NAME:______________________________

GRADE:______________________________

SCHOOL CODE:________________________

2006-2007 UAB MATH TALENT SEARCH

This is a two hour contest. Answers are to be written in the spaces provided on the test sheet. You **MUST** justify your answers in order to get full credit. Your work (including full justification) should be shown on the extra paper which is attached.

PROBLEM 1 (10 pts) What is the smallest sum one could get by adding three different numbers from the set $\{7, 25, -1, 12, -3\}$?

YOUR ANSWER:

PROBLEM 2 (20 pts) How many integers are there between $\sqrt{5}$ and $\sqrt{27}$?

YOUR ANSWER:

PROBLEM 3 (30 pts) In a triangle ABC, E and D are interior points of AC and BC, respectively. AF bisects the angle CAD, and BF bisects the angle CBE. It is known that the angle AEB equals 80 degrees, and the angle ADB equals 70 degrees. What is the angle AFB in degrees?

YOUR ANSWER:

PROBLEM 4 (50 pts) Let the radius of the circle inscribed in a rhombus with diagonals of length 18 and 80 be equal to r. What is $41r$ equal to?

YOUR ANSWER:

PROBLEM 5 (80 pts)The numbers $A > 0, c, d, u, v$ are non-zero constants. The graphs of $f(x) = |Ax + c| + d$ and $g(x) = -|Ax + u| + v$ intersect exactly at two points $(1000, 4000)$ and $(3000, 10000)$. What is $\frac{u+c}{A}$ equal to?

YOUR ANSWER:

PROBLEM 6 For a positive integer $i \leq 1000$ we define $f(i)$ as twice the average of all multiples b of i such that $1 \leq b \leq 1000$.

a) (110 pts) For what $i, 1 \leq i \leq 500$ the maximal value of $f(i)$ is assumed?

YOUR ANSWER:

b) (130 pts) For each number $j > 1000$ let $h(j)$ be the number of i's such that $f(i) = j$. For which of the numbers $1001, 1002, 1003$ is the maximal value of $h(i)$ assumed?

YOUR ANSWER:

PROBLEM 7 (150 pts) A huge Egyptian palace was discovered by archeologists (a palace is a one-floor building divided into many rectangular rooms without hallways). The number of doors in each room in the house is $0, 1$ or 2. A door can be either an *outside* one (if it leads outside the house), or an *inside* one (if it connects two neighboring rooms). A room with exactly one door is called a *dead end*. It is known that there are 2006 outside doors in the palace, however counting all the dead ends was hard, so it is only known that the number of dead ends in the palace is between 2007 and 2050. What is the minimal possible number of dead ends in the palace? **YOU MUST FULLY JUSTIFY YOUR ANSWER; ANSWERS WITHOUT JUSTIFICATION WILL NOT BE CONSIDERED!**

YOUR ANSWER:

2006-2007 UAB MTS: SOLUTIONS

PROBLEM 1 (10 pts) What is the smallest sum one could get by adding three different numbers from the set $\{7, 25, -1, 12, -3\}$?

Solution: To get the smallest sum we need to sum up the smallest numbers from the set, that is $-3, -1$ and 7.

So the answer is **(-3)+(-1)+7=3**.

PROBLEM 2 (20 pts) How many integers are there between $\sqrt{5}$ and $\sqrt{27}$?

Solution: Since $4 < 5 < 9$ then $2 < \sqrt{5} < 3$. Since $25 < 27 < 36$ then $5 < \sqrt{27} < 6$. Hence the integers between $\sqrt{5}$ and $\sqrt{27}$ are $3, 4, 5$

So the answer is **3**.

PROBLEM 3 (30 pts) In a triangle ABC, E and D are interior points of AC and BC, respectively. AF bisects the angle CAD, and BF bisects the angle CBE. It is known that the angle AEB equals 80 degrees, and the angle ADB equals 70 degrees. What is the angle AFB in degrees?

Solution: Denote the angle CAF by x and the angle CBF by y. Also, denote the angle DAB by a and the angle ABE by b. Then it follows that the angle AFB equals $180 - a - x - b - y$. Now, the angle AEB equals 80 degrees while on the other hand from the triangle AEB it follows that the angle AEB equals $180 - b - a - 2x$. Similarly we see that the angle ADB equals 70 degrees while on the other hand it equals $180 - a - b - 2y$. If we sum up the formulas for the angles AEB and ADB, we get $70 + 80 = 150$ while on the other hand we get $360 - 2a - 2b - 2x - 2y = 2(180 - a - x - b - y)$, that is twice the angle AFB. Therefore, the angle AFB equals 75 degrees.

So, the answer is **75 degrees**.

PROBLEM 4 (50 pts) Let the radius of the circle inscribed in a rhombus with diagonals of length 18 and 80 be equal to r. What is $41r$ equal to?

