NAME:______________________________

GRADE:______________________________

SCHOOL CODE:______________________

2005 UAB MATH TALENT SEARCH 10th GRADE

This is a two hour and a half contest. Answers are to be written in the spaces provided on the test sheet. Your work **must** be shown on the extra paper which we supply. In case of a tie, the judging panel will use your work to choose the winner.

PROBLEM 1 (10 points). In a quadrilateral ABCD AB=4 and CD=5. Find the perimeter of the quadrilateral KPHM where K is the midpoint of BC, H is the midpoint of AD, P is the midpoint of AC, and M is the midpoint of BD.

YOUR ANSWER:

PROBLEM 2 (20 points). Five apples, five pears, and one orange cost 78 cents. One apple, five pears, and five oranges cost $1.18. Each apple costs x cents, each pear costs y cents, and each orange costs z cents. It is known that $y > 5$. What is y?

YOUR ANSWER:

PROBLEM 3 (30 points). Solve the equation $(x^2 - 1)^2 = 4x + 1$ and find the sum of cubic powers of all its real roots.

YOUR ANSWER:

PROBLEM 4 (40 points). The point O is the center of the circle inscribed into a triangle ABC. It is known that the angle BOC is three times the angle A. Find the angle A in degrees.

YOUR ANSWER:

over, please

PROBLEM 5 (50 points). Find the smallest natural n such that a) its decimal representation has 6 as the last digit (i.e. in the slot for "ones"); b) if this last digit (that is 6) is erased and moved in front of the remaining digits, the resulting number is four times larger than n.

YOUR ANSWER:

PROBLEM 6 (70 points). Consider all three-digit numbers N such that N is divisible by 11, and $N/11$ is equal to the sum of the squares of the digits of N. Denote the greatest among all such numbers by a and the least by b. What is $a - b$?

YOUR ANSWER:

PROBLEM 7 (90 points). The given system of equations is $x+y+z=\sqrt{3.5}, x^2+y^2+z^2=6.5, xy=z^2$. Compute out the value of $-z\sqrt{14}$.

YOUR ANSWER:

PROBLEM 8 (110 points). Five points in a plane are situated so that no two of the straight lines joining them are parallel, perpendicular, or coincident. From each point perpendiculars lines are drawn to all the lines joining the other four points. All points of intersections of these perpendicular lines are counted (if several perpendicular lines intersect at one point this point is counted only once). What is the maximal number of intersections that these perpendiculars can have?

YOUR ANSWER:

2005 UAB MTS, 10th GRADE: SOLUTIONS

PROBLEM 1 (10 points). In a quadrilateral ABCD AB=4 and CD=5. Find the perimeter of the quadrilateral KPHM where K is the midpoint of BC, H is the midpoint of AD, P is the midpoint of AC, and M is the midpoint of BD.

Solution: HP is the midline of the triangle ABD, so its length is 2.5. Similarly, PK=2, KM=2.5, MH=2. Hence the perimeter of KPHM is 9.

So, the answer is **9**.

PROBLEM 2 (20 points). Five apples, five pears, and one orange cost 78 cents. One apple, five pears, and five oranges cost \$1.18. Each apple costs x cents, each pear costs y cents, and each orange costs z cents. It is known that $y > 5$. What is y?

Solution: Denote the price of an apple by x, the price of a pear by y, and the price of an orange by z. Then we get two equations:

$$5x + 5y + z = 78$$

and

$$x + 5y + 5z.$$

Subtracting the former from the latter we get $4z - 4x = 40$, and so $z = x + 10$. Replacing z by $x + 10$ in the first equation, we get $6x + 5y = 68$ and therefore $y = \dfrac{68 - 6x}{5}$. Since x, y, z are positive integers, we can easily see that the only two possible values of y are either 10 or 4. Since it is known that $y > 5$ we conclude that $y = 10$.

So, the answer is **10**.

PROBLEM 3 (30 points). Solve the equation $(x^2 - 1)^2 = 4x + 1$ and find the sum of cubic powers of all its real roots.

Solution: After simplifications we get the following equivalent equation: $x^4 - 2x^2 - 4x = 0$. After factoring we get $x(x - 2)(x^2 + 2x + 2) = 0$. Hence the roots are $0, 2$.

So the answer is **8**.

PROBLEM 4 (40 points). The point O is the center of the circle inscribed into a triangle ABC. It is known that the angle BOC is three times the angle A. Find the angle A in degrees.

Solution: Suppose that the angle $ABC = \beta$, the angle $ACB = \gamma$. It is known that BO and CO are the bisectrices of the angles B and C of the triangle ABC. Therefore the angle $OBC = \beta/2$, and the angle $OCB = \gamma/2$. We conclude that the angle $BOC = 180 - (\beta/2 + \gamma/2)$. On the other hand, it is given that the angle BOC is three times the angle A. Thus,

$$3[180 - (\beta + \gamma)] = 180 - (\beta/2 + \gamma/2),$$

which implies that

$$540 - 3(\beta + \gamma) = 180 - 1/2(\beta + \gamma),$$

and therefore $2.5(\beta + \gamma) = 360$ and $\beta + \gamma = 144$, which implies that $A = 36$ degrees.

So, the answer is **36**.

PROBLEM 5 (50 points). Find the smallest natural n such that a) its decimal representation has 6 as the last digit (i.e. in the slot for "ones"); b) if this last digit (that is 6) is erased and moved in front of the remaining digits, the resulting number is four times larger than n.

