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Rotation numbers for certain maps of an n-od [1]

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Abstract

We study maps of an n-od with the branching point fixed and show that sometimes it is possible to introduce rotation numbers and prove theorems similar to those known for the circle and the interval. We obtain additional results for $n = 3$. They explain the form of the sets of periods of periodic points for triod maps. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

An important problem in dynamical systems theory is that of coexistence of various kinds of behavior, in particular in the case of *periodic orbits* (called *cycles* in what follows). A number of results in this direction were obtained for one-dimensional maps with the first being a famous Sharkovskiĭ's theorem [11]. To state it let us first introduce the *Sharkovskiĭ ordering* for the set $\mathbb{N}$ of positive integers:

$$3 \succ 5 \succ 7 \succ \cdots \succ 2\cdot 3 \succ 2\cdot 5 \succ 2\cdot 7 \succ \cdots$$
$$\succ 2^2\cdot 3 \succ 2^2\cdot 5 \succ 2^2\cdot 7 \succ \cdots \succ 8 \succ 4 \succ 2 \succ 1.$$

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Denote by $S(k)$ the set of all integers m such that $k \succ m$, together with k, by $S(2^\infty)$ the set $\{1, 2, 4, \ldots\}$ and by $\mathbb{N}'$ the set $\mathbb{N} \cup \{2^\infty\}$. Let also $\mathrm{Per}(f)$ be the set of periods of cycles of f (by a *period* we mean the least period).

Theorem 1.1 [11]. *If $f : [0, 1] \to [0, 1]$ is a continuous map, $m \succ n$ and $m \in \mathrm{Per}(f)$, then $n \in \mathrm{Per}(f)$; hence there exists $k \in \mathbb{N}'$ with $\mathrm{Per}(f) = S(k)$. Moreover, for every $k \in \mathbb{N}'$ there exists a continuous map $f : [0, 1] \to [0, 1]$ with $\mathrm{Per}(f) = S(k)$.*

Here certain types of dynamical behavior, once exhibited by a map, *force* some other types, thus the question of coexistence becomes a question of forcing.

Now we state the related results of [9]. Consider a circle map f of degree 1. Choose its lifting F and observe that if $x \in S^1$ is periodic of period n and X is its lifting then $F^n(X) = X + m$ where m does not depend on the choice of X. Define the *rotation pair* of x as (m, n) and the *rotation number* of X as m/n. Note that they can also be defined by means of the "displacement" $\varphi(x) = F(X) - X$; the rotation pair of x is $(\sum_{i=0}^{n-1} \varphi(f^i(x)), n)$ and the rotation number is $\sum_{i=0}^{n-1} \varphi(f^i(x))/n$.

It is useful to represent rotation pairs differently. Think of a rotation pair (mp, mq) with p, q coprime as a pair (t, m), where $t = p/q$ is a rational number and m a positive integer. We call the latter pair a *modified rotation pair* and the number mq its *period*. Then think of the real line with a prong attached at each rational point and the set $\mathbb{N}'$ marked on this prong in the Sharkovskiĭ ordering (1 is closest to the real line and 3 is furthest from it). All points of the real line are marked 0; at irrational points we can think of degenerate prongs with only 0 on them. The union of all prongs and the real line is denoted by $\mathbb{M}$. Thus, a modified rotation pair (t, m) corresponds to the specific element of $\mathbb{M}$, namely to the number m on the prong attached at t. However, no rotation pair corresponds to $(t, 2^\infty)$ or to $(t, 0)$.

With all this in mind, it is natural to speak of the *hull* $[(t_1, m_1), (t_2, m_2)]$ of two elements of $\mathbb{M}$ (in particular, (t_1, m_1) and (t_2, m_2) may be modified rotation pairs themselves). It consists of all modified rotation pairs (t, m) with either t strictly between t_1 and t_2 or $t = t_i$ and $m \in S(m_i)$ for $i = 1$ or 2.

Let $\mathrm{mrp}(f)$ be the set of modified rotation pairs of all cycles of f. Clearly, $\mathrm{mrp}(f) \subset \mathbb{M}$. Moreover, the following theorem holds.

Theorem 1.2 [9]. *Let f be a degree 1 map of the circle. Then there are elements (t_1, m_1) and (t_2, m_2) of $\mathbb{M}$ such that $\mathrm{mrp}(f) = [(t_1, m_1), (t_2, m_2)]$, and if t_i is rational then $m_i \neq 0$ for $i = 1, 2$. Moreover, for any set of the above form there exists a degree 1 map f of the circle with $\mathrm{mrp}(f)$ equal to this set.*

Similar results hold for interval maps. Let $f : I \to I$ be continuous, let P be a cycle of f of period $q > 1$ and let m be the number of points $x \in P$ such that $(f(x) - x)(f^2(x) - f(x)) < 0$. Then $(m/2, q)$ is called the *over-rotation pair* of P and the number $m/(2q)$ is called the *over-rotation number* of P (see [8]). Again, a specific displacement generates them. Set $\Phi_{or}(f)(x)$ to be 1/2 if $(f^2(x) - f(x))(f(x) - x) \leqslant 0$ and 0 otherwise. Then the over-rotation pair of a periodic point x of period n is $(\sum_{i=0}^{n-1} \Phi_{or}(f)(f^i(x)), n)$. Observe that the function Φ_{or} is invariant under non-reversing orientation conjugacies: if f and g are ψ-conjugate and ψ does not reverse the orientation then $\Phi_{or}(f)(x) = \Phi_{or}(g)(\psi(x))$.

