NAME:__________________________________

GRADE:_________________________________

SCHOOL NAME:_______________________

2018-2019 UAB MATH TALENT SEARCH

This is a two hour contest. **There will be no credit if the answer is incorrect.** Full credit will be awarded for a correct answer with complete justification. At most half credit will be given for a correct answer without complete justification. Your work (including full justification) should be shown on the extra paper which is attached.

PROBLEM 1 The first Tuesday of a month David spent in Auburn. The first Tuesday after the first Monday of this month he spent in Birmingham. In the next month he spent the first Tuesday in Muscle Shoals, and the first Tuesday after the first Monday in Cullman. In what month was he visiting Auburn?

YOUR ANSWER:

PROBLEM 2 50 boxers participate in a boxing tournament. After each fight the loser leaves the tournament. The winner is the boxer who has not lost any fights. What is the least number of fights necessary to determine the winner? what is the greatest number of fights necessary to determine the winner?

YOUR ANSWER: The least number is ; the greatest number is

PROBLEM 3 The Digitland has islands $0, 1, \ldots, N$ connected with bridges so that you can reach any island from any island. A tourist walked through all islands. It is given that she walked over each bridge exactly once. It is also known that she visited island 3 three times, did not start from it and did not end her journey on this island either. How many bridges connect island 3 with other islands?

YOUR ANSWER:

over, please

PROBLEM 4 Consider a figure which consists of three squares A, B and C on the plane, attached to each other as follows: A is located under B so that the top side of A coincides with the bottom side of B, and C is located to the right of B so that the right side of B coincides with the left side of C. Cut this figure into 4 congruent and connected (i.e., consisting of one piece) figures. You are allowed to make cuts each of which follows a connected broken line.

YOUR ANSWER: the picture below shows a correct way to do this:

PROBLEM 5 An absent minded professor X lives in a city where phone numbers consist of 7 digits each. Professor X can only remember symmetric phone numbers (so-called *palindromes*) that read the same both from left to right and from right to left (e.g., 1234321). What is a probability of the fact that X will be able to remember a randomly chosen phone number?

YOUR ANSWER:

PROBLEM 6 The number 123456789 is given. The following action can be performed with its digits: two adjacent digits are chosen, such that neither one is equal to 0, then both digits are decreased by 1, and then they are swapped. What is the least number one can obtain out of 123456789 after several actions like that?

YOUR ANSWER:

PROBLEM 7 Consider the function defined on the interval $[0, 3]$ by the following split formula:

$$f(x) = \begin{cases} \text{f(x)=3x} & \text{if } 0 \leq x \leq 1 \\ \text{f(x)=-3x+6} & \text{if } 1 < x \leq 2 \\ \text{f(x)=3x-6} & \text{if } 2 < x \leq 3 \end{cases}$$

How many solutions does the equation $f(f(x)) = x$ have?

YOUR ANSWER:

2018-2019 UAB MTS: SOLUTIONS

PROBLEM 1 The first Tuesday of a month David spent in Auburn. The first Tuesday after the first Monday of this month he spent in Birmingham. In the next month he spent the first Tuesday in Muscle Shoals, and the first Tuesday after the first Monday in Cullman. Can you decide on what dates was he in all these places?

Solution: A month can have two distinct days for the first Tuesday and for the first Tuesday after the first Monday only if its first day is Tuesday. Hence the two months discussed in the problem must begin on Tuesday. Since these are consecutive months, it follows that the earlier of these months must have a number of days which is a multiple of 7. Thus, this is the month of February.

The answer is: **February.** □

PROBLEM 2 50 boxers participate in a boxing tournament. After each fight the loser leaves the tournament. The winner is the boxer who has not lost any fights. What is the least number of fights necessary to determine the winner? what is the greatest number of fights necessary to determine the winner?

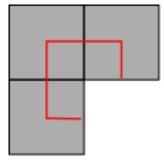
Solution: After each fight the number of participants decreases by 1. In the end there is one winner. Hence in any case there will be exactly 49 fights before the winner is determined.

