NAME:________________________________

GRADE:______________________________

SCHOOL NAME:______________________

2013-2014 UAB MATH TALENT SEARCH

This is a two hour contest. Answers are to be written in the spaces provided on the test sheet. There will be no credit if the answer is incorrect. You **MUST** justify your answers in order to get full credit; otherwise, partial credit or no credit will be awarded according to the decision made by the judges. Your work (including full justifications) should be shown on the extra paper which is attached. The problems are listed in increasing order of difficulty.

PROBLEM 1 (30 pts) Find the greatest integer which is less than 2013 and has an odd number of positive integer divisors.

YOUR ANSWER:

PROBLEM 2 (50 pts) List all possible ways to represent 2013 as the difference of two squares of positive integers.

YOUR ANSWER:

PROBLEM 3 (70 pts) List all integers a such that the equation $x^4 - (a+4)x^2 + (2a+3) = 0$ has an integer solution.

YOUR ANSWER:

over, please

PROBLEM 4 (110 pts) Three judges are judging figure skating competition among 10 competitors. Each judge assigns each competitor a number of points from 1 through 10 so that the numbers of points assigned to different contestants are distinct. The competitor with the least number of points wins (it is known that the winner got less points than the others - that is, there were no ties for the first place). What is the maximal number of points the winner could receive?

YOUR ANSWER:

PROBLEM 5 (160 pts) Given a fraction $\frac{a}{b}$ with a, b being positive integers with no common divisors we consider a decimal fraction $\overline{b.a}$ (for example, for $\frac{3}{7}$ we consider 7.3 and for $\frac{165}{37}$ we consider 37.165). Find all fractions $\frac{a}{b}$ such that $\overline{b.a} = \frac{a}{b}$.

YOUR ANSWER:

PROBLEM 6 (240 pts) Ten houses, denoted $H_1, H_2, \ldots, H_{10}$, are standing along a straight road from the left to the right. The distances from H_1 (the leftmost house) to $H_2, H_3, \ldots, H_{10}$ are 1, 3, 6, 11, 21, 25, 30, 37, 39 miles. What location of the well along the same road will guarantee that the sum of distances of the houses to the well is minimal possible and what is that sum?

YOUR ANSWER:

2013-2014 UAB MTS: SOLUTIONS

PROBLEM 1 (30 pts) Find the greatest integer which is less than 2013 and has an odd number of positive integer divisors.

Solution: If a positive integer N is not a full square then N has an even number of positive integer divisors because to each divisor x we can associate a divisor $N/x \neq x$ and this way break down the family of all positive integer divisors of N into pairs. On the other hand, if a positive integer $N = m^2$ is a full square then in addition to the just described family of pairs we will have a number m in the list of all positive integer divisors of N (its counterpart N/m equals m and cannot be listed alongside m). Hence a positive integer N has an odd number of positive integer divisors if and only if N is a full square.

The answer is **1936** as $1936 = 44^2$ is the greatest full square which is less than 2013. □

PROBLEM 2 (50 pts) List all possible ways to represent 2013 as the difference of two squares of positive integers.

Solution: Suppose that $x^2 - y^2 = (x-y)(x+y) = 2013$. It is easy to see that the prime factorization of 2013 is $2013 = 3 \cdot 11 \cdot 61$. Since $x + y > x - y$ are integers, the possibilities for these numbers are:

(1) $x - y = 1, x + y = 2013$ which yields $x = 1007, y = 1006$;
(1) $x - y = 3, x + y = 11 \cdot 61 = 671$ which yields $x = 337, y = 334$;
(2) $x - y = 11, x + y = 3 \cdot 61 = 183$ which yields $x = 97, y = 86$;
(3) $x - y = 3 \cdot 11 = 33, x + y = 61$ which yields $x = 47, y = 14$.

The answer is as follows:
$\mathbf{x = 1007, y = 1006; x = 337, y = 334;}$
$\mathbf{x = 97, y = 86; x = 47, y = 14.}$ □

PROBLEM 3 (70 pts) List all integers a such that the equation $x^4 - (a+4)x^2 + (2a+3) = 0$ has an integer solution.

Solution: According to the quadratic formula,

$$x^2 = \frac{a + 4 \pm \sqrt{a^2+4}}{2};$$

since there exists an integer solution of our equation, this implies that $a^2 + 4$ is a full square, and it follows from the fact that a is an integer that $a = 0$ (otherwise it is easy to verify that $a^2 + 4$ cannot be a full square). On the other hand, it is easy to check that for $a = 0$ our equation $x^4 - 4x^2 + 3 = 0$ has integer solutions 1 and -1.

The answer is **0**. □

PROBLEM 4 (110 pts) Three judges are judging figure skating competition among 10 competitors. Each judge assigns each competitor a number of points from 1 through 10 so that the numbers of points assigned to different contestants are distinct. The competitor with the least number of points wins (it is known that the winner got less points than the others - that is, there were no ties for the first place). What is the maximal number of points the winner could receive?

Solution: Let us show that the number of points of the winner cannot be greater than 15. Indeed, let us count the overall number of points at this competition. On the one hand, it equals $3 \cdot (1 + 2 + \cdots + 10) = 3 \cdot 55 = 165$. On the other hand, it can be obtained by adding up the points of all competitors. If the winner received the number of points 16 or more, then by the assumption each other competitor received at least 17 points. Thus, the overall sum will have to be at least $16 + 9 \cdot 17 = 169$, a contradiction. Hence the winner's number of points is at most 15 .

