LOCATION OF SIEGEL CAPTURE POLYNOMIALS IN PARAMETER SPACES

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Dedicated to the memory of Anatole Katok

Abstract. We study the set of cubic polynomials f with a marked fixed point. If f has a Siegel disk at the marked fixed point, and if this disk contains an eventual image of a critical point, we call f a *IS-capture polynomial* ("IS" stands for Invariant Siegel). We study the location of IS-capture polynomials in the parameter space of all marked cubic polynomials modulo affine conjugacy. In particular, we show that any IS-capture polynomial is on the boundary of a unique bounded hyperbolic component determined by the rational lamination of the map. We also relate IS-capture polynomials to the cubic Principal Hyperbolic Domain and its closure (by definition, the *cubic Principal Hyperbolic Domain* consists of cubic hyperbolic polynomials with Jordan curve Julia sets).

1. Introduction

A complex polynomial P of any degree is said to be *hyperbolic* if all of its critical points belong to the basins of attracting or superattracting periodic cycles. The set of all hyperbolic polynomials in any particular parameter space is open. Components of this set are called *hyperbolic components*. The dynamics of hyperbolic complex polynomials is well understood. According to the famous Fatou conjecture [Fat20], hyperbolic polynomials are dense in the parameter space of all complex polynomials. This explains why hyperbolic components play a prominent role in complex dynamics.

According to a general result of J. Milnor [Mil12], every bounded hyperbolic component in the moduli space of degree d polynomials is an open topological cell of complex dimension $d-1$. Hence it is fair to say that the structure of such hyperbolic domains is known.

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However, in degrees greater than 2, the same cannot be said about the closures of hyperbolic components. Arguably in the most important case of the cubic Principal Hyperbolic Domain PHD_3, the description of its boundary has proved to be rather elusive. For example, in a recent paper by L. Petersen and T. Lei [LP09] it is shown that the boundary of PHD_3 has a very intricate "fractal" structure that is not fully understood. Thus, understanding the boundaries of hyperbolic components in particular, understanding of the boundary of PHD_3, is an important open problem.

Qualitative changes in the dynamics of polynomials take place on the boundary of the connectedness locus. It is known that boundaries of hyperbolic components are contained in the boundary of the entire connectedness locus. This provides an additional incentive for studying boundaries of hyperbolic components.

In our paper we consider these issues in the cubic case. More precisely, we consider the parameter space of cubic polynomials with a marked fixed point. The corresponding connectedness locus contains many complex analytic disks in its boundary. A typical example is provided by *IS-capture polynomials*, i.e., polynomials that have an invariant Siegel domain around the marked fixed point and a critical point which is eventually mapped into it. We prove that an IS-capture polynomial f belongs to the boundary of a *unique* bounded hyperbolic component with the same *rational lamination* as f. We also prove that f belongs to a complex analytic disk lying in the boundary of this hyperbolic component. We give a combinatorial description of all hyperbolic components whose boundaries can contain IS-capture polynomials.

In [BOPT14a] it was proven that all polynomials from PHD_3 satisfy some simple conditions. Our standing conjecture is that these conditions are not only necessary but actually sufficient for a polynomial to belong to the closure of PHD_3. Working in this direction, we showed in [BOPT14b] that bounded components complementary to the intersection of $\overline{\mathrm{PHD}}_3$ with slices of the parameter space can be either of so-called *queer* type, or must contain an IS-capture polynomial. The results of the present paper together with those of [BOPT16a] allow one to rule out the case of IS-capture polynomials.

Finally, we show that if a polynomial P which does not belong to the closure of PHD_3 and has multiplier $\lambda = e^{2\pi i\theta}$ at its fixed point w, where θ is not a Brjuno number, then w is a Cremer fixed point of P. Thus, a counterexample to the Douady conjecture in the cubic case *must* be a polynomial from the boundary of PHD_3.

2. Detailed statement of the results

We write $\mathbb{C}$ for the plane of complex numbers. The *Julia set* of a polynomial $f : \mathbb{C} \to \mathbb{C}$ is denoted by $J(f)$, and the *filled Julia set* of f by $K(f)$. For quadratic polynomials, a crucial object of study is the *Mandelbrot set* $\mathcal{M}_2$. Let $P_c(z)$ be a quadratic polynomial defined by the formula $P_c(z) = z^2 + c$. Clearly, 0 is the only critical point of the polynomial P_c in $\mathbb{C}$. By definition, $c \in \mathcal{M}_2$ if the orbit of 0 under P_c is bounded. Equivalently, $c \in \mathcal{M}_2$ if and only if the filled Julia set $K(P_c)$ is connected. If $c \notin \mathcal{M}_2$, then the set $K(P_c)$ is a Cantor set.

By *classes* of polynomials we mean affine conjugacy classes. The class of f is denoted by $[f]$. Complex numbers c are in one-to-one correspondence with classes of quadratic polynomials. A higher-degree analog of the set $\mathcal{M}_2$ is the *degree d connectedness locus* $\mathcal{M}_d$, i.e., the set of classes of degree d polynomials f all of whose critical points do not escape or, equivalently, whose Julia set $J(f)$ is connected.

The structure of the Mandelbrot set is described in the seminal work of Thurston [Thu85] (see also [DH8485]). In particular, [Thu85] gives a full description of how distinct hyperbolic components of $\mathcal{M}_2$ are located with respect to each other and what kind of dynamics is exhibited by polynomials from their boundaries. However, for degrees $d > 2$ studying the set $\mathcal{M}_d$ has proven to be a difficult task. The combinatorial structure of $\mathcal{M}_d$ remains elusive despite some recent progress (see [BOPT17a] for the general case and [BOPT17b, BOPT17c] for the case of cubic polynomials with only repelling periodic points).

The central and, arguably, the simplest part of the Mandelbrot set is the (*quadratic*) *Principal Hyperbolic Domain* denoted by PHD_2. It is the set of all parameter values c such that the polynomial P_c has an attracting fixed point. All these polynomials have Jordan curve Julia sets. The closure $\overline{\mathrm{PHD}}_2$ of PHD_2 consists of all parameter values c such that P_c has a non-repelling fixed point. It is sometimes called the *filled Main Cardioid*. Its boundary $\mathrm{Bd}(\mathrm{PHD}_2)$ is a plane algebraic curve, a cardioid called the *Main Cardioid*. As follows from the Douady–Hubbard parameter landing theorem and from the "no ghost limbs" theorem by Yoccoz [DH8485, Hub93], the Mandelbrot set itself can be thought of as the union of $\overline{\mathrm{PHD}}_2$ and *limbs*, connected components of $\mathcal{M}_2 \setminus \overline{\mathrm{PHD}}_2$, parameterized by reduced rational fractions $p/q \in (0, 1)$.

It is natural to consider analogs of the Main Cardioid for higher degree polynomials, in particular for cubic polynomials. This motivates our interest to the boundary of the *cubic Principal Hyperbolic Domain* PHD_3 and to a closely related set, the so-called *Main Cubioid*, that was studied in a few recent papers ([BOPT14a, BOPT14b, BOPT16a, BOPT16b]). In this framework an important task is to describe whether polynomials with certain dynamical properties belong to the boundary of the Main Cubioid. This is one of the problems addressed in the present paper.

Let us now concentrate on cubic polynomials. Let $\mathcal{F}$ be the space of polynomials $f_{\lambda,b}$ given by the formula

$$f_{\lambda,b}(z) = \lambda z + bz^2 + z^3, \quad \lambda \in \mathbb{C}, \quad b \in \mathbb{C}.$$

The space $\mathcal{F}$ is adapted to studying polynomials with a marked fixed point. Any such polynomial is affinely conjugate to one from $\mathcal{F}$ under a conjugacy sending the marked fixed point to 0. All polynomials $g \in \mathcal{F}$ have 0 as a fixed point. Let the λ-*slice* $\mathcal{F}_\lambda$ of $\mathcal{F}$ be the space of all polynomials $g \in \mathcal{F}$ with $g'(0) = \lambda$. It is well known that two polynomials $f_{\lambda,b}$ and $f_{\lambda,b'}$ are conjugate by a Möbius transformation $M(z)$ that fixes 0 if and only if $M(z) = \pm z$ and $b' = \pm b$. We will deal with $f \in \mathcal{F}_\lambda$ for some λ and consider only perturbations of f in $\mathcal{F}$. Set $\mathcal{F}_{at} = \bigcup_{|\lambda|<1} \mathcal{F}_\lambda$ (the subscript *at* stands for **at**tracting).[1] Let us emphasize that $\mathcal{F}_{at}$ is the family of polynomials from $\mathcal{F}$ that have the point 0 as an attracting fixed point. For each $g \in \mathcal{F}_{at}$, let $A(g)$ be the immediate basin of attraction of 0. Denote by $\mathcal{F}_{nr}$ the set of all polynomials $f = f_{\lambda,b} \in \mathcal{F}$ such that 0 is **n**on-**r**epelling for f (so that $|\lambda| \leqslant 1$).

Suppose that a is a fixed point of a polynomial f of any degree. Assume that $f'(a) = e^{2\pi i\theta}$ where θ is irrational. Then a is said to be an *irrationally indifferent* fixed point. If f is *linearizable* (i.e., analytically conjugate to a rotation) in a neighborhood of a, the point a is called a *Siegel* fixed point. In this case the rotation in question is well defined and is the rotation by $2\pi\theta$ so that θ is called the *rotation number*. Moreover, this is equivalent to the existence of an *orientation preserving topological* conjugacy between f in a neighborhood of a and the rotation by $2\pi\theta$ of the unit disk. If a is a Siegel fixed point, the biggest neighborhood of a on which f is linearizable exists and is called the *Siegel disk* around a. If f is not linearizable in any neighborhood of a then the point a is called a *Cremer* fixed point.

Definition 2.1 (Siegel captures)**.** Suppose that a polynomial $f \in \mathcal{F}$ has a Siegel disk $\Delta(f)$ around 0. If a critical point of f is *eventually* mapped to $\Delta(f)$, then this critical point is denoted by $\mathrm{ca}(f)$ (here "ca" stands for "captured"), and f is called an *IS-capture polynomial*, or simply an *IS-capture* (here "I" stands for "invariant" and "S" stands

[1]The set $\mathcal{F}_{at}$ was denoted by $\mathcal{A}$ in [BOPT14b, BOPT16b]. We adopt a more consistent notation in this paper.

for "Siegel"). By [Man93], there exists a recurrent critical point $\mathrm{re}(f)$ of f (here "re" stands for "recurrent") whose limit set contains $\mathrm{Bd}(\Delta(f))$. It follows that the critical points $\mathrm{ca}(f)$ and $\mathrm{re}(f)$ are well-defined and distinct (evidently, $\mathrm{ca}(f)$ is not recurrent).

Remark 2.2. Generically, maps in the family $\mathcal{F}$ have three fixed points. Any of these points, not only 0, could have a Siegel disk around it that captures a critical point. However, let us stress that we only speak of IS-captures when 0 is the Siegel fixed point whose Siegel disk captures a critical point.

In this paper, we study the location of IS-captures in $\mathcal{F}$ relative to hyperbolic components. An important role here is played by the set $\mathcal{P}^\circ$ of $\mathcal{F}$ of all hyperbolic polynomials $f \in \mathcal{F}$ such that $f \in \mathcal{F}_{at}$ and $J(f)$ is a Jordan curve. Equivalently, $f \in \mathcal{F}_{at}$ belongs to $\mathcal{P}^\circ$ if and only if $A(f)$, the immediate basin of attraction of 0, contains both critical points of f. Evidently, $\mathcal{P}^\circ$ is open in $\mathcal{F}$. To see that $\mathcal{P}^\circ$ is one hyperbolic component of $\mathcal{F}$, not only of $\mathcal{F}_{at}$, observe that polynomials $f_{b,\lambda} = z^3 + bz^2 + \lambda z$ with $|\lambda| = 1$ are not hyperbolic and that by Corollary 4.13, the set $\mathcal{P}^\circ$ is connected.

Definition 2.3. The set $\mathcal{P}^\circ$ is called the *principal hyperbolic component* of $\mathcal{F}$. We say that a hyperbolic polynomial $f \in \mathcal{F}_{at}$ is an *IA-capture polynomial* (IA stands for *Invariant Attracting*) if a critical point of f, denoted by $\omega_2(f)$, is eventually mapped to $A(f)$ but does not lie in $A(f)$ (then the remaining critical point $\omega_1(f)$ belongs to $A(f)$, and no critical point of f belongs to $J(f)$). A hyperbolic component $\mathcal{U}$ of $\mathcal{F}$ is of *IA-capture type* if $\mathcal{U}$ contains an IA-capture polynomial. Hyperbolic components of IA-capture type will also be called *IA-capture components*.

Similarly to Remark 2.2, we emphasize that IA-capture polynomials have 0 as their attracting fixed point. Evidently, both critical points $\omega_1(f), \omega_2(f)$ are well-defined for an IA-capture polynomial f. Observe also that, similarly to the above, the fact that polynomials $f_{b,\lambda} = z^3 + bz^2 + \lambda z$ with $|\lambda| = 1$ are not hyperbolic implies that any hyperbolic component $\mathcal{U}$ of $\mathcal{F}$ of IA-capture type is contained in $\mathcal{F}_{at}$. Thus, the principal hyperbolic component $\mathcal{P}^\circ$ of $\mathcal{F}$ and the hyperbolic components of $\mathcal{F}$ of IA-capture type are subsets of $\mathcal{F}_{at}$.

