Maps with no a priori bounds

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Dedicated to the memory of Yu. I. Lyubich and O. M. Sharkovskiy

Abstract

The modulus of a polynomial-like (PL) map is an important invariant that controls distortion of the straightening map and, hence, geometry of the corresponding PL Julia set. Lower bounds on the modulus, called *complex a priori bounds*, are known in a great variety of contexts. For any rational function we complement this by an upper bound for moduli of PL maps in the satellite case that depends *only on the relative period and the degree of the PL map.* This rules out a priori bounds in the satellite case with unbounded relative periods. We also apply our tools to obtain lower bounds for hyperbolic lengths of geodesics in the infinitely renormalizable case, and to show that moduli of annuli must converge to **0** for a sequence of arbitrary renormalizations, under several conditions all of which are shown to be necessary.

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1 Introduction

Since Sullivan's work on Feigenbaum's universality [30], complex a priori bounds for polynomial-like renormalizations play a key role in polynomial dynamics. Such bounds, often hard to prove, are established and crucially used for various classes of polynomials (see, e.g., [3, 7, 8, 10–15, 20, 21, 23, 27, 30];). They imply that the Julia set $J(f)$ is locally connected and, in a lot of cases, rigidity of the corresponding maps in the considered family (if the latter is the quadratic family, this means the MLC conjecture at the corresponding parameters). Thus, bounds are an important tool in complex dynamics.

On the other hand, counterexamples by Douady and Hubbard of satellite infinitely renormalizable quadratic polynomials with non-locally connected Julia sets show that the bounds do not always hold (see [26, 29] for qualitative versions and [16–18] and [2] for quantitative ones). These and similar examples of satellite renormalizations with "no bounds" are based on the polynomial-like (PL) connected Julia sets keeping definite size, deal with *polynomials*, and require a *sequence* of consecutive satellite renormalizations with fast convergence to zero of the rotation numbers at the α-fixed points of renormalizations ([17, 18], see also [2]). In particular, in all those examples the relative periods of the renormalizations (i.e., the denominators of the rotation numbers) converge to infinity *very fast*. The authors are not aware of other cases where the "no bounds" condition have been established. So it was unclear how general the phenomenon was, and no explicit upper bounds were previously known.

Motivated by the desire to clarify the mechanism behind the "no bound" phenomenon, we study renormalizations of *any rational function*. Given a PL map $P : U \to V$ (see [4]) with connected filled Julia set K^*, call $U \setminus K^*$ a *root annulus*, and $V \setminus \overline{U}$ a *fundamental annulus* (of K^*). Notice that complex a priori bounds mean lower bounds on the modulus of a fundamental annulus. Let f be a rational function such that $f^q : U \to V$ is a PL map of degree d^* with connected Julia set K^*. The cycle $\mathcal{K} = \{K^*, f(K^*), \dots, f^{q-1}(K^*)\}$ is called the *PL cycle* of f. Suppose that a point $\alpha \in K^*$ of period $q/s < q$ is the only intersection point of K^* and $\bigcup_{0<i<q} f^i(K^*)$; then s is called the *relative period* of $\mathcal{K}$, and the orbit of α is called the *base cycle* of $\mathcal{K}$. See Definitions 2.3–2.4 for more details. The Main Theorem is proven in Section 2.2.

Main Theorem. *Let f be a rational function of degree $d \geqslant 2$ with a PL cycle $\mathcal{K}$ of relative period $s \geqslant 2$. Then, for any PL $f^q : U \to V$ of degree d^* of any filled PL Julia set $K^* \in \mathcal{K}$,*

$$\operatorname{mod}(V \setminus U) \leqslant (d^* - 1)\operatorname{mod}(U \setminus K^*) < \frac{d^*(d^*-1)\pi}{\ln(4(s+1))} \leqslant \frac{2^{2d-2}(2^{2d-2}-1)\pi}{\ln(4(s+1))},$$

and, if f is a polynomial, then

$$\operatorname{mod}(V \setminus U) \leqslant (d^* - 1)\operatorname{mod}(U \setminus K^*) < \frac{d^*(d^*-1)\pi}{\ln(4(s+1))} \leqslant \frac{2^{d-1}(2^{d-1}-1)\pi}{\ln(4(s+1))}.$$

Observe, that an upper bound on $\operatorname{mod}(V \setminus U)$ would not automatically imply an upper bound on $\operatorname{mod}(U \setminus K^*)$.

As an example of how the Main Theorem can be used, consider the family of infinitely renormalizable polynomials $f_c(z) = z^2 + c$ such that all renormalizations are satellite. To each such f_c one associates a sequence of relative rotation numbers of consecutive renormalizations. Then, by the Main Theorem, for a generic subset of the space of all such sequences the associated polynomials admit no a priori bounds. Here, "generic" refers to Baire category, that is, the corresponding subset is a countable intersection of open dense subsets of $(\mathbb{Q}/\mathbb{Z})^{\mathbb{N}}$.

In this context we want to also mention a recent preprint [6] (see also references therein) that generalizes [10] to the satellite case and completes the proof of the fact that a priori bounds hold for *all* infinitely renormalizable quadratic polynomials $f_c(z) = z^2+c$ with bounded combinatorics (i.e., such that for some number $\overline{p}$ the period of the next renormalization is less than $\overline{p}$ times the period of the previous renormalization). Together with the Main Theorem this provides a criterion for the existence of a priori bounds in the case of infinitely renormalizable quadratic polynomials with satellite renormalizations.

Another application of our tools deals with geometry of the postcritical set in the infinitely renormalizable case. Write $\mathbb{P}^1$ for the projective line over the field $\mathbb{C}$ of complex numbers, i.e., for the Riemann sphere. For a compactum $Y \subset \mathbb{P}^1$ such that $X = \mathbb{P}^1 \setminus Y$ is a hyperbolic Riemann surface, let $\ell_X(\gamma)$ be the hyperbolic length of a Jordan curve $\gamma \subset X$ in X. Given a geodesic γ in a hyperbolic Riemann surface X, say that an annulus A in X is *homotopic* to γ if a simple closed curve in A that is a deformation retract of A is homotopic to γ in X. Bounded geometry of a postcritical infinitely renormalizable set Y (with respect to a given sequence of renormalizations) means that for some $L > 1$, each level of renormalization, each PL Julia set K^* of this level, and any simple closed geodesic in $X = \mathbb{P}^1 \setminus Y$ isotopic in X to the boundary of a tight Jordan neighborhood of K^*, we have $\ell_X(\gamma) \in [1/L, L]$.

Recently, D. Cheraghi [virtual workshop "Many faces of renormalization", Simons Center for Geometry and Physics, 2021] and M. Pedramfar [Workshop "On geometric complexity of Julia sets", 2018, Będlewo, Poland] announced their joint results on the geometry and topology of a postcritical infinitely renormalizable set $\mathcal{PC}_f$ for a quadratic polynomial $f(z)$ where all renormalizations are satellite. Then, a *(combinatorial) rational rotation number* $k/s \in (-1/2, 1/2]$ is associated with a PL cycle, describing how its PL Julia sets rotate about their base points. The results of D. Cheraghi and M. Pedramfar concern the cases when all such rotation numbers are of *high type* (see [9]) and state, in particular, that the postcritical set $\mathcal{PC}_f$ of such a polynomial f has bounded geometry if and only if the set of rotation numbers is bounded away from 0. Recall that for a rational function f, $\mathcal{PC}_f$ is the closure of the union of forward orbits of all critical values of f.

