NAME:______________________________

GRADE:______________________________

SCHOOL CODE:____________________

2008-2009 UAB MATH TALENT SEARCH

This is a two hour contest. Answers are to be written in the spaces provided on the test sheet. There will be no credit if the answer is incorrect. You **MUST** justify your answers in order to get full credit; otherwise, partial credit or no credit will be awarded according to the decision made by the judges. Your work (including full justifications) should be shown on the extra paper which is attached. The problems are listed in increasing order of difficulty.

PROBLEM 1 (5 pts) It is given that $9^9 + 9^9 + 9^9 + 9^9 + 9^9 + 9^9 + 9^9 + 9^9 + 9^9 = 9^k$. Find k.

YOUR ANSWER:

PROBLEM 2 (10 pts) Four points A, B, C, D are chosen on a straight line. It is given that $AB = 1, BC = 2, CD = 4$. List all possible values of AD.

YOUR ANSWER:

PROBLEM 3 (20 pts) It is given that in a triangle ABC we have $AB = BC$. The point M is chosen on BC so that $BM = MA = AC$. Find the angle B in degrees.

YOUR ANSWER:

PROBLEM 4 (30 pts) A flat country has several cities such that the distances between them are all distinct. On January 1, 2009 in the morning a plane flies from each city to the closest city. What is the maximal number of planes which can land in one city?

YOUR ANSWER:

over, please

PROBLEM 5 (1) (25 pts) Compute $\sqrt{20 \cdot 19 \cdot 18 \cdot 17 + 1}$.

(2) (25 pts) Express $\sqrt{(k+1) \cdot k \cdot (k-1) \cdot (k-2) + 1}$ as a polynomial of k.

YOUR ANSWER:

PROBLEM 6 (70 pts) Seven points are marked on the unit circle. (a) How many distinct **convex** polygons with vertices at some (or all) these points can be drawn? (b) How many distinct polygons with vertices at some (or all) these points can be drawn (here we do not require the convexity of polygons)?

YOUR ANSWER:

PROBLEM 7 (90 pts) Find all numbers k among $1, 2, 3, 4, 5, 6, 7, 8$ such that $m^4 + k$ is not prime for any $m, |m| \neq 1$.

YOUR ANSWER:

PROBLEM 8 (120 pts) Find all natural numbers n such that $2 < n < 2009$ which have the following property: any number $1 < m < n$ which has no common divisors with n is prime (a prime number is a number p which has no divisors other than 1 and itself; 1 is not considered prime). Sum these numbers up. Write the answer below in the space provided.

YOUR ANSWER:

2008-2009 UAB MTS: SOLUTIONS

PROBLEM 1 (5 pts) It is given that $9^9 + 9^9 + 9^9 + 9^9 + 9^9 + 9^9 + 9^9 + 9^9 + 9^9 = 9^k$. Find k.

Solution: We sum up 9^9 nine times, so we get $9 \cdot 9^9 = 9^{10}$.

So the answer is **10**.

PROBLEM 2 (10 pts) Four points A, B, C, D are chosen on a straight line. It is given that $AB = 1, BC = 2, CD = 4$. List all possible values of AD.

Solution: We are only concerned about the distance between A and D, but not about its direction. Hence we may assume that $A = 0, B = 1$ thus fixing the direction of AB as positive. Then there are two choices for directions of the segments BC and CD each. Hence there are four ways the points C, D can be located. They a listed below:
$C = 3, D = 7$, so in this case $AD = 7$;
$C = 3, D = -1$, so in this case $AD = 1$;
$C = -1, D = 3$, so in this case $AD = 3$;
$C = -1, D = -5$, so in this case $AD = 5$.

So the answer is **1, 3, 5, 7**.

PROBLEM 3 (20 pts) It is given that in a triangle ABC we have $AB = BC$. The point M is chosen on BC so that $BM = MA = AC$. Find the angle ABC in degrees.

Solution: Since $AB = BC$, then $\angle ABM = \angle BAM = x$. Hence $\angle AMC = 2x$. Since $AM = AC$, then $\angle AMC = \angle ACM = 2x$. Since $AB = BC$, then $\angle BAC = \angle BCA = \angle MCA = 2x$. Thus, $\angle BAC + \angle ACB + \angle CBA = 180° = 5x$, hence $\angle ABC = 36°$.