We also need the concepts of rational lamination and full lamination. Denote by $\mathbb{D}$ the open unit disk in the complex plane centered at the origin and by $\mathbb{S}$ the unit circle which is the boundary of $\mathbb{D}$. We will identify $\mathbb{R}/\mathbb{Z}$ with $\mathbb{S}$ via $x \mapsto e^{2\pi i x}$.

Let f be a polynomial of degree greater than 1 and connected Julia set. In this case all external rays with rational arguments land. Given two rational angles $\alpha, \beta \in \mathbb{R}/\mathbb{Z}$, we declare $\alpha \sim_r \beta$ iff the landing points of the corresponding external rays coincide. This defines an equivalence relation on $\mathbb{Q}/\mathbb{Z}$. The equivalence classes are finite (see Theorems 3.5 and 3.6 with references). We then consider the collection $\mathcal{L}^r_f$ of all edges of the convex hulls of all equivalence classes and call it the *rational lamination* of f.

If the Julia set $J(f)$ is locally connected, then all external rays land. Given any two angles $\alpha, \beta \in \mathbb{R}/\mathbb{Z}$ we declare that $\alpha \sim \beta$ iff the landing points coincide. This defines an equivalence relation on $\mathbb{R}/\mathbb{Z}$, and in this case too the equivalence classes are finite (see Theorems 3.5 and 3.6 and Theorem 1.1 of [Kiw02]). The collection of all edges of the convex hulls of all classes is denoted $\mathcal{L}_f$ and is called the *(full) lamination* of f. We will refer to the elements of $\mathcal{L}_f$ as *leaves.*

We include in each lamination the singletons $\{e^{2\pi i\alpha}\}$ and call them *degenerate leaves,* with $\alpha \in \mathbb{Q}/\mathbb{Z}$ for $\mathcal{L}^r_f$, resp. $\alpha \in \mathbb{R}/\mathbb{Z}$ for $\mathcal{L}_f$. The set $\mathcal{C}$ of all possible chords of the unit disk and singletons in the unit circle is equipped with a natural topology that associates to a chord $\overline{ab}$ of $\mathbb{S}$ with endpoints a, $b \in \mathbb{S}$ the pair $\{a, b\}$ of the symmetric product $\mathbb{S} \times \mathbb{S}/(a, b) \sim (b, a)$.

Clearly, in the case when $J(f)$ is locally connected we have $\mathcal{L}^r_f \subset \mathcal{L}_f$ and, since $\mathcal{L}_f$ is closed (see Section 3), $\overline{\mathcal{L}^r_f} \subset \mathcal{L}_f$. Contrary to what one may expect, it is *not always* true that $\overline{\mathcal{L}^r_f} = \mathcal{L}_f$. A typical example is the case of a quadratic polynomial Q with invariant Siegel domain and locally connected Julia set. Then $\mathcal{L}^r_Q$ consists only of degenerate leaves and, therefore, $\overline{\mathcal{L}^r_Q}$ cannot be distinguished from the rational lamination of z^2 (abusing the language we will call such a lamination the *empty* lamination). For IS-capture polynomials, we relate rational and full laminations in Subsection 3. Recall that a polynomial with connected Julia set that belongs to a hyperbolic component has a locally connected Julia set and, hence, a well-defined lamination.

Theorem A. *If $f \in \mathcal{F}$ is an IS-capture polynomial, then there is a unique bounded hyperbolic component $\mathcal{U}$ in $\mathcal{F}$, whose boundary contains f. Moreover, $\mathcal{U} \subset \mathcal{F}_{at}$, we have $\mathcal{L}_P = \overline{\mathcal{L}^r_P}$ for all $P \in \mathcal{U}$, and there are two possibilities:*

1. *the Julia set of f contains no periodic cutpoints, then $\mathcal{U} = \mathcal{P}^\circ$;*
2. *the Julia set of f has a repelling periodic cutpoint, then $\mathcal{U}$ is of IA-capture type.*

A polynomial is said to be *J-stable with respect to a family of polynomials* if its Julia set admits an equivariant holomorphic motion over some neighborhood of the map in the given family [Lyu83, MSS83]. Say that $f \in \mathcal{F}_\lambda$ is *λ-stable* if it is J-stable with respect to $\mathcal{F}_\lambda$ with $\lambda = f'(0)$, otherwise f is called *λ-unstable*. A component of the set of λ-stable polynomials in $\mathcal{F}_\lambda$ is called an *IS-capture component* if some (equivalently, all) polynomials from this component are IS-capture polynomials. Thus IS-capture components are complex one-dimensional analytic disks in the two-dimensional space $\mathcal{F}$. Every such disk is contained in a slice $\mathcal{F}_\lambda$ represented as a straight (complex) line in coordinates (λ, b) of $\mathcal{F}$.

Theorem B. *Every IS-capture polynomial belongs to some IS-capture component. Every IS-capture component is contained in the boundary of a unique hyperbolic component $\mathcal{U}$ of $\mathcal{F}$. Moreover, $\mathcal{U} = \mathcal{P}^\circ$ or $\mathcal{U}$ is of IA-capture type. Conversely, let $\mathcal{U}$ be either an IA-capture component or $\mathcal{P}^\circ$. Then the boundary of $\mathcal{U}$ contains uncountably many IS-capture components lying in $\mathcal{F}_\lambda$, where $\lambda = e^{2\pi i\theta}$, and θ runs through all Brjuno numbers in $\mathbb{R}/\mathbb{Z}$.*

The first claim is contained in [Zak99, Theorem 5.3]. A more precise formulation of the second part of Theorem B is contained in Theorem 6.5. We will apply Theorem A to the study of $\mathcal{P}$, the closure of $\mathcal{P}^\circ$ in $\mathcal{F}$. The following are some properties of polynomials in $\mathcal{P}$.

Theorem 2.4 ([BOPT14a])**.** *If $f = f_{\lambda,b} \in \mathcal{P}$, then $|\lambda| \leqslant 1$, the Julia set $J(f)$ is connected, f has no repelling periodic cutpoints in $J(f)$, and all its non-repelling periodic points, except possibly 0, have multiplier 1.*

These properties extend almost verbatim to the higher degree case [BOPT14a]. Theorem 2.4 motivates Definition 2.5.

Definition 2.5 ([BOPT14a])**.** Let $\mathcal{CU}$ be the family of cubic polynomials $f \in \bigcup_{|\lambda|\leqslant 1} \mathcal{F}_\lambda$ such that $J(f)$ is connected, f has no repelling periodic cutpoints in $J(f)$, and all its non-repelling periodic points, except possibly 0, have multiplier 1. The family $\mathcal{CU}$ is called the *Main Cubioid* of $\mathcal{F}$.

Note that $\mathcal{P}^\circ$ and $\mathcal{CU}$ are subsets of $\mathcal{F}$ that play a similar role to the principal hyperbolic component PHD_3 and the main cubioid CU in the (unmarked) moduli space of cubic polynomials. However, the difference is that, when defining $\mathcal{P}^\circ$ and $\mathcal{CU}$, we take into account the special role of the marked fixed point 0 for polynomials in $\mathcal{F}$. As a consequence, the sets $\mathcal{P}^\circ$ and $\mathcal{CU}$ are not stable under arbitrary affine conjugacies. By Theorem 2.4, this definition immediately implies that

$$\mathcal{P} \subset \mathcal{CU}.$$

For a compact set $X \subset \mathbb{C}$, define the *topological hull* $\mathrm{TH}(X)$ of X as the union of X with all bounded components of $\mathbb{C} \setminus X$. We write $\mathcal{P}_\lambda$ for the λ-slice of $\mathcal{P}$, i.e., for the set $\mathcal{P} \cap \mathcal{F}_\lambda$.

Theorem C. *IS-capture polynomials do not belong to $\mathcal{CU} \setminus \mathcal{P}$. If $\mathcal{W}$ is a component of $\mathrm{TH}(\mathcal{P}_\lambda) \setminus \mathcal{P}_\lambda$ and $f \in \mathcal{W}$, then the following holds.*

1. *Any such polynomial f is λ-stable.*
2. *Critical points of f are distinct and belong to $J(f)$.*
3. *The Julia set $J(f)$ has positive Lebesgue measure and carries an invariant line field.*

Parts of Theorem B follow from [Zak99, Theorem 3.4].

In Section 7, we obtain corollaries of Theorem B that help distinguish between Siegel and Cremer fixed points of a given multiplier.

3. Rays and laminations

We will make use of the concepts of the full/rational *lamination* associated to a polynomial with connected Julia set. These concepts are due to Thurston [Thu85] and Kiwi [Kiw97, Kiw01, Kiw04]. In fact, in [Thu85] full laminations are defined independently of polynomials as a combinatorial concept and are often studied in that setting (see, e.g., [BMOV13]). Laminations are important tools of combinatorial complex polynomial dynamics. Some of these tools are applicable to polynomials of arbitrary degree, including those with non-locally connected Julia sets. However, for the sake of brevity in this paper we avoid unnecessary generality and define full lamination only in the case when P has a locally connected Julia set.

3.1. **Rays.** Studying periodic external rays of polynomials is a powerful tool in complex dynamics. Given a polynomial f with connected Julia set we denote by $R_f(\alpha)$ the external ray of f with argument α. (According to our convention, arguments of external rays are elements of $\mathbb{R}/\mathbb{Z}$ rather than $\mathbb{R}/2\pi\mathbb{Z}$.) The arguments of external rays depend on the choice of a Böttcher coordinate near infinity. For an arbitrary cubic polynomial, such coordinate is defined up to a sign, i.e., up to the involution $z \mapsto -z$. However, for $f \in \mathcal{F}$, we can distinguish a linearizing coordinate asymptotic to the identity. We assume that, whenever $f \in \mathcal{F}$, the linearizing coordinate near infinity is chosen in this way.

We begin this subsection by quoting known results concerning rational rays.

Lemma 3.1 (see, e.g., [Mil06], Section 18). *Let f be a polynomial. All external rays of f with (pre)periodic arguments land. The landing points eventually map to periodic points that are parabolic or repelling. If $J(f)$ is connected then all rays landing at points that are eventually mapped to parabolic or repelling periodic points have rational arguments.*

We call an external ray *smooth* if it does not contain an escaping (pre)critical point. Let us now state results on continuity of smooth periodic rays landing at repelling periodic points. The next lemma can be found in [GM93] (Lemma B.1) or [DH8485] (Lecture VIII, Section II, Proposition 3).

Lemma 3.2. *Let f be a polynomial, and z be a repelling periodic point of f. If a* smooth periodic *ray $R_f(\theta)$ lands at z, then, for every polynomial g sufficiently close to f, the ray $R_g(\theta)$ lands at a repelling periodic point w close to z, and w depends holomorphically on g.*

A useful corollary of this lemma is stated below.

Corollary 3.3 (Lemma 4.7 [BOPT14b]). *Suppose that $h_n \to h$ is an infinite sequence of polynomials of degree d with connected Julia sets, and $\{\alpha, \beta\}$ is a pair of periodic arguments such that the external rays $R_{h_n}(\alpha)$, $R_{h_n}(\beta)$ land at the same repelling periodic point x_n of h_n. If the external rays $R_h(\alpha)$, $R_h(\beta)$ do not land at the same periodic point of h, then one of these two rays must land at a parabolic point of h.*

Lemma 4.7 of [BOPT14b] is more general and includes (with provisions) the case when Julia sets of polynomials h_n are disconnected.

The following result is purely topological and is based on local behavior of polynomials at points of the plane. Given a polynomial f with connected Julia set $J(f)$ and a point $z \in J(f)$, denote by A_z the set of arguments of rays landing at z. It is known [Hub93] that A_z is finite. Given a finite set $X \subset \mathbb{S}$, the points $a, b, c \in X$ are said to be *consecutive* if the positively oriented arcs (a, b) and (b, c) are disjoint from X (observe that the order of points in this definition is essential).

Theorem 3.4 (cf Lemma 18.1 [Mil06]). *Let f be a polynomial of degree $d > 1$ whose Julia set $J(f)$ is connected. (We do not assume that $J(f)$ is locally connected.) Let $z \in J(f)$ be a point such that $A_z \neq \varnothing$. Then $\sigma_d|_{A_z}$ is a k-to-1 map between A_z and $A_{f(z)}$, and, if z is non-critical, then $k = 1$. Moreover, there are two possibilities.*

(1) The set $\sigma_d(A_z) = A_{f(z)}$ is a singleton.
(2) Given any three consecutive points a, b, c in A_z, the points $\sigma_d(a)$, $\sigma_d(b)$ and $\sigma_d(c)$ form a triple of consecutive points in $A_{f(z)}$.

The next result is classical and has a proof using the Schwarz-Pick metric in [DH8485]. Recall that the (pre)periodic external rays are exactly those whose arguments are rational.

