NAME:______________________________

GRADE:______________________________

SCHOOL NAME:________________________

2016-2017 UAB MATH TALENT SEARCH

This is a two hour contest. Answers are to be written in the spaces provided on the test sheet. **There will be no credit if the answer is incorrect.** Full credit will be awarded for a correct answer with complete justification. At most half credit will be given for a correct answer without complete justification. Your work (including full justification) should be shown on the extra paper which is attached. Please clearly indicate which problems you are solving. Problems are included on pages 2 and 3.

PROBLEM 1 (20 pts) Numbers $c \neq 0, d \neq 0$ are given. Moreover, it is known that $\frac{x/c}{d} = \frac{c}{x/d}$. Express x in terms of c and d.

YOUR ANSWER:

PROBLEM 2 (30 pts) An acute isosceles triangle ABC is inscribed in a circle (here $|AB| = |AC|$). At points B and C tangent lines to the circle are drawn, meeting at point D. Its given that $\angle ABC = \angle ACB = 10\angle BDC$. Find $\angle BDC$.

YOUR ANSWER:

PROBLEM 3 (40 pts) Solve the puzzle

$$AAAA - BBB + CC - K = 1234$$

where A, B, C, K are digits (possibly including 0) and $AAAA, BBB, CC$ and K are, respectively, 4-, 3-, 2- and 1-digit numbers, and distinct letters correspond to distinct digits.

YOUR ANSWER:

over, please

PROBLEM 4 (50 pts) Let x be a natural number. The following five claims are given:

(1) $2x > 70$;
(2) $x < 100$;
(3) $3x > 25$;
(4) $x \geq 10$;
(5) $x > 5$.

It is known that among these claims there are three true ones and two false ones. What is x equal to?

YOUR ANSWER:

PROBLEM 5 (70 pts) There are two wise men named Sam and Paul. Someone chose three natural (positive integer) numbers a, b, c and informed Sam of their sum and Paul of their product. Then Sam said to Paul:
"I am wise! If I knew that your number is greater than mine I would have named all the three numbers!"
and Paul responded:
"I am wiser than you are! I have just been told that my number is less than yours, and I can say that the numbers are so, so and so!"
What are the numbers a, b and c equal to? Give the answer assuming that $a \geq b \geq c$.

YOUR ANSWER:

PROBLEM 6 (110 pts) For how many integers $N, 1 \leq N \leq 2016$, are the numbers $N^2 - 2$ and $N + 5$ not coprime?

YOUR ANSWER:

PROBLEM 7 (150 pts) Let $S_n = a^n + b^n$ where $a = [7 + \sqrt{41}], b = [7 - \sqrt{41}]$, and $n = 0, 1, 2, \ldots$. What is the unit digit of S_{2016}?

YOUR ANSWER:

2016-2017 UAB MTS: SOLUTIONS

PROBLEM 1 (20 pts) Numbers $c \neq 0, d \neq 0$ are given. Moreover, it is known that $\frac{x/c}{d} = \frac{c}{x/d}$. Express x in terms of c and d.

Solution: After algebraic simplifications, we see that $x^2 = c^2d^2$ that implies that $x = \pm cd$.

The answer is $\mathbf{x} = \mathbf{cd}$ and $\mathbf{x} = -\mathbf{cd}$. $\square$

PROBLEM 2 (30 pts) An acute isosceles triangle ABC is inscribed in a circle (here $|AB| = |AC|$). At points B and C tangent lines to the circle are drawn, meeting at point D. Its given that $\angle ABC = \angle ACB = 10\angle BDC$. Find $\angle BDC$.