The answer is: **49.** □

PROBLEM 3 The Digitland has islands $0, 1, \ldots, N$ connected with bridges so that you can reach any island from any island. A tourist walked through all islands. It is given that she walked over each bridge exactly once. It is also known that she visited island 3 three times, did not start from it and did not end her journey on this island either. How many bridges connect island 3 with other islands?

Solution: Since the tourist had to come and leave the island 3 walking over distinct bridges, there must be 6 bridges connecting island 3 with other islands.

The answer is: **6 bridges.** □



PROBLEM 4 Consider a figure which consists of three squares A, B and C on the plane, attached to each other as follows: A is located under B so that the top side of A coincides with the bottom side of B, and C is located to the right of B so that the right side of B coincides with the left side of C. Cut this figure into 4 equal figures.

Solution: See the figure above:

PROBLEM 5 An absent minded professor X lives in a city where phone numbers consist of 7 digits each. Professor X can only remember symmetric phone numbers (so-called *palindromes*) that read the same both from left to right and from right to left (e.g., 1234321). What is the probability of the fact that X will be able to remember a randomly chosen phone number?

Solution: For each first 4 digits there are 1000 phon numbers with them and only one of them is a palindrome. Therefore the desired probability is $\frac{1}{1000}$.

The answer is $\mathbf{0.001}$. □

PROBLEM 6 The number 123456789 is given. The following action can be performed with its digits: two adjacent digits are chosen, such that neither one is equal to 0, then both digits are decreased by 1, and then they are swapped. What is the least number one can obtain out of 123456789 after several actions like that?

Solution: It is easy to see that after each action the parity of a digit standing at a certain place does not change. Hence *any* number that we get in the end of any sequence of actions will look as follows:

(odd)(even)(odd)(even)(odd)(even)(odd)(even)(odd)

and, therefore, the desired minimal number is greater than or equal to 101010101. To obtain the number 101010101, we need to apply the action described in the problem twice to digits 2 and 3, four times to digits 4 and 5, six times to the digits 6 and 7, and, finally, eight times to the digits 8 and 9.

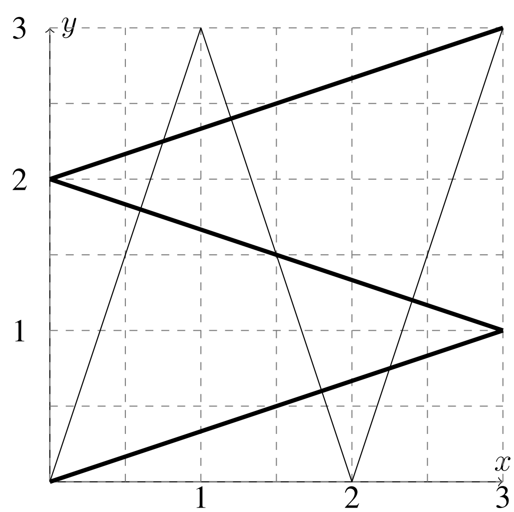
The answer is $\mathbf{101010101}$. □

PROBLEM 7 Consider the function defined on the interval $[0,3]$ by the following split formula:

$$f(x) = \begin{cases} \text{f(x)=3x} & \text{if } 0 \leq x \leq 1 \\ \text{f(x)=-3x+6} & \text{if } 1 < x \leq 2 \\ \text{f(x)=3x-6} & \text{if } 2 < x \leq 3 \end{cases}$$

How many solutions does the equation $f(f(x)) = x$ have?

Solution: Suppose that t solves the equation $f(f(x)) = x$. Set $f(t) = y$. Then $f(t) = y$ and $f(y) = t$. It follows that every t that solves our equation is the x-coordinate of a point of intersection of the graphs $y = f(x)$ and $x = f(y)$ (namely, the point $(t, f(t))$). The graphs can be easily sketched yielding the answer 9 (see figure below).



The answer is **9**. $\square$