NAME:________________________________

GRADE:________________________________

SCHOOL NAME:_________________________

2019-2020 UAB MATH TALENT SEARCH

This is a two hour contest. **There will be no credit if the answer is incorrect.** Full credit will be awarded for a correct answer with complete justification. At most half credit will be given for a correct answer without complete justification. Your work (including full justification) should be shown on the extra paper which is attached.

PROBLEM 1 (10 pts) Two positive integers a and b are such that $a + b = \frac{a}{b} + \frac{b}{a}$. What is the value of $a^2 + b^2$?

YOUR ANSWER: ________________________

PROBLEM 2 (20 pts) How many positive integers less than 1000 have the property that the sum of the digits of each such number is divisible by 7 and the number itself is divisible by 3?

YOUR ANSWER: ________________________

PROBLEM 3 (40 pts) For each positive integer n, consider the greatest common divisor h_n of the two numbers $n! + 1$ and $(n + 1)!$. For $n < 20$, find the largest value of h_n.

YOUR ANSWER: ________________________

PROBLEM 4 (70 pts) Integers $1, 2, \ldots, n$ are written on a board. Two numbers $m \neq k$ such that $1 < m < n, 1 < k < n$ are removed and the average of the remaining numbers is found to be 17. What is the maximum sum of the two removed numbers?

YOUR ANSWER: ________________________

over, please

PROBLEM 5 (90pts) There are 12 people in a room. Some are honest (always tell the truth), some are liars (always lie). We shall call them N1, N2 etc. Here is what they said:
N1: "there are no honest people here"; N2: "there are no more than 1 honest person here"; N3: "there are no more than 2 honest people here"; ...; N12: "there are no more than 11 honest people here".
How many honest people are in the room?

YOUR ANSWER: ______________________

PROBLEM 6 (120 pts) In a triangle ABC we have $BA = 10$, $BC = 24$, and the median $BD = 13$. The circle inscribed in the triangle ABD touches BD at the point M. The circle inscribed in the triangle BDC touches BD at the point N. Find the length of the segment MN.

YOUR ANSWER: ______________________

PROBLEM 7 (150 pts) There are 100 numbered boxes, from 1 to 100, of which 99 are empty and one box contains a prize. You can send a moderator several pieces of paper, each with a YES-NO question. The moderator then mixes your questions and answers them in a random order simply saying:YES! or NO! without specifying which question he/she is answering. What is the minimal number of questions that would guarantee that you win the prize? Could you suggest such questions?

YOUR ANSWER: ______________________

2019-2020 UAB MTS: SOLUTIONS

PROBLEM 1 (10 pts) Two positive integers a and b are such that $a + b = \frac{a}{b} + \frac{b}{a}$. What is the value of $a^2 + b^2$?

Solution: After simple algebraic simplifications we get that $a^2(b-1) + b^2(a-1) = 0$. Since $a^2 \geq 1$ and $b^2 \geq 1$, this is only possible if $a = b = 1$.

The answer is **2**. $\square$

PROBLEM 2 (20 pts) For each positive integer n, consider the greatest common divisor h_n of the two numbers $n! + 1$ and $(n+1)!$. For $n < 100$, find the largest value of h_n.

Solution: The number $n! + 1$ is not divisible by any number $k = 2, 3, \dots, n$ as it has the remainder 1 when we divide $n!+1$ by k. Hence $h_n = 1$ (if $n+1$ is not prime) and $h_n = n+1$ (if $n+1$ is prime). Therefore the largest value of h_n will be obtained if $n+1$ is the greatest possible prime number under the assumption that $n < 100$. Evidently, this is the case when $n = 96$ and $h_n = n + 1 = 97$.

The answer is: **97**. $\square$

PROBLEM 2 (20 pts) How many positive integers less than 1000 have the property that the sum of the digits of each such number is divisible by 7 and the number itself is divisible by 3?

Solution: Let a number $\overline{abc}$ (i.e., with digits a, b and c) has the desired properties. Then $a + b + c$ is divisible by both 3 and 7 which implies that $a+b+c = 21$ (clearly, $a+b+c \leq 27 < 42$). Since $b \leq 9$ and $c \leq 9$, $b + c \leq 18$ which implies that $a \geq 3$. To each value of $a = 3, 4, \dots, 9$ the sum $b+c$ must be equal to $21-a$, i.e. to $18, 17, \dots, 12$ respectively. To each sum $s = 18, 17, \dots, 12$ there are $1, 2, \dots, 7$ ways to represent it as $b+c$, respectively (to see that, begin with $b = 9$ and go down to the least available value of b which is equal to $s - 9$). Hence overall there are $1 + 2 + \cdots + 7 = 28$ desired numbers.

