NAME: ____________________________
GRADE: __________________________
SCHOOL NAME: ____________________

2019-2020 UAB MATH TALENT SEARCH

This is a two hour contest. **There will be no credit if the answer is incorrect.** Full credit will be awarded for a correct answer with complete justification. At most half credit will be given for a correct answer without complete justification. Your work (including full justification) should be shown on the extra paper which is attached.

**PROBLEM 1** (10 pts) Two positive integers $a$ and $b$ are such that $a + b = \frac{a}{b} + \frac{b}{a}$. What is the value of $a^2 + b^2$?

*YOUR ANSWER: ____________________________*

**PROBLEM 2** (20 pts) How many positive integers less than 1000 have the property that the sum of the digits of each such number is divisible by 7 and the number itself is divisible by 3?

*YOUR ANSWER: ____________________________*

**PROBLEM 3** (40 pts) For each positive integer $n$, consider the greatest common divisor $h_n$ of the two numbers $n! + 1$ and $(n + 1)!$. For $n < 20$, find the largest value of $h_n$.

*YOUR ANSWER: ____________________________*

**PROBLEM 4** (70 pts) Integers $1, 2, \ldots, n$ are written on a board. Two numbers $m \neq k$ such that $1 < m < n, 1 < k < n$ are removed and the average of the remaining numbers is found to be 17. What is the maximum sum of the two removed numbers?

*YOUR ANSWER: ____________________________*
**PROBLEM 5** (90pts) There are 12 people in a room. Some are honest (always tell the truth), some are liars (always lie). We shall call them N1, N2 etc. Here is what they said:

N1: “there are no honest people here”; N2: “there are no more than 1 honest person here”; N3: “there are no more than 2 honest people here”; …; N12: “there are no more than 11 honest people here”.

How many honest people are in the room?

YOUR ANSWER: _______________________

**PROBLEM 6** (120 pts) In a triangle $ABC$ we have $BA = 10$, $BC = 24$, and the median $BD = 13$. The circle inscribed in the triangle $ABD$ touches $BD$ at the point $M$. The circle inscribed in the triangle $BDC$ touches $BD$ at the point $N$. Find the length of the segment $MN$.

YOUR ANSWER: _______________________

**PROBLEM 7** (150 pts) There are 100 numbered boxes, from 1 to 100, of which 99 are empty and one box contains a prize. You can send a moderator several pieces of paper, each with a YES-NO question. The moderator then mixes your questions and answers them in a random order simply saying: YES! or NO! without specifying which question he/she is answering. What is the minimal number of questions that would guarantee that you win the prize? Could you suggest such questions?

YOUR ANSWER: _______________________
2019-2020 UAB MTS: SOLUTIONS

PROBLEM 1 (10 pts) Two positive integers $a$ and $b$ are such that $a + b = \frac{a}{b} + \frac{b}{a}$. What is the value of $a^2 + b^2$?

Solution: After simple algebraic simplifications we get that $a^2(b - 1) + b^2(a - 1) = 0$. Since $a^2 \geq 1$ and $b^2 \geq 1$, this is only possible if $a = b = 1$.

The answer is 2.

PROBLEM 2 (20 pts) For each positive integer $n$, consider the greatest common divisor $h_n$ of the two numbers $n! + 1$ and $(n + 1)!$. For $n < 100$, find the largest value of $h_n$.

Solution: The number $n! + 1$ is not divisible by any number $k = 2, 3, \ldots, n$ as it has the remainder 1 when we divide $n! + 1$ by $k$. Hence $h_n = 1$ (if $n + 1$ is not prime) and $h_n = n + 1$ (if $n + 1$ is prime). Therefore the largest value of $h_n$ will be obtained if $n + 1$ is the greatest possible prime number under the assumption that $n < 100$. Evidently, this is the case when $n = 96$ and $h_n = n + 1 = 97$.

The answer is: 97.