Since the number m above is even, positive, and does not exceed $q/2$ then in an over-rotation pair (p, q) both p and q are integers and $0 < p/q \leqslant 1/2$. Just like we did before we can again transform all rotation pairs of cycles of f into modified rotation pairs and denote the set of all modified rotation pairs of cycles of f by $\mathrm{mrp}(f)$. Then again $\mathrm{mrp}(f) \subset \mathbb{M}$ and the following theorem holds.

Theorem 1.3 [8]. *If $f : [0,1] \to [0,1]$ is a continuous map with some cycles of period greater than 1 then $\mathrm{mrp}(f) = [(t_1, m_1), (1,2)]$ for some $(t_1, m_1) \in \mathbb{M}$. Moreover, for every $(t_1, m_1) \in \mathbb{M}$ there exists a continuous map $f : [0,1] \to [0,1]$ with $\mathrm{mrp}(f) = [(t_1, m_1), (1,2)]$.*

Theorems 1.1, 1.2 and 1.3 are examples of the situation when knowing a little about a map (the period of a cycle, the rotation pair or two rotation pairs of two cycles) we can say a lot about the variety of its cycles. Observe, that here the number of known parameters does not depend on the periods of points. We would like to obtain similar informative results for other dynamical systems. We call them “informative” because relying upon them one gets a lot of information from just a little information. Theorems 1.1–1.3 fit into a general scheme [6,12] described below; in the present paper we apply this scheme to n-od maps.

Let $f : K \to K$ be a continuous map of a compact space K into itself and let φ be a function (*displacement*) defined on K or its subset. If x is a periodic point of period n and displacement is defined at all points of its orbit then we call the pair of numbers $\mathrm{rp}_\varphi(x) = (\sum_{i=0}^{n-1} \varphi(f^i(x)), n) = (s, n)$ the *φ-rotation pair* of x and the number $\rho_\varphi(x) = s/n$ the *φ-rotation number* of x. The closure $I_\varphi(f)$ of the set of all φ-rotation numbers of periodic points of f is called the *φ-rotation set* of f, the set of all their φ-rotation pairs transformed into modified rotation pairs is denoted by $\mathrm{mrp}_\varphi(f)$. If the displacement φ is fixed then we often omit it from the notation.

All these objects are called *functional rotation* pairs, numbers and sets. If for some maps f and appropriate choice of displacement φ the set $\mathrm{mrp}_\varphi(f)$ is a hull then it means that knowing just a little (the ends of the hull) we can get a lot of information about the cycles of a map. We also work in a different situation. Namely, we consider maps with cycles of certain type and prove that then for some displacement φ the set $\mathrm{mrp}_\varphi(f)$ is a hull; that is, in this case we do not fix the displacement up front, rather try to choose it for certain types of maps so that $\mathrm{mrp}_\varphi(f)$ is a hull. In either case we say that *rotation theory is constructed*.

In this paper we introduce displacement and rotation numbers and pairs for maps of n-od fixing its branching point and obtain for these maps some results similar to Theorems 1.2 and 1.3. Still, the situation for the n-od is more complicated and those results are not full analogs of Theorems 1.2 and 1.3.

The paper is organized as follows. In Sections 2 and 3 we introduce our main tools. In Section 4 we introduce a class of cycles on the n-od X which we call *non-passing*; with such cycles P and P-monotone maps we always associate a specific displacement φ. Then by the general construction we get sets of all modified (φ-)rotation pairs of cycles of f denoted by $\mathrm{mrp}(f)$ and the following theorem holds (we state it here in a weaker form than in Section 4). A map $f : X \to X$ of the n-od into itself is called P-linear for its cycle P if it fixes the branching point a of X, is affine on every component of $[P] \setminus (P \cup \{a\})$ and constant on every component of $X \setminus [P]$, where $[P]$ is the smallest connected set containing P.

Theorem 4.5′. *Let $f : X \to X$ be a P-linear map where P is a non-passing cycle. Then there are there are elements (t_1, m_1) and (t_2, m_2) of $\mathbb{M}$ such that* $\mathrm{mrp}(f) = [(t_1, m_1), (t_2, m_2)]$.

