NAME:______________________________

GRADE:______________________________

SCHOOL NAME:______________________

2011-2012 UAB MATH TALENT SEARCH

This is a two hour contest. Answers are to be written in the spaces provided on the test sheet. There will be no credit if the answer is incorrect. You **MUST** justify your answers in order to get full credit; otherwise, partial credit or no credit will be awarded according to the decision made by the judges. Your work (including full justifications) should be shown on the extra paper which is attached. The problems are listed in increasing order of difficulty.

PROBLEM 1 (10 pts) On the side AD of a parallelogram ABCD a point P is chosen so that AP:AD=1:n. Let Q be the point of intersection of AC and BP. Express in terms of n the ratio AQ:AC.

YOUR ANSWER:

PROBLEM 2 (30 pts) A square floor is tiled with congruent square tiles. The tiles on the two diagonals are yellow. The rest of the tiles are white. If there are x yellow tiles, give the formulas for the overall number of tiles on the floor depending on whether x is odd or even.

YOUR ANSWER IF x IS ODD:

YOUR ANSWER IF x IS EVEN:

PROBLEM 3 (60 pts) A box is filled with big and small balls which are either green or yellow. It is known that among all balls the proportion of big balls is α ($0 \leq \alpha \leq 1$). It is also known that among the small balls the proportion of small yellow balls is β ($0 \leq \beta \leq 1$). Express in terms of α and β the proportion of small green balls among all balls.

YOUR ANSWER:

over, please

PROBLEM 4 (120 pts) An increasing sequence of natural numbers $b_1, b_2, \ldots$ is such that $b_{n+2} = b_n + b_{n+1}$ for all $n \geq 1$. It is known that $b_7 = 120$. Find b_8.

YOUR ANSWER:

PROBLEM 5 (210 pts) Consider all numbers $a_n = 19 \cdot 8^n + 17, n = 0, 1, \ldots$. For each such number let p_n be its smallest prime factor. What is the maximal value of p_n?

YOUR ANSWER:

PROBLEM 6 (300 pts) Find all triples of natural numbers $x \leq y \leq z$ such that the product of any two of them plus one is a multiple of the third one.

YOUR ANSWER:

2011-2012 UAB MTS: SOLUTIONS

PROBLEM 1 (10 pts) On the side AD of a parallelogram ABCD a point P is chosen so that AP:AD=1:n. Let Q be the point of intersection of AC and BP. Express in terms of n the ratio AQ:AC.

Solution: Draw through the point C a straight line parallel to BP. Denote the point of intersection of this line and the straight line containing AD by X. Then the triangle ABP equals the triangle CDX. Hence DX=AP and it follows that AP:AX=1:(n+1). On the other hand, the fact that CX is parallel to BP implies that AQ:AC=AP:AX. Therefore AQ:AC=1:(n+1).

So the answer is **1:(n+1)**. □

PROBLEM 2 (30 pts) A square floor is tiled with congruent square tiles. The tiles on the two diagonals are yellow. The rest of the tiles are white. If there are x yellow tiles, give the formulas for the overall number of tiles on the floor depending on whether x is odd or even.

Solution: Consider first the case when the side of the floor is $n \times n$ with $n = 2m$ being even. In this case two diagonals of the floor are disjoint, and each diagonal consists of n tiles. Hence overall there are $2n = x$ yellow tiles. This implies that $n = x/2$ and hence the overall number of tiles is $n^2 = x^2/4$. This formula applies if x is even.

Assume now that $n = 2m + 1$ is odd. Then there exists a unique central tile located at the geometric center of the floor. This tile is shared by both diagonals of the floor. Hence in this case $x = 4m + 1$ (each diagonal has two segments of m tiles each and the central tile). We conclude that in terms of x the length of the side of the floor can be expressed as $n = 2m+1 = (x+1)/2$. Hence the overall number of tiles in this case is given by $(x + 1)^2/4$.

So the answer is $\mathbf{(x + 1)^2/4}$ if x is odd and $\mathbf{x^2/4}$ if x is even. □

PROBLEM 3 (60 pts) A box is filled with big and small balls which are either green or yellow. It is known that among all balls the proportion of big balls is α $(0 \leq \alpha \leq 1)$. It is also known that among the small balls the proportion of small yellow balls is β $(0 \leq \beta \leq 1)$. Express in terms of α and β the proportion of small green balls among all balls.

