NAME:______________________________

GRADE:______________________________

SCHOOL NAME:______________________

2022-2023 UAB MATH TALENT SEARCH

This is a two hour contest. **There will be no credit if the answer is incorrect.** Full credit will be awarded for a correct answer with complete justification. At most half credit will be given for a correct answer without complete justification. Your work (including full justification) should be shown on the extra paper which is attached.

PROBLEM 1 (10 pts) Five coins are on the table in a row. The middle one is heads up, the other ones are tails up. It is allowed to flip three adjacent coins. Can one make them all with heads up after several such actions? If yes, then in how many steps?

YOUR ANSWER: Yes, in **2** steps.

PROBLEM 2 (20 pts) Alex, Bob and Chris were drinking tea. Alex and Bob together drank 11 cups. Bob and Chris together drank 15 cups. Chris and Alex together drank 14 cups. How many cups did Alex, Bob and Chris drink together?

YOUR ANSWER: **20.**

PROBLEM 3 (50 pts) There are 127 matches in a box. Judy and Mary are playing the following game. In one turn a player can take out of the box at most one half of all the matches that are currently in the box. Players take turns. The player who cannot take the next turn loses (so, the game stops if there is exactly one match left). Who wins, Judy (who makes the first move), or Mary? What strategy should the winner adopt?

YOUR ANSWER: **Mary.**

over, please

PROBLEM 4 (70 pts) A river flows into a lake. 183 elephants can drink the entire lake up in 1 day. 37 elephants can drink the entire lake up in 5 days. How long will it take one elephant to drink the entire lake up?

YOUR ANSWER: **365** days.

PROBLEM 5 (110 pts) You are supposed to paint each edge of a cube in a color so that for each pair of different colors there are two edges of these colors that share a vertex. What is the greatest number of colors that one can use while still satisfying the requirements?

YOUR ANSWER: **6**.

PROBLEM 6 (160 pts) Solve the following equation. Notice that all the numbers involved are real, and roots are supposed to be non-negative.

$$(\sqrt{\frac{1+2x\sqrt{1-x^2}}{2}}) + 2x^2 = 1$$

YOUR ANSWER: $\mathbf{x_1} = -\frac{\mathbf{1}}{\sqrt{\mathbf{2}}}$, $\mathbf{x_2} = \sqrt{\mathbf{1/2} - \sqrt{\mathbf{3}}/\mathbf{4}}$.

2022-2023 UAB MTS: SOLUTIONS

PROBLEM 1 (10 pts) Five coins are on the table in a row. The middle one is heads up, the other ones are tails up. It is allowed to flip three adjacent coins. Can one make them all with heads up after several such actions?

Solution: First flip the coins # 1, 2, 3. Then flip the coins # 3, 4, 5. This will make all coins with heads up as desired.

So, the answer is "Yes, in **2** steps". □

PROBLEM 2 (20 pts) Alex, Bob and Chris were drinking tea. Alex and Bob together drank 11 cups. Bob and Chris together drank 15 cups. Chris and Alex together drank 14 cups. How many cups did Alex, Bob and Chris drink together?

Solution: If we add up 11, 15 and 14, we will have the number of cups of tea that three friends drank together in which each cup consumed by each person is counted twice. Thus, the answer is $\dfrac{11+15+14}{2} = 20$.

The answer is: **20**. □

PROBLEM 3 (50 pts) There are 127 matches in a box. Judy and Mary are playing the following game. In one turn a player can take out of the box at most one half of all the matches that are currently in the box. Players take turns. The player who cannot take the next turn loses (so, the game stops if there is exactly one match left). Who wins, Judy (who makes the first move), or Mary? What strategy should the winner adopt?

Solution: The initial number of matches is $127 = 2^7 - 1$. No matter how many matches Judy takes out of the box in the beginning, Mary can always complement that move by her move so that after that there will be $63 = 2^6 - 1$ matches left. Indeed, Judy can take between 1 and 6t3 matches out, so the number of remaining matches is from 64 to 126. Clearly, this allows Mary to take out the desired number of matches leaving Judy with 63 matches in the box. Repeating this strategy, Mary will in the end leave Judy with exactly one match on Judy's turn. In this way Mary can always win the game.

The answer is **Mary**. □

PROBLEM 4 (70 pts) A river flows into a lake. 183 elephants can drink the entire lake up in 1 day. 37 elephants can drink the entire lake up in 5 days. How long will it take one elephant to drink the entire lake up?

Solution: Let A is the number of liters of water in the lake. In one day y liters of water are added to A by the river. So, without elephants there would have been $A + y$ water in the lake by the end of the day. If one elephant drinks x liters in one day, we get that $A + y = 183x$. Hence, $x = \dfrac{A+y}{183}$.

Using this in the second case we have that $\dfrac{5 \cdot 37 \cdot (A+y)}{183} = A + 5y$.

Standard algebraic arguments implies then that $A = 365y$ and $x = 2y$. To see how many days one elephant needs to drink all water from the lake, we need to solve the equation $Nx = A + Ny$, or, in terms of y, $2Ny = (365 + N)y$. This easily implies that $N = 365$.

The answer is **365** days. □

PROBLEM 5 (110 pts) You are supposed to paint each edge of a cube in a color so that for each pair of different colors there are two edges of these colors that share a vertex. What is the greatest number of colors that one can use while still satisfying the requirements?

Solution: It is easy to see that there are ways to paint edges in 6 colors satisfying the requirements. Let us show that for 7 or more colors the desired painting is impossible. Indeed, if there are 7 or more colors then inevitably there a color (say, purple) so that there is a UNIQUE edge painted in that color. There are exactly 4 edges that share a vertex with the purple edge. These edges are painted with at most 4 colors. Hence there exists a color X which is neither purple, nor one of those colors. It follows, that for a pair (purple, X) there are no two edges of these colors that share a vertex, a contradiction.

The answer is: **6**. □

PROBLEM 6 (160 pts) Solve the following equation. Notice that all the numbers involved are real, and roots are supposed to be non-negative.

$$(\sqrt{\frac{1+2x\sqrt{1-x^2}}{2}})+2x^2=1$$

Solution: Observe that the solution must satisfy the inequality $2x^2 \le 1$ and so $|x| \le 1/\sqrt{2}$. To solve the equation for x we begin by bringing $2x^2$ to the other side and then squaring both sides. This yields

$$1+2x\sqrt{1-x^2}=2(1-2x^2)^2.$$

It is easy to verify using algebra that the right hand side of the above equation equals

$$2-2(2x\sqrt{1-x^2})^2$$

and hence we can set $2x\sqrt{1-x^2}=t$ and rewrite our equation in terms of t as follows:

$$1+t=2-2t^2$$

which implies that either $t=-1$ or $t=1/2$.

If $2x\sqrt{1-x^2}=-1$ then $x<0$ and we can square both sides that yields the following:

$$4x^4-4x^2+1=0$$

and hence $x^2=1/2$ and $x=-1/\sqrt{2}$.

If $2x\sqrt{1-x^2}=1/2$ then $x>0$. We can again quare both sides which yields that, if we set $x^2=u$, that $4u^2-4u+1/4=0$ and, hence, $x^2=u=1/2\pm\sqrt{3}/4$. However the square root of $1/2+\sqrt{3}/4$ is greater than $1/\sqrt{2}$ (impossible by the remark in the beginning of the proof). This gives yet another solution of the original equation which is

$$\sqrt{1/2-\sqrt{3}/4}.$$

The answer is $\mathbf{x_1=-1/\sqrt{2}}$ and $\mathbf{x_2=\sqrt{1/2-\sqrt{3}/4}}$. ∎