In Sections 5 and 6 we apply our tools to triod maps. In particular we show that if a P-linear map f of the triod X has only one fixed point (namely, its branching point) and no periodic points of period 2 then the results similar to Theorems 1.2, 1.3 and 4.5 can be proven for f. We then deduce well-known results about coexistence of periods for triod maps [1] from this.

Note that in [4] results about coexistence of periods similar to those of [1] were obtained. A nice interpretation of this description of sets of periods of a continuous self map of the n-od was given in [3], where it was shown that the sets of periods of such a map can be expressed as the unions of “initial segments” of the linear orderings associated to all rationals in the interval $(0, 1)$ with denominator at most n defined in certain subsets of natural numbers. However, this phenomenon was only observed but not explained. Rotation theory fully explains it in the case of interval [8] and, as we show in this paper, in the case of triod. We hope that appropriate version of rotation theory for n-od will explain the results of [4,3] as well.

2. Patterns

We consider the set $\mathcal{U}_n$ of all continuous maps of an n-od X into itself for which the central point a of X is fixed. We write $x > y$ if x and y lie on the same branch of X and x is further from a than y. We write also $x \geqslant y$ if $x > y$ or $x = y$.

Call two cycles P, Q on X *equivalent* if there exists a homeomorphism $h : [P] \to [Q]$ conjugating P and Q and fixing branches of X. The class of equivalence of a cycle P is called the *pattern* of P. This definition is a slight variation of the standard one (see, e.g., [1,2]) since we treat branches as distinguishable.

The cycle $\{a\}$ and its pattern are very special. In many cases when we speak about some properties of patterns or some constructions involving patterns, it constitutes an exception and often we treat $\{a\}$ as having no pattern at all. Repeating such statements each time when it is necessary makes the paper less readable. Thus we adopt the method often used with the empty set: forget about this nasty exception. We hope this will not cause any confusion.

We use the standard terminology for patterns. A cycle P of a map $f \in \mathcal{U}_n$ *exhibits* a pattern A (or *is of pattern A*, or is a *representative* of the pattern A in f) if P belongs to the equivalence class A. A pattern A *forces* a pattern B if and only if any map $f \in \mathcal{U}_n$ with a cycle of pattern A has also a cycle of pattern B.

The following theorems are minor variations of the ones proved in [1,2], and the proofs are practically the same. We say that a cycle P has a *block structure over* a cycle Q if it can be divided into subsets (*blocks*) $P_1, \dots, P_m$ of the same cardinality, where m is the period of Q, the sets $[P_i]$ are pairwise disjoint, none of them contains a, each of them contains one point x_i of Q, and $f(P_i) = P_j$ whenever $f(x_i) = x_j$. We use the same terminology for patterns. In particular, a pattern A has a *block structure over* a pattern B if there exists a cycle P of pattern A with a block structure over a cycle Q of pattern B.

Theorem 2.1. *If a pattern A forces a pattern $B \neq A$ then B does not force A.*

Theorem 2.2. *Let f be a P-linear map where P is of pattern A. Then a pattern B is forced by A if and only if f has a cycle Q of pattern B.*

Theorem 2.3. *If a pattern A has a block structure over B and A forces C then either C has a block structure over B or B forces C. Moreover, if P is a representative of A in a P-linear map $f \in \mathcal{U}_n$ and A has a block structure over B then P has blocks P_i corresponding to this structure and whenever A forces a pattern C with block structure over B then there is a representative Q of C in f contained in the union of the convex hulls of these blocks.*

We need more terminology. If P has a block structure over Q and blocks consist of two points each, we call P a *doubling* of Q. A cycle that is a doubling of another cycle is called a *doubling*. We use the same terminology for patterns.

We call a cycle (and its pattern) *primitive* if each of its points lies on a different branch of X.

If f is Q-linear for a cycle Q and $P \neq Q$ is a cycle of f of period m, each point of P is repelling for f^m, except two cases: either Q is a doubling of P, or Q is primitive (and then P has the same pattern). With those two exceptions, we call P *positive* or *negative*, according to whether f^n preserves or reverses orientation at points of P. Now we prove an analog of Theorem 9.12 of [10].

Theorem 2.4. *Assume that a pattern A forces a pattern B of period m, A has no block structure over B, and B is not a doubling. Then for every $k > 1$ A forces a pattern of period km with a block structure over B.*

Proof. Let f be the Q-linear map for a cycle Q with pattern A. By Theorem 2.1, patterns of cycles of f are exactly those that are forced by A. Hence f has a cycle P of pattern B. Since A has no block structure over B, the cycles P and Q have different patterns and Q is not a doubling of P. Therefore P is repelling, and either negative or positive.

Assume first that P is negative and try to find a positive representative of B. Choose $x \in P$ such that there is no $y \in P$ with $y > x$ (i.e., x is the farthest from a point of P on its branch). Moreover, if P is not primitive, choose x from a branch on which there is more than one point of P. Start to move x towards a and look what happens with its images. To describe the movement, we introduce the real variable t (time) and speak of $x(t)$ depending affinely on t, with $x(0) = x$ and $x(1) = a$.