Theorem 3.5 (Proposition 2, Section II, Lecture VIII [DH8485])**.** *Let f be a polynomial of degree greater than one with connected Julia set. Then all rational external rays for f land, and their landing points are (pre)periodic points eventually mapped to repelling or parabolic periodic points.*

The next theorem is a form of converse of Theorem 3.5. It is due to A. Douady.

Theorem 3.6 (Theorem I.A [Hub93])**.** *Let f be a polynomial of degree $d > 1$ whose Julia set $J(f)$ is connected. Let $z \in J(f)$ be a repelling or parabolic periodic point. Then:*

i. The point $z \in J(f)$ is the landing point of at least one periodic external ray.
ii. Every external ray landing at z is periodic.
iii. All periodic external rays landing at z have the same period.
iv. There are finitely many external rays landing at z.

Once one proves the first claim, the others follow from it, Theorem 3.4 and properties of the d-tupling map. In [Hub93] there is another proof, using the Yoccoz inequality.

The following nice theorem will not be used in its full strength; we add it for the sake of completeness. A *wandering point* in $J(f)$ is a point whose orbit is infinite: this is the opposite of being (pre)periodic.

Theorem 3.7 ([Kiw02])**.** *Let f be a polynomial of degree $d > 1$ with locally connected Julia set $J(f)$. Then there exists an integer $k = k(d)$ independent of f, such that every wandering point $z \in J(f)$ can be the landing point of at most k external rays.*

3.2. **Full lamination.** For a (finite or infinite) set $A \subset \mathbb{S}$, denote by $\mathrm{CH}(A)$ its (closed Euclidian) convex hull. A *chord* $\overline{ab}$ between any two points $a, b \in \mathbb{S}$ is $\mathrm{CH}(\{a, b\})$ and contains the endpoints a and b. If $b = a$ the chord is called *degenerate*. Consider a closed set $A \subset \mathbb{S}$ and its convex hull $\mathrm{CH}(A)$. An *edge* of $\mathrm{CH}(A)$ is a closed straight segment I connecting two points of $\mathbb{S}$ such that $I \subset \mathrm{Bd}(\mathrm{CH}(A))$. Define the map $\sigma_d : \mathbb{S} \to \mathbb{S}, \mathbb{S} \subset \mathbb{C}$, by $\sigma_d(s) = s^d$. Then the (σ_d-)*image* of a chord $\overline{ab}$ is by definition the chord $\overline{\sigma_d(a)\sigma_d(b)}$. A ($\sigma_d$-)*critical chord* is a *non-degenerate* chord whose endpoints have the same σ_d-image.

Consider a locally connected Julia set $J(f)$ of a polynomial f of degree d. We can associate to f an equivalence relation $\sim_f$ on $\mathbb{S}$ as follows. Two points α and β in $\mathbb{S}$ are $\sim_f$-equivalent if $R_f(\alpha)$ and $R_f(\beta)$ land at the same point. Then $J(f)$ is homeomorphic to $\mathbb{S}/\sim_f$. By Theorems 3.6 and 3.7, any $\sim_f$-class is finite.

Recall that an equivalence relation $\sim$, or any binary relation, on a set X can be modeled by its *graph*, the subset of pairs $(a, b) \in X \times X$ such that $a \sim b$. If f has a locally connected Julia set, then all external rays land, and the landing point depends continuously on the external angle, thus the graph of $\sim_f$ is a closed subset of $\mathbb{S} \times \mathbb{S}$.

As mentioned in Section 2, the set $\mathcal{C}$ of all possible degenerate and non-degenerate chords is naturally topologized by associating to a chord $\overline{ab}$ the pair $\{a, b\}$ from the symmetric square of $\mathbb{S}$. Lemma 3.8 follows from the fact that the graph of $\sim_f$ is closed as well as the fact that $\sim_f$-classes are finite and pairwise disjoint (we leave the proof to the reader).

Lemma 3.8. *If f has a locally connected Julia set, then $\mathcal{L}_f$ is a closed subset of the space of all chords of the unit disk with the above topology.*

Moreover, $\sigma_d : \mathbb{S} \to \mathbb{S}$ descends to a map $\sigma_d/\sim_f$: $\mathbb{S}/\sim_f \to \mathbb{S}/\sim_f$, and there exists a homeomorphism $\phi : J(f) \to \mathbb{S}/\sim_f$ conjugating $P|_{J(f)}$ with $\sigma_d/\sim_f$. This shows that the equivalence relation $\sim_f$ gives a precise model of the dynamics of $f|_{J(f)}$. However such a precise model can be obtained only in the locally connected case.

With every $\sim_f$-class G', we associate its convex hull $G = \mathrm{CH}(G')$. The *geodesic lamination* $\mathcal{L}_{\sim_f} = \mathcal{L}_f$ is defined as the set of edges of all such polygons G together with all singletons in $\mathbb{S}$. We refer to $\mathcal{L}_f$ as the *(full) geodesic lamination associated with f*. Elements of $\mathcal{L}_f$ are called *leaves*. A leaf is *degenerate* if it coincides with a point in $\mathbb{S}$; otherwise it is *non-degenerate*. If $\ell = \overline{ab}$ is a leaf, then, by Theorem 3.4, the chord $\overline{\sigma_d(a)\sigma_d(b)}$ is again a (possibly degenerate) leaf. It is denoted $\sigma_d(\ell)$. A *critical leaf* is a leaf that is a critical chord.

A *gap* of $\mathcal{L}_f$ is defined as the closure of a component of $\mathbb{D} \setminus \bigcup \mathcal{L}_f$. For any gap G of $\mathcal{L}_f$, we define G' as $G \cap \mathbb{S}$ and $\sigma_d(G)$ as $\mathrm{CH}(\sigma_d(G'))$. A gap G is said to be *invariant* if $\sigma_d(G) = G$. If $\mathcal{L}_f$ has a gap G such that G' is infinite, then the union of the convex hulls of all $\sim_f$-classes does not equal $\overline{\mathbb{D}}$.

An equivalence relation $\sim$ on $\mathbb{S}$, similar to $\sim_f$ above, can be introduced with no references to polynomials.

Definition 3.9 (Laminational equivalence relations)**.** An equivalence relation $\sim$ on the unit circle $\mathbb{S}$ is said to be *laminational* if:
(E1) the graph of $\sim$ is a closed subset in $\mathbb{S} \times \mathbb{S}$;
(E2) convex hulls of distinct equivalence classes are disjoint;
(E3) each equivalence class of $\sim$ is finite.

By an *edge* of a $\sim$-class we mean an edge of its convex hull.

Definition 3.10 (Laminational equivalences and dynamics)**.** A laminational equivalence relation $\sim$ is (σ_d-)*invariant* if:
(D1) $\sim$ is *forward invariant*: for a class $\mathbf{g}$, the set $\sigma_d(\mathbf{g})$ is a class too;
(D2) for any $\sim$-class $\mathbf{g}$, the map $\tau = \sigma_d|_{\mathbf{g}}$ extends to $\mathbb{S}$ as an orientation preserving covering map $\hat{\tau}$ such that $\mathbf{g}$ is the full preimage of $\tau(\mathbf{g})$ under the covering map $\hat{\tau}$.

To each laminational equivalence relation $\sim$ we associate the corresponding *geodesic lamination* $\mathcal{L}_\sim$ defined as the collection of all edges of convex hulls of $\sim$-classes together with all points of $\mathbb{S}$. The terminology introduced for laminations $\mathcal{L}_f$ applies to laminations $\mathcal{L}_\sim$ too. Abusing the language, we will call the lamination all of whose leaves are singletons in $\mathbb{S}$ the *empty lamination*.

3.3. **Invariant gaps of cubic laminations.** Let $\mathcal{L}_\sim$ be a cubic lamination. The *degree* of a gap G of $\mathcal{L}_\sim$ is defined as the maximal number of disjoint critical chords that can fit in G and that are not on the boundary of G, plus 1, except for the case when G is a triangle with critical edges in which case the degree of G is 3. Recall that chords include their endpoints, so by disjoint critical chords we mean chords whose endpoints are distinct. Degree 2 (respectively, 3) gaps are said to be *quadratic* (respectively, *cubic*).

By [BOPT14a], a quadratic σ_3-invariant gap G has a unique longest edge $M(G)$ called the *major* (of G). The major $M(G)$ can be either critical (then G is said to be of *regular critical type*) or periodic (then G is said to be of *periodic* type). For every edge $\ell = \overline{ab}$ of G, let $H_\ell(G)$ be the arc of the unit circle $\mathbb{S}$ with endpoints a and b and no points of G in $H_\ell(G)$. Then the major $M(G)$ can be singled out by the fact that the length of $H_{M(G)}(G)$ is greater than or equal to $1/3$ (here, the length of a circle arc is normalized so that the total length of $\mathbb{S}$ is equal to 1). We may summarize the above in the following theorem (recall that the sets $\mathcal{F}_{at}$ and $\mathcal{P}^\circ$ were defined in Section 2; for each $g \in \mathcal{F}_{at}$, we write $A(g)$ for the immediate basin of attraction of 0).

Theorem 3.11 ([BOPT16a])**.** *Consider a polynomial $f \in \mathcal{F}_{at} \setminus \mathcal{P}^\circ$ with locally connected Julia set $J(f)$. Then the geodesic lamination $\mathcal{L}_f$ has a quadratic invariant gap G, and there are two possibilities.*

(1) *The major $M(G)$ of G is critical, the corresponding critical point of f belongs to* $\mathrm{Bd}(A(f))$*, and periodic cutpoints of $J(f)$ do not exist.*
(2) *The major $M(G)$ of G is periodic, and the corresponding point of $J(f)$ is a repelling or parabolic periodic cutpoint of $J(f)$.*

Corollary 3.12 easily follows.

Corollary 3.12. *Suppose that $f \in \mathcal{F}_\lambda$, where $|\lambda| < 1$, is an IA-capture polynomial. Then $J(f)$ is locally connected, the geodesic lamination $\mathcal{L}_f$ has a quadratic invariant gap G with periodic major $M(G)$, the Julia set $J(f)$ contains a periodic repelling cutpoint associated to $M(G)$, and $f \in \mathcal{F}_\lambda \setminus \mathcal{P}$.*

Proof. Since f is hyperbolic, $J(f)$ is locally connected so that Theorem 3.11 applies to f. Evidently, neither critical point of f belongs to $J(f)$. Hence case (1) of Theorem 3.11 does not apply to f while case (2) does apply. The cutpoint cannot be parabolic for otherwise f would not be hyperbolic. This proves all claims of the corollary except for the last one. To see that $f \in \mathcal{F}_\lambda \setminus \mathcal{P}$ it remains to apply Lemma 3.2 which implies that small perturbations of f will have a periodic cutpoint in their Julia sets and, therefore, cannot belong to $\mathcal{P}^\circ$. □

3.4. **Rational lamination.** A less informative but more universal concept is that of the *rational lamination* $\mathcal{L}^r_f$ associated to a polynomial f. It was introduced by Kiwi (see [Kiw97, Kiw01, Kiw04]) and is based upon the work of Goldberg and Milnor [GM93].

Definition 3.13 ([Kiw97, Kiw01, Kiw04])**.** Let f be a polynomial of degree greater than 1 with connected Julia set. Consider all its external rays with rational arguments. If rays with distinct arguments land at the same point in the plane, declare them equivalent so that classes of equivalence are subsets of $\mathbb{S}$. Then the convex hulls of the thus defined classes of equivalence are pairwise disjoint. The set formed by the edges of the convex hulls of all classes is called the *rational (geodesic) lamination* of f and is denoted by $\mathcal{L}^r_f$.

Note that in the case $J(f)$ is locally connected, the equivalence relation above is just the restriction of the equivalence relation $\sim_f$ to the set of rational arguments. Images of leaves of $\mathcal{L}^r_f$ are leaves of $\mathcal{L}^r_f$. While the rational lamination $\mathcal{L}^r_f$ contains valuable information about the dynamics of f, it does not define the polynomial, as we already noted in Section 2.

Lemma 3.14. *Let f be a polynomial of degree $d \geqslant 2$ with connected Julia set. If a chord is a limit of leaves $\ell_i \in \mathcal{L}^r_f$ and one of its endpoints is periodic, then its other endpoint is periodic of the same period.*

Proof. We may assume that all leaves from the lemma are non-degenerate. By Theorem 3.6, we may assume that $\ell_i \in \mathcal{L}^r_f$, that $\ell_i \neq \overline{ab}$ for any i, and that $\ell_i \to \overline{ab}$. We may also assume that a is of period m and b is either non-periodic or of period greater than m; then $\sigma_d^m(b) \neq b$, and we may assume that $a \leqslant \sigma_d^m(b) < b$ in the sense of the counterclockwise order on $\mathbb{S}$. A point in $\mathbb{S}$ belongs to at most two non-degenerate leaves, hence we may assume that $\ell_i = \overline{x_i y_i}$ with $x_i \to a, y_i \to b$ and $x_i \neq a$, $y_i \neq b$. Then $a < x_i < b$, as otherwise $\sigma_d^m(\ell_i)$ crosses ℓ_i as σ_d^m at a repels and does not change orientation. Since distinct leaves ℓ_i do not cross, then $a < x_i < y_i < b$. However, then, if $1 \ll i \ll j$, the leaf $\sigma_d^m(\ell_j)$ crosses ℓ_i, a contradiction. □

If the Julia set of f is locally connected, there is an alternative proof of Lemma 3.14: the endpoints of ℓ_i are equivalent under $\sim_f$, and thus $a \sim_f b$ since $\sim_f$ is closed. The claim then follows from Theorem 3.6.