Our results complement these. However, unlike in the definition of bounded geometry, we require that a simple geodesic γ goes not only around the chunk of a postcritical infinitely renormalizable set $\mathcal{PC}_f$ contained in a PL Julia set K^* but also around $\{\alpha\} = \mathcal{B} \cap K^*$ (recall that $\mathcal{B}$ is the corresponding base cycle of $\mathcal{K}$). Corollary A follows from the more general, but also more technical, Theorem 3.5.

Corollary A. *Consider a polynomial f with a satellite PL cycle $\mathcal{K}$ of relative period $s > 1$ that takes part in the formation of an infinitely-renormalizable set S (see Def. 3.1). Suppose that all (finite) critical points of f belong to S, and set $X = \mathbb{P}^1 \setminus (\mathcal{B} \cup \mathcal{PC}_f)$. Let γ be a simple closed geodesic in X isotopic in X to the outer boundary of some fundamental annulus of a PL Julia set $K^* \in \mathcal{K}$. Then*

$$\ell_X(\gamma) \geqslant \ln\left(\frac{4\ln(s+1)}{(d^*)^2\pi}\right).$$

Thus, if all critical points of a rational function f belong to an infinitely renormalizable set then adding the base cycle to the postcritical set makes the situation with the length of simple closed geodesics universally opposite to that described by Cheraghi and Pedramfar. Notice that in our setting the assumptions on f are less restrictive. Corollary A is proven in Section 3.

The Main Theorem shows that for rational functions f_i of degree d with satellite PL cycles of relative periods $s_i \to \infty$, the moduli of root annuli tend (uniformly in s_i) to zero. As we show in Theorem B in Section 4, this conclusion holds not only in the satellite case. Moreover, no conditions in Theorem B can be dropped (see Proposition 4.2 and the short discussion before it).

Theorem B. *Let $\{f_n\}$ be a sequence of degree $d \geqslant 2$ rational functions that converges to a rational function f of degree d. Assume that for each n there is a renormalization $f_n^{q_n} : U_n \to V_n$ of period q_n with connected PL Julia set K_n, and $K_n \to K$ in the Hausdorff metric. Suppose that the following holds:*

(1) *there are at least two points in K,*
(2) *the sequence q_n tends to infinity,*
(3) *no parabolic periodic domain of f contains K (e.g., if f has no parabolic points).*

Then $\operatorname{mod}(U_n \setminus K_n) \to 0$.

Let us comment briefly on our methods. The proof of the Main Theorem uses the solution of Teichmueller's extremal problem and is otherwise elementary. This allows us to prove a completely general (depending only on the degree and the relative period) upper bound for a single satellite renormalization.

Corollary A follows from the more technical Lemma 3.4 and Theorem 3.5 that extend some estimates used in the proof of the Main Theorem and rely upon results of [22] relating hyperbolic lengths of closed geodesics and moduli of the associated annuli.

Finally, unlike the Main Theorem, Theorem B deals with *arbitrary renormalizations* and contains restrictive conditions that guarantee convergence to zero of moduli of the corresponding annuli. The proof of the fact that these conditions are necessary (see Proposition 4.2) is based on the existence of universal complex bounds for real unimodal maps [15], [8], [21] and the "parabolic implosion" technique involving Lavaurs' map (see, e.g., [32]). Our results are applicable to the case, when a fundamental annulus of a PL map under consideration contains critical values of f.

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2 Proof of the Main Theorem

Let $\mathbb{P}^1$ be the Riemann sphere. For a compact subset $X \subset \mathbb{C}$, let $\mathrm{diam}_e(X)$ be its Euclidian diameter. For a finite set $\mathcal{P} \subset \mathbb{C}$, let $\delta(\mathcal{P})$ be the shortest Euclidian distance between two distinct points of $\mathcal{P}$. Write C_f for the set of critical points of the map f.

2.1 The Teichmueller extremal problem

Section 2.1 describes classical geometric inequalities arising from extremal problems of conformal geometry (see, e.g., [1], Sections 4.11 – 4.12, for details). Recall the *Teichmueller extremal problem*: find the maximal value of $\mathrm{mod}(U \setminus Z)$, where an open Jordan domain $U \subset \mathbb{C}$ and a full continuum Z are such that $0, -1 \in Z$ while U does not contain the disk $\{z \in \mathbb{C} \mid |z| \leqslant \varepsilon\}$. In other words, the distance between the complement of U and 0 is less than or equal to ε. It reads that the maximal $m = \mathrm{mod}(U \setminus Z)$ is attained when $Z = [-1, 0]$ while $U = \mathbb{C} \setminus [\varepsilon, \infty)$. For this optimal choice of U and Z, write $\tau^{-1}(\varepsilon)$ for the modulus of $U \setminus Z$. Here τ^{-1} is an increasing function defined on $\mathbb{R}_{>0}$, that can be expressed in terms of elliptic integrals. Let τ be the inverse function of τ^{-1}; then

$$\tau(m) > 16e^{-\pi/2m}.$$

This inequality is asymptotically sharp for small values of m and ε.

Based upon the above, one can restate the Teichmueller extremal problem as follows: if U and Z are as above, and $m = \mathrm{mod}(U \setminus Z)$, then the disk $\{z \in \mathbb{C} || z| < \tau(m)\}$ is contained in U.

Lemma 2.1. *If $Z \subset \mathbb{C}$ is a full continuum and $V \supset Z$ is an open Jordan domain, then the minimal distance between a point $x \in Z$ and a point $y \in \mathbb{C} \setminus V$ is greater than $\frac{1}{2}\tau(\mathrm{mod}(V \setminus Z)) \cdot \mathrm{diam}_e(Z)$.*

Proof. Note that, in any compact metric space, the distance from any point to the point farthest from it is at least half the diameter. In particular, the distance from any point $x \in Z$ to the point $z \in Z$ farthest from x is at least $\mathrm{diam}_e(Z)/2$.

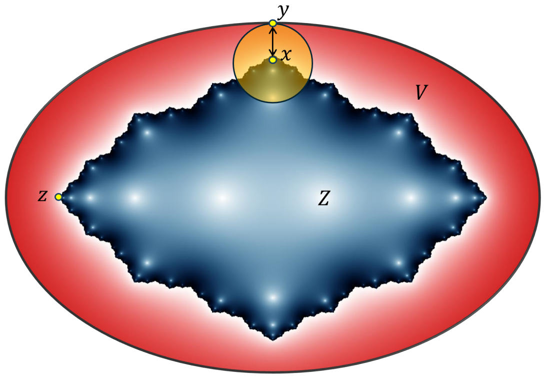

Fig. 1 This figure illustrates the proof of Lemma 2.1 and uses the notation from the lemma.

