leq 0 \) then \( T_\sigma \circ \varphi_- (I_-) = [X + \sigma, X + 1 + \sigma] \subset \varphi_+([c_1, x_0]) \), i.e., for each \( \sigma \leq 0 \), the map \( g_\sigma : I_- \to [c_1, x_0] \) is a well-defined orientation preserving homeomorphism on its image \( [g_\sigma(F(0))], g_\sigma(0) \) where \( g_\sigma(F(0)) = F(g_\sigma(0)) \). When \( \sigma \) monotonically moves from 0 to the left, the end points \( g_\sigma(F(0)), g_\sigma(0) \) of the image \( g_\sigma(I_-) \) move monotonically to the right. There is a unique \( \sigma_0 < 0 \) such that \( g_{\sigma_0}(F(0)) = c_{-1} \).

For every \( \sigma \in [\sigma_0, 0] \), there exists a unique solution \( q_\sigma \in I_- \) of the equation \( g_\sigma(x) = c_{-1} \), so that \( g_\sigma([q_\sigma, 0]) = [c_{-1}, g_\sigma(0)] \). Note that \( q_\sigma \) increases from \( q_0 \) to \( q_{\sigma_0} \) as \( \sigma \) decreases from 0 to \( \sigma_0 \).

Let \( G_\sigma = f_\sigma^2 \circ g_\sigma \). As \( f_\sigma^2 \) increases on \([c_{-1}, 0]\), for every \( \sigma \in [\sigma_0, 0] \), \( G_\sigma : [g_\sigma(0), 0] \to [c_1, f_\sigma^2(g_\sigma(0))] \) is an orientation preserving homeomorphism. The map \( g_\sigma \) extends immediately by symmetry to an even map on \([F(0), -F(0)]\) (which is again denoted by \( g_\sigma \)). Therefore, we get a unimodal map \( G_\sigma \) on \([q_{\sigma_0}, -q_\sigma]\) for each \( \sigma \in \sigma_0 \).

The following hold: (a) \( G_\sigma \) is increasing on \([q_{\sigma_0}, 0]\) and is an even function on \([q_\sigma, -q_\sigma]\) for \( \sigma_0 \leq \sigma \leq 0 \), (b) \( G_\sigma(q_\sigma) = c_1 < q_\sigma < 0 \) for \( \sigma_0 \leq \sigma \leq 0 \), (c) \( G_0(0) = 0 \) and \( G_{\sigma_0}(0) = -c_{-2} > -F(0) > -q_{\sigma_0} > 0 \), (d) \( G_\sigma(0) \) decreases from \( G_{\sigma_0}(0) > 0 \) to \( G_0(0) = 0 \) as \( \sigma \) increases from \( \sigma_0 \) to 0.

Indeed, (a)-(b) hold by the construction and \( G_0(0) = f_\sigma^2(c_{-2}) = 0 \). Now, \( G_{\sigma_0}(0) = f_\sigma^2(F^{-1}(c_{-1})) \) is a point of \( f_\sigma^{-1}(\{c_{-1}\}) \) where \( c_{-1} < 0 \) and \( f_\sigma(c_{-1}) = 0 \). On the other hand, \( c_1 < c_{-1} < c_{-2} < F^{-1}(c_{-1}) < 0 \) and \( f_\sigma^2 \) increases on \([c_{-1}, 0]\), hence, \( 0 < G_{\sigma_0}(0) \). There is just one positive point of \( f_\sigma^{-1}(\{c_{-1}\}) \), which is \(-c_{-2}\). To finish with (c), it remains to note that \( c_{-2} < x_0 < F(0) < q_{\sigma_0} \). Finally, (d) follows from (c).

Now, (5.1) allows us to extend \( \varphi_- \) from \( U_- \) to an analytic function to the basin of attraction \( \Delta \) of the parabolic 3-cycle of \( f_\sigma \) while the inverse map \( \varphi_+^{-1} \) extends by (5.2) from \( \varphi_+(U_+) \) to an entire function. Hence, \( g_\sigma \) extends to an analytic function to \( \Delta \). The main purpose of introducing \( g_\sigma \) (and our use of it) is the following theorem due to Douady and Lavaurs [3] (stated in the particular case of 3-cycle) as follows: for every \( \sigma \in \mathbb{R} \) there exists a sequence \( a_n \searrow a \) and an increasing sequence of positive integers \( N_n \) such that \( g_\sigma(z) = \lim_{n \to \infty} f_\sigma^{3N_n}(z) \) uniformly on compact subsets of \( \Delta \).