So, the answer is **36°**.

However there is also a degenerate case. Indeed, if the triangle ABC is equilateral and $M = C$ then all conditions of the problem are satisfied. Hence the second correct answer is **60°**. Both answers were treated as correct and deserving full credit.

PROBLEM 4 (30 pts) A flat country has several cities such that the distances between them are all distinct. On January 1, 2009 in the morning a plane flies from each city to the closest city. What is the maximal number of planes which can land in one city?

Solution: If there are 6 cities in the country one of which is at the center of a circle of radius R and the remaining 5 cities are located very close to the vertices of the right pentagon inscribed into the circle, then the distances between all the cities can be made distinct, for each city on the circle the closest city will be the "central" city, hence 5 planes will land at the central city.

Suppose that there are 6 planes landing at one city. Denote it by A and denote the cities whose planes land at A by $B_1, \ldots, B_6$. We may assume that segments $AB_1, \ldots, AB_6$ are ordered in the counterclockwise direction. The angles formed by these segments then add up to $360°$, hence at least one of them is less than or equal to $60°$. Suppose that $\angle B_1AB_2 \leq 60°$. Then, since all angles in the triangle B_1AB_2 are distinct (otherwise two sides of the triangle will have the same length, a contradiction with the assumptions) we see that there will be at least one angle of this triangle greater than $\angle B_1AB_2$. Hence either AB_1 or AB_2 is shorter than B_1B_2, a contradiction with the assumption that the planes from all B_i's must land in A.
So, the answer is **5**.

PROBLEM 5 (1) (25 pts) Compute $\sqrt{20 \cdot 19 \cdot 18 \cdot 17 + 1}$.
(2) (25 pts) Express $\sqrt{(k+1) \cdot k \cdot (k-1) \cdot (k-2) + 1}$ as a polynomial of k.

Solution: We solve the second part which easily allows one to get the answer to the first part too. We have

$$a = (k+1) \cdot k \cdot (k-1) \cdot (k-2) + 1 = [(k+1)(k-2)] \cdot [k(k-1)] + 1$$

and so

$$a = (k^2 - k - 2)(k^2 - k) + 1 = (k^2 - k)^2 - 2(k^2 - k) + 1 = (k^2 - k - 1)^2$$

which gives the answer to (2) and then to (1).
So the answer to (1) is **341**, and the answer to (2) is $\boldsymbol{k^2 - k - 1}$.

PROBLEM 6 (70 pts) Seven points are marked on the unit circle.
(1) How many distinct **convex** polygons with vertices at some (or all)

these points can be drawn? (2) How many distinct polygons with vertices at some (or all) these points can be drawn (here we do not require the convexity of polygons)?

Solution: To get a polygon, one needs to choose $k \geq 3$ points out of 7 points and connect them with segments of straight lines. Every such polygon is well-defined by the choice and necessarily convex, hence the answers to the part (1) and the part (2) coincide. To see in how how many ways one can choose 3 or more points out of 7 observe that by the study of binomial coefficients the desired number is

$$\frac{7 \cdot 6 \cdot 5}{3!} + \frac{7 \cdot 6 \cdot 5 \cdot 4}{4!} + \cdots + \frac{7!}{7!}$$

which equals

$$2^7 - 1 - 7 - \frac{7 \cdot 6}{2!} = 99.$$

So the answer to both (1) and (2) is **99**.

However if a participant considered self-intersecting polygons as admissible while solving (2), this was viewed as a legitimate point of view and, provided the answer was correct, was graded with full credit too. The solution then changes as follows. First of all, we take the position that given any sequences of non-repeating vertices we can connect them appropriately (as sequence prescribes) and get a self-intersecting polygon. Here we ignore the fact that in some cases (if the points on the circle are located in a specific way) more than two sides of the polygon may have intersection at one point. In other words, we consider 7 points on the circle located *generically*. Then our self-intersecting polygon is actually simply a closed broken line.