To prove the lemma let $x = 0$. By the above, we can scale Z by the factor $\gamma \leqslant 2/\mathrm{diam}_e(Z)$ so that the point $z \in Z$ of maximal $|z|$ maps to -1. Choose $y \in \mathbb{C} \setminus V$ so that $|y|$ is minimal over $\mathbb{C} \setminus V$. By the restatement of the Teichmüller extremal problem, $|y| \geqslant \tau(\mathrm{mod}(V \setminus Z))$. Scaling everything back, we see that

$$\frac{|y|}{\gamma} \geqslant \frac{\tau(\mathrm{mod}(V \setminus Z))}{\gamma} \geqslant \frac{\tau(\mathrm{mod}(V \setminus Z)) \cdot \mathrm{diam}_e(Z)}{2},$$

as claimed. □

Recall that $\delta(P)$ is the shortest Euclidian distance between two distinct points of a finite set $\mathcal{P}$. Lemma 2.2 estimates $\delta(\mathcal{P})$ for finite subsets $\mathcal{P} \subset \mathbb{C}$.

Lemma 2.2. *Let $\mathcal{P} \subset \mathbb{C}$ be a set of $t > 1$ points. Then $\delta(\mathcal{P}) < \frac{2}{\sqrt{t}}\mathrm{diam}_e(\mathcal{P})$.*

Proof. Since $1 < \frac{2}{\sqrt{t}}$ for $t \leqslant 3$ we may assume that $t \geqslant 4$. We may also assume that $\mathrm{diam}_e(\mathcal{P}) = 1$. Since the open disks of radius $\delta/2$ centered at all points of $\mathcal{P}$ are pairwise disjoint and contained in a convex set of diameter $1+\delta$, the total area $t\pi\delta^2/4$ of these disks is less than $\pi(1+\delta)^2/4$, the maximal area of a planar set of diameter $1+\delta$ (see p. 239, ex. 610a of [31]). Thus, $\frac{t\delta^2}{4} < \frac{(1+\delta)^2}{4}$ which implies that $\delta < \frac{1+\sqrt{t}}{t-1} \leqslant \frac{2}{\sqrt{t}}$ where the last inequality is based on the fact that $t \geqslant 4$. □

The next theorem is instrumental for the proof of the Main Theorem.

Static Theorem. *Let $\mathcal{P} = \{\alpha, z_1, \dots, z_t\} \subset \mathbb{P}^1$, $t \geqslant 3$. Then there are distinct points z_r, z_l such that any annulus $A \subset \mathbb{P}^1$ separating $\{\alpha, z_r\}$ from $\mathcal{P} \setminus \{\alpha, z_r\}$ or $\{\alpha, z_l\}$ from $\mathcal{P} \setminus \{\alpha, z_l\}$ satisfies $\mathrm{mod}(A) < \frac{\pi}{\ln(4t)}$. Thus, for any choice of $w \in \mathcal{P}$, there exists $z_k \in \mathcal{P}, z_k \neq w$ such that, for any annulus $A \subset \mathbb{P}^1$ separating $\{\alpha, z_k\}$ from $\mathcal{P} \setminus \{\alpha, z_k\}$, we have $\mathrm{mod}(A) < \frac{\pi}{\ln(4t)}$.*

Proof. Using a suitable Moebius transformation and the conformal invariance of the modulus we assume that $\alpha = \infty$ and show that there are two points $z_r \neq z_l \in \mathcal{P}$

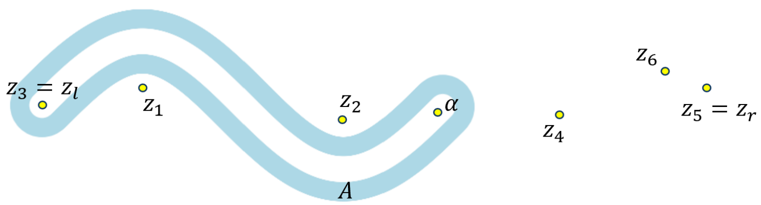

Fig. 2 This figure illustrates the proof of the Static Theorem and uses the notation from the theorem. Observe that after we apply the suitable Moebius transformation which maps α to infinity, the images of points $z_3 = z_l$ and $z_5 = z_r$ become the closest among images of all points of the set $\mathcal{P}$.

such that for $z' = z_r$ or $z' = z_l$ and for every annulus A in $\mathbb{C}$ separating $\{z', \infty\}$ from $\mathcal{P} \setminus \{z'\}$ we have $\mathrm{mod}(A) < \frac{\pi}{\ln(4t)}$. We may assume that $\mathrm{diam}_e(\mathcal{P} \setminus \{\infty\}) = 1$.

Let $z_r, z_l \in \mathcal{P}$ such that $|z_r - z_l| = \delta(\mathcal{P} \setminus \{\infty\})$ and show that they are the desired points (it suffices to prove it for z_r). Consider an annulus A separating $\{z_r, \infty\}$ from $\mathcal{P} \setminus \{z_r, \infty\}$. Then $z_l \in \mathcal{P} \setminus \{\infty, z_r\} \subset Z$ where Z is the bounded complementary component of A. Hence $\mathrm{diam}_e(Z) \geqslant \mathrm{diam}_e(\mathcal{P} \setminus \{\infty, z_r\}) \geqslant \frac{1}{2}$ as otherwise $|z_r - z_l| < \frac{1}{2}$ by the choice of z_r, z_l and the Euclidian distance between any two points of $\mathcal{P} \setminus \{\infty\}$ is less than $1 = \mathrm{diam}_e(\mathcal{P} \setminus \infty)$ which is absurd. By Lemma 2.1,

$$\frac{\tau(\mathrm{mod}(A)) \cdot 1/2}{2} \leqslant \frac{\tau(\mathrm{mod}(A)) \cdot \mathrm{diam}_e(Z)}{2} \leqslant |z_r - z_l| \leqslant \frac{2}{\sqrt{t}}$$

and so $\tau(\mathrm{mod}(A)) \leqslant \frac{8}{\sqrt{t}}$; since $16e^{-\pi/2\mathrm{mod}(A)} < \tau(\mathrm{mod}(A))$, then $\mathrm{mod}(A) < \frac{\pi}{\ln(4t)}$ as claimed. If now w is given (as in the assumptions of the theorem) we can choose z_i to be one of the points z_r, z_l not equal to w; clearly, z_i will have the desired property. □

2.2 Moduli of annuli

We need some basic notation and definitions first.

Definition 2.3 (Renormalization and related concepts)**.** A rational function $f : \mathbb{P}^1 \to \mathbb{P}^1$ is *renormalizable* if there are an integer $q > 0$ and Jordan domains $U \Subset V$ such that $f^q : U \to V$ is a degree d^* polynomial-like (PL) map with *connected* Julia set K^* (recall that K^* is the set of all points $z \in U$ whose f^q-orbits are contained in U). The annulus $A = V \setminus \overline{U}$ is called a *fundamental annulus of* K^*, and the annulus $U \setminus K^*$ is called a *root annulus (of* K^**)*. The collection of sets

$$\mathcal{K} = \{K^*, f(K^*), \dots, f^{q-1}(K^*)\}$$

is called a *PL cycle*; we always assume that $f^i(K^*) \neq f^j(K^*)$ for all $0 \leqslant i \neq j < q$. Note that each $f^i(K^*)$, $i \geqslant 0$, is a connected filled Julia set of a PL map $f^q : U_i \to V_i$ for some Jordan domains U_i and V_i. These sets, denoted K^*, are called *members of* $\mathcal{K}$ or *PL sets*.