Definition 3.15 ([BMOV13])**.** A collection of chords $\mathcal{L}$ is *sibling σ_d-invariant* provided that:

(1) for each $\ell \in \mathcal{L}$, we have $\sigma_d(\ell) \in \mathcal{L}$,
(2) for each $\ell \in \mathcal{L}$ there exists $\ell_1 \in \mathcal{L}$ so that $\sigma_d(\ell_1) = \ell$.
(3) for each $\ell \in \mathcal{L}$ so that $\sigma_d(\ell)$ is a non-degenerate leaf, there exist d **disjoint** leaves $\ell_1, \ldots, \ell_d$ in $\mathcal{L}$ so that $\ell = \ell_1$ and $\sigma_d(\ell_i) = \sigma_d(\ell)$ for all $i = 1, \ldots, d$.

We will use the following lemma.

Lemma 3.16 ([Kiw97, Kiw01])**.** *For a polynomial f with connected Julia set the rational lamination $\mathcal{L}^r_f$ is sibling invariant.*

Sketch of a proof. Conditions (1) and (2) from Definition 3.15 are straightforward. Assume that $\{z_1, \ldots, z_k\} \subset J(f)$ is the full preimage of a point $z \in J(f)$. Denote by $A_1, \ldots, A_k$ and A the sets of arguments of external rays landing at $z_1, \ldots, z_k$ and z, respectively. Condition (3) from Definition 3.15 is immediate for an edge ℓ of the convex hull $\mathrm{CH}(A)$ except when some z_i's are critical. However then, too, one can find r_i pairwise disjoint edges of the convex hull $\mathrm{CH}(A_i)$ of A_i where r_i is the order of z_i so that each edge from this collection maps to ℓ. This completes the proof. □

The next lemma studies properties of infinite gaps of $\overline{\mathcal{L}^r_f}$.

Lemma 3.17. *Edges of infinite gaps of $\overline{\mathcal{L}^r_f}$ are (pre)periodic.*

Proof. Any leaf on the boundary of an infinite gap of any lamination $\mathcal{L}$ is eventually mapped to a critical leaf or to a periodic leaf, cf. [BOPT17a, Lemma 4.5]. However if an infinite gap G has a critical edge ℓ, then by the properties of laminations this leaf must be isolated in $\mathcal{L}$. If $\mathcal{L} = \overline{\mathcal{L}^r_f}$, then the above is only possible if $\ell \in \mathcal{L}^r_f$ is (pre)periodic. $\square$

We are ready to prove the next lemma.

Lemma 3.18. *If f is a polynomial of degree $d \geqslant 2$ with locally connected Julia set and there is no bounded Fatou domain of f whose boundary contains a critical point with infinite orbit, then $\overline{\mathcal{L}^r_f} = \mathcal{L}_f$.*

Proof. Recall that always $\overline{\mathcal{L}^r_f} \subset \mathcal{L}_f$. Suppose that $\overline{\mathcal{L}^r_f} \subsetneqq \mathcal{L}_f$. By Lemma 3.16, the collection $\mathcal{L}^r_f$ is sibling invariant. Moreover, let x and y be rational arguments. By Theorems 3.5 and 3.6, if $x \sim y$ and x is periodic for σ_3, then y is periodic of the same period. By Lemma 3.14 it follows that there are no critical leaves in $\overline{\mathcal{L}^r_f}$ with a periodic endpoint. Moreover, it follows also that if $x \in \mathbb{S}$ is periodic and $\overline{xy} \neq \overline{xz}$ are leaves of $\overline{\mathcal{L}^r_f}$, then $\sigma_d(\overline{xy}) \neq \sigma_d(\overline{xz})$. Sibling invariant collections of leaves with these properties are called *proper*; such collections as well as their closures are studied in [BMOV13]. In particular, it follows from Theorem 4.9 of [BMOV13] that $\overline{\mathcal{L}^r_f}$ is a lamination associated with an equivalence relation, say, $\approx$, on the unit circle. This means that $\overline{\mathcal{L}^r_f}$ is formed by the edges of the convex hulls of all $\approx$-classes. Recall that $\mathcal{L}_f$ is generated by a specific equivalence relation on $\mathbb{S}$ denoted by $\sim_f$.

Now, by the assumption $\overline{\mathcal{L}^r_f} \subsetneqq \mathcal{L}_f$. This implies that there is a gap $\widehat{G}$ of $\overline{\mathcal{L}^r_f}$ that contains leaves of $\mathcal{L}_f$ inside (so that only the endpoints of these leaves belong to the boundary of $\widehat{G}$). The gap $\widehat{G}$ cannot be finite because then all its vertices must be $\sim_f$-equivalent, and leaves of $\mathcal{L}_f$ cannot intersect the interior of $\widehat{G}$. Suppose that $\widehat{G}$ is infinite. We claim that there are no infinite gaps H of $\mathcal{L}_f$ properly contained in $\widehat{G}$. Indeed, suppose otherwise. Then an edge ℓ of H must be contained in the interior of $\widehat{G}$ (except for its endpoints). Observe that any edge of an infinite gap of any lamination is either (pre)critical or (pre)periodic (cf. [BOPT17a, Lemma 4.5]). Since $\ell \in \mathcal{L}_f \setminus \overline{\mathcal{L}^r_f}$, this implies that ℓ is (pre)critical with infinite orbit, a contradiction with the assumption of the lemma. Thus, all gaps of $\mathcal{L}_f$ in $\widehat{G}$ are finite.

By [Kiw02, Theorem 1.1], all infinite gaps are (pre)periodic. Hence for some n the infinite gap $G = \sigma^n_d(\widehat{G})$ is periodic. By the previous paragraph all gaps of $\mathcal{L}_f$ in G are finite. Then the quotient space $(G \cap \mathbb{S})/\sim_f$ is a so-called *dendrite*, which carries a self-map induced by

σ_d^p where p is the minimal period of G. Theorem 7.2.7 from [BFMOT12] implies that there are infinitely many periodic cutpoints in this dendrite, hence G contains leaves of $\mathcal{L}_f^r$, a contradiction. □

4. Preliminaries to Theorem A

In this section, we list various preliminary results. Some of them are well known and therefore given without proof.

4.1. **A perturbation lemma.** Consider a sequence $\lambda_n \in \mathbb{D}$ converging to $\lambda \in \mathbb{S}$. We say that λ_n converges to λ *non-tangentially* if all λ_n belong to a cone with the following properties. The vertex of the cone is λ. The axis of symmetry of the cone is the radius (radial line) through λ. The angle between the edges of the cone and its axis of symmetry is less than $\pi/2$. For an open set $U \subset \mathbb{C}$ and a holomorphic map $g : U \to \mathbb{C}$ with attracting fixed point 0, let $A(g)$ be the immediate basin of attraction of 0 with respect to g. Recall a part of Corollary 2 from [BP08], based on ideas of [Yoc95, Proposition 1, page 66]:

Lemma 4.1 (Corollary 2 of [BP08])**.** *Suppose that $\lambda_n \in \mathbb{D}$ converge non-tangentially to $\lambda \in \mathbb{S}$. Let $U \subset \mathbb{C}$ be an open set, and $f : U \to \mathbb{C}$ be a holomorphic map with $f(0) = 0$ and $f'(0) = \lambda$. Assume that f has a Siegel disk Δ around 0. If the sequence $f_n : U \to \mathbb{C}$ satisfies $f_n(0) = 0$, $f_n'(0) = \lambda_n$, and for every compact subset $K \subset \Delta$*

$$\max_{z\in K} |f_n(z) - f(z)| = O(|\lambda - \lambda_n|), \quad n \to \infty,$$

then any compact set $\widetilde{K} \subset \Delta$ is contained in $A(f_n)$ for n large enough.

We now go back to our family $\mathcal{F}$. Below, we define some special perturbations of polynomials in $\mathcal{F}_{nr}$. Let $f(z) = f_{\lambda,b}(z) = \lambda z + bz^2 + z^3 \in \mathcal{F}_{nr}$ so that $|\lambda| \leqslant 1$. Then denote by f_ε the polynomial

$$f_{(1-\varepsilon)\lambda,b}(z) = (1-\varepsilon)\lambda z + bz^2 + z^3 \in \mathcal{F}_{at}, \tag{4.1.1}$$

where $\varepsilon > 0$. The following is an easy corollary of Lemma 4.1.

Corollary 4.2. *If $f = \lambda z + bz^2 + z^3$ has a Siegel disk $\Delta(f)$ around 0, then, for every compact set $\widetilde{K} \subset \Delta(f)$, there exists $\delta(\widetilde{K}) > 0$ such that every polynomial f_ε has the property $\widetilde{K} \subset A(f_\varepsilon)$ for any $0 < \varepsilon < \delta(\widetilde{K})$.*

Proof. Assume the contrary. Then there exists a sequence $\varepsilon_n \to 0$ with $\widetilde{K} \not\subset A(f_{\varepsilon_n})$. Set $\lambda_n = (1-\varepsilon_n)\lambda$; then λ_n converge to λ non-tangentially. To use Lemma 4.1, observe that for a compact set $K \subset \Delta(f)$

$$\max_{z\in K} |f_{\varepsilon_n}(z) - f(z)| = O(|\lambda - \lambda_n|), \quad n \to \infty$$

because the left-hand side equals $\varepsilon_n \max_{z\in K}|z|$ while $|\lambda-\lambda_n|=\varepsilon_n$. This yields a contradiction with Lemma 4.1 and proves the corollary. $\square$

4.2. **Blaschke products.** Here we deal with the dynamics of Blaschke products. As we do not need Blaschke products of higher degrees and for the sake of simplicity we only consider quadratic Blaschke products with fixed point 0. For a complex number a, we let $\overline{a}$ denote the complex conjugate of a.

Definition 4.3 (Blaschke products)**.** Let b and s be complex numbers such that $0<|b|<1$ and $|s|=1$. Then the formula

$$B_{b,s}(z)=sz\frac{b-z}{1-\overline{b}z} \tag{4.2.1}$$

defines a *quadratic Blaschke product with fixed point* 0. It is not hard to see that the Blaschke product (4.2.1) is conjugate by a rotation to a so-called *normalized quadratic Blaschke product* Q_a of the form

$$Q_a(z)=z\frac{a-z}{1-\overline{a}z}; \tag{4.2.2}$$

for some complex number a with $|a|<1$.

Our normalized Blaschke product Q_a differs by a sign from the traditional one in which the numerator is $z-a$, not $a-z$. It is well known that Q_a is a quadratic rational function that preserves $\mathbb{D}$, its complement $\mathbb{C}\setminus\mathbb{D}$, and the unit circle $\mathbb{S}$. Moreover,

$$Q_a'(z)=\frac{\overline{a}z^2-2z+a}{(1-\overline{a}z)^2}, \tag{4.2.3}$$

which implies that $Q_a'(0)=a$; an easy computation shows that the multiplier of the fixed point at ∞ is $\overline{a}$. Thus, both 0 and infinity are attracting fixed points of Q_a. Set $\mathbb{D}_r=\{|z|<r\}$; then, by the Schwarz Lemma (or directly), we have $Q_a(\overline{\mathbb{D}}_r)\subset\mathbb{D}_r$. Similarly, $|Q_a(z)|>|z|$ if $|z|>1$. Hence the Julia set of Q_a is $\mathbb{S}$. In fact, Q_a is expanding on $\mathbb{S}$, see [Tis00]. For the sake of completeness, we now sketch the proof. Assume that there exists a point $w\in\mathbb{S}$ such that $|Q_a'(w)|\leqslant 1$. Set $s=\frac{w}{Q_a(w)}$ and consider $T(z)=sQ_a(z)$; then $|T'(w)|\leqslant 1$ and $T(w)=w$ is a non-repelling fixed point of T. Since the Julia set of any quadratic Blaschke product, in particular, of $T(z)$, is $\mathbb{S}$, the point w cannot be attracting. By the Snail Lemma, w must then be parabolic with multiplier 1. However, this contradicts the fact that points near $\mathbb{S}$ are attracted to 0 and ∞ (and repelled away from $\mathbb{S}$) by T.

Solving the quadratic equation $\overline{a}z^2 - 2z + a = 0$, we see that the critical points of $Q_a(z)$ are given by

$$c_a = \frac{1 - \sqrt{1 - |a|^2}}{\overline{a}} \quad \text{and} \quad d_a = \frac{1 + \sqrt{1 - |a|^2}}{\overline{a}}$$

It is easy to see, that

$$c_a = \frac{1 - \sqrt{1 - |a|^2}}{\overline{a}} = a\frac{1 - \sqrt{1 - |a|^2}}{|a|^2} = \frac{a}{1 + \sqrt{1 - |a|^2}} \tag{4.2.4}$$

is the unique critical point of Q_a that belongs to $\mathbb{D}$. Also, by (4.2.4) a and c_a belong to the same radial segment of $\overline{\mathbb{D}}$ so that c_a is located between 0 and a. Observe that if $a \to s \in \mathbb{S}$, then $c_a \to s$ too. To describe the limit behavior of the entire orbit of c_a as $a \to s \in \mathbb{S}$, we need Lemma 4.4. For a complex number w, set $\mathrm{R}_w(z) = wz$.