Now, given m there are $7 \cdot \cdots \cdot (7 - m + 1) = a_m$ ways to choose an ordered sequence of m points out of 7. Speaking of the polygon this defines, we need to take into account that the same polygon is obtained if the sequence is cyclically shifted (thus, 1245, 2451, 4512 and 5124 define the same polygon) and if the order is reversed (thus, the same polygon as above is also defined by 4 sequences 1542, 5421, 4215 and 2154). This implies that a_m has to be divided by $2m$ to give us the number b_m of distinct self-intersecting m-gons. Also, we do not consider degenerate polygons (such as points and segments) and polygons here. The computations show that $b_3 = 35$, $b_4 = 115$, $b_5 = 252$,

$b_6 = 420$ and $b_7 = 360$ and so the answer to the problem in this interpretation is $35 + 115 + 252 + 420 + 360 = \mathbf{1172}$.

PROBLEM 7 (90 pts) Find all numbers k among $1, 2, 3, 4, 5, 6, 7, 8$ such that $m^4 + k$ is not prime for any $m, |m| \neq 1$.

Solution: Consider a few cases depending on the value of k.
1. If $k = 1$ then $2^4 + 1 = 17$ is prime.
2. If $k = 2$ then $0^4 + 2 = 2$ is prime.
3. If $k = 3$ then $0^4 + 3 = 3$ is prime.
4. If $k = 4$ then

$$m^4+4 = [(m^2)^2+2\cdot m^2\cdot 2+2^2]-2\cdot m^2\cdot 2 = (m^2+2)^2-(2m)^2 = (m^2+2-2m)(m^2+2+2m)$$

and since $m \geq 2$ then $m^2 + 2 - 2m \geq 2$ and clearly $m^2 + 2 + 2m \geq 2$ which implies that $m^4 + 4$ is not prime.
5. If $k = 5$ then $0^4 + 5 = 5$ is prime.
6. If $k = 6$ then $5^4 + 6 = 631$ is prime.
7. If $k = 7$ then $0^4 + 7 = 23$ is prime.
8. If $k = 8$ then $3^4 + 8 = 89$ is prime.

So the answer is **4**.

PROBLEM 8 (120 pts) Find all natural numbers n such that $2 < n < 2009$ which have the following property: any number $1 < m < n$ which has no common divisors with n is prime (a prime number is a number p which has no divisors other than 1 and itself; 1 is not considered prime). Sum these numbers up. Write the answer below in the space provided.

Solution: Call numbers with the desired property *independent*. Also, two numbers without common divisors are called *coprime*. Consider an independent natural number n. Take a prime number p such that $p^2 < n$. Then such number p must be a divisor of n since otherwise n is not independent (p^2 and n are coprime and p^2 is not prime). Now, consider numbers from 3 to 6. Then 3 is independent (2 is the only number less than 3 coprime with 3, and 2 is prime), and 4 is independent (3 is the only number less than 4 coprime with 4, and 3 is prime). However, 5 is not independent (4 is coprime with 5 and 4 is not prime). Now, 6 is independent (the only number less than 6 and coprime with 6 is 5, and 5 is prime). Similarly, 7 is not independent, 8 is independent, and 9 is not independent.

Now, take any number greater than 9. If it is not a multiple of $6 = 2 \cdot 3$ then either 4 or 9 is coprime with it while not being prime. Hence only multiples of 6 can be independent. It is easy to check that $12, 18, 24$ and 30 are independent. Take any number greater than 30. If it is not divisible by 5 then it is not independent since then 25 and n are coprime even though 25 is not prime. Hence any independent $n > 30$ must be a multiple of $2 \cdot 3 \cdot 5 = 30$. Take a number $n \geq 50$. Then by the same arguments applied to n and 7 we see that to be independent n must be a multiple of $2 \cdot 3 \cdot 5 \cdot 7 = 210$. Since any such n is greater than 208, then the same arguments applied to n and 11 show that to be independent n must be a multiple of $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 = 2310$. Hence the list of all independent numbers less than or equal to 2009 is $3, 4, 6, 8, 12, 18, 24, 30$, and the sum of all these numbers is

$$3 + 4 + 6 + 8 + 12 + 18 + 24 + 30 = 105.$$

So the answer is **105**.