Definition 2.4 (Satellite PL cycles)**.** For a rational function f, let $\mathcal{K}$ be a PL cycle of period q and write $\mathcal{K}^+$ for the union of all members of $\mathcal{K}$. Consider a repelling f-cycle

$\{\alpha_0, \ldots, \alpha_{r-1}\} = \mathcal{B}$ of period $r < q$ in $\mathcal{K}^+$, and suppose that $\mathcal{K}^+$ has r connected components, each consisting of s members of $\mathcal{K}$ with a point of $\mathcal{B}$ in common. We assume that different members of $\mathcal{K}$ can intersect at most in a single point of $\mathcal{B}$. If points of $\mathcal{B}$ are non-separating points for members of $\mathcal{K}$, that is, if $K^* \setminus \mathcal{B}$ are connected for all $K^* \in \mathcal{K}$, then $\mathcal{K}$ and is said to be *satellite* with *base period* r and *relative period* $s = \frac{q}{r} > 1$. Otherwise, i.e., when points of $\mathcal{B}$ separate PL sets from $\mathcal{K}$, one talks of a *cross renormalization.* The cycle $\mathcal{B}$ is called the *base cycle (of $\mathcal{K}$).*

The notation in Definitions 2.3 and 2.4 is used throughout. Note that the base cycle $\mathcal{B}$ can be recovered from $\mathcal{K}$: namely, $\mathcal{B}$ is the union of $K_i^* \cap K_j^*$ for all pairs of distinct K_i^*, $K_j^* \in \mathcal{K}$. Recall that C_{f^q} stands for the set of critical points of f^q.

Lemma 2.5. *Let f be a rational function with a satellite PL cycle $\mathcal{K}$ and $s \geqslant 2$. Fix $\alpha \in \mathcal{B}$ and let $\{K_i^* = f^{r(i-1)}(K_1^*)\}_{i=1}^s$ be all elements of $\mathcal{K}$ attached to α. Suppose that for each $i \in \{1, \ldots, s\}$, there is a proper map $f^q : U_i \to V_i$ of degree $d^* = \deg(f^q|_{K_1^*})$, where U_i, V_i are Jordan domains containing K_i^*. Then the following holds.*

(1) *For each j the set $U_j \setminus K_j^*$ contains no critical points of f^q.*
(2) *No point of $C_{f^q} \cap K_j^* \subset U_j$ belongs to U_i for $i \neq j$.*
(3) *There exists a critical point w of f^2 (and, hence, of f^q) that does not belong to $\bigcup U_i$.*
(4) *If, for each $i = 1, \ldots, s$, there is a continuum $Z_i \subset U_i$ such that $(C_{f^q} \cap K_i^*) \cup \{\alpha\} \subset Z_i$ then there exists j with*

$$\mathrm{mod}(U_j \setminus Z_j) < \frac{\pi}{\ln(4(s+1))}.$$

Observe that the inclusion $U_i \subset V_i$ is not assumed. Lemma 2.5 is also applicable to cross renormalization. A typical situation in which we apply the lemma is when a Jordan domain V of a PL map $f^q : U \to V$ is pulled back to create pairs of Jordan domains U_i, V_i.

Proof. (1) This claim follows from the fact that maps $f^q : U_i \to V_i$ are of degree $d^* = \deg(f^q|_{K_1^*})$. By the Riemann–Hurwitz formula, extra critical points in U_i would make the degree of $f^q : U_i \to V_i$ bigger.

(2) This follows from (1) and the setup.

(3) By (2), it suffices to prove that there must exist a critical point w of f^2 that does not belong to $\mathcal{K}^+$. Indeed, assume the contrary, i.e., $C_{f^2} = C_f \cup f^{-1}(C_f) \subset \mathcal{K}^+$. Let the degree of f be equal to $d \geqslant 2$. First, note that a critical point of any iterate f^i can belong to at most one element of $\mathcal{K}$ because any two members of $\mathcal{K}$ can intersect each other only at a point of the repelling cycle $\mathcal{B}$. Recall also that f permutes all q elements of $\mathcal{K}$. Now, choose $c_0 \in C_f$. It belongs to some $X_0 \in \mathcal{K}$. Hence, the full preimage $f^{-1}(c_0)$ is contained in some other $X_{-1} \in \mathcal{K}$, and the degree of the map $f : X_{-1} \to X_0$ is d. (Otherwise, there are two distinct members of $\mathcal{K}$ that map onto X_0 under f; a contradiction with $\mathcal{K}$ being a cycle).

As $d > 1$ and since X_{-1}, X_0 are connected Julia sets of some PL maps, there must be another point $c_{-1} \in C_f \cap X_{-1}$. Replace X_0 by X_{-1} and c_0 by c_{-1}, repeat the argument $q-1$ times. Along the way, we obtain the sets X_0, X_{-1}, X_{-2}, $\ldots$, $X_{-(q-1)}$,

$X_{-q} \in \mathcal{K}$, where $X_{-q} = X_0$ and $\deg(f : X_{-i} \to X_{-i+1}) = d$ for $i = 1, 2, \ldots, q$. Then $\deg(f^q : X_0 \to X_0) = d^q = \deg(f^q : \hat{\mathbb{C}} \to \hat{\mathbb{C}})$; recall also that X_0 is a filled PL Julia set for f^q. Thus, X_0 as well as its boundary $\partial X_0 \subset J(f)$ are completely invariant under f^q. This implies $\partial X_0 = J(f^q) = J(f)$, which is absurd.

(4) Choose arbitrary points $z_j \in C_{f^q} \cap K_j^* \subset U_j$. By (2), $z_j \notin U_i$ for any $i \neq j$. By (3), $w \notin U_i$ for any i. Set $\mathcal{P} = \{\alpha, z_1, \ldots, z_s, w\}$. It follows that the annulus $U_i \setminus Z_i$ separates $\{\alpha, z_i\}$ from all z_j with $j \neq i$ and from w. Now (4) follows from Static Theorem. □

Observe that the inequality in part (4) of Lemma 2.5 features $s+1$ in the denominator thanks to (3). Without (3), we would have shown that $\mathrm{mod}(U_j \setminus Z_j) < \frac{\pi}{\ln(4s)}$. On the other hand, having a lot of critical points of f^q outside $\bigcup U_i$ does not help one to improve the inequality because of the nature of Static Theorem. Therefore, even though Lemma 2.5(3) actually implies that there are much more than one critical point of f^q outside of $\bigcup U_i$, this observation does not help one to improve the inequality from Lemma 2.5(4).

Fix an element K^* of a PL cycle $\mathcal{K}$. Let $V \setminus \overline{U}$ be a fundamental annulus of K^*. Let U_i, V_i be iterated pullbacks of U and V containing $K_i^* = f^i(K^*)$, for $i = 0, 1, ..., q-1$. Observe that the map $f^q : U_i \to V_i$ is a PL map with PL set K_i^* and $U_i \setminus K_i^*$ is a root annulus of K_i^* (here $i = 0, \ldots, q-1$).

Lemma 2.6. *Let K^* be a satellite PL set for a degree d rational function f. Then $d^* \leqslant 2^{2d-2}$, and for each $i = 0, \ldots, q-1$,*

$$\frac{\mathrm{mod}(U \setminus K^*)}{d^*} \leqslant \mathrm{mod}(U_i \setminus K_i^*) \leqslant \mathrm{mod}(U \setminus K^*).$$

If f is a polynomial of degree d then $d^* \leqslant 2^{d-1}$.

