NAME:______________________________

GRADE:______________________________

SCHOOL NAME:______________________

2017-2018 UAB MATH TALENT SEARCH

This is a two hour contest. **There will be no credit if the answer is incorrect.** Full credit will be awarded for a correct answer with complete justification. At most half credit will be given for a correct answer without complete justification. Your work (including full justification) should be shown on the extra paper which is attached.

PROBLEM 1 (20 pts) You toss a coin 3 times. Find the probability that: (a) you get three heads, (b) that you get either three heads or three tails.

YOUR ANSWER:

PROBLEM 2 (30 pts) The island Fiveoclock is inhabited by Lemons (who always lie) and Teapots (who always tell the truth). A traveler to Fiveoclock meets a group of 5 people (A, B, C, D, E). A says: exactly three of us are Teapots; B says: exactly four of us are Teapots; C says: exactly one of us is a Teapot; D says: exactly two of us are Teapots; E says: all five of us are Teapots. Who of them are Teapots?

YOUR ANSWER:

PROBLEM 3 (40 pts) Find all fractions a/b with a and b coprime (i.e, without common factors greater than 1) such that the fraction doubles if we increase both top and bottom of it by 10.

YOUR ANSWER:

PROBLEM 4 (50 pts) A 179×57 rectangle is divided into 1×1 squares. If we draw a diagonal in this rectangle, how many squares will it intersect?

YOUR ANSWER:

over, please

PROBLEM 5 (70pts) It is given that each elephant drinks the same amount of water per day. On January 1, 1900, 183 elephants came to a lake and drank all its water. After that it started raining and by December 31, 1900, the lake was full again. On January 1, 1901, 37 elephants came and drank some water from the lake. When Merlin saw it, he decided to add some (the same) amount of water to the lake every night. The 37 elephants kept coming, Merlin kept adding water, and by the end of January 5, 1901, the lake was empty. Then the rain started, and by December 31, 1901, the lake was full. On January 1, 1902 one elephant came to the lake. How long will it take the elephant to drink all the water from the lake? Merlin will keep adding the same amount of water as before every night, but will it help the lake?

YOUR ANSWER:

PROBLEM 6 (110 pts) An isosceles right triangle ABC with right angle C is given. A rectangle MNKL is inscribed in ABC so that vertices M, N belong to AB (MN is a side of MNKL), K belongs to BC, and L belongs AC. It is given that $|AB| = 45$ and that $\frac{|MN|}{|NK|} = \frac{5}{2}$. Find the lengths of the sides of MNKL.

YOUR ANSWER:

PROBLEM 7 (150 pts) In a triangle ABC the median BK, the angle bisector BE, and the height AD are drawn. It is known that BK and BE divide AD in three equal segments and that $|AB| = 4$. Find the length of AC. **Hint:** the height AD is located outside the triangle ABC.

YOUR ANSWER:

2017-2018 UAB MTS: SOLUTIONS

PROBLEM 1 (20 pts) You toss a coin 3 times. Find the probability that: (a) you get three heads, (b) that you get the same result in all three tosses (i.e., either three heads or three tails.)

Solution: There are two options (heads or tails) for each tossing. Hence overall there are 8 possible outcomes. Three heads is one of them. Hence the answer to (a) is $\frac{1}{8}$. In the case (b) out of 8 outcomes only 2 are good for us. Hence the answer to (b) is $\frac{1}{4}$.

The answers are: $(a)\mathbf{\frac{1}{8}}$ and $(b)\mathbf{\frac{1}{4}}$. □

PROBLEM 2 (30 pts) The island Fiveoclock is inhabited by Lemons (who always lie) and Teapots (who always tell the truth). A traveler to Fiveoclock meets a group of 5 people (A, B, C, D, E). A says: exactly three of us are Teapots; B says: exactly four of us are Teapots; C says: exactly one of us is a Teapot; D says: exactly two of us are Teapots; E says: all five of us are Teapots. Who of them are Teapots?

Solution: Notice that these five people contradict each other. Hence at most one of them tells the truth. Therefore the only possibility is that C is a Teapot and all others are Lemons.

The answer is: **C is a Teapot, all others are Lemons.** □

PROBLEM 3 (40 pts) Find all fractions a/b with a and b coprime (i.e, without common factors greater than 1) such that the fraction doubles if we increase both top and bottom of it by 10.

Solution: By the assumptions,

$$\frac{2a}{b} = \frac{a+10}{b+10}$$

which implies (after simplifications) that

$$a(b+20) = 10b.$$

Since a and b are coprime, a must be a divisor of 10; thus, $a = 1, 2, 5, 10$ are all the possibilities. If $a = 1$ we get $b + 20 = 10b$ which does not integer solutions. Similarly, if $a = 2$ then $b = 5$; if $a = 5$ then $b = 20$, however a and b must be coprime, so in this case there are no solutions;

finally, if $a = 10$ then we get $b + 20 = b$ that has no solutions.