PROBLEM 2 (20 pts) How many positive integers less than 1000 have the property that the sum of the digits of each such number is divisible by 7 and the number itself is divisible by 3?

Solution: Let a number $\overline{abc}$ (i.e., with digits $a, b$ and $c$) has the desired properties. Then $a + b + c$ is divisible by both 3 and 7 which implies that $a + b + c = 21$ (clearly, $a + b + c \leq 27 < 42$). Since $b \leq 9$ and $c \leq 9$, $b + c \leq 18$ which implies that $a \geq 3$. To each value of $a = 3, 4, \ldots, 9$ the sum $b + c$ must be equal to $21 - a$, i.e. to $18, 17, \ldots, 12$ respectively. To each sum $s = 18, 17, \ldots, 12$ there are $1, 2, \ldots, 7$ ways to represent it as $b + c$, respectively (to see that, begin with $b = 9$ and go down to the least available value of $b$ which is equal to $s - 9$). Hence overall there are $1 + 2 + \cdots + 7 = 28$ desired numbers.

The answer is 28.

PROBLEM 3 (40 pts) For each positive integer $n$, consider the greatest common divisor $h_n$ of the two numbers $n! + 1$ and $(n + 1)!$. For
\[n < 20, \text{ find the largest value of } h_n.\]

**Solution:** The number \(n! + 1\) is not divisible by any number \(k = 2, 3, \ldots, n\) as it has the remainder 1 when we divide \(n! + 1\) by \(k\). Hence \(h_n = 1\) (if \(n + 1\) is not prime) and \(h_n = n + 1\) (if \(n + 1\) is prime). Therefore the largest value of \(h_n\) may be obtained if \(n + 1\) is the greatest possible prime number under the assumption that \(n < 100\) (even in that case one will need to verify that \(n! + 1\) is divisible by \(n + 1\)). Since \(n < 20\), one should start from \(n = 18\) because \(18 + 1 = 19\) is a prime number. And, indeed, one can verify that \(18! + 1\) is divisible by 19.

The answer is: 18. \(\square\)

**PROBLEM 4** (70 pts) Integers 1, 2, \ldots, \(n\) are written on a board. Two numbers \(m, k\) such that \(1 < m < n, 1 < k < n\) are removed and the average of the remaining numbers is found to be 17. What is the maximum sum of the two removed numbers?

**Solution:** We are given that

\[
\frac{n(n+1)}{2} - m - k = 17
\]

which implies, after simple algebraic simplifications, that \(n^2 - 33n + 68 = 2(m + k)\); since \(m\) and \(k\) are two distinct integers between 1 and \(n\) we see that \(n^2 - 33n + 68 \leq 2(n - 1 + n - 2) = 4n - 6\) and, therefore, \(n^2 - 37n + 74 \leq 0\). Solving this quadratic inequalities and taking into account that \(n\) is an integer, we conclude that \(3 \leq n \leq 34\). We need to maximize the number \(m + k\) which, as we have already seen, equals \(\frac{1}{2} \cdot (n^2 - 33n + 68)\). Thus, we need to maximize \(n^2 - 33n + 68\) knowing that \(3 \leq n \leq 34\). It follows that the maximum value of \(\frac{1}{2} \cdot (n^2 - 33n + 68) = m + k\) is assumed when \(n = 34\) and is 51.

The answer is 51. \(\square\)

**PROBLEM 5** (90 pts) There are 12 people in a room. Some are honest (always tell the truth), some are liars (always lie). We shall call them N1, N2 etc. Here is what they said: N1: “there are no honest people here”; N2: “there are no more than 1 honest person here”; N3: “there are no more than 2 honest people here”; \ldots; N12: “there are no more than 11 honest people here”.

\[n < 20, \text{ find the largest value of } h_n.\]

**Solution:** The number \(n! + 1\) is not divisible by any number \(k = 2, 3, \ldots, n\) as it has the remainder 1 when we divide \(n! + 1\) by \(k\). Hence \(h_n = 1\) (if \(n + 1\) is not prime) and \(h_n = n + 1\) (if \(n + 1\) is prime). Therefore the largest value of \(h_n\) may be obtained if \(n + 1\) is the greatest possible prime number under the assumption that \(n < 100\) (even in that case one will need to verify that \(n! + 1\) is divisible by \(n + 1\)). Since \(n < 20\), one should start from \(n = 18\) because \(18 + 1 = 19\) is a prime number. And, indeed, one can verify that \(18! + 1\) is divisible by 19.