Proof. As before, we write s for the relative period of K^*. Indeed, d^* is the product of $d_i^* = \deg(f : f^i(K^*) \to f^{i+1}(K^*))$ over $i = 0, \ldots, q-1$. The function f has $2d-2$ critical points, counting multiplicities. Positive integers d_i^* satisfy the inequality

$$(d_0^* - 1) + \cdots + (d_{q-1}^* - 1) \leqslant 2d - 2.$$

Using that $1 + x \leqslant 2^x$ for any integer $x \geqslant 0$, we deduce that

$$d_0^* \ldots d_{q-1}^* \leqslant 2^{(d_0^*-1)+\cdots+(d_{q-1}^*-1)} \leqslant 2^{2d-2}.$$

The latter inequality is sharp: equality is attained when $2d-2$ numbers d_i^* are equal to 2, and all remaining d_j^* are equal to 1. The case of a polynomial is similar. □

Proof of the Main Theorem. Let f be a rational function of degree $d \geqslant 2$ with a satellite PL cycle $\mathcal{K}$ of relative period $s \geqslant 2$. For any root annulus A of a filled PL Julia set $K^* \in \mathcal{K}$ consider its pullback root annuli of pullbacks of K^* containing the same base point as K^* itself. These root annuli together with the corresponding pullbacks of K^* satisfy the assumptions of Lemma 2.5 which implies that there is one of them of modulus which is less than or equal to $\frac{\pi}{\ln(4(s+1))}$. Then the inequalities on the moduli of $A = U \setminus K^*$ and $V \setminus U$ claimed in the statement of the Main Theorem follow respectively from Lemma 2.6 and the Groetsch inequality (the latter implies that $\text{mod}(V \setminus U) \leqslant (d^* - 1)\text{mod}(U \setminus K^*)$). ☐

3 Geometry of infinitely renormalizable sets

Let us now consider the geometry of infinitely-renormalizable sets. Given a rational function f, denote by $\mathcal{PC}_f$ the closure of the union of forward orbits of all critical values of f and call it the *postcritical set (of f)*.

Definition 3.1 (Infinitely renormalizable sets)**.** If a sequence of PL cycles $\mathcal{K}_n$ of f is such that $\mathcal{K}^+_{n+1} \subset \mathcal{K}^+_n$, and the period of $\mathcal{K}_{n+1}$ is greater than the period of $\mathcal{K}_n$, then the set $S = \bigcap_n \mathcal{K}^+_n$ is called an *infinitely renormalizable* set for f. Consider all critical points of f that belong to S; then the union of the closures of the orbits of their images is called a *postcritical infinitely renormalizable set.*

Let us now prove a couple of topological lemmas.

Lemma 3.2. *Let W_0 and W_1 be Jordan domains and let f be a branch covering map of $\mathbb{P}^1$ to itself such that $f : W_0 \to W_1$ is a degree k branched covering. Consider a Jordan curve Γ isotopic to ∂W_1 rel. $f(C_f)$ and let V be the Jordan domain enclosed by Γ that contains $f(C_f \cap W_0)$. Then the f-pullback U of V that contains $C_f \cap W_0$ is a Jordan domain that maps onto V as a branched covering map of degree k with critical set $C_f \cap U = C_f \cap W_0$.*

Observe that Γ does not have to lie in $\overline{W}_1$, and C_f is the set of all critical points of f, not only those contained in W_0.

Proof. One can assume that the isotopy connecting ∂W_1 with Γ consists of Jordan curves Γ_t parameterized by $t \in [0, 1]$. The curve Γ_t never crosses the critical values of f. Therefore, the pullbacks of Γ_t also form an isotopy γ_t parameterized by $t \in [0, 1]$ such that $\gamma_0 = \partial W_0$ and $\gamma_1 \subset f^{-1}(\Gamma)$. All γ_t enclose the same set of critical points, and all are Jordan curves. In particular, this is true for γ_1, which proves the desired. ☐

The next lemma is a consequence of well-known facts.

Lemma 3.3. *Let U and V be Jordan domains and let $f : U \to V$ be a degree k branched covering. Consider a full continuum $Z \subset V$ such that $f(C_f) \subset Z$. Then the full preimage of Z under f is a full continuum containing C_f.*

Proof. Set $Z' = f^{-1}(Z)$. The set $V \setminus Z$ is a topological annulus, and $f : U \setminus Z' \to V \setminus Z$ is a degree k unbranched covering of this annulus. All covering spaces of $V \setminus Z$ are classified by subgroups of $\pi_1(V \setminus Z) = \mathbb{Z}$, and, in particular, such a covering is determined (up to covering equivalence) by its degree. It follows that $U \setminus Z'$ is also an annulus, hence Z' is a full continuum. ☐

Now we use these lemmas to study moduli of annuli whose outer boundaries are defined through an isotopy.

Lemma 3.4. *Let f be a rational function with a satellite PL cycle $\mathcal{K}$ of degree d^* and relative period $s \geqslant 2$, let $K^* \in \mathcal{K}$, and let $V \setminus \overline{U}$ be a fundamental annulus with $U \supset K^*$. Denote the point $\mathcal{B} \cap K^*$ by α. Set $\mathcal{F} = \{\alpha\} \cup f^{2q}(C_{f^{2q}})$. Let Γ_t be an isotopy rel. $\mathcal{F}$ that transforms the Jordan curve $\partial V = \Gamma_0$ to a Jordan curve Γ_1 that encloses a Jordan domain W. Let $Z \subset W$ be a full continuum containing $\{\alpha\} \cup f^{2q}(C_{f^{2q}} \cap K^*)$. Then*

$$\operatorname{mod}(W \setminus Z) < \frac{(d^*)^2 \pi}{\ln(4(s+1))}.$$

Observe that $\{\alpha\} \cup f^{2q}(C_{f^{2q}} \cap K^*) \subset W$ by the assumption on the isotopy. However, both U and W may contain other elements of $\mathcal{F}$. Also, W is an arbitrary Jordan domain that can be obtained as described in the Lemma; it does not have to be the range of a PL map. Finally, the result is also applicable to cross renormalization.

Proof. We may assume that $r = 1$ and, hence, $s = q$. Denote by W_i the f^i-pullback of W containing α (here $1 \leqslant i \leqslant 2q$). By Lemma 3.2, the sets $W_i, 1 \leqslant i \leqslant 2q$ are Jordan domains. We claim that the critical points of $f^{2q}|_{W_{2q}}$ belong to K^*. Indeed, the f^{2q}-pullback V_{2q} of V containing α contains only critical points of f^{2q} that belong to K^*. If there exists a critical point β of $f^{2q}|_{W_{2q}}$ that does not belong to K^* then $f^{2q}(\beta) \in \Gamma_t$ for some $t \in [0, 1]$, a contradiction with the properties of Γ_t (which is an isotopy rel. $\mathcal{F}$). So, the critical points of $f^{2q}|_{W_{2q}}$ belong to K^*. Observe that the f^q-pullback of V containing α coincides with U. Moreover, the conditions on the isotopy will allow us to apply Lemma 2.5.

Now, since Z is a full continuum containing $\{\alpha\} \cup f^{2q}(C_{f^{2q}} \cap K^*)$, then Lemma 3.3 implies that the f^i-pullbacks of Z contained in the sets W_i, are full continua (here $1 \leqslant i \leqslant 2q$), and the annuli $W_i \setminus Z_i$ are f^i-pullbacks of the annulus $W \setminus Z$. Finally, let $K_i^* \in \mathcal{K}$ be the f^i-pullbacks of K^* (here and in what follows K_i^* should be understood with i taken modulo $s = q$).