Let us show that it is possible to have the distribution of points such that the winner will get 15 points and all others will get 16 or more. Assuming that contestants are listed under numbers $1, 2, \ldots, 10$ here is an example of how it could happen:

$1, 2, 3, 4, 5, 6, 7, 8, 9, 10$ are assigned by the first judge,

$10, 8, 9, 3, 5, 1, 2, 7, 4, 6$ are assigned by the second judge,

$6, 7, 4, 9, 5, 10, 8, 2, 3, 1$ are assigned by the third judge.

It is easy to see that the fifth contestant received 15 points $(5+5+5)$ while all other contestants received 16 or 17 points.

The answer is **15**. □

PROBLEM 5 (160 pts) Given a fraction $\frac{a}{b}$ with a, b being positive integers with no common divisors we consider a decimal fraction $\overline{b.a}$ (for example, for $\frac{3}{7}$ we consider 7.3 and for $\frac{165}{37}$ we consider 37.165). Find all fractions $\frac{a}{b}$ such that $\overline{b.a} = \frac{a}{b}$.

Solution: Assume that a has n digits. Then we have that

$$\overline{b.a} = \frac{a}{b} \Leftrightarrow \frac{a}{b} = b + a \cdot 10^{-n} \Leftrightarrow 10^n(a - b^2) = a \cdot b$$

which implies that $a > b$. Since a and b have no common divisors, $a - b^2$ has no common divisors with $a \cdot b$. Hence the above implies that $a - b^2 = 1$ and $a \cdot b = 10^n$. Since a and b have no common divisors and $a > b$, we have that either (1) $a = 5^n, b = 2^n$, or (2) $a = 10^n$ and $b = 1$. Clearly, (2) fails, and (1) implies that $n = 1$ and thus $a = 5$ and $b = 2$.

The answer is $\mathbf{\frac{5}{2}}$. □

PROBLEM 6 (240 pts) Ten houses, denoted $H_1, H_2, \ldots, H_{10}$, are standing along a straight road from the left to the right. The distances from H_1 (the leftmost house) to $H_2, H_3, \ldots, H_{10}$ are 1, 3, 6, 11, 21, $25, 30, 37, 39$ miles. What location of the well on the plane will guarantee that the sum of distances of the houses to the well is minimal possible and what is that sum?

Solution: Let us solve the problem in general assuming that there are N houses overall. We view houses and points along a real line R. Denote by I_j the segment between H_j and H_{j+1} and denote the length of I_j by l_j. For each point W on the plane let $d(W)$ be the sum of distances between W and the houses. We need to minimize $d(W)$.

It is easy to see that the optimal W must be located on R because otherwise we can project our point to R and find a new location better than the previous one (as a result of this transformation each distance involved in $d(W)$ only decreases). Similar argument shows that optimal locations of W must be between H_1 and H_N.

Now, choose a point $W \in I_j$ which is t miles away from H_j (and hence $s = l_j - t$ miles away from H_{j+1}). Computing the sum of distances from W to all houses, we break it down to two sums: the "left" sum where we sum up the distances from W to $H_1, \ldots, H_j$ and the "right" sum where we sum up the distances from W to $H_{j+1}, \ldots, H_N$.

It follows that in the left sum we include the same constant summand which consists of the sum of distances from H_j to H_{j-1}, to H_{j_2}, …, to H_1. In addition to that we include into the left sum the number $j \cdot t$ because in the left sum the distance from T to H_j is counted j times. Similarly, in the right sum there is constant summand plus $(N - j) \cdot s$.

Hence $d(W)$ equals a constant summand plus $jt + s(N-j)$. Assume that $j < N-j$ (i.e., $2j < N$). Then it follows that

$$jt + s(N-j) = jt + (l_j - t)(N-j) = l_j(N-j) + t(2j-N)$$

which shows that the minimal value $d(W)$ will be assumed if t takes its maximal (under the assumptions) value l_j. In other words, if $j < \frac{N}{2}$ then to decrease $d(T)$ we should move W to H_{j+1}. Applying the same arguments now on the interval I_{j+1} we see that if $j+1$ is still less than $\frac{N}{2}$ we should continue moving W to the left, i.e. *closer to the **median** place* on the entire interval $[H_1, H_N]$, i.e. such place that there is the same number of houses to the left and to the right of that place. The same conclusion can be made if we consider houses H_j such that $j > \frac{N}{2}$ (or, equivalently, $N - j < \frac{N}{2}$).

Clearly, if $N = 2k+1$ is odd then there is only one median place in this setting, namely the house H_{k+1}. If $N = 2k$ then there is the whole interval $I_k = [H_k, H_{k+1}]$ consisting of points such that there is equal number of houses to the left and to the right of them. These are the optimal locations of the point W minimizing $d(W)$. IN the setting of the problem $N = 10$ and choosing a point W anywhere in the interval $[H_5, H_6]$ (e.g. at H_5) we see that the minimal value of $d(W)$ is

$$(5+8+10+11) + (10+14+19+26+28) = 131\,\text{miles}.$$

The answer is **131** miles: the optimal location of the well is between the fifth and the sixth house. $\square$