As \( [F(0), -F(0)] \subset \Delta \) and \( f_\sigma \) has a negative Schwarzian derivative \( Sf_\sigma < 0 \) on \( \mathbb{R} \) this theorem implies, in particular, that \( Sg_\sigma \leq 0 \) on \([F(0), -F(0)]\). (This also follows directly from the fact that \( g_\sigma^{-1} \) extends from the real interval to a univalent function of the upper half plane into
itself, see [14].) As $G_\sigma = f_\alpha^2 \circ g_\sigma$ and $Sf_\alpha^2 < 0$ on $\mathbb{R}$, then $SG_\sigma < 0$ on $[F(0), -F(0)]$. Now, $G_\sigma$ has on $[F(0), -F(0)]$ precisely 3 critical points: 0 and two symmetric critical points at $\pm q_\sigma$. Along with relations $G_\sigma(q_\sigma) < q_\sigma < 0 < G_\sigma(0)$ for $\sigma_0 \leq \sigma < 0$, $G_\sigma(0) = 0$, and general properties of maps with the negative Schwarzian [17] we get that, for each $\sigma \in [\sigma_0, 0]$ the map $G_\sigma$ has a unique (orientation preserving) fixed point $\beta_\sigma \in (q_\sigma, 0)$. Moreover, it is repelling. So we have a unimodal restriction of $G_\sigma$ to $L_\sigma := [\beta_\sigma, -\beta_\sigma]$. Now, as $\beta_0 < 0 = G_\sigma(0) < \beta_0$ while $-\beta_\sigma < G_\sigma(0)$, by continuity, there is a Chebyshev parameter $\sigma_{Ch} \in (\sigma_0, 0)$, i.e., such that $G_{\sigma_{Ch}}^2(0) = \beta_{Ch}$. Let us fix any $\sigma_* \in (\sigma_{Ch}, 0)$ such that $G_* := G_{\sigma_*} : L_* \to L_*$ where $L_* = L_{\sigma_*}$ has no attracting or neutral fixed point or 2-cycle. For example, any $\sigma_*$ close enough $\sigma_{Ch}$ would work. Notice that $L_* \subset [F(0), -F(0)]$. By the earlier stated theorem of Douady and Lavaurs, there exists a sequence $a_n \searrow a$ and an increasing sequence of positive integers $N_n$ such that $g_\sigma_\alpha(z) = \lim_{n \to \infty} f_{a_n}^{3N_n}(z)$ uniformly in some complex neighborhood of $[F(0), -F(0)]$. Let $q_\sigma = 3N_n + 2$. Then, for each large enough $n$, the map $f_{a_n}$ has a symmetric periodic interval $L_n \ni 0$ of period $q_\sigma$ and $L_n \to L_*$ as $n \to \infty$. Besides, $f_{a_n}^{2q_\sigma}$ has no attracting or neutral fixed point on $L_n$. It follows that the sequence $(a_n)$ is as required.

Notice that in Example 5.2, the number of components of the orbit of $K_\alpha$ of size at least $k$ asymptotically, as $n \to \infty$, is $Ck^{-1/2}$ with some $C > 0$.

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(Alexander Blokh and Lex Oversteegen) DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ALABAMA AT BIRMINGHAM, BIRMINGHAM, AL 35294

(Genadi Levin) INSTITUTE OF MATHEMATICS, THE HEBREW UNIVERSITY OF JERUSALEM, GIVAT RAM 91904, JERUSALEM, ISRAEL. TEMPORARY ADDRESS: INSTITUTE OF MATHEMATICS OF PAN, SNIADECKICH 8, WARSAW 00-656, POLAND

(Vladlen Timorin) FACULTY OF MATHEMATICS, HSE UNIVERSITY, 6 USACHEVA STR., MOSCOW, RUSSIA, 119048

Email address, Alexander Blokh: ablokh@uab.edu
Email address, Genadi Levin: genady.levin@mail.huji.ac.il
Email address, Lex Oversteegen: overstee@uab.edu
Email address, Vladlen Timorin: vtimorin@hse.ru