We will now consider Jordan domains $W_{q+1}, W_{q+2}, \dots, W_{2q}$ that contain full continua $Z_{q+1}, Z_{q+2}, \dots, Z_{2q}$, respectively. By the previous paragraph and by Lemma 3.3, the map $f^q : W_{q+i} \to W_i$ is a branch covering map of degree d^* for any $i, 1 \leqslant i \leqslant q$, and all points of $C_{f^q} \cap K^*_{q+i}$ belong to Z_{q+i}. Applying Lemma 2.5 to this collection of Jordan domains and full continua, we find i, $1 \leqslant i \leqslant q$ with

$$\operatorname{mod}(W_{q+i} \setminus Z_{q+i}) < \frac{\pi}{\ln(4(s+1))}.$$

On the other hand, the map $f^{q+i}|_{W_{q+i}}$ is of degree at most $(d^*)^2$. Properties of moduli of annuli then imply that

$$\operatorname{mod}(W \setminus Z) < \frac{(d^*)^2 \pi}{\ln(4(s+1))}$$

as desired. □

Consider a simple closed geodesic γ in a hyperbolic Riemann surface X. By [22], there always exists an annulus $A \subset X$ homotopic to γ, such that

$$\frac{1}{\ell(\gamma)/2 \cdot e^{\ell(\gamma)/2}} \leqslant \frac{2}{\ell(\gamma)} \arcsin\left(e^{-\ell(\gamma)/2}\right) \leqslant \mathrm{mod}(A) \leqslant \frac{\pi}{\ell(\gamma)}. \qquad (\ell|m)$$

Theorem 3.5. *Let f be a rational function with a satellite PL cycle $\mathcal{K}$ of degree d^* and relative period $s > 1$. Let $K^* \in \mathcal{K}$, denote the point $\mathcal{B} \cap K^*$ by α, and set $\mathcal{F} = \{\alpha\} \cup f^{2q}(C_{f^{2q}})$. Suppose that $T \supset \mathcal{F}$ is a closed set. Let $\widetilde{X}_T = \mathbb{P}^1 \setminus T$. If γ is a Jordan curve in $\widetilde{X}_T$ isotopic to the outer boundary of a fundamental annulus around K^* rel. T then its hyperbolic length $\ell_{\widetilde{X}_T}(\gamma)$ in $\widetilde{X}_T$ satisfies the inequality*

$$\frac{2}{e^{\ell_{\widetilde{X}_T}(\gamma)/2} \cdot \ell_{\widetilde{X}_T}(\gamma)} \leqslant \frac{(d^*)^2\pi}{\ln(4(s+1))} \quad \Rightarrow \quad \ell_{\widetilde{X}_T}(\gamma) \geqslant \ln\left(\frac{4\ln(s+1)}{(d^*)^2\pi}\right).$$

Proof. Notice that if $\frac{1}{xe^x} < C$ for some C then $\frac{1}{e^{2x}} < \frac{1}{xe^x} < C$ and, hence, $x > \frac{\ln \frac{1}{C}}{2}$; thus, the second inequality is indeed a formal consequence of the first.

The theorem follows from Lemma 3.4, inequality $(\ell|m)$ displayed before Theorem 3.5, and a simple inequality discussed in the first paragraph of the proof. Note: in the estimates, we use the annuli A between K^* and Jordan curves homotopic to γ in $\widetilde{X}_T$. These annuli may have nonempty intersections with T. In this case, an annulus homotopic to γ in $\widetilde{X}_T$ is necessarily smaller than a suitable A, which makes its modulus even smaller and the corresponding hyperbolic length $\ell_{\widetilde{X}_T}(\gamma)$ even larger. □

We are ready to prove Corollary A.

Proof of Corollary A. Consider a polynomial f with a satellite PL cycle $\mathcal{K}$ of relative period $s > 1$ which is a part of an infinitely-renormalizable set S. Recall that $\mathcal{B}$ is the base cycle of $\mathcal{K}$. Suppose that all (finite) critical points of f belong to S, and set $X = \mathbb{P}^1 \setminus (\mathcal{B} \cup \mathcal{PC}_f)$. Let γ be a simple closed geodesic in X isotopic in X to the outer boundary of a fundamental annulus of a PL Julia set $K^* \in \mathcal{K}^+$. Theorem 3.5 implies now the desired inequality. This proves Corollary A. □

Theorem 3.5 as well as Corollary A are also applicable to cross renormalization.

4 Theorem B and a counterexample

Proof of Theorem B. Clearly, it is enough to prove that we have $\mathrm{mod}(V_n \setminus K_n) \to 0$. By way of contradiction, assume that $\mathrm{mod}(V_n \setminus K_n)$ stay away from 0 for an infinite subsequence of numbers n. Pulling V_n and U_n back under $f_n^{q_n}$ and then renaming V_n by W_n, U_n by V_n, and the pullback by U_n, we may assume that there are also Jordan domains W_n with $f_n^{q_n} : V_n \to W_n$ being PL-maps. By Lemma 2.1, a neighborhood of K is contained in all sets U_n. Passing to a subsequence, assume that there is a domain U such that for all n we have $K_n \subset U \Subset U_n$, and $\mathrm{mod}(U \setminus K_n) \geqslant m$ for some $m > 0$.

Lemma 4.1. *The set U is contained in the Fatou set of f.*

Proof. The *exceptional set* of f is the maximal finite subset $E_f \subset \mathbb{P}^1$ with the property $f^{-1}(E) \subset E$. By [25, Lemma 4.9], E_f consists of one or two points. Moreover, if E_f is nonempty, then f^2 is Moebius conjugate to a polynomial. Assume, by way of contradiction, that $U \cap J(f) \neq \varnothing$. Then, by [25] (see Corollary 14.2 and the following remark), the complement of $f^j(U)$ is a subset of an arbitrarily small neighborhood of E_f, for all sufficiently large j. Since $q_n \to \infty$, we may assume that q_1 is already sufficiently large, so that $f^{q_1}(U) \cup O(E_f) = \mathbb{P}^1$ for a small neighborhood $O(E_f)$ of E_f (if E_f is empty, then $f^{q_1}(U) = \mathbb{P}^1$). We may also assume that $q_1 < q_2 < \cdots < q_n < \dots$. Consider two cases.

(1) $E_f = \varnothing$; then $f^{q_1}(U) = \mathbb{P}^1$, and since $f_n^{q_1} \to f^{q_1}$ as $n \to \infty$, by compactness of $\mathbb{P}^1$ it follows that $f_n^{q_n}(U) = \mathbb{P}^1$ for large n, a contradiction with $f_n^{q_n}(U) \Subset V_n$.

(2) $E_f \neq \varnothing$; then we may assume that f is a polynomial, possibly replacing f with f^2 and f_n with f_n^2. In this case $f^{q_1}(U)$ is the whole of $\mathbb{C}$ except, perhaps, for a small neighborhood of ∞ and $V_n \supset f_n^{q_1}(U) \supset J(f_n)$ for large n. This is a contradiction as $f_n^{q_1}(K_n)$ is a proper subset of $J(f_n)$ and no points of $J(f_n)$ escape V_n under iterates of f_n. □

It follows that $U \subset \Omega$ where Ω is a Fatou component of f. The set Ω cannot be a component of the basin of an attracting cycle of f as otherwise, for large n, the PL set $K_n \subset U$ would be in the basin of an attracting cycle of f_n, a contradiction. By assumption 3 of the theorem, Ω is not in the basin of a parabolic cycle of f as well. Thus Ω is eventually mapped to a periodic rotation domain (Siegel disk or Herman ring) under f. It is safe to assume that Ω is itself a periodic rotation domain of period p.