Let us make some observations concerning the movement of the points. First of all, notice that $f^m(x(0)) = x(0)$ and that for small t we have $f^m(x(t)) > x(0) > x(t)$, which follows from the fact that P is negative. Thus, the point $f^m(x(t))$ starts to move initially away from x, but then it has to turn back and collide with $x(t)$ for some $T \in (0, 1]$ for the first time. Then by continuity we have $f^m(x(t)) > x(t)$ for $0 < t < T$. Also, by the choice of the point $x(0)$ for any $1 \leqslant k < m$ either $x(0) > f^k(x(0))$ or the point $f^k(x(0))$ does not belong to the same branch of X as $x(0)$.

We claim that for $0 < t < T$ there is no collision between points $f^i(x(t))$ and $f^j(x(t))$ for $0 \leqslant i < j \leqslant m$ and no collision between $f^i(x(t))$ and a for $0 \leqslant i \leqslant m$. Suppose that there is such a collision. Since the times of collision form a closed set and there is no collision for small t, if there is a collision for some $0 < t < T$ then there is the smallest time $0 < s < T$ when a collision occurs. If $f^i(x(s)) = a$ then also $f^m(x(s)) = a$, but this contradicts $f^m(x(s)) > x(s)$. If $f^i(x(s)) = f^j(x(s))$ for some $0 \leqslant i < j \leqslant m$ then $f^k(x(s)) = f^m(x(s))$ for $k = m - j + i$, so $f^k(x(s)) > x(s)$. Since $0 < k < m$, either $x(0) > f^k(x(0))$ or $f^k(x(0))$ and $x(0)$ lie on different branches. Hence there is a collision between $x(t)$ and $f^k(x(t))$ for some t smaller than s, a contradiction. This proves the claim. Note that it follows from the claim that it is impossible to have $f^i(x(t)) \geqslant x(t)$ for $0 < i < m$ and $0 \leqslant t < T$.

From the above claim it follows that the ordering of the images of $x(t)$ on the branches stay the same all the time. Denote the orbit of $y = x(T)$ by P'. If the period of y is m then P' has the same pattern as P. Moreover, since for all $t < T$ we have $f^m(x(t)) > x(t) > x(T)$ then P' is positive. Let us see what happens if the period of P' is $k < m$ and $T < 1$. Since $f^m(y) = y$, k divides m. If P' is positive (that is, f^k preserves orientation at the points of P') then for $t < T$, but close to T, we have $f^k(x(t)) > x(t)$, a contradiction. If P' is negative then for such t we have $f^{2k}(x(t)) > x(t)$ and hence $2k = m$. Thus, at T we have a collision of pairs $f^i(x(T))$, $f^{m/2+i}(x(T))$, and before T the relative order of points $f^j(x(t))$ stays the same. This means that P is a doubling of P'. However, we assumed that B (and therefore P) is not a doubling. Hence, the situation described above cannot occur.

Another possibility is $T = 1$. Then $y = a$. We claim that in this case P is primitive. Indeed, otherwise, by our choice of x, there is $0 < i < m$ such that $x(0) > f^i(x(0))$. For some $t < T$ we have $x(t) = f^i(x(0))$, so $f^{m-i}(x(t)) = f^m(x(0)) = x(0) > x(t)$, a contradiction.

In such a way we proved that f has a positive cycle R of pattern B. If B is primitive, R may degenerate to $\{a\}$. This means that instead of points of R we have to look at germs of branches at a. Fortunately, this does not make much difference for us, and the rest of the proof is essentially the same for non-degenerate and degenerate cases. For simplicity we write the proof for the non-degenerate case; the reader can easily verify that it works for the degenerate case too.

We use a similar technique as in the first part of the proof. Namely, we choose a point $x \in R$ such that there is no $y \in R$ with $y > x$ and then move it away from a. Since f^m

preserves orientation at x, the point $f^m(x)$ moves initially also away from a. We stop when either we get with $x(t)$ to the end of the branch or with $f^m(x(t))$ to $x(0)$ (observe the difference at this moment with the construction from the beginning of the proof). Note also that all the time (except the initial time 0 and perhaps the final time T) we have $f^m(x(t)) > x(0)$.

We claim that if $s, t \in (0, T)$ and $0 < i < j \leqslant m$ then $f^i(x(s)) \neq f^j(x(t))$. Indeed, if $f^i(x(s)) = f^j(x(t))$ then $f^k(x(s)) = f^m(x(t)) > x(0)$ for $k = m - j + i$ while on the other hand $x(0) > f^k(x(0))$. Hence, there is $u \in (0, s)$ with $f^k(x(u)) = x(0)$. Clearly $f^{j-i}(x(0)) > x(0)$ is impossible since $j - i < m$. On the other hand we must have $f^{j-i}(x(0)) = f^{m-k}(x(0)) = f^m(x(u)) > x(0)$, a contradiction. This proves the claim.