Lemma 4.4. *Suppose that $s \in \mathbb{S}$ and $K \subset \mathbb{C} \setminus \{s\}$ is a compact set. Then the maps Q_a converge to R_s uniformly on K as $a \to s$.*

Proof. Since $|s| = 1$, we have $s\overline{s} = 1$. Therefore $s - z = s - s\overline{s}z = s(1 - \overline{s}z)$. Dividing on both sides by $1 - \overline{s}z$, we see that $\frac{s-z}{1-\overline{s}z} = s$ for all $z \neq \frac{1}{\overline{s}} = s$. Since $K \subset \mathbb{C} \setminus \{s\}$ is a compact set, standard continuity arguments imply the conclusions of the lemma. $\square$

This does not yet yield the limit behavior of the orbit of c_a as $a \to s \in \mathbb{S}$ as then $c_a \to s$ too, and Lemma 4.4 does not apply.

Lemma 4.5. *Suppose that $s = e^{2\pi i\theta}$, where θ is irrational. Let ε be a positive real number and m be a positive integer. Then there exists $\delta > 0$ such that for any $a \in \mathbb{D}$ with $|s - a| < \delta$ we have $|Q_a^i(c_a)| > 1 - \varepsilon$ for all $i = 0, 1, \dots, m$.*

In other words, if $a = Q'_a(0)$ is close to s, then the orbit of c_a stays close to the unit circle for any given period of time. The conclusions of the lemma are sensitive with respect to the point whose trajectory we consider. For example, $Q_a(a) = 0$ so that the orbit of a under Q_a is $(a, 0, 0, \dots)$ and, thus, the limit behavior of the orbits of a and of c_a are very different even though both a and c_a converge to $s = e^{2\pi i\theta}$.

Proof. We will use the following notation and terminology. Given a *small* arc $T \subset \mathbb{S}$ with endpoints of arguments α and β, denote by U_T a "polar rectangle" built upon T with vertices (in polar coordinates) given by $(1 - |T|, \alpha)$, $(1 + |T|, \alpha)$, $(1 + |T|, \beta)$, $(1 - |T|, \beta)$.

Simple computations show that

$$Q_a(c_a) = \frac{(1-\sqrt{1-|a|^2})^2}{\overline{a}^2} = c_a^2 \tag{4.2.5}$$

Since θ is irrational, there exists a closed arc $I \subset \mathbb{S}$ symmetric with respect to s such that I, $\mathrm{R}_s(I)$, $\mathrm{R}_s^2(I)$, $\dots$, $\mathrm{R}_s^m(I)$ are pairwise disjoint circle arcs. By Lemma 4.4, we can choose a small arc $T \subset \mathrm{R}_s(I)$ centered at s^2 such that for all a sufficiently close to s we have that $Q_a^i(U_T) \subset U_{\mathrm{R}_s^{i+1}(I)}$ for all $i = 0, \dots, m-1$. We can then choose a small neighborhood W of s so that $\zeta^2 \subset U_T$ provided that $\zeta \in W$; by (4.2.4) and (4.2.5) this implies that for any a sufficiently close to s we have $c_a \in W$ and $Q_a^j(c_a) \in U_{\mathrm{R}_s^j(I)}$ for every $j = 1, \dots, m$. $\square$

4.3. **Modulus.** The notion of the modulus of an annulus is widely used in complex dynamics. We need several well known facts concerning annuli and their moduli.

Definition 4.6. A *round* annulus $A(r, R) \subset \mathbb{C}$ is an open annulus formed by two concentric circles of radii $r < R$. A *topological* annulus $U \setminus K$ is formed by a simply connected domain $U \subset \mathbb{C}$ and a *non-separating* (i.e., such that $\mathbb{C} \setminus K$ is connected) continuum $K \subset U$. If K is not a singleton and $U \neq \mathbb{C}$, then we will call $U \setminus K$ *non-degenerate.*

An important fact concerning annuli is the following well known version of the Riemann Mapping Theorem.

Theorem 4.7 (see, e.g., Theorem 10, Section 5, Chapter 6 [Ahl79])**.** *Any non-degenerate annulus is conformally equivalent to a non-degenerate round annulus.*

Let us also mention classical Schottky's Theorem.

Theorem 4.8 ([Sch877])**.** *Two round annuli $A(r, R)$ and $A(r', R')$ are conformally equivalent if and only if $\frac{R}{r} = \frac{R'}{r'}$.*

We are ready to define the modulus.

Definition 4.9. For a round annulus $A(r, R)$ its *modulus* $m(A(r, R))$ is defined as the number $\frac{\ln(R)-\ln(r)}{2\pi}$. Given a topological annulus $\widehat{A}$ that is conformally equivalent to the round annulus $A = A(r, R)$, we set $m(\widehat{A}) = m(A) = \frac{\ln(R)-\ln(r)}{2\pi}$.

Observe that by Schottky's Theorem the modulus of an annulus is well defined and invariant under conformal equivalence. We need Theorem 4.10 concerning this concept; below $d(X, Y)$ denotes the infimum of the distance between points $x \in X$ and $y \in Y$ for sets $X, Y \subset \mathbb{C}$.

Theorem 4.10. *Suppose that $A \subset A'$ are two annuli such that A is not null-homotopic in A'. Then $m(A) \leqslant m(A')$. Moreover, there exists a function $\psi : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ such that $d(K, \mathbb{S}) \geqslant \psi(m(\mathbb{D} \setminus K))$ for any non-separating continuum $K \subset \mathbb{D}$.*

The first part of Theorem 4.10 is well known and can be found in various textbooks; the second part easily follows, e.g., from [McM94, Theorem 2.4] or from [Ahl06, Problem I of Section A, Chapter III].

4.4. **Hyperbolic components.** We will make use of the following result [McS98, Corollary 2.10]:

Lemma 4.11. *Let f be a hyperbolic rational function. Then the set $[f]_{top}$ of rational functions topologically conjugate to f coincides with the set of rational functions qc-conjugate to f and is connected.*

Suppose now that f and g are hyperbolic polynomials in $\mathcal{F}$ with connected Julia sets. Recall that then $J(f)$, $J(g)$ are locally connected. A *critical orbit relation* for f is a constraint of the form $f^n(c) = f^m(d)$, $m \neq n$, where c and d are critical points of f, not necessarily different. As in Section 3, we can associate geodesic laminations $\mathcal{L}_f$ and $\mathcal{L}_g$ with f and g, respectively.

Lemma 4.12. *Let f and g be two degree $d > 1$ hyperbolic polynomials with connected Julia sets such that $\mathcal{L}_f = \mathcal{L}_g$. If f and g have no critical orbit relations, then f and g are topologically conjugate.*

See [McS98] for very similar statements. The same methods prove Lemma 4.12. It follows that $g \in [f]_{top}$. Note however that, in the cubic case, the intersection of $[f]_{top}$ with $\mathcal{F}$ may be disconnected.

Corollary 4.13. *If polynomials f and g belong to the same bounded hyperbolic component of $\mathcal{F}$, then $\mathcal{L}_f = \mathcal{L}_g$. On the other hand, suppose that f, $g \in \mathcal{F}_{at}$ are hyperbolic polynomials with connected Julia sets such that $\mathcal{L}_f = \mathcal{L}_g = \mathcal{L}$. If f and g have no attracting fixed points except 0, then f, g belong to the same hyperbolic component of $\mathcal{F}$.*

Proof. The first claim is a variation of a well-known property of hyperbolic components; it is left to the reader. To prove the rest, we may assume that neither f nor g has critical orbit relations. Indeed, otherwise we can slightly perturb f and g within their hyperbolic components of $\mathcal{F}$ so that the perturbed maps have no critical orbit relations. Then f and g are topologically conjugate by Lemma 4.12. Suppose that $f = f_{\lambda_f, b_f} = z^3 + b_f z^2 + \lambda_f z$ and $g = g_{\lambda_g, b_g} = z^3 + b_g z^2 + \lambda_g z$.

By Lemma 4.11, there is a continuous family f_t, $t \in [0, 1]$ of cubic rational functions qc-conjugate to f such that $f_0 = f$ and $f_1 = g$.

Indeed, a qc-conjugacy between f and g takes the standard complex structure on the dynamical plane of g to some invariant qc-structure on the dynamical plane of f. The latter is represented by a Beltrami differential ν. Considering the family of Beltrami differentials $\nu_t = t\nu$ and using the Ahlfors–Bers theorem, we obtain a family f_t with the desired properties. Observe that all rational functions f_t are hyperbolic.

Let M_t be a complex affine transformation such that $h_t = M_t \circ f_t \circ M_t^{-1} \in \mathcal{F}$. Since $[0,1]$ is simply connected, we may choose M_t to depend continuously on t and so that $M_0 = id$. Let $\mathcal{U}$ be the hyperbolic component of $\mathcal{F}$ containing f. Then $h_t \in \mathcal{U}$ for all t by continuity; in particular, $h_1 \in \mathcal{U}$. On the other hand, $h_1 = M_1 \circ g \circ M_1^{-1} \in \mathcal{F}$ and g are affinely conjugate. This implies that either $h_1 = g$ or $h_1 = z^3 - b_g z^2 + \lambda_g z$. In the former case, we are done. In the latter case, observe that h_1 and g have the same linearizing coordinate near infinity (this follows from the fact that $z \mapsto z^3$ commutes with the involution $z \mapsto -z$) while the orbits of g are obtained from the orbits of h_1 by $z \mapsto -z$. Therefore, the geodesic lamination of g differs from the geodesic lamination of h_1 by a half-turn.

On the other hand, by our construction $\mathcal{L}$ coincides with the geodesic lamination of h_1. Thus, $\mathcal{L}$ is invariant with respect to the rotation by 180 degrees about the center of the unit disk. Then, by [BOPT16a], the major of an invariant quadratic gap G in $\mathcal{L}$ corresponding to the basin of immediate attraction of 0 (of either f or g) is $\overline{0\frac{1}{2}}$. This implies that there are two invariant attracting domains of g (or f), corresponding to G and the 180-degree rotation of G with respect to the center of the unit disk. A contradiction with the assumption that g (and f) has only one attracting fixed point. The statement now follows. □

5. Proof of Theorem A

Let $\mathcal{W}$ be a component of $\mathrm{TH}(\mathcal{P}_\lambda) \setminus \mathcal{P}_\lambda$, where $|\lambda| \leqslant 1$. It is called a *queer domain* (or is said to be of *queer* type) if there exists a polynomial $f \in \mathcal{W}$ so that all of its critical points are in $J(f)$. Polynomials from such $\mathcal{W}$ are also said to be of *queer type*. Observe that IS-polynomials and polynomials of queer type have connected Julia sets. If f is an IS-polynomial, then $\mathrm{ca}(f)$ is a critical point of f that does not belong to $J(f)$, hence f is *not* a polynomial of queer type.

The following theorem relies on [Zak99, Theorem 3.4], where the most difficult case is worked out.

Theorem 5.1 ([BOPT14b])**.** *Let $\mathcal{W}$ be a component of* $\mathrm{TH}(\mathcal{P}_\lambda) \setminus \mathcal{P}_\lambda$ *of queer type. Then, for any polynomial $f \in \mathcal{W}$, the Julia set $J(f)$ has positive Lebesgue measure and carries an invariant line field.*

Properties of polynomials from $\mathcal{P}$ listed in Theorem 2.4 are inherited by polynomials from the topological hulls $\mathrm{TH}(\mathcal{P}_\lambda)$.

Theorem 5.2 ([BOPT14a]). *Suppose that* $|\lambda| \leqslant 1$. *We have*

$$\mathrm{TH}(\mathcal{P}_\lambda) \subset \mathcal{CU}.$$

Moreover, all components of the set $\mathrm{TH}(\mathcal{P}_\lambda) \setminus \mathcal{P}_\lambda$*, where* $|\lambda| \leqslant 1$*, consist of* λ*-stable polynomials.*

In [BOPT14b], we consider components of the set $\mathrm{TH}(\mathcal{P}_\lambda) \setminus \mathcal{P}_\lambda$, where $|\lambda| \leqslant 1$. Let us describe some results of [BOPT14b, BOPT16b]. A cubic polynomial $f \in \mathcal{F}_\lambda \setminus \mathcal{P} = \mathcal{F}_\lambda \setminus \mathcal{P}_\lambda$ with $|\lambda| \leqslant 1$ is said to be *potentially renormalizable*. Perturbing a potentially renormalizable polynomial $f \in \mathcal{F}_{nr} \setminus \mathcal{P}$ to a polynomial $g \in \mathcal{F}_{at}$, we see that $g|_{A(g)}$ is two-to-one (otherwise $f \in \mathcal{P}$) and, hence, g has two distinct critical points. By Lemma 5.3 below this property is inherited by a potentially renormalizable polynomial.