Solution: Denote the rhombus in question by ABCD, and let their diagonals AC (of length 18) and BD (of length 80) intersect at the

point F. The triangle AFB is the right triangle with sides AF of length 18/2=9 and BF of length 80/2=40. Hence by Pythagorean Theorem the length of AB is 41. The radius r of the inscribed circle is the height of AFB from the vertex F to AB. We can compute the area of ABF in two ways: 1) as one half of the product of the lengths of AF and BF, 2) as one half of the product of the length of AB and r. Hence we see that $r = \frac{9 \cdot 40}{4}1$.
So the answer is **360**.

PROBLEM 5 (80 pts)The numbers $A > 0, c, d, u, v$ are non-zero constants. The graphs of $f(x) = |Ax + c| + d$ and $g(x) = -|Ax + u| + v$ intersect exactly at two points $(1000, 4000)$ and $(3000, 10000)$. What is $\frac{u + c}{A}$ equal to?

Solution: The intersection of the graphs in question is a parallelogram, say, P. The vertices of the graphs of f and g are opposite vertices of P while the other two opposite vertices of P are the points $(1000, 4000)$ and $(3000, 10000)$. The vertex of the graph like that of the function f or g is always the point at whose x-coordinate the absolute value ingredient in the formula is equal to 0. Thus the vertex F of the graph of f is the point with x-coordinate which satisfies the equation $Ax + c = 0$, and so $x_F = -c/A$. The vertex G of the graph of g is the point with x-coordinate which satisfies the equation $Ax+u = 0$, so that $x_G = -u/A$. Now, the diagonals of the parallelogram intersect at their midpoints. Hence the sum of the x-coordinates of one pair of opposite vertices of P equals the sum of x-coordinates of the other pair of opposite vertices of P. In other words, $x_F + x_G = 1000 + 3000$. Since by the above formulas $x_F + x_G = -(u + c)/A$ we see that $(u + c)/A = -4000$.

So the answer is **-4000**.

PROBLEM 6 For a positive integer $i \leq 1000$ we define $f(i)$ as twice the average of all multiples b of i such that $1 \leq b \leq 1000$.

a) (110 pts) For what $i, 1 \leq i \leq 500$ the maximal value of $f(i)$ is assumed?

b) (130 pts) For each number $j > 1000$ let $h(j)$ be the number of i's such that $f(i) = j$. For which of the numbers $1001, 1002, 1003$ is the maximal value of $h(i)$ assumed?

Solution: (a) The average of a set of consecutive multiples of any number can be evaluated by computing out the average of the greatest and the smallest number in the set. Hence the function $f(i)$ is the sum of i and the greatest multiple of i not exceeding 1000. Therefore $f(i)$ is simply the smallest multiple of i which is greater than 1000. Since 500 is a factor of 1000 we see that $f(500) = 1500$. On the other hand if $i < 500$ then the smallest multiple of i greater than 1000 is less than 1500.

So the answer is **500**.

(b) As follows from our equivalent definition of $f(i)$, one can factor j and count the number of its factors i such that $j - i \leq 1000$; this number is $h(j)$. Doing it with $1001, 1002, 1003$ we see that

$$f(1) = f(7) = f(11) = f(13) = f(77) = f(91) = f(143) = 1001,$$

$$f(2) = f(3) = f(167) = f(6) = f(334) = f(501) = 1002,$$

$$f(17) = f(59) = 1003,$$

and hence $h(1001) = 7, h(1002) = 6$ and $h(1003) = 2$.

So the answer is **1001**.

PROBLEM 7 (150 pts) A huge Egyptian palace was discovered by archeologists (a palace is a one-floor building divided into many rectangular rooms without hallways). The number of doors in each room in the house is $0, 1$ or 2. A door can be either an *outside* one (if it leads outside the house), or an *inside* one (if it connects two neighboring rooms). A room with exactly one door is called a *dead end*. It is known that there are 2006 outside doors in the palace, however counting all the dead ends was hard, so it is only known that the number of dead ends in the palace is between 2007 and 2050. What is the minimal possible number of dead ends in the palace? **YOU MUST FULLY JUSTIFY YOUR ANSWER; ANSWERS WITHOUT JUSTIFICATION WILL NOT BE CONSIDERED!**

Solution: Let a *walk* be a path through the palace such that 1) any door can be passed only once, 2) a walk starts either by entering the house from outside or from a dead end, 3) a walk terminates only in one of two cases: when we are outside, or when we are in a dead end. The assumptions determine each walk uniquely, once it is started. Now, there are several types of walks; let us list them below.

(1) A walk from an outside door to an outside door. Let there be i such walks.

(2) A walk from an outside door to a dead end. Let there be j such walks.

(3) A walk from a dead end to a dead end. Let there be k such walks.

Clearly, all outside doors and dead ends are involved in walks because every outside door and each dead end can serve as the beginning of a walk. Since the walks can be only of the three above listed types, we see that there are $2i + j$ outside doors and $2k + j$ dead ends. Hence the parity of the number of dead ends and the number of outside doors is the same. Since the number of outside doors is 2006 we see that of the two possibilities for the number of dead ends only 2008 satisfy this condition. It is easy to come up with an example: imagine a palace with 2006 outside doors each of which leads into a dead end right away such that in this palace there is also a pair of dead ends connected immediately to each other.

So the answer is **2008**.