Consequently, if $J = (x(0), x(T))$ then the intervals $f^i(J)$, $j = 1, \ldots, m$, are pairwise disjoint. Moreover, since a is a fixed point, these intervals do not contain a (if for some $s \in (0, T)$ we have $f^m(x(s)) = a$ then for some $u < s$ we have $f^m(x(u)) = x(0)$, which contradicts the choice of T). Therefore any cycle contained in $\bigcup_{i=0}^{m-1} f^i(J)$ has a block structure over R. As we just explained, the f^m-image of no point from the interior of J is $x(0)$ and therefore $f^m(J)$ lies (non-strictly) farther away from a than $x(0)$.

By the definition of T, either $x(T)$ is an end of a branch, or $f^m(x(T)) = x(0)$. Let us show that in any case there is a point of Q inside $\bigcup_{i=0}^{m-1} f^i(J)$. Indeed, in the former case it follows from the fact that the pattern B is forced by the pattern A and so there must be a point of Q farther away from a than $x(0)$, and this point is in J because $x(T)$ is the end of the branch. In the latter case notice that $f^m(x(t))$ does not move in a monotone way. Since f is Q-linear, it implies that there is a point of Q in $\bigcup_{i=0}^{m-1} f^i(J)$.

If $f^m(J) \subset J$ then Q is contained in $\bigcup_{i=0}^{m-1} f^i(J)$. Hence Q has a block structure over R, contrary to our assumptions. Therefore $f^m(J) \supsetneq J$, $x(T)$ cannot be the end of the branch and always $f^m(x(T)) = x(0)$. Also, $f^m(J) \supsetneq J$ implies that we can find a point $y \in J$ with $f^m(y) \geqslant x(T)$. Therefore if we set $J_1 = (x(0), y)$ and $J_2 = (y, x(T))$ then both $f^m(J_1)$ and $f^m(J_2)$ contain J. The standard technique (see, e.g., [5,2]) allows us to find for every k a periodic point of f^m of period k, belonging to in J. This point is periodic for f of period km. Its orbit is contained in $\bigcup_{i=0}^{m-1} f^i(J)$, so it has a block structure over R. □

When we study patterns with a block structure over other patterns, we often have to look at the iterate of our map that maps a block into itself. If the initial map is P-linear for a cycle P with a given pattern then this leads to the investigation of an interval map $g : I \to I$ which is Q-linear for some invariant finite set Q (not necessarily a cycle).

Lemma 2.5. *Let $g : I \to I$ be a continuous interval map which is Q-linear for a finite invariant set Q. Then either g has a cycle of period which is not a power of 2 or it has only cycles of finitely many periods.*

Proof. Let us look at the standard oriented graph G whose vertices are closures of components of $I \setminus Q$ and arrows correspond to g-covering. Then there is a correspondence between the loops of G and the cycles of g (see, e.g., [5,2]). If G has no distinct elementary loops that pass through the same vertex then there are only finitely many loops in G, and thus there are only finitely many periods of cycles of g. If G has such loops then the number of paths in G of length m grows exponentially with m and therefore g has positive topological entropy. Hence, g has a cycle of period which is not a power of 2. □

3. Oriented graphs

We will say that *there is an arrow from x to y* and write $x \to y$, if there is z such that $x \geqslant z$ and $f(z) \geqslant y$. Thus, any finite set $P \subset X \setminus \{a\}$ gives us an oriented graph, whose vertices are elements of P and arrows are defined as above. When we refer to a loop then, unless stated otherwise, we mean a loop in this type of graph. Also, we use the standard definition of f-covering (see, e.g., [2]). Namely, we say that an interval I *f-covers* an interval J if $f(I) \supset J$. Then we speak of a *chain* of intervals $I_0 \to I_1 \to \cdots$ if every previous interval in the chain f-covers the next one. We also speak of loops of intervals. The following two lemmas are the basic tool in the rest of the paper.

Lemma 3.1. *Let $x_0 \to x_1 \to \cdots \to x_{m-1} \to x_0$ be a loop. Assume that there is $z \in X \setminus \{a\}$ and $x_i > z$ such that $f(x_i)$ and $f(z)$ lie on different branches of X. Then there is a point $y \in X \setminus \{a\}$ such that $f^m(y) = y$ and $x_k \geqslant f^k(y)$ for $k = 0, 1, \ldots, m-1$.*

Proof. There is a point $w \in [z, x_i]$ such that $f(w) = a$. Then, using the standard technique of f-covering (see, e.g., [1,2]), we get a loop of intervals $[a, x_0] \to \cdots \to [w, x_i] \to [a, x_{i+1}] \to \cdots \to [a, x_{m-1}] \to [a, x_0]$, which gives us a desired point y. □