Solution: Denote by H the center of the circle; also, set $\angle BDC = \theta$. Then we have that $\angle HCD = \angle HBD = \pi/2$. Hence $\theta + \angle BHC = \pi$. Therefore $\angle BHC = \pi - \theta$. On the other hand, by the assumptions $\angle ACB = \angle ABC = 10\theta$ which implies that $\angle BAC = \pi - 20\theta$. Now, properties of angles inscribed in circles imply that in fact $\angle BHC = 2\angle BAC = 2(\pi - 20\theta) = 2\pi - 40\theta$. We conclude that $\pi - \theta = 2\pi - 40\theta$. Therefore $\theta = \frac{\pi}{39}$.

The answer is $\frac{\pi}{\mathbf{39}}$. $\square$

PROBLEM 3 (40 pts) Solve the puzzle

$$AAAA - BBB + CC - K = 1234$$

where A, B, C, K are digits (possibly including 0) and $AAAA, BBB, CC$ and K are, respectively, 4-, 3-, 2- and 1-digit numbers, and distinct letters correspond to distinct digits.

Solution:

Clearly, $AAAA > 1234$. Hence $A > 1$. On the other hand, $AAAA - 1234 = BBB - CC + K \leq 999 + 8 = 1007$ which implies that $AAAA \leq 1234 + 1007 = 2241$. Hence $A = 2$ and the problem becomes to solve an equation $2222 - BBB + CC - K = 1234$ or $988 = BBB - CC + K$. Since $BBB - CC + K < BBB$, we see that $988 < BBB$ and that $B = 9$. This yields an equation $988 = 999 - CC + K$ or $11 = CC - K$. It follows that $C = 1$ and $K = 0$.

The answer is $\mathbf{2222 - 999 + 11 - 0 = 1234}$. $\square$

PROBLEM 4 (50 pts) Let x be a natural number. The following five claims are given:

(1) $2x > 70$;
(2) $x < 100$;
(3) $3x > 25$;
(4) $x \geq 10$;
(5) $x > 5$.

It is known that among these claims there are three true ones and two false ones. What is x equal to?

Solution: Consider various cases. Observe that $2x > 70$ implies that $x \geq 10$, that in turn implies that $3x > 25$, that in turn implies that $x > 5$. Hence if $2x > 70$ then at least 4 claims hold, a contradiction. Thus, $2x > 70$ is false and $x \leq 35$. This implies that $x < 100$ is true. Therefore of the three remaining inequalities $3x > 25$, $x \geq 10$ and $x > 5$ two must be true and one must be false. By the above, if $x \geq 10$ holds, then $3x > 25$ and $x > 5$ hold as well. Therefore in that case all three inequalities are true, a contradiction. It follows that $x \geq 10$ is false (and hence $x < 10$) while $3x > 25$ and $x > 5$ is true. This leaves only the option $x = 9$.

The answer is $\mathbf{x = 9}$. $\square$

PROBLEM 5 (70 pts) There are two wise men named Sam and Paul. Someone chose three natural (positive integer) numbers a, b, c and informed Sam of their sum and Paul of their product. Then Sam said to Paul:
"I am wise! If I knew that your number is greater than mine I would have named all the three numbers!"
and Paul responded:
"I am wiser than you are! I have just been told that my number is less than yours, and I can say that the numbers are so, so and so!"
What are the numbers a, b and c equal to? Give the answer assuming that $a \geq b \geq c$.

Solution: Set $s = a + b + c$ and $p = abc$. Then Sam knows the value of s and Paul knows the value of p. The claim made by Sam means that if one knows the value of s and knows also that $p > s$ then one can find a, b and c in a unique fashion. What can s then be equal to? Clearly, the least value of $s = a + b + c$ is 3. However then $a = b = c = 1$ and their product $abc = 1$ can never by greater than 3. Hence $s \neq 3$. Similarly, $s \neq 4$ (as then $a = b = 1, c = 2$, and $abc = 2 < a+b+c = s = 4$). Suppose that $s = 5$; then the possibilities are $2, 2, 1$ or $3, 1, 1$ that yield $p = 2, p = 3$, a contradiction with the assumption according to which there is a possibility for p to be greater than s.