As $\operatorname{mod}(U \setminus K_n) \geqslant m$, there is $\delta > 0$ such that $\operatorname{dist}(\partial U, K_n) > \delta$ for all n. Now, since $f^p : \Omega \to \Omega$ is conjugate to an irrational rotation, one can fix $s > 0$ such that, for each $z \in \overline{U}$ we have $\operatorname{dist}(z, f^{sp}(z)) < \delta/2$. As $f_n \to f$, it follows that for every n large enough, $f_n^{sp}(K_n) \subset U \subset U_n$ and $f_n^{sp}(K_n) \neq K_n$. The sets K_n and $K_n' = f_n^{sp}(K_n)$ are in the same PL cycle and $K_n' \setminus K_n$ is a nonempty set of points which do not escape $U \subset U_n$ under iterates of f_n, a contradiction again. □

All conditions (1) – (3) are essential for the conclusion of Theorem B. Namely, it is clear that the conclusion breaks down without condition (2). As for conditions (1) and (3), counterexamples can be found in the real unimodal family $f_c(z) = z^d + c$, with $c \in \mathbb{R}$, for every even positive d. Indeed, in that case it is known that there is a universal complex bound for a specific choice of domains for any renormalization of f_c of period s whenever f_c^{2s} has no attracting or neutral fixed point, see Theorem A of [15] (complemented by the following paragraph), see also [8, 21]. Thus it is enough to find corresponding examples of real f_c with periodic intervals. In particular, taking $f_n = f$ where f is an infinitely renormalizable real quadratic map f, we see that the conclusion of Theorem B breaks down without condition (1). As for (3), the following counterexample shows that this assumption is also necessary.

Proposition 4.2. *One can choose a sequence $f_{a_n}(z) = z^2 + a_n$ of real quadratic polynomials with connected Julia sets as follows. For each n there is a renormalization $f_{a_n}^{q_n} : U_n \to V_n$ of period q_n with a single critical point at 0, connected PL set K_n, such* *that $K_n \to K$ in the Hausdorff metric where K is non-degenerate, $q_n \to \infty$ while, for some $\delta > 0$, the modulus of each fundamental annulus $V_n \setminus U_n$ is bigger than δ.*

Proof. It is enough to choose a sequence $a_n \in [-2, 0]$ such that:

- (i) the sequence a_n is decreasing, and $a_n \searrow a$;
- (ii) the map f_a has a parabolic 3-cycle of multiplier 1;
- (iii) there is a symmetric w.r.t. 0 and f_{a_n}-periodic interval L_n of period q_n such that $f_{a_n}^{q_n} : L_n \to L_n$ is a unimodal map;
- (iv) maps $f_{a_n}^{2q_n}$ have no attracting/parabolic fixed points;
- (v) periods q_n tend to infinity;
- (vi) diameters of L_n stay away from 0.

Indeed, then, by applying the above universal complex bounds, for the sequence f_{a_n} and a special choice of renormalizations $f_{a_n}^{q_n} : U_n \to V_n$ where $L_n \subset U_n \Subset V_n$, there exists $\delta > 0$ such that $\mathrm{mod}(V_n \setminus U_n) > \delta$.

The sequence (a_n) can be defined using Lavaurs maps. Here is a detailed construction motivated by [19]. Start with a map f_a satisfying (ii). Set $F = f_a^3$, and let $x_0 < 0$ be the point of the parabolic 3-cycle of f that contains 0 in its immediate basin of attraction. Locally near the point x_0, we have $F(z) = z + A(z - x_0)^2 + B(z - x_0)^3 + ...$ where $A < 0$. Let $[c_{-2}, 0]$ be the maximal interval containing x_0 on which F is increasing. Here $c_{-2} < 0$, $f_a^2(c_{-2}) = 0$ and $F(c_{-2}) = c_1 := f_a(0)$ where $c_1 < c_{-2}$. The interval (c_1, c_{-2}) contains a point c_{-1} such that $f_a(c_{-1}) = 0$ so that f_a^2 is decreasing on $[c_1, c_{-1}]$ and increasing on $[c_{-1}, 0]$. Thus we have the following order of points: $c_1 < c_{-1} < c_{-2} < x_0$. Consider the fundamental interval $I_- = [F(0), 0]$ of the immediate attracting basin of x_0. This means that the sets $F^n(I_-)$ for $n \geqslant 0$ have disjoint interiors, and their union covers the interval $(x_0, 0]$ all of whose points converge to x_0 under forward iterations of F. Also, consider the interval $I_+ = [c_1, c_{-2}]$. This is a fundamental interval for the backward iteration of $F^{-1} : [c_1, x_0] \to [c_{-2}, x_0]$ which, from now on, denote the inverse branch of strictly increasing $F : [c_{-2}, x_0] \to [c_1, x_0]$.

Now, consider attracting (φ_-) and repelling (φ_+) Fatou coordinates of F near x_0, see, e.g., [25, 28]. Recall briefly the definition. There are two topological disks $D_\pm$ in a small neighborhood of x_0 and the point x_0 at their boundaries such that $F(D_-) \subset D_-$, $D_+ \subset F(D_+)$, while φ_-, φ_+ map D_-, $F(D_+)$ conformally on a right, respectively, left half-plane conjugating F to T_1 where $T_\sigma : z \mapsto z + \sigma$ denotes the translation by σ. That is,

$$\varphi_-(F(z)) = T_1(\varphi_-(z)) \tag{4.1}$$

for $z \in D_-$ and, correspondingly,

$$\varphi_+(F(z)) = T_1(\varphi_+(z)) \tag{4.2}$$

for $z \in D_+$. By symmetry (a being real), we may assume that $D_\pm$ are symmetric w.r.t. $\mathbb{R}$ and $\varphi_\pm(\bar{z}) = \overline{\varphi_\pm(z)}$.

Then φ_- extends by (4.1) to an orientation reversing homeomorphism $\varphi_- : (x_0, 0] \to [\varphi(0), +\infty)$ and φ_+ extends by (4.2) to an orientation reversing homeomorphism $\varphi_+ : [c_1, x_0) \to (-\infty, \varphi_+(c_1)]$. So, $\varphi_-(I_-) = [\varphi_-(0), \varphi_-(F(0))]$ with $\varphi_-(F(0)) = \varphi_-(0) + 1$ and $\varphi_+(I_+) = [\varphi_+(c_{-2}), \varphi_+(c_1)]$ with $\varphi_+(c_1)) = \varphi_+(c_{-2}) + 1$.