Lemma 5.3 ([BOPT14b]). *A potentially renormalizable polynomial has two distinct critical points.*

A critical point c of a potentially renormalizable polynomial f is said to be *principal* if there is a neighborhood $\mathcal{U}$ of f in $\mathcal{F}$ and a holomorphic function $\omega_1 : \mathcal{U} \to \mathbb{C}$ defined on $\mathcal{U}$ such that $c = \omega_1(f)$, and, for every $g \in \mathcal{U} \cap \mathcal{F}_{at}$, the point $\omega_1(g)$ is the critical point of g contained in $A(g)$.

Theorem 5.4 ([BOPT14b]). *A potentially renormalizable polynomial has a unique principal critical point.*

By Theorem 5.4, if $f \in \mathcal{F}_{nr}$ is potentially renormalizable, then the point $\omega_1(f)$ is well-defined; let the other critical point of f be $\omega_2(f)$. It is easy to see that $\omega_1(f) \in K(f)$. It is proven in [BOPT16a] (see Theorem 3.11 and Corollary 3.12 where the appropriate results of [BOPT16a] are summarized) that an IA-capture polynomial g has a repelling periodic cutpoint of the Julia set $J(g)$. Hence an IA-capture polynomial g is not in $\mathcal{CU}$, thus not in $\mathcal{P}$, i.e., it is potentially renormalizable, and the notation for its critical points $\omega_1(g)$, $\omega_2(g)$, introduced in Definition 2.3, is consistent with the just introduced notation for all potentially renormalizable polynomials.

Recall that, by Theorem 5.2, all polynomials in a component $\mathcal{W}$ of $\mathrm{TH}(\mathcal{P}_\lambda) \setminus \mathcal{P}_\lambda$ are conjugate on their Julia set. Moreover, if some polynomial in $\mathcal{W}$ is an IS-capture, then it is easy to see that so are all polynomials in $\mathcal{W}$. This inspires the following definition. Let $\mathcal{W}$ be a component of λ-stable polynomials, where $|\lambda| \leqslant 1$. Then $\mathcal{W}$ is said to be of *IS-capture type* if any $f \in \mathcal{W}$ is an IS-capture polynomial. We also say in this case that $\mathcal{W}$ is an *IS-capture component*. It is easy to construct examples of IS-captures in $\mathcal{F}_\lambda \setminus \mathrm{TH}(\mathcal{P}_\lambda)$.

Theorem 5.5 ([BOPT14b]). *Let $\mathcal{W}$ be a component of $\mathrm{TH}(\mathcal{P}_\lambda) \setminus \mathcal{P}_\lambda$, where $|\lambda| \leqslant 1$. Then $\mathcal{W}$ is either of IS-*capture type *or of* queer *type.*

By Theorem C, which we prove later on, the first possibility listed in Theorem 5.5 is impossible.

Let us introduce notation, which will be used in the rest of this section. Let f be an IS-capture polynomial. As before, we write $\Delta(f)$ for the Siegel disk of f. Recall that f has two distinct critical points $\mathrm{re}(f)$ and $\mathrm{ca}(f)$ ("re" from "recurrent" and "ca" from "captured"), see Definition 2.1. The point $\mathrm{re}(f)$ is recurrent, and the closure of its forward orbit includes $\mathrm{Bd}(\Delta(f))$. The point $\mathrm{ca}(f)$ is eventually mapped to $\Delta(f)$. Let $m_f > 0$ be the smallest positive integer for which we have $f^{m_f}(\mathrm{ca}(f)) \in \Delta(f)$. Observe that, given sufficiently small $\varepsilon > 0$, for all polynomials g close enough to f, there exist a unique critical point $\mathrm{re}(g)$ of g that is ε-close to $\mathrm{re}(f)$ and a unique critical point $\mathrm{ca}(g)$ of g that is ε-close to $\mathrm{ca}(f)$. Notice that the functions $\mathrm{re}(g)$ and $\mathrm{ca}(g)$ are holomorphic functions of the coefficients of g. However $\mathrm{re}(g)$ is not necessarily recurrent, and g may not have a Siegel invariant domain.

Lemma 5.6 is based on special perturbations (4.1.1) introduced right before Corollary 4.2. Namely, recall that for $f(z) = f_{\lambda,b}(z) = \lambda z + bz^2 + z^3 \in \mathcal{F}_{nr}$ with $|\lambda| \leqslant 1$ we denote by f_ε the polynomial $f_{(1-\varepsilon)\lambda,b}(z) = (1-\varepsilon)\lambda z + bz^2 + z^3 \in \mathcal{F}_{at}$, where $\varepsilon > 0$ is a small positive number.

Lemma 5.6. *Suppose that f is an IS-capture polynomial. Then, for sufficiently small $\varepsilon > 0$, we have $\mathrm{re}(f_\varepsilon) \in A(f_\varepsilon)$. In particular, if $f_\varepsilon \notin \mathcal{P}^\circ$, then $\mathrm{ca}(f_\varepsilon) \notin A(f_\varepsilon)$.*

Proof. Set $f = f_{\lambda,b}$. Then $\lambda = e^{2\pi i\theta}$, where θ is irrational. Take a closed Jordan disk K and an open Jordan disk U such that

$$0 \in K \subset U \subset \overline{U} \subset \Delta(f).$$

We may assume that $f^{m_f}(\mathrm{ca}(f))$ lies in the interior of K.

Observe that if $f_\varepsilon \in \mathcal{P}^\circ$ then $\mathrm{re}(f_\varepsilon) \in A(f_\varepsilon)$ as desired. In particular, if for sufficiently small $\varepsilon > 0$ we have that $f_\varepsilon \in \mathcal{P}^\circ$, then we are done. Thus we need to consider the case when there are positive values of ε arbitrarily close to 0 and such that $f_\varepsilon \notin \mathcal{P}^\circ$. We need to show that $\mathrm{re}(f_\varepsilon) \in A(f_\varepsilon)$ for all these values of ε. Observe that in any case at least one critical point must belong to $A(f_\varepsilon)$ for all $\varepsilon > 0$. Hence, if $\mathrm{ca}(f_\varepsilon) \notin A(f_\varepsilon)$ for some $\varepsilon > 0$, then $\mathrm{re}(f_\varepsilon) \in A(f_\varepsilon)$ for this ε as desired. Thus, to prove the lemma it would suffice to prove the following claim.

Claim. *For sufficiently small $\varepsilon > 0$, if $f_\varepsilon \notin \mathcal{P}^\circ$ then $\mathrm{ca}(f_\varepsilon) \notin A(f_\varepsilon)$.*

Proof of the Claim. Suppose that there are positive values of ε arbitrarily close to 0 and such that $f_\varepsilon \notin \mathcal{P}^\circ$. Moreover, suppose by way of contradiction that the Claim fails. Then there exists a sequence $\varepsilon_n \to 0$ with $f_{\varepsilon_n} \notin \mathcal{P}^\circ$ and $\mathrm{ca}(f_{\varepsilon_n}) \in A(f_{\varepsilon_n})$. Since $f_{\varepsilon_n} \notin \mathcal{P}^\circ$, then $\mathrm{ca}(f_{\varepsilon_n})$ is the only critical point in $A(f_{\varepsilon_n})$. A Riemann map $\varphi : A(f_{\varepsilon_n}) \to \mathbb{D}$ with $\varphi(0) = 0$ conjugates $f_{\varepsilon_n}|_{A(f_{\varepsilon_n})}$ with a normalized quadratic Blaschke product Q_{a_n}, where $a_n \in \mathbb{D}$. Then $\varphi(\mathrm{ca}(f_{\varepsilon_n})) = c_{a_n}$ is the unique critical point of Q_{a_n} in $\mathbb{D}$. This yields the following contradiction.

(i) By Lemma 4.5, the point $Q_{a_n}^{m_f}(c_{a_n})$ approaches the unit circle as $\varepsilon_n \to 0$.
(ii) By Corollary 4.2 and by continuity, the point $Q_{a_n}^{m_f}(c_{a_n})$ is bounded away from the unit circle as $\varepsilon_n \to 0$.

A more detailed proof follows.

(i) Clearly, the multiplier $(1-\varepsilon_n)\lambda$ of f_{ε_n} at 0 converges to $\lambda = e^{2\pi i\theta}$. It follows that the multiplier of $Q_{a_{\varepsilon_n}}$ at 0 also converges to λ. By Lemma 4.5, the point $Q_{a_n}^{m_f}(c_{a_n})$ approaches the unit circle as $\varepsilon_n \to 0$.

(ii) On the other hand, take a polynomial f_ε with small $\varepsilon > 0$. By Corollary 4.2, we have $\overline{U} \subset A(f_\varepsilon)$ for all sufficiently small $\varepsilon > 0$. By continuity, $f_\varepsilon^{m_f}(\mathrm{ca}(f_\varepsilon)) \in K$ if $\varepsilon > 0$ is sufficiently small. Thus, the point $f_{\varepsilon_n}^{m_f}(\mathrm{ca}(f_{\varepsilon_n}))$ is separated from $\mathrm{Bd}(A(f_{\varepsilon_n}))$ by the annulus $U \setminus K$ of a definite positive modulus. It follows, by the conformal invariance of the modulus, that the point $Q_{a_{\varepsilon_n}}^{m_f}(c_{a_n})$ must also be separated from $\mathbb{S}$ by an annulus of a definite positive modulus. However, this contradicts Theorem 4.10 and the conclusions of (i) above. $\square$

Recall (Definition 2.3) that for an IA-capture polynomial f we denote by $\omega_1(f)$ its critical point that belongs to $A(f)$ and by $\omega_2(f)$ its critical point that does not belong to $A(f)$ but eventually (after one or more iterations) maps into $A(f)$. Observe that our notation for critical points $\omega_1(f)$ and $\omega_2(f)$ is consistent with Definition 2.3. Finally, recall that by potentially renormalizable polynomials we mean polynomials in $\mathcal{F}$ that do not belong to $\mathcal{P} = \overline{\mathcal{P}^\circ}$.

Corollary 5.7. *Suppose that f is an IS-capture polynomial. If f is potentially renormalizable, then $\omega_1(f) = \mathrm{re}(f)$ and $\omega_2(f) = \mathrm{ca}(f)$.*

Proof. Since f is potentially renormalizable, all maps f_ε of f are outside $\mathcal{P}^\circ$ if ε is small. By definition and Lemma 5.6, $\mathrm{re}(f) = \omega_1(f)$ and $\mathrm{ca}(f) = \omega_1(f)$. $\square$

According to Definition 2.3, a polynomial $f \in \mathcal{F}_{at}$ is said to be an *IA-capture (polynomial)* (IA from "Invariant Attracting") if it has a critical point that does not belong to the basin of immediate attraction

$A(f)$ of 0 but maps to $A(f)$ under a finite iteration of f. A hyperbolic component in $\mathcal{F}$ is said to be an *IA-capture component* if it contains an IA-capture polynomial; it is easy to see that then all polynomials in it are IA-capture polynomials. It is well known that all polynomials in an IA-capture component are topologically conjugate on their Julia sets. Moreover, if $\mathcal{W}$ is a hyperbolic component non-disjoint from $\mathcal{F}_{at}$ such that polynomials in $\mathcal{W}$ have a critical point which maps into a cycle of attracting Fatou domains but does not belong to it, then $\mathcal{W} \subset \mathcal{F}_{at}$ is an IA-capture component consisting of polynomials f with an *invariant* attracting Fatou domain $A(f) \ni 0$, a well-defined critical point $\omega_1(f) \in A(f)$ and a well-defined critical point $\omega_2(f) = \mathrm{ca}(f) \notin A(f)$ such that for some minimal $m_f > 0$ we have $f^{m_f}(\omega_2(f)) \in A(f)$.

Theorem 5.8. *If $f \in \mathcal{F}_{nr}$ is an IS-capture polynomial, then f belongs to the boundary of exactly one bounded hyperbolic component $\mathcal{W}$ in $\mathcal{F}_{at}$. Every polynomial $g \in \mathcal{W}$ has a locally connected Julia set so that $\mathcal{L}_g = \overline{\mathcal{L}^r_g}$, and $\mathcal{W}$ is either $\mathcal{P}^\circ$, or an IA-capture component.*

Proof. First we consider maps f_ε. By Lemma 5.6, for some $\delta > 0$ and any $\varepsilon > 0$ with $\varepsilon < \delta$, we have $\mathrm{re}(f) \in A(f_\varepsilon)$. By Corollary 4.2 and continuity, $f^{m_f}(\mathrm{ca}(f_\varepsilon)) \in A(f_\varepsilon)$. Thus, f_ε is hyperbolic, and there is a unique hyperbolic component $\mathcal{U}$ of $\mathcal{F}$ containing all polynomials f_ε with $\varepsilon < \delta$. Clearly, $\mathcal{U}$ is either $\mathcal{P}^\circ$, or an IA-capture component.