Lemma 3.2. *The following properties hold.*

(1) *Let $x_0 \to x_1 \to \cdots \to x_{m-1} \to x_0$ be a loop. Then there is a point $y \in X \setminus \{a\}$ such that $f^m(y) = y$ and for every $k = 0, 1, \ldots, m-1$ the points x_k and $f^k(y)$ lie on the same branch of X.*

(2) *Let f be a P-linear map for some cycle $P \neq \{a\}$. Suppose that $y \neq a$ is a periodic point of f of period q. Then there exists a loop $x_0 \to x_1 \to \cdots \to x_{q-1} \to x_0$ such that $x_i \geqslant f^i(y)$ for all i.*

Proof. (1) If the assumptions of Lemma 3.1 are satisfied, we use it. Otherwise, if $z \in X \setminus \{a\}$ is sufficiently close to a and $x_0 > z$, then $x_k > f^k(z)$ for $k = 0, 1, \ldots, m-1$. If $z \geqslant f^m(z)$ then we get the loop of intervals $[z, x_0] \to [f(z), x_1] \to \cdots \to [f^{m-1}(z), x_{m-1}] \to [z, x_0]$, which gives us a desired point y. If $f^m(z) > z$ then we move z (formally, we consider $z(t)$ where t is "time") away from a for as long as $f^k(z) \neq a$ for $k = 0, 1, \ldots, m-1$ and $f^m(z) > z$. The condition $f^m(z) > z$ has to break first, since if $f^k(z) = a$ then $z > a = f^m(z)$. Then $y = z$ is our point.

(2) We may assume that $y \notin P \cup \{a\}$. Then for any i, $0 \leqslant i \leqslant q-1$, there exists a well-defined component J_i of $X \setminus (P \cup \{a\})$ containing $f^i(y)$. Denote by x_i the endpoint of J_i for which $x_i > f^i(y)$. Then $x_i \in P$. By the definition of a P-linear map for one of the endpoints z of J_i we have $f(z) > f^{i+1}(y)$, and since $f(z) \in P$, we get

$f(z) \geqslant x_{i+1}$. Hence, there is an arrow $x_i \to x_{i+1}$. Since $x_q = x_0$, we obtained the desired loop $x_0 \to x_1 \to \cdots \to x_{q-1} \to x_0$. □

Lemma 3.2 and Theorem 2.2 allow us to speak of loops in the graph G given by a pattern A which correspond to patterns forced by A. Together with Theorem 2.4, they also provide tools for studying sets of periods for maps of n-od. Moreover, later we introduce the notion of rotation pair for such maps and rely upon Lemma 3.2 in studying them by means of graph G.

Now we start to build a rotation theory for our maps. Suppose we have $f \in \mathcal{U}_n$ and a finite set $P \subset X \setminus \{a\}$, and the oriented graph G given by P is transitive (that is, there is a path from every vertex to every vertex). This is the case for instance if P is a cycle.

Denote the set of all arrows of G by A. We will call the set of all points of P that are contained in a given branch of X also a *branch*. Thus, the set P is divided into branches. Denote the set of all branches of P by B. Now think about X as being embedded into the plane with the central point at the origin and branches being segments of straight lines. To go from one branch to another we have to turn by some angle. This angle is defined up to a multiple of 2π, but we choose one value. This value, divided by 2π, will be the displacement assigned to the transition between the two branches. We can formalize this in the following way. We choose functions $\zeta : B \to \mathbb{R}$ (*position*) and $\psi : B \times B \to \mathbb{R}$ (*displacement*) such that for any branches b, c the number $\zeta(b) + \psi(b, c)$ differs from $\zeta(c)$ by an integer. Moreover, we require that if $b \neq c$ then $\zeta(b) - \zeta(c)$ is not an integer.

Once the displacement has been defined on $B \times B$, it induces in the natural way a function $\varphi : A \to \mathbb{R}$, which we also call *displacement*. Namely, if $u \in b$ and $v \in c$, then $\varphi(u \to v) = \psi(b, c)$. Note that although set P and the graph G coming with P motivate us to introduce displacement, in fact the latter may be introduced for transitions between branches of X which in turn would induce the displacements ψ and φ as above.

Note that our graph G has a special property, connected with the branch structure. Namely, if u and v belong to the same branch, one of them is further from a than the other one. This is visible in G in the following way. If $v \geqslant u$ then whenever there is an arrow $w \to v$, there is also an arrow $w \to u$, and whenever there is an arrow $u \to w$, there is also an arrow $v \to w$.

For a loop Γ in G denote by $\varphi(\Gamma)$ the sum of the values of the displacement φ along the loop. In the model with X embedded into the plane, this number tells us how many times we revolved around the origin. Thus, $\varphi(\Gamma)$ is an integer. We denote the length of Γ by $|\Gamma|$. As usual, $(\varphi(\Gamma), |\Gamma|)$ is called the *rotation pair* of Γ and $\varphi(\Gamma)/|\Gamma|$ the *rotation number* of Γ. The closure of the set of rotation numbers of all loops of G is called the *rotation set* of G and denoted $L(G)$.