The answer is $\frac{\mathbf{2}}{\mathbf{5}}$ of $\mathbf{a = 2, b = 5}$. □

PROBLEM 4 (50 pts) A 179×57 rectangle is divided into 1×1 squares. If we draw a diagonal in this rectangle, how many squares will it intersect?

Solution: It is easy to check that $57 = 3 \cdot 19$ and 179 is a prime number. Therefore they are coprime. It follows that diagonal in question does not pass through any vertices of the grid until it hits the opposite vertex of the rectangle. Hence there are $1 + 178 + 56 + 1 = 236$ points of intersection of the diagonal and lines of the grid in the rectangle, including the initial and the terminal points. Hence there are 235 segments into which these points partition the diagonal. Since each segment corresponds to the intersection of the diagonal and one square, we see that there are 235 squares intersected by the diagonal.
The answer is **235.** □

PROBLEM 5 (70 pts) It is given that each elephant drinks the same amount of water per day. On January 1, 1900, 183 elephants came to a lake and drank all its water. After that it started raining and by December 31, 1900, the lake was full again. On January 1, 1901, 37 elephants came and drank some water from the lake. When Merlin saw it, he decided to add some (the same) amount of water to the lake every night. The 37 elephants kept coming, Merlin kept adding water, and by the end of January 5, 1901, the lake was empty. Then the rain started, and by December 31, 1901, the lake was full. On January 1, 1902 one elephant came to the lake. How long will it take the elephant to drink all the water from the lake? Merlin will keep adding the same amount of water as before every night, but will it help the lake?

Solution: Denote by L the amount of water in the lake. Then it follows from the first assumption that that one elephant drinks $L \cdot \frac{l}{183}$ a day. On the other hand the second assumptions translates into the following equation:

$$5 \cdot 37 \cdot L \cdot \frac{1}{183} = L + 4M$$

where M is the amount of water added to the lake by Merlin every night. Solving this for M we see that $M = \frac{L}{366}$.

Now consider the case of one elephant. Suppose that one elephant needs E days to empty the lake (this is with the assumption that every night Merlin will add $M = \frac{L}{366}$ gallons of water every night. Then the equation for E is as follows:

$$E \cdot \frac{L}{183} = L + (E-1) \cdot \frac{L}{366}$$

that easily yields that $E = 365$.

The answer is **365 days.** □

PROBLEM 6 (110 pts) An isosceles right triangle ABC with right angle C is given. A rectangle MNKL is inscribed in ABC so that vertices M, N belong to AB (MN is a side of MNKL), K belongs to BC, and L belongs AC. It is given that $|AB| = 45$ and that $\frac{|MN|}{|NK|} = \frac{5}{2}$. Find the lengths of the sides of MNKL.

Solution: Assume that on the hypotenuse AB the vertices are ordered as follows: A, M, N, B. Consider the triangle BNK. Clearly, this is a right isosceles triangle. Hence $|BN| = |NK|$. Similarly, $|ML| = |AM|$. By the assumptions we may assume that $|NK| = |LM| = 2x$ while $|MN| = 5x$ for some x. Hence $|LM| + |MN| + |NK| = 9x = 45$ and $x = 5$. Thus the sides of $MNKL$ are $2 \cdot 5 = 10$ and $5 \cdot 5 = 25$.

The answer is $|\mathbf{MN}| = \mathbf{25}, |\mathbf{NK}| = \mathbf{10}$. □

PROBLEM 7 (150 pts) In a triangle ABC the median BK, the angle bisector BE, and the height AD are drawn. It is known that BK and BE divide AD in three equal segments and that $|AB| = 4$. Find the length of AC. **Hint:** the height AD is located outside the triangle ABC.

Solution: It is well-known that bisectrix of a triangle divides the side to which it is drawn proportionally to the sides forming the angle. Denote the point of intersection of BK with AD by X; denote the point of intersection of BE with AD by Y. Since $|BD| < |AB|$ (in a right triangle the hypotenuse is longer than the other sides), we see that the order in which points X and Y are located on AD is as follows:A, X, Y, C. Therefore the order in which K and E are located on AC is as follows: A, K, E, C. It is given that $\frac{|AY|}{|YD|} = 2$. Hence $\frac{|AB|}{|BD|} = 2$ and so $|BD| = 2$. Since ABD is a right triangle, we can use Pythagorean Theorem which shows that $|AD| = 2\sqrt{3}$. Observe that AD in fact must be contained outside the triangle ABC because otherwise BK cannot divide AC in half, still to simplify solving the problem we include this observation as a hint.

Now, consider the triangle ACY. The segment KY connects midpoints of AC and AY, hence it is parallel to the segment YC. Since Y is the midpoint of XD, it follows that C must be the midpoint of BD. Hence $|CD| = 1$. Since AC is the hypothenuse of the right triangle ACD and the two sides of this triangle are CD with $|CD| = 1$ and AD with $|AD| = 2\sqrt{3}$, we conclude that $|AC| = \sqrt{1+12} = \sqrt{13}$.

The answer is $\boldsymbol{\sqrt{13}}$. $\square$