Solution: Suppose that the desired number n has k+1 digits and write it in the form $10N + 6$; then the transformed number is $6 \times 10^k + N$. The problem requires that

$$6 \times 10^k + N = 4(10N + 6),$$

which, when simplified, becomes

$$f(k) = 2 \times 10^k - 8 = 13N.$$

Thus we need to find the least k such that $f(k)$ is divisible by 13. If we consider the values of $f(k)$ for $k = 0, 1, 2, \ldots$ we see that the least k such that $f(k)$ is divisible by 13 is 5, and the answer then is $N = 15384$ and so $n = 153846$.

So, the answer is **153846**.

PROBLEM 6 (70 points). Consider all three-digit numbers N such that N is divisible by 11, and $N/11$ is equal to the sum of the squares of the digits of N. Denote the greatest among all such numbers by a and the least by b. What is $a - b$?

Solution: Let $N = 100h + 10t + u$; it is given that $N = 11(h^2 + t^2 + u^2)$. Observe, that $N = (99h + 11t) + (h - t + u)$ which implies that $h - t + u$ is divisible by 11. Since h, t, u are digits we conclude that either $h - t + u = 0$ or $h - t + u = 11$.
(1) Suppose that $h - t + u = 0$. Then the main equation

$$(99h + 11t) + (h - t + u) = 11(h^2 + t^2 + u^2)$$

becomes (after simplification)

$$10h + u = 2(h^2 + uh + u^2),$$

so u is even. The last equation can be rewritten as quadratic in h:

$$2h^2 + (2u - 10)h + 2u^2 - u = 0.$$

Since h is an integer, the discriminant $4(25 - 8u - 3u^2)$ of this quadratic equation must be a perfect square. But easy to check that this is true only if $u = 0$! Thus $u = 0$, the equation for h now reads $2h^2 - 10h = 0$, so $h = 5, t = h + u = 5$, and $N = 550$.
(2) Suppose that $h - t + u = 11$; then $t = h + u - 11$. Then the main equation

$$(99h + 11t) + (h - t + u) = 11(h^2 + t^2 + u^2)$$

easily yields

$$10h + u - 10 = 2[h^2 + uh + u^2 - 11(h + u)] + 121,$$

so u must be odd. Again we rewrite the last equation in the form

$$2h^2 + (2u - 32)h + 2u^2 - 23u + 131 = 0$$

and see that its discriminant $4(-3u^2 + 14u - 6)$ must be a perfect square. Since u is odd, we see that 1 does not yield a perfect square, 3 which yields the square 36, and find that all larger odd u yield a negative discriminant. If $u = 3$, our quadratic equation in h becomes $2h^2 - 26h + 80 = 2(h - 5)(h - 8) = 0$. When $h = 5, t = 5 + 3 - 11 = -3$, not admissible. When $h = 8, t = 8 + 3 - 11 = 0$, so $N = 803$.
So, the answer is 803-550=**253**.

PROBLEM 7 (90 points). The given system of equations is $x+y+z=\sqrt{3.5}, x^2+y^2+z^2=6.5, xy=z^2$. Compute out the value of $-z\sqrt{14}$.

Solution: Consider a bit more general system of equations given by $x+y+z=a, x^2+y^2+z^2=b^2, xy=z^2$ where a,b are parameters. Replacing z^2 by xy in the second equation we have $x^2+xy+y^2=b^2$. Next, observe that from the first equation it follows that $x+y-a=-z$; squaring this, we get that $(x+y-a)^2=(-z)^2=xy$ which implies that $x^2+xy+y^2-2a(x+y)=-a^2$.
Subtracting the expression for $-a^2$ from that for b^2 we get $x+y=\dfrac{a^2+b^2}{2a}$; hence $z=a-(x+y)=\dfrac{a^2-b^2}{2a}$. In other words, the value of z is the same regardless of what solution is found. In our case $a=\sqrt{3.5}$ and $b^2=6.5$. Hence $z=\dfrac{3.5-6.5}{2\sqrt{3.5}}=\dfrac{-3}{\sqrt{14}}$ so that $-z\sqrt{14}=3$.

So, the answer is **3**.

PROBLEM 8 (110 points). Five points in a plane are situated so that no two of the straight lines joining them are parallel, perpendicular, or coincident. From each point perpendiculars lines are drawn to all the lines joining the other four points. All points of intersections of these perpendicular lines are counted (if several perpendicular lines intersect at one point this point is counted only once). What is the maximal number of intersections that these perpendiculars can have?

Solution: Fix one of the points, say, A. There are $\dfrac{4\times 3}{2}=6$ lines connecting other points perpendiculars to which must be drawn, all passing through A. Since there are 5 points, there are 5×6 such perpendiculars. Therefore, the maximal number of intersections between them is $\dfrac{30\times 29}{2}=435$. However, not all these points are distinct.
First, consider one of the 10 lines determined by the original 5 points. There are 3 points outside this line, and perpendiculars to this line passing through those points are parallel and thus disjoint. Therefore, 3 possible intersections among them are lost. Since this holds for each of the 10 lines connecting the original 5 points, we lose 30 points of intersection. We are left with $435-30=405$ possibly distinct points of intersection.
Second, there are $\dfrac{5\times 4\times 3}{1\times 2\times 3}$ triangles formed by the original 5 points. In each of them heights intersect at one point, not at 3 distinct points. So we lose $2\times 10=20$ points of intersection because of that which

yields the figure $405 - 20 = 385$.

Finally, each original point has 6 perpendiculars passing through it - that is, these six perpendiculars produce only one point of intersection instead of $\frac{6 \times 5}{2} = 15$. This decreases the number of distinct points of intersection by yet 14 and happens at each of the original 5 points. So the overall number decreases by $5 \times 14 = 70$. This yields $385 - 70 = 315$ as the answer.
So, the answer is **315**.