By way of contradiction, assume now that $\mathcal{U}$ and $\mathcal{V}$ are different bounded hyperbolic components in $\mathcal{F}_{at}$ whose boundaries contain f. All polynomials in $\mathcal{U}$ have locally connected Julia sets, are conjugate on their Julia sets, and give rise to the same cubic invariant lamination $\mathcal{L}_\mathcal{U}$; similarly, all polynomials in $\mathcal{V}$ give rise to the same cubic lamination $\mathcal{L}_\mathcal{V}$ (cf. Corollary 4.13). Since, for a hyperbolic polynomial, the iterated forward images of a critical point cannot lie on the boundary of a Fatou component, then, by Lemma 3.18, we have $\mathcal{L}_\mathcal{U} = \overline{\mathcal{L}^r_\mathcal{U}}$ and $\mathcal{L}_\mathcal{V} = \overline{\mathcal{L}^r_\mathcal{V}}$ where $\mathcal{L}^r_\mathcal{U}$ and $\mathcal{L}^r_\mathcal{V}$ are the corresponding rational laminations.

Consider a leaf $\ell \in \mathcal{L}^r_f$. It corresponds to a (pre)periodic point in $J(f)$. Since all periodic points in $J(f)$ are repelling, then, by Lemma 3.2, we have $\ell \in \mathcal{L}_\mathcal{U}$ and $\ell \in \mathcal{L}_\mathcal{V}$. Since this holds for any $\ell \in \mathcal{L}^r_f$, we conclude that $\mathcal{L}^r_f \subset \mathcal{L}^r_\mathcal{U}$ and $\mathcal{L}^r_f \subset \mathcal{L}^r_\mathcal{V}$. Now consider a leaf $\overline{\alpha\beta} \in \mathcal{L}^r_\mathcal{U}$. Then $R_g(\alpha)$, $R_g(\beta)$ land at the same (pre)periodic point x_g, for every $g \in \mathcal{U}$. The periodic cycle, into which the point x_g eventually maps, is repelling. Consider a sequence $g_n \in \mathcal{U}$ converging to f. By Corollary 3.3 applied to this sequence, we have $\overline{\alpha\beta} \in \mathcal{L}^r_f$. Since $\overline{\alpha\beta}$ is an arbitrary leaf of $\mathcal{L}^r_\mathcal{U}$, we conclude that $\mathcal{L}^r_\mathcal{U} \subset \mathcal{L}^r_f$. Similarly, $\mathcal{L}^r_\mathcal{V} \subset \mathcal{L}^r_f$. Together with the opposite inclusions proved earlier, this implies that

$\mathcal{L}^r_{\mathcal{U}} = \mathcal{L}^r_{\mathcal{V}} = \mathcal{L}^r_f$. By the first paragraph, it follows that $\mathcal{L}_{\mathcal{U}} = \mathcal{L}_{\mathcal{V}}$. Finally, by Corollary 4.13, we have $\mathcal{U} = \mathcal{V} = \mathcal{W}$. □

Proof of Theorem A. Let $f \in \mathcal{F}_\lambda$ be an IS-capture polynomial. By Theorem 5.8, there is a unique bounded hyperbolic component $\mathcal{U}$ in $\mathcal{F}_{at}$ with $f \in \mathrm{Bd}(\mathcal{U})$. A priori, there could exist a different hyperbolic component $\mathcal{V}$ outside of $\mathcal{F}_{at}$ with $f \in \mathrm{Bd}(\mathcal{V})$. Since for $g \in \mathcal{V}$ the fixed point 0 is repelling, there is a periodic angle θ such that $R_g(\theta)$ lands at 0 for all $g \in \mathcal{V}$. Consider a sequence $g_n \in \mathcal{V}$ converging to f. By Lemma 3.1, the ray $R_f(\theta)$ lands at a periodic point $y \neq 0$ (recall that 0 is a Siegel point). By Lemma 3.2, the point y is parabolic. However, an IS-capture has no parabolic periodic points, a contradiction. Thus, $\mathcal{U}$ is the only bounded hyperbolic component in $\mathcal{F}$ containing f in its boundary. It remains to observe that, if $\mathcal{U}$ is an IA-capture, then, by Corollary 3.12, the polynomial f has a repelling periodic cutpoint in its Julia set. □

6. Existence of IS-capture components

In this section, we find IS-capture components on the boundary of $\mathcal{P}^\circ$ as well as on the boundaries of IA-capture components. Thus we will prove Theorem B.

Let $\mathcal{U}$ be an IA-capture component in $\mathcal{F}$. Then, for every $f \in \mathcal{U}$, we write $A(f)$ for the immediate attracting basin of 0. There is a unique critical point $\omega_2(f)$ not in $A(f)$, and we have $f^{m_f}(\omega_2(f)) \in A(f)$ for some positive integer m_f. We may assume that m_f is the smallest positive integer with this property. Observe that m_f does not depend on f; it depends only on $\mathcal{U}$. We call this integer the *preperiod* of $\mathcal{U}$.

Lemma 6.1. *Let $\mathcal{U}$ be a hyperbolic component in $\mathcal{F}$ that is either $\mathcal{P}^\circ$ or an IA-capture component. In the latter case, let m be the preperiod of $\mathcal{U}$; in the former case, set $m = 2$. For every Brjuno $\theta \in \mathbb{R}/\mathbb{Z}$ and every $n \geqslant m$, there exists a map $f \in \mathrm{Bd}(\mathcal{U}) \cap \mathcal{F}_\lambda$, where $\lambda = e^{2\pi i\theta}$ and $f^n(c) = 0$ for some critical point c of f. Additionally, it can be arranged that $f^k(c) \neq 0$ for $k < n$.*

Let $\mathcal{X}_n$ be the set of all polynomials $f \in \mathcal{F}$ such that $f^n(c) = 0$ for some critical point c of f, and n is the smallest non-negative integer with this property. It is clear that $\mathcal{X}_n$ is a complex algebraic curve in $\mathcal{F} = \mathbb{C}^2$. Define a function μ on $\mathcal{X}_n$ as $\mu(f) = f'(0)$.

Lemma 6.2. *Let $\mathcal{U}$ be an IA-capture component. Consider a slice $\mathcal{F}_\lambda$ with $\lambda \neq 0$ such that $\mathcal{F}_\lambda \cap \mathcal{U} \neq \varnothing$; then clearly $|\lambda| < 1$. Take any integer $n \geqslant m$, where m is preperiod of $\mathcal{U}$. There is a polynomial $f_! \in \mathcal{F}_\lambda \cap \mathcal{U}$*

such that $f_!^n(c_!) = 0$ *for some critical point* $c_!$ *of* $f_!$, *and* $f_!^k(c_!) \neq 0$ *for* $k < n$.

Proof. The proof is a standard qc-deformation argument, cf. [BF14]. Take any $f \in \mathcal{F}_\lambda \cap \mathcal{U}$. Then there is a critical point c of f with $f^m(c) \in A(f)$. The point $v = f(c)$ is contained in a strictly preperiodic Fatou component V of f such that $f^{m-1}(V) = A(f)$. Consider a C^1-homeomorphism $h : \mathbb{C} \to \mathbb{C}$ that coincides with the identity outside of some compact subset of V. Taking iterated $h \circ f$-pullbacks of the standard complex structure in iterated pullbacks of V, we obtain an $h \circ f$-invariant complex structure on $\mathbb{C}$ that coincides with the standard one outside of iterated pullbacks of V. By the Measurable Riemann Mapping theorem, $h \circ f$ is conjugate to a rational function f_h by a qc-conjugacy fixing ∞. Since ∞ is a fixed critical point of f_h of multiplicity 2, we conclude that f_h is a polynomial. We may also arrange that $f_h \in \mathcal{F}$ by an affine change of variables. In a small neighborhood of 0, we have $h \circ f = f$, and f is conformally conjugate to f_h. Therefore, f and f_h have the same multiplier at 0, and $f_h \in \mathcal{F}_\lambda$. Note that f_h depends continuously on h, and $f_h = f$ for $h = id$. Thus any connected set of homeomorphisms h gives rise to a connected subset of $\mathcal{F}_\lambda$ lying entirely in $\mathcal{U}$.

We now consider a connected set $\mathcal{H}$ of homeomorphisms as above (i.e., all $h \in \mathcal{H}$ equal the identity outside of some compact subset of V). Let $\mathcal{D}$ be the corresponding set of maps f_h, where h runs through $\mathcal{H}$. Clearly, $\mathcal{D}$ is connected. For $g = f_h \in \mathcal{D}$, define v_g as the image of $h(v)$ under the conjugacy between $h \circ f$ and f_h. Then v_g is a critical value of g. We can choose a homeomorphism $h_!$ so that $f^{n-1}(h_!(v)) = 0$ and that $f^{k-1}(h_!(v)) \neq 0$ for $k < n$. Moreover, we can arrange that $f^{m-1}(h_!(v))$ is any given f^{n-m}-preimage of 0 in $A(f)$. This chosen homeomorphism $h_!$ can be included into a connected set $\mathcal{H}$ of homeomorphisms. The corresponding polynomial $f_! = f_{h_!}$ has a critical point $c_!$ corresponding to the critical point c of $h_! \circ f$. Set $v_! = f_!(c_!)$ to be the corresponding critical value; clearly, it corresponds to the critical value $h_!(v)$ of $h_! \circ f$. We have $f_!^n(c_!) = 0$ and $f_!^k(c_!) \neq 0$ for $k < n$. On the other hand, $f_!$ belongs to a connected set $\mathcal{D}$ of hyperbolic polynomials; therefore, $f_! \in \mathcal{F}_\lambda \cap \mathcal{U}$. □

The component $\mathcal{P}^\circ$ has been extensively studied in [PT09]. In particular, the following is an immediate corollary of the parameterization of $\mathcal{P}^\circ$ obtained in [PT09]:

Lemma 6.3. *Let* λ *be any complex number with* $|\lambda| < 1$, *and* n *be any integer that is at least 2. Then* $\mathcal{P}^\circ \cap \mathcal{F}_\lambda$ *contains a polynomial* $f_!$ *with* *the following properties:* $f_!^n(c_!) = 0$ *for some critical point* $c_!$ *of* $f_!$*, and* $f_!^k(c_!) \neq 0$ *for* $k < n$*.*

Thus, both in the case $\mathcal{U} = \mathcal{P}^\circ$ and in the case where $\mathcal{U}$ is an IA-capture component, we found a certain map $f_! \in \mathcal{U}$.

Proof of Lemma 6.1. Recall that the function $\mu : \mathcal{X}_n \to \mathbb{C}$ was defined by the formula $\mu(f) = f'(0)$. We claim that $\mu(\mathcal{X}_n \cap \mathcal{U})$ coincides with $\mathbb{D}$, possibly with finitely many punctures. In the case $\mathcal{U} = \mathcal{P}^\circ$, this follows from Lemma 6.3. Thus it suffices to assume that $\mathcal{U}$ is an IA-capture component. The inclusion $\mu(\mathcal{X}_n \cap \mathcal{U}) \subset \mathbb{D}$ is obvious. It now suffices to show that $\mu(\mathcal{X}_n \cap \mathcal{U})$ is open and closed in $\mathbb{D}$. It is open by the Open Mapping Theorem and since μ is holomorphic. Suppose now that λ belongs to the boundary of $\mu(\mathcal{X}_n \cap \mathcal{U})$ in $\mathbb{D}$ but not to $\mu(\mathcal{X}_n \cap \mathcal{U})$. Then there is a polynomial $f \in \mathcal{F}_\lambda \cap \overline{\mathcal{X}_n \cap \mathcal{U}}$. In other words, there is a sequence $f_i \in \mathcal{X}_n \cap \mathcal{U}$ with $f_i \to f \in \mathcal{F}_\lambda$ as $i \to \infty$. For every i, there is a critical point c_i of f_i with $f_i^n(c_i) = 0$. Passing to a subsequence, we may assume that $c_i \to c$ as $i \to \infty$, where c is a critical point of f, and $f^n(c) = 0$. On the other hand, $|\lambda| < 1$, hence f is hyperbolic. A hyperbolic polynomial belongs to the closure of a hyperbolic component $\mathcal{U}$ only if it belongs to $\mathcal{U}$. Therefore, $f \in \mathcal{U}$, but then by definition we have $f \in \mathcal{X}_n \cap \mathcal{U}$ unless f is a puncture of $\mathcal{X}_n$ (which means that $f^k(c) = 0$ for some $k < n$). The latter case is ruled out for the following reason. By [BP08], there is $\delta > 0$ such that the basin $A(f_i)$ for all large i contains a δ-disk around 0. This implies that $f^k(c) \neq 0$ for $k < n$. It follows that $\mu(f)$ as f runs through $\overline{\mathcal{X}}_n$ takes all values in $\mathbb{S}$, in particular, all values of the form $e^{2\pi i\theta}$, where θ is Brjuno.