Solution: Denote the number of balls of any specific kind by the pair of first letters of the properties of the balls. Thus, for example, bg is the number of big green balls, and the other numbers are by, sg and sy. Moreover, we may assume that all these numbers are expressed relative to the overall number of balls which implies that $bg+by+sg+sy=1$. We are given that $bg+by=\alpha$; hence $sg+sy=1-bg-by=1-\alpha$. We are also given that $\frac{sy}{sg+sy}=\beta$; hence $\frac{sg}{sg+sy}=1-\beta$.

Thus, the number of small green balls relative to the number of all balls is $(\mathbf{1}-\alpha)(\mathbf{1}-\beta)$. □

PROBLEM 4 (120 pts) An increasing sequence of natural numbers $b_1, b_2, \ldots$ is such that $b_{n+2}=b_n+b_{n+1}$ for all $n \geq 1$. It is known that $b_7=120$. Find b_8.

Solution: Set $b_1=x, b_2=y$. It is given that $x \leq y$. Moreover, $b_3=x+y$, $b_4=x+2y$, $b_5=2x+3y$, $b_6=3x+5y$, $b_7=5x+8y$, $b_8=8x+13y$. Thus, $b_7=5x+8y=120$. Since $5x=8(15-y)$, we see that x is a multiple of 8. Similarly, y is a multiple of 5. Set $y=5k$ and $x=8j$. Then $b_7=40j+40k=120$ and $j+k=3$. Also, x must be less than y. Hence $j=1$ and $k=2$ which implies that $x=8$ and $y=10$. We conclude that $b_8=8x+13y=194$.

So the answer is **194**. □

PROBLEM 5 (210 pts) Consider all numbers $a_n=19\cdot 8^n+17, n=0,1,\ldots$. For each such number let p_n be its smallest prime factor. What is the maximal value of p_n?

Solution: We will use the following standard notation: if a and b have the same remainder when divided by m we will write $d \equiv b \mod m$. It is easy to check that $x \equiv y \mod m$ is equivalent to the fact that $x-y \equiv 0 \mod m$, i.e. that $x-y$ is a multiple of m. Let us show that if $P(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$ is a polynomial with integer coefficients, then $x \equiv y \mod m$ implies that $P(x) \equiv P(y) \mod m$. Indeed, it is easy to check that $x^k-y^k=(x-y)Q_k(x,y)$ where $Q_k(x,y)=x^{k-1}+x^{k-2}y+x^{k-3}y^2+\cdots+y^{k-1}$ (e.g., $Q_1(x,y)=1, Q_2(x,y)=x+y, Q_3(x,y)=x^2+xy+y^2$ etc). Hence

$$P(x)-P(y)=a_n(x^n-y^n)+\cdots+a_1(x-y)=$$

$$= (x - y)[a_n Q_n(x, y) + \cdots + a_2 Q_2(x, y) + a_1 Q_1(x, y)]$$

which implies that if $x - y$ is a factor of $P(x) - P(y)$. If $x - y$ is a multiple of m, then so is $P(x) - P(y)$ as desired.

Now let us go back to our problem and consider several cases depending on the parity of n.

(1) Suppose that $n = 2k$ is even. Then $8^n \equiv (-1)^n \equiv 1 \mod 3$. Hence $a_n \equiv 19 \cdot 1 + 17 \equiv 0 \mod 3$. Hence in this case the minimal prime factor of a_n, denoted above by p_n, is less than of equal to 3.

(2) Suppose that $n = 2k + 1$. Then $a_n = 19 \cdot 8 \cdot 64^k + 17 = 152 \cdot 64^k$. Consider two subcases.

(2a) Suppose that $k = 2m + 1$. Then $a_n \equiv 152 \cdot (-1)^{2m+1} + 17 \mod 5 \equiv -135 \equiv 0 \mod 5$. Thus, in this case the number p_n is less than or equal to 5.