The answer is **28**. $\square$

PROBLEM 3 (40 pts) For each positive integer n, consider the greatest common divisor h_n of the two numbers $n! + 1$ and $(n+1)!$. For

$n < 20$, find the largest value of h_n.

Solution: The number $n! + 1$ is not divisible by any number $k = 2, 3, \ldots, n$ as it has the remainder 1 when we divide $n! + 1$ by k. Hence $h_n = 1$ (if $n+1$ is not prime) and $h_n = n+1$ (if $n+1$ is prime). Therefore the largest value of h_n may be obtained if $n + 1$ is the greatest possible prime number under the assumption that $n < 100$ (even in that case one will need to verify that $n! + 1$ is divisible by $n+1$). Since $n < 20$, one should start from $n = 18$ because $18 + 1 = 19$ is a prime number. And, indeed, one can verify that $18! + 1$ is divisible by 19.

The answer is: **18**. □

PROBLEM 4 (70 pts) Integers $1, 2, \ldots, n$ are written on a board. Two numbers m, k such that $1 < m < n, 1 < k < n$ are removed and the average of the remaining numbers is found to be 17. What is the maximum sum of the two removed numbers?

Solution: We are given that

$$\frac{\frac{n(n+1)}{2} - m - k}{n - 2} = 17$$

which implies, after simple algebraic simplifications, that $n^2 - 33n + 68 = 2(m + k)$; since m and k are two distinct integers between 1 and n we see that $n^2 - 33n + 68 \leq 2(n - 1 + n - 2) = 4n - 6$ and, therefore, $n^2 - 37n + 74 \leq 0$. Solving this quadratic ineqaulities and taking into account that n is an integer, we conclude that $3 \leq n \leq 34$. We need to maximize the number $m + k$ which, as we have already seen, equals $\frac{1}{2} \cdot (n^2 - 33n + 68)$. Thus, we need to maximize $n^2 - 33n + 68$ knowing that $3 \leq n \leq 34$. It follows that the maximum value of $\frac{1}{2} \cdot (n^2 - 33n + 68) = m + k$ is assumed when $n = 34$ and is 51.

The answer is **51.** □

PROBLEM 5 (90 pts) There are 12 people in a room. Some are honest (always tell the truth), some are liars (always lie). We shall call them N1, N2 etc. Here is what they said:
N1: "there are no honest people here"; N2: "there are no more than 1 honest person here"; N3: "there are no more than 2 honest people here"; . . . ; N12: "there are no more than 11 honest people here".

How many honest people are in the room?

Solution: Suppose that there are X honest people in the room. Then N1, N2, ..., NX are lying (even NX says that there are no more than $X-1$ honest people in the room while in fact there are X of them). Hence, $X = 12 - X$ and $X = 6$.

The answer is, **there are 6 honest people in the room.** □

PROBLEM 6 (120 pts) In a triangle ABC we have $BA = 10$, $BC = 24$, and the median $BD = 13$. The circle inscribed in the triangle ABD touches BD at the point M. The circle inscribed in the triangle BDC touches BD at the point N. Find the length of the segment MN.

Solution: We will use the following well-known result from triangle geometry. Suppose that a circle S is inscribed in a triangle XYZ. Moreover, suppose that it touches XY at the point T. Then $BT = \frac{XY+XZ-YZ}{2}$. Applying this fact to the triangle ABD and the segment BM we see that $BM = \frac{AB+BD-AD}{2} = \frac{23-AD}{2}$. Analogously, $BN = \frac{BC+BD-CD}{2} = \frac{37-CD}{2}$. Since BD is a median of ABC, $AD = CD = x$. Subtracting BM from BN, we get that the length of MN is $MN = |BN - BM| = \frac{37-x}{2} - \frac{23-x}{2} = 7$.

The answer is $\mathbf{MN = 7}$. □

PROBLEM 7 (150 pts) There are 100 numbered boxes, from 1 to 100, of which 99 are empty and one box contains a prize. You can send a moderator N pieces of paper, each with a YES-NO question. The moderator then mixes your questions and answers them in a random order simply saying:YES! or NO! etc. What is the minimal number of questions that would guarantee that you win the prize? Could you suggest such questions?

Solution: You should be able to figure out in which box the prize is only from the number of answers YES you will hear. In other words, there must exist a function, $p(y)$, from the number y of YES-answers to your questions to the number $p(y)$ of the box with the prize. If you ask N questions, the number of answers YES can be any number between

0 and N. Hence there are $N+1$ possibilities for the number y of YES-answers, and $N+1$ possible values of $p(y)$. Hence if $N+1 < 100$ your strategy will inevitably fail.

Here is a strategy that allows your to win the prize in at most 99 questions: you should ask questions "Does the box with prize have the number greater than or equal to N?" for $N = 1, 2, \ldots, 99$. Then the number of YES-answers equals the number of the box with the prize.

The answer is **99**. $\square$