Choose a point $f \in \overline{\mathcal{X}}_n \cap \mathcal{U}$ with $\mu(f) = e^{2\pi i\theta}$, where θ is Brjuno. It is clear that f is on the boundary of $\mathcal{U}$. We will now prove that f is IS-capture. Indeed, $f'(0) = \lambda = e^{2\pi i\theta}$ and θ is Brjuno, hence f has a Siegel disk Δ around 0 (we distinguish between the function μ and its particular value λ). On the other hand, since $f \in \overline{\mathcal{X}}_n$, there is a critical point c of f such that $f^n(c) = 0$. We have in fact $f \in \mathcal{X}_n$ (and $f^k(c) \neq 0$ for $k < n$) for the same reason as above. By definition, this means that f is an IS-capture polynomial. $\square$

The following statement is proved as Theorem 5.3 in [Zak99] for a different parameterization of basically the same slices. The only difference with [Zak99] is that Zakeri considers critically marked cubic polynomials.

Lemma 6.4. *Suppose that $f \in \mathcal{F}_\lambda$, where $|\lambda| = 1$, and f has a Siegel disk Δ around 0. If $f^n(c) \in \Delta$ for some critical point c of f, then there is an IS-capture component in $\mathcal{F}_\lambda$ containing f.*

Proof. The proof is based on the same qc-deformation argument as the proof of Lemma 6.2. We will use the notation introduced in Lemma 6.2, in particular, v, $\mathcal{H}$, $\mathcal{D}$ and f_h. Then $\mathcal{D} = \{f_h \,|h \in \mathcal{H}\}$ a connected subset of $\mathcal{F}_\lambda$ consisting of IS-capture polynomials. Recall that $v = f(c)$ is a critical value of f. We choose the set $\mathcal{H}$ of homeomorphisms so that $D = \{h(v) \,|\, h \in \mathcal{H}\}$ is open.

For every $g \in \mathcal{D}$, we let Δ_g be the Siegel disk of g around 0. We let V_g denote the Fatou component of g containing a critical value and such that $g^{n-1}(V_g) = \Delta_g$. These properties define V_g in a unique way. We will also write v_g for the critical value of g contained in V_g. Note that, if $g = f_h$, then v_g is the image of $h(v)$ under the conjugacy between $h \circ f$ and f_h. Consider the Riemann map $\phi_g : \Delta_g \to \mathbb{D}$ such that $\phi(0) = 0$ and $\phi'(0) \in \mathbb{R}_{>0}$. The map $g \mapsto \phi_g(g^{n-1}(v_g))$ takes $\mathcal{D}$ to the open set $\phi_f(f^{n-1}(D))$. Indeed, the image of f_h under this map is $\phi_f(f^{n-1}(h(v)))$. Thus, an analytic map takes $\mathcal{D}$ to some open set. It follows that $\mathcal{D}$ contains an open subset of $\mathcal{F}_\lambda$. Since $\mathcal{D}$ consists of IS-capture polynomials, it is contained in some IS-capture component. □

Finally, we can prove the main theorem of this section.

Theorem 6.5. *Let $\mathcal{U}$ be a hyperbolic component of $\mathcal{F}$ that is either $\mathcal{P}^\circ$ or an IA-capture component. In the latter case, set m to be the preperiod of $\mathcal{U}$; in the former case set $m = 2$. For every Bjuno $\theta \in \mathbb{R}/\mathbb{Z}$ and every $n \geqslant m$, there exists an IS-capture component $\mathcal{D}$ in $\mathrm{Bd}(\mathcal{U}) \cap \mathcal{F}_\lambda$ with $\lambda = e^{2\pi i\theta}$ such that, for all $g \in \mathcal{D}$, we have $g^n(c_g) \in \Delta_g$ for some critical point c_g of g. Here Δ_g is the Siegel disk of g around 0.*

Proof. By Lemma 6.1, for any Brjuno $\theta \in \mathbb{R}/\mathbb{Z}$ and any $n \geqslant m$, there is a cubic polynomial f with the following properties:

(1) we have $f \in \mathcal{F}_\lambda$, where $\lambda = e^{2\pi i\theta}$;
(2) there is a critical point c of f with $f^n(c) = 0$;
(3) we have $f^k(c) \neq 0$ for $k < n$.

By Lemma 6.4, there is an IS-capture component $\mathcal{D}$ in $\mathcal{F}_\lambda$ containing f. By Theorem A, the component $\mathcal{D}$ belongs to the boundary of a unique hyperbolic component $\mathcal{V}$ of $\mathcal{F}$. Moreover, by Theorem 5.8, the polynomial f lies on the boundary of a unique hyperbolic component. But f is in the boundary of $\mathcal{U}$. It follows that $\mathcal{V} = \mathcal{U}$, hence $\mathcal{D}$ is contained in the boundary of $\mathcal{U}$. □

Theorem 6.5 establishes the existence of many analytic disks on the boundary of the cubic connectedness locus. Observe that Lemma 6.4 and Theorem 6.5 imply Theorem B.

We conclude this section with a remark which relates our results concerning IA-capture components and laminations. A cubic invariant lamination $\mathcal{L}$ is said to be an *IA-capture lamination* if the following assumptions hold:

(1) there is an invariant Fatou gap A such that $\sigma_3|_{A\cap\mathbb{S}}$ is two-to-one;
(2) there is a Fatou gap $V \neq A$ such that $\sigma_3|_{V\cap\mathbb{S}}$ is two-to-one;
(3) we have $\sigma_3^{m_{\mathcal{L}}}(V) = A$, where $m_{\mathcal{L}} = m$ is the minimal integer with this property.

The number m is called the *preperiod of* $\mathcal{L}$. It is well-known (and easy to see) that any IA-lamination is the closure of its restriction upon all the rational angles (i.e., the closure of the corresponding rational lamination).

It follows from the appendix to [Mil12] written by Poirier that, for each IA-capture lamination $\mathcal{L}$, there exists a unique IA-capture component $\mathcal{U}_{\mathcal{L}} \subset \mathcal{F}$ with the following property. No matter which $f \in \mathcal{U}_{\mathcal{L}}$ we take, the lamination generated by f coincides with $\mathcal{L}$. The result of [Mil12] is stated in the language of Hubbard trees and so-called reduced mapping schemes, however, a straightforward translation of this result into the language of laminations yields the claim stated above. Similarly, if $\mathcal{L}$ is the empty lamination, then we set $\mathcal{U}_{\mathcal{L}} = \mathcal{P}^\circ$. Evidently, Theorem 6.5 can be restated to emphasize the role of IA-capture laminations, e.g., as follows.

Theorem 6.5′. *Let $\mathcal{L}$ be the empty lamination or an IA-capture lamination. In the latter case, set m to be the preperiod of $\mathcal{L}$; in the former case set $m = 2$. For every Brjuno $\theta \in \mathbb{R}/\mathbb{Z}$ and every $n \geqslant m$, the hyperbolic component $\mathcal{U}_{\mathcal{L}}$ contains an IS-capture component $\mathcal{D}$ in $\mathrm{Bd}(\mathcal{U}_{\mathcal{L}}) \cap \mathcal{F}_\lambda$ with $\lambda = e^{2\pi i\theta}$ such that, for all $g \in \mathcal{D}$, we have $g^{\circ n}(c_g) \in \Delta_g$ for some critical point c_g of g, and n is the least such integer. Here Δ_g is the Siegel disk of g around 0.*

7. The main cubioid of $\mathcal{F}$

In this section, we prove Theorem C and obtain corollaries related to the problem of distinguishing between Siegel and Cremer fixed points. Recall that the Main Cubioid $\mathcal{CU}$ was introduced in Definition 2.5.

Proof of Theorem B. Suppose that $f \in \mathcal{F}_\lambda$ with $|\lambda| = 1$ is a cubic IS-capture polynomial. By way of contradiction, assume that $f \in \mathcal{CU} \setminus \mathcal{P}$.

By Theorem 5.8, all polynomials f_ε for small $\varepsilon > 0$ belong to some IA-capture component $\mathcal{U}$ (since $f \notin \mathcal{P}$, we have $f_\varepsilon \notin \mathcal{P}^\circ$ for small ε). On the other hand, then, by Theorem A, the map f contains a repelling periodic cutpoint in its Julia set, a contradiction with $f \in \mathcal{CU}$. The rest of Theorem B now follows from Theorem 5.5. $\square$

For the sake of completeness we also prove the next lemma.

Lemma 7.1. *The only hyperbolic component of $\mathcal{F}$ intersecting $\mathcal{CU}$ is $\mathcal{P}^\circ$.*

Proof. Assume, to the contrary, that there exists a hyperbolic component $\mathcal{V} \neq \mathcal{P}^\circ$ intersecting $\mathcal{CU}$. Set $\mathcal{V}_\lambda = \mathcal{V} \cap \mathcal{F}_\lambda$ and $\mathcal{CU}_\lambda = \mathcal{CU} \cap \mathcal{F}_\lambda$. Choose λ with $\mathcal{V}_\lambda \cap \mathcal{CU}_\lambda \neq \varnothing$. We must have $|\lambda| \leqslant 1$ since otherwise $\mathcal{CU}_\lambda = \varnothing$. From $\mathcal{V}_\lambda \neq \varnothing$, it follows that $\mathcal{V} \cap \mathcal{F}_{at} \neq \varnothing$. But then $\mathcal{V} \subset \mathcal{F}_{at}$ and $|\lambda| < 1$. Note also that, since polynomials in $\mathcal{CU}$ have connected Julia sets, all polynomials in $\mathcal{V}$ have connected Julia sets, i.e., the component $\mathcal{V}$ is bounded.

Take $g \in \mathcal{V}_\lambda \cap \mathcal{CU}_\lambda$. Then $J(g)$ is locally connected; let $\mathcal{L}$ be the corresponding geodesic lamination. There is a gap G of $\mathcal{L}$ corresponding to $A(g)$. By Theorem 3.11, the major M of G is either critical or periodic. The former implies that a critical point of g belongs to $\mathrm{Bd}(A(g))$, a contradiction. Therefore, $M = \overline{\alpha\beta}$ is periodic. The rays $R_g(\alpha)$, $R_g(\beta)$ land at the same periodic point x of g. Since g is hyperbolic, x must be repelling. Thus g has a repelling periodic cutpoint of $J(g)$, a contradiction with $g \in \mathcal{CU}$. $\square$

A question as to whether a fixed irrationally indifferent point of a polynomial is Cremer or Siegel depending on the multiplier at this point is addressed in a conjecture by A. Douady. Let us now state a related to it corollary based upon results of Perez-Marco.

Definition 7.2 (Brjuno numbers)**.** The set $\mathcal{B}$ is the set of irrational numbers θ such that $\sum \frac{\ln q_{n+1}}{q_n} < \infty$, where $\frac{p_n}{q_n} \to \theta$ is the sequence of approximations given by the continued fraction expansion of θ. Numbers from $\mathcal{B}$ are called *Brjuno* numbers.

The following is a classical result by A. D. Brjuno [Brj71].

Theorem 7.3 ([Brj71])**.** *If a is an irrationally indifferent fixed point of a polynomial f with multiplier $e^{2\pi i\theta}$ and $\theta \in \mathcal{B}$, then the point a is a Siegel fixed point.*

Another classical result, due to J.-C. Yoccoz, states that in the quadratic case Theorem 7.3 is sharp.

Theorem 7.4 ([Yoc95]). *In the situation of Theorem 7.3, if f is quadratic and $\theta \notin \mathcal{B}$ is not a Brjuno number, then a is a Cremer fixed point of f.*

A conjecture by A. Douady states that Theorem 7.4 holds for higher degree polynomials too. Below we verify this for cubic polynomials $f_{\lambda,b} = \lambda z + bz^2 + z^3$ except for polynomials that belong to the set $\mathcal{P}_\lambda$. An important ingredient of our arguments is a result of R. Perez-Marco [Per01]; again for brevity we state only a relevant corollary of Perez-Marco's theorem reduced to our spaces of polynomials (the actual results of [Per01] are much stronger and much more general).

Corollary 7.5 (Corollary 1 [Per01]). *Suppose that $\lambda = e^{2\pi i\theta}$ and θ is irrational. Then the set of parameters b for which $f_{\lambda,b}$ has 0 as a Siegel fixed point is either the entire $\mathcal{F}_\lambda$, or, otherwise, has Hausdorff dimension 0 (in particular, it has empty interior).*

Combining these results with our tools we prove Corollary 7.6.

Corollary 7.6. *If $\theta \notin \mathcal{B}$ is not a Brjuno number and $\lambda = e^{2\pi i\theta}$, then the fact that $f \in \mathcal{F}_\lambda \setminus \mathcal{P}_\lambda$ implies that 0 is a Cremer fixed point of f.*

Proof. Suppose first that $f = f_{\lambda,b} \notin \mathrm{TH}(\mathcal{P}_\lambda)$. Then, by [BOPT16b], the map is immediately renormalizable; moreover, 0 belongs to the filled quadratic-like Julia set $K^* \subset K(f)$ of f. By Theorem 7.4, this implies that 0 is a Cremer point of f . By Corollary 7.5, it follows then that the set of parameters b for which $f_{\lambda,b}$ has 0 as a Siegel point has empty interior. Since, by [BOPT16b], in each component of $\mathrm{TH}(\mathcal{P}_\lambda) \setminus \mathcal{P}_\lambda$ the polynomials are conjugate, then polynomials in those bounded domains cannot have 0 as their fixed Siegel point. This completes the proof. □

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