(2b) suppose that $k = 2m$. Then$a_n \equiv 152 \cdot (-1)^{2m} + 17 \mod 13 \equiv 169 \equiv 0 \mod 13$. Thus, in this case the number p_n is less than or equal to 13.

This analysis shows that $p_n \leq 13$ for any n. Now, $a_1 = 19 \cdot 8 + 17 = 169$ which implies that $p_1 = 13$. We conclude that the maximal value of p_n is assumed if $n = 1$ and is equal to 13.

So the answer is **13**. $\square$

PROBLEM 6 (300 pts) Find all triples of natural numbers $x \leq y \leq z$ such that the product of any two of them plus one is a multiple of the third one.

Solution: We are given that $xy + 1$ is a multiple of z, $xz + 1$ is a multiple of y and $yz + 1$ is a multiple of x. Hence $a = (xy + 1)(xz + 1)(yz + 1)$ is a multiple of xyz. Expanding parentheses we have that

$$a = x^2y^2z^2 + xyz^2 + xy^2z + yz + x^2yz + xz + xy + 1$$

is a multiple of xyz. If we now subtract $x^2y^2z^2 + xyz^2 + xy^2z + x^2yz$ from a, we will get $xy + xz + yz + 1$ which is still a multiple of xyz. Dividing both sides by xyz we see that the sum $s = s(x, y, z) = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{xyz} \geq 1$ is an integer.

This shows that numbers x, y, z cannot be all big because then s will be smaller than 1. Indeed, assume that $4 \leq x$ (recall that $x \leq y \leq z$). Then we have that $s(x, y, z) \leq s(4, 4, 4) = \frac{49}{64} < 1$, a contradiction. Hence either $x = 1$, or $x = 2$, or $x = 3$. Consider these three cases.

(1) $x = 1$. Then $xy + 1 = y + 1 = k_z z$ for some integer $k_z \geq 1$ so that $y + 1 \geq z$. Since we assume that $x \leq y \leq z$, we see that $y \leq z \leq y + 1$. Hence either $z = y$, $z = y + 1$. Suppose that $y = z$. Then since $y + 1$ is a multiple of $z = y$ we conclude that $y = z = 1$. Suppose that $z = y + 1$. Then since $z + 1 = y + 2$ is a multiple of y we conclude that $y = 1$ and $z = 2$. These yields two triples solving the problem: 1) $x = 1, y = 1, z = 1$, and 2) $x = 1, y = 1, z = $.

(1) $x = 2$. Then since $yz + 1$ is a multiple of $x =$ we have that y and z are odd. We have $s(2, 3, 3) = \frac{11}{9}$ is not an integer; $s(2, 3, 5) = \frac{16}{15}$ is not an integer; $s(2, 3, 7) = 1$ is an integer. To verify the conditions of the problem, we see that $2 \cdot 3 + 1 = 7$ is a multiple of 7, $2 \cdot 7 + 1 = 15$ is a multiple of 3 and $3 \cdot 7 + 1 = 22$ is a multiple of 2. Thus, $x = 2, y = 3, z = 7$ solves the problem.

Let us show that there are no other solutionis. Indeed, clearly $s(2, 3, 9) < 1$, hence we now need to consider the case when $x = 2$ and $y = 5$. Then we have $s(2, 5, 5) = \frac{23}{25} < 1$; since for all other choices of y and z the value of $s(x, y, z)$ only decreases and hence cannot be integer, this completes considering case (1).

(2) $x = 3$. Then $yz + 1$ must be a multiple of 3. Hence neither y nor z can be equal to 3. Moreover, $y = z = 4$ is impossible for the same reason. The first case which needs to be considered is when $y = 4, y = 5$ as then $yz + 1 = 4 \cdot 5 + 1 = 21$ is a multiple of 3. However in this case we have $s(3, 4, 5) = \frac{47}{60} < 1$. Since for all other choices of y and z the value of $s(x, y, z)$ only decreases and hence cannot be integer, this completes considering case (2).

So, the answers are as follows: 1)$\{\mathbf{x = 1, y = 1, z = 1}\}$, 2)$\{\mathbf{x = 1, y = 1, z = 2}\}$, 3)$\{\mathbf{x = 2, y = 3, z = 7}\}$. $\square$