By [12], the rotation set of G is equal to the smallest interval containing the rotation numbers of all elementary loops of G (a loop is called *elementary* if it passes through every vertex at most once). Moreover, every rational number from $L(G)$ is the rotation number of some loop of G. The next question usually asked in such situations is whether if p/q belongs to $L(G)$ and p, q are coprime then there exists a loop with rotation pair

(p, q). In general the answer is "no". However, there is a special situation when the answer is "yes".

For every arrow $u \to v$ of G, where u belongs to the branch b, and an integer m, think of a car $(u \to v)_m$ driving from $m + \zeta(b)$ to $m + \zeta(b) + \varphi(u \to v)$. A car $(u' \to v')_{m'}$ (with $u' \in b'$) *passes* a car $(u \to v)_m$ if $m' + \zeta(b') < m + \zeta(b)$, while $m' + \zeta(b') + \varphi(u' \to v') > m + \zeta(b) + \varphi(u \to v)$. If this does not happen for any pair of cars corresponding to arrows in G, we say that our displacement is *non-passing*. Although the definition formally involves infinitely many objects, passing cannot occur if $|m - m'|$ is too large. Therefore checking whether a displacement is non-passing involves only finitely many operations.

Proposition 3.3. *The following properties hold.*

(1) *Assume that G is transitive and the displacement is non-passing. Then for every pair of integers (p, q) with $q > 0$ and p, q coprime, such that $p/q \in L(G)$, there exists a loop in G with rotation pair (p, q).*

(2) *Suppose that (r, s) and (r', s') are rotation pairs of two non-disjoint loops in G and that (r, s) and (r', s') are Farey neighbors (that is, $|rs' - r's| = 1$). Then for every pair of integers (u, t) with $t > 0$ such that u/t lies between r/s and r'/s' there exists a loop in G with rotation pair (u, t).*

Proof. (1) As we already know, there exists a loop in G with rotation number p/q. Take such a loop Γ of minimal length. Then the rotation pair of Γ is (mp, mq) for some positive integer m. We have to show that $m = 1$.

Assume that $m > 1$. Let the consecutive arrows in Γ be $\alpha_0, \alpha_1, \ldots, \alpha_{mq-1}$, and set $\alpha_{mq} = \alpha_0$, $\alpha_{mq+1} = \alpha_1, \ldots$. Look at the sums $s_i = \varphi(\alpha_i) + \varphi(\alpha_{i+1}) + \cdots + \varphi(\alpha_{i+q-1})$. We claim that at least one of these sums is equal to p. Indeed, if none of them is, then (since mp is the sum of m such sums) there are i's with $s_i < p$ and with $s_i > p$. Therefore there is j with $s_j < p$ and $s_{j+1} > p$. Let the arrow α_j go from the branch b to c, and the arrow α_{j+q} from the branch b' to c'. Then $\zeta(b) + s_j = \zeta(b') + k$ for some integer k. The car $(b, c)_p$ drives from $p + \zeta(b)$ to $p + \zeta(b) + \psi(b, c)$, while the car $(b', c')_k$ drives from $s_j + \zeta(b)$ to $s_j + \zeta(b) + \psi(b', c')$. Since $s_j < p$, we have $s_j + \zeta(b) < p + \zeta(b)$. On the other hand,

$$s_j + \psi(b', c') = s_j + \varphi(a_{j+q}) = s_{j+1} + \varphi(a_j) = s_{j+1} + \psi(b, c) > p + \psi(b, c),$$

and hence $s_j + \zeta(b) + \psi(b', c') > p + \zeta(b) + \psi(b, c)$. Therefore the car $(b', c')_k$ passes the car $(b, c)_p$, a contradiction. This proves the claim.

Hence, there is i such that $s_i = p$. Denote the vertex at which the arrow α_i begins by u and the vertex at which the arrow α_{i+q-1} ends by v. Since the values of the position function ζ on different branches do not differ by an integer, u and v belong to the same branch of X. Therefore either $v \geqslant u$ or $u \geqslant v$. In the first case the arrows $\alpha_i, \alpha_{i+1}, \ldots, \alpha_{i+q-2}, \alpha_{i+q-1}$, form a loop of rotation pair (p, q). In the second case in a similar way we get a loop of rotation pair $((m-1)p, (m-1)q)$. In both cases we get a contradiction, since we assumed that Γ is the shortest loop of rotation number p/q. This completes the proof of the statement (1) of the proposition.

(2) It is well known that if (r, s) and (r', s') are Farey neighbors and u/t lies between r/s and r'/s' then there exist positive integers a, b such that $ar + br' = u$, $as + bs' = t$. Hence the concatenation of a copies of the first loop and b copies of the second loop, each starting at the common vertex of both loops, is a loop of rotation pair (u, t), as desired. □

We finish this section with a technical lemma.