The answer is: 18. \(\square\)

**PROBLEM 4** (70 pts) Integers 1, 2, \ldots, \(n\) are written on a board. Two numbers \(m, k\) such that \(1 < m < n, 1 < k < n\) are removed and the average of the remaining numbers is found to be 17. What is the maximum sum of the two removed numbers?

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The answer is 51. \(\square\)

**PROBLEM 5** (90 pts) There are 12 people in a room. Some are honest (always tell the truth), some are liars (always lie). We shall call them N1, N2 etc. Here is what they said: N1: “there are no honest people here”; N2: “there are no more than 1 honest person here”; N3: “there are no more than 2 honest people here”; \ldots; N12: “there are no more than 11 honest people here”.

\[n < 20, \text{ find the largest value of } h_n.\]
How many honest people are in the room?

Solution: Suppose that there are $X$ honest people in the room. Then $N_1, N_2, \ldots, N_X$ are lying (even $N_X$ says that there are no more than $X - 1$ honest people in the room while in fact there are $X$ of them). Hence, $X = 12 - X$ and $X = 6$.

The answer is, **there are 6 honest people in the room.**

**PROBLEM 6** (120 pts) In a triangle $ABC$ we have $BA = 10$, $BC = 24$, and the median $BD = 13$. The circle inscribed in the triangle $ABD$ touches $BD$ at the point $M$. The circle inscribed in the triangle $BDC$ touches $BD$ at the point $N$. Find the length of the segment $MN$.

Solution: We will use the following well-known result from triangle geometry. Suppose that a circle $S$ is inscribed in a triangle $XYZ$. Moreover, suppose that it touches $XY$ at the point $T$. Then $BT = \frac{XY + XZ - YZ}{2}$. Applying this fact to the triangle $ABD$ and the segment $BM$ we see that $BM = \frac{AB + BD - AD}{2} = \frac{23 - AD}{2}$. Analogously, $BN = \frac{BC + BD - CD}{2} = \frac{37 - CD}{2}$. Since $BD$ is a median of $ABC$, $AD = CD = x$. Subtracting $BM$ from $BN$, we get that the length of $MN$ is $MN = |BN - BM| = \frac{37 - x}{2} - \frac{23 - x}{2} = 7$.

The answer is $MN = 7$.

**PROBLEM 7** (150 pts) There are 100 numbered boxes, from 1 to 100, of which 99 are empty and one box contains a prize. You can send a moderator $N$ pieces of paper, each with a YES-NO question. The moderator then mixes your questions and answers them in a random order simply saying: YES! or NO! etc. What is the minimal number of questions that would guarantee that you win the prize? Could you suggest such questions?

Solution: You should be able to figure out in which box the prize is only from the number of answers YES you will hear. In other words, there must exist a function, $p(y)$, from the number $y$ of YES-answers to your questions to the number $p(y)$ of the box with the prize. If you ask $N$ questions, the number of answers YES can be any number between
0 and $N$. Hence there are $N + 1$ possibilities for the number $y$ of YES-answers, and $N + 1$ possible values of $p(y)$. Hence if $N + 1 < 100$ your strategy will inevitably fail.

Here is a strategy that allows your to win the prize in at most 99 questions: you should ask questions “Does the box with prize have the number greater than or equal to $N$?” for $N = 1, 2, \ldots, 99$. Then the number of YES-answers equals the number of the box with the prize.

The answer is 99. □