Now, suppose that $s = 6$. Moreover, suppose that in addition to that it is known that $p > s$. Does this completely determine a, b and c? Well, assuming that $a \geq b \geq c$ we see that the possibilities for a, b and c are as follows.

(i) $a = 4, b = c = 1$; then $p = 4 < s$, so these a, b, c are not the ones Sam has in mind.

(ii) $a = 3, b = 2, c = 1$; then $p = 6 \leq s$, so these a, b, c are not the ones Sam has in mind.

(iii) $a = b = c = 2$; then $p = 8 > s$. Since these a, b, c are the unique combination satisfying the desired conditions, it follows that $s = 6$ and $a = b = c$ are the numbers that Sam meant.

We claim that this is the only possibility for s. Indeed, suppose that $s \geq 7$. Then there are two distinct possibilities in each of which we will have $p > s$. Namely, we can have numbers $a = s - 4, b = c = 2$ and then $p = 2 \cdot 2 \cdot (s - 4) > s$ (the inequality is easy to verify based upon the fact that $s \geq 7$), or the numbers $s - 5, 3, 2$ implying again that $p = 2 \cdot 3 \cdot (s - 5) > s$ (again, easy to verify based upon the fact that $s \geq 7$). Thus, knowing the value of $s \geq 7$ and the fact that $p > s$ does not allow for a unique collection of numbers a, b, c satisfying the listed conditions. We conclude that $s = 6$ and the possibilities for the values of a, b, c are the ones listed above under (i), (ii) and (iii).

Since according to Paul the true inequality satisfied by s and p is that $p < s$, the only case from the listed three cases is (i).

The answer is $\mathbf{a = 4, b = 1, c = 1}$. $\square$

PROBLEM 6 (110 pts) For how many integers $N, 1 \leq N \leq 2016$, are the numbers $N^2 - 2$ and $N + 5$ not coprime?

Solution: Denote by d the greatest common divisor of numbers $N^2 - 2$ and $N + 5$. Then the number

$$(N^2 - 2) - (N + 5)(N - 5) = (N^2 - 2) - (N^2 - 25) = 23$$

is also divisible by d. It follows that $N^2 - 2$ and $N + 5$ are not coprime if and only if $N + 5$ is divisible by 23 in which case $N^2 - 2$ and $N + 5$ have a common divisor 23. Thus, the desired numbers are $N = 23 \cdot 1 - 5, 23 \cdot 2 - 5, \ldots, 23 \cdot 87 - 5 = 1996$; this is the entire list as $23 \cdot 88 - 5 = 2019 > 2016$.

The answer is **87**. □

PROBLEM 7 (150 pts) Let $S_n = a^n + b^n$ where $a = [7 + \sqrt{41}], b = [7 - \sqrt{41}]$, and $n = 0, 1, 2, \ldots$. What is the unit digit of S_{2016}?

Solution: Let us begin by expressing $S_n + 1$ through S_n and S_{n-1}. To this end observe that

$$(a^n + b^n)(a + b) = a^{n+1} + b^{n+1} + ab(a^{n-1} + b^{n-1})$$

which shows that

$$a^{n+1} + b^{n+1} = (a^n + b^n)(a + b) - ab(a^{n-1} + b^{n_1};$$

in our particular case this implies that

$$S_{n+1} = 14S_n - 8S_{n-1}$$

(observe that $a + b = 14$ and $ab = 49 - 41 = 8$).

Since we are only concerned about the last digit i_n of S_n, we may list last digits of numbers $S_0, S_1, S_2, \ldots$; the list looks as follows:
$i_0 = 2$;
$i_1 = 4$;
$i_2 = 0$;
$i_3 = 8$;
$i_4 = 2$;
$i_5 = 4$;
after which the sequence repeats itself. It follows, that last digits of all numbers S_n such that n is a multiple of 4, are all equal to 2. In particular, the last digit of S_{2016} is 2.

The answer is **2**. □