Lemma 3.4. *Assume that G is transitive, p, q are coprime, and p/q belongs to the interior of $L(G)$. Then there exists a loop in G with rotation number p/q which does not correspond to a pattern with a block structure over any pattern of rotation pair (p, q).*

Proof. By [12], there are loops Γ_1, Γ_2 in G with rotation numbers t_1, t_2, passing through a common vertex v, and positive integers m_1, m_2 such that $t_1 < p/q < t_2$ and the concatenation of m_1 copies of Γ_1 and m_2 copies of Γ_2 is a loop of rotation number p/q. Let Γ be the concatenation of qm_1 copies of Γ_1 followed by qm_2 copies of Γ_2. If the pattern corresponding to Γ has a block structure over a pattern of rotation pair (p, q) then the displacement corresponding to any q consecutive arrows in Γ must be p. Therefore the first qm_1 copies of Γ_1 form a loop of rotation number p/q, while on the other hand its rotation number must be $t_1 \neq p/q$, a contradiction. This completes the proof. □

4. Sets of rotation pairs for cycles

We are mainly interested in the rotation numbers and rotation pairs of cycles (or patterns) forced by a given cycle (or pattern). Let $f \in \mathcal{U}_n$, let φ be a displacement as in the preceding section. We will denote the set of all pairs (f, φ) by $\mathcal{V}_n$. Let P be a cycle of f of pattern A. We will denote the set of rotation numbers of all cycles forced by P by $\mathrm{rn}(P)$. Clearly, it is equal to the set of rotation numbers $\mathrm{rn}(A)$ of all patterns forced by A. Similarly, we denote the set of rotation pairs of all cycles forced by P by $\mathrm{rp}(P)$, and it is equal to the set of rotation pairs $\mathrm{rp}(A)$ of all patterns forced by A.

Now it should be clear why in the definition of equivalence of cycles we distinguish branches. Namely, the displacement may distinguish branches, so if we do not do it, we cannot give above definitions for patterns.

In the general case we can characterize quite well the set $\mathrm{rn}(A)$. For the sake of brevity we will say that A is a *pattern for* $(f, \varphi) \in \mathcal{V}_n$ when A is a pattern represented in f and considered with the displacement φ (so, e.g., it makes sense to speak of non-passing patterns for (f, φ)).

Theorem 4.1. *Let A be a pattern for $(f, \varphi) \in \mathcal{V}_n$. Then* $\mathrm{rn}(A)$ *is the intersection of a closed interval $L(A)$ (perhaps degenerate) with the set $\mathbb{Q}$ of rational numbers. The endpoints of $L(A)$ are rotation numbers of some patterns forced by A of period not exceeding the period of A.*

Proof. As we noticed in the preceding section, the oriented graph G given by the pattern A is transitive, and thus by [12], its rotation set is a closed interval I (perhaps degenerate) with endpoints equal to the rotation numbers of some elementary loops in G, and for every rational number from I there is a loop in G with that rotation number. The number of vertices of G is equal to the period of A and an elementary loop cannot have larger length. Thus, by Theorem 2.2 and Lemma 3.2 applied to a P-linear map for a representative P of A, the theorem follows. □

Now we would like to know more about $\mathrm{rp}(A)$. By the definition, for every $p/q \in \mathrm{rn}(A)$ with p, q coprime, there is $m \geqslant 1$ such that $(mp, mq) \in \mathrm{rp}(A)$. Then we can use Theorem 2.4 to get other numbers m with this property. However, without additional assumptions we will not get all of them. The situation is much better for non-passing displacements. Let us stress that whether a given displacement is non-passing, may depend strongly on the graph G, that is on the pattern A. Therefore, when we fix a displacement function for the transitions between the branches of X, we may speak of *non-passing patterns* (and cycles). Clearly, a pattern forced by a non-passing one is also non-passing.

Lemma 4.2. *If A is a non-passing pattern for $(f, \varphi) \in \mathcal{V}_n$ forcing a pattern of rotation number p/q with p, q coprime then it forces a pattern of rotation pair (p, q).*

Proof. It follows from Theorem 2.2, Proposition 3.3(1) and Lemma 3.2 applied to a P-linear map f for a representative P of A. □

Theorem 4.3. *Let A be a non-passing pattern for $(f, \varphi) \in \mathcal{V}_n$. Then for every coprime p, q with p/q from the interior of the interval $L(A)$ and every $m \geqslant 1$ there is a pattern B forced by A with rotation pair (mp, mq). The same holds if p/q is an endpoint of $L(A)$ but is not equal to the rotation number of A.*

Proof. Let p, q be coprime with $p/q \in L(A)$. If $p/q \in \mathrm{int}(L(A))$ is equal to the rotation number of A, then by Lemma 3