Notice that the Fatou coordinates $\varphi_\pm$ are unique up to post-composition by a real translation. This allows us to fix the choice of $\varphi_\pm$ in such a way that $X := \varphi_-(0) = \varphi_+(c_{-2})$ which means that $\varphi_-((x_0, 0]) = [X, +\infty)$ while $\varphi_+([c_1, x_0)) = (-\infty, X+1]$. Hence, $\varphi_-(I_-) = \varphi_+(I_+) = [X, X+1]$ and the following map is a well-defined orientation preserving homeomorphism: $g_0 := \varphi_+^{-1} \circ \varphi_- : I_- \to I_+$. More generally, let

$$g_\sigma = \varphi_+^{-1} \circ T_\sigma \circ \varphi_- \tag{4.3}$$

be the Lavaurs map. If $\sigma \leqslant 0$ then $T_\sigma \circ \varphi_-(I_-) = [X+\sigma, X+1+\sigma] \subset \varphi_+([c_1, x_0))$, i.e., for each $\sigma \leqslant 0$, the map $g_\sigma : I_- \to [c_1, x_0)$ is a well-defined orientation preserving homeomorphism onto its image $[g_\sigma(F(0))), g_\sigma(0)]$ where $g_\sigma(F(0)) = F(g_\sigma(0))$. When σ monotonically moves from 0 to the left, the endpoints $g_\sigma(F(0))$, $g_\sigma(0)$ of the image $g_\sigma(I_-)$ move monotonically to the right. There is a unique $\sigma_0 < 0$ such that $g_{\sigma_0}(F(0)) = c_{-1}$. For every $\sigma \in [\sigma_0, 0]$, there exists a unique solution $q_\sigma \in I_-$ of the equation $g_\sigma(x) = c_{-1}$, so that $g_\sigma([q_\sigma, 0]) = [c_{-1}, g_\sigma(0)]$. Note that q_σ increases from q_0 to q_{σ_0} as σ decreases from 0 to σ_0.

Let $G_\sigma = f_a^2 \circ g_\sigma$. As f_a^2 is increasing on $[c_{-1}, 0]$, for every $\sigma \in [\sigma_0, 0]$, the map $G_\sigma : [q_\sigma, 0] \to [c_1, f_a^2(g_\sigma(0))]$ is an orientation preserving homeomorphism. The map g_σ extends immediately by symmetry to an even map on $[F(0), -F(0)]$ (which is again denoted by g_σ). Therefore, we get a unimodal map G_σ on $[q_\sigma, -q_\sigma]$, for each $\sigma \in [\sigma_0, 0]$.

The following holds: (a) G_σ is increasing on $[q_\sigma, 0]$ and is an even function on $[q_\sigma, -q_\sigma]$, for $\sigma_0 \leqslant \sigma \leqslant 0$, (b) $G_\sigma(q_\sigma) = c_1 < q_\sigma < 0$ for $\sigma_0 \leqslant \sigma \leqslant 0$, (c) $G_0(0) = 0$ and $G_{\sigma_0}(0) = -c_{-2} > -F(0) > -q_{\sigma_0} > 0$, (d) $G_\sigma(0)$ decreases from $G_{\sigma_0}(0) > 0$ to $G_0(0) = 0$ as σ increases from σ_0 to 0.

Indeed, (a) – (b) hold by the construction and $G_0(0) = f_a^2(c_{-2}) = 0$. Now, $G_{\sigma_0}(0) = f_a^2(F^{-1}(c_{-1}))$ is a point of $f_a^{-1}(\{c_{-1}\})$ where $c_{-1} < 0$ and $f_a(c_{-1}) = 0$. On the other hand, $c_1 < c_{-1} < c_{-2} < F^{-1}(c_{-1}) < 0$ and f_a^2 increases on $[c_{-1}, 0]$, hence, $0 < G_{\sigma_0}(0)$. There is just one positive point of $f_a^{-1}(\{c_{-1}\})$, which is $-c_{-2}$. To finish with (c), it remains to note that $c_{-2} < x_0 < F(0) < q_{\sigma_0}$. Finally, (d) follows from (c).

Now, (4.1) allows us to extend φ_- from D_- to an analytic function in the component Δ attached to x_0 of the basin of attraction of the parabolic 3-cycle of f_a while the inverse map φ_+^{-1} extends by (4.2) from $\varphi_+(F(D_+))$ to an entire function. Hence, g_σ extends to an analytic function to Δ. The main purpose of introducing g_σ (and our use of it) is the following theorem due to Douady and Lavaurs [5] stated (in the particular case of 3-cycle) as follows: *for every $\sigma \in \mathbb{R}$ there exists a sequence $a_n \searrow a$ and an increasing sequence of positive integers N_n such that $g_\sigma(z) = \lim_{n\to\infty} f_{a_n}^{3N_n}(z)$ uniformly on compact subsets of Δ.*

As $[F(0), -F(0)] \subset \Delta$ and f_a has a negative Schwarzian derivative $Sf_a < 0$ on $\mathbb{R}$ this theorem implies, in particular, that $Sg_\sigma \leqslant 0$ on $[F(0), -F(0)]$. (This also follows directly from the fact that g_σ^{-1} extends from the real interval to a univalent function of the upper half plane into itself, see [19].) As $G_\sigma = f_a^2 \circ g_\sigma$ and $Sf_a^2 < 0$ on $\mathbb{R}$, then $SG_\sigma < 0$ on $[F(0), -F(0)]$. Now, G_σ has on $[F(0), -F(0)]$ precisely 3 critical points: 0 and two symmetric critical points at $\pm q_\sigma$. Along with relations $G_\sigma(q_\sigma) < q_\sigma < 0 < G_\sigma(0)$ for $\sigma_0 \leqslant \sigma < 0$, $G_0(0) = 0$, and general properties of maps with the negative Schwarzian [24] we obtain that, for each $\sigma \in [\sigma_0, 0]$ the map G_σ has a unique (orientation preserving) fixed point $\beta_\sigma \in (q_\sigma, 0)$. Moreover, it is repelling. So we have a unimodal restriction of G_σ to $L_\sigma := [\beta_\sigma, -\beta_\sigma]$. Now, as $\beta_0 < 0 = G_0(0) < -\beta_0$ while $-\beta_{\sigma_0} < G_{\sigma_0}(0)$, by continuity, there is a Chebyshev parameter $\sigma_{Ch} \in (\sigma_0, 0)$, i.e., such that $G^2_{\sigma_{Ch}}(0) = \beta_{Ch}$. Let us fix any $\sigma_* \in (\sigma_{Ch}, 0)$ such that $G_* := G_{\sigma_*} : L_* \to L_*$ where $L_* = L_{\sigma_*}$ has no attracting or neutral fixed point or 2-cycle. For example, any σ_* close enough σ_{Ch} would work. Notice that $L_* \subset [F(0), -F(0)]$. By the earlier stated theorem of Douady and Lavaurs, there exists a sequence $a_n \searrow a$ and an increasing sequence of positive integers N_n such that $g_{\sigma_*}(z) = \lim_{n\to\infty} f^{3N_n}_{a_n}(z)$ uniformly in some complex neighborhood of $[F(0), -F(0)]$. Let $q_n = 3N_n + 2$. Then, for each large enough n, the map f_{a_n} has a symmetric periodic interval $L_n \ni 0$ of period q_n and $L_n \to L_*$ as $n \to \infty$. Besides, $f^{2q_n}_{a_n}$ has no attracting or neutral fixed point on L_n. It follows that the sequence (a_n) is as required. Notice that in the above example the number of components of the orbit of K_n of size at least k asymptotically, as $n \to \infty$, is $Ck^{-1/2}$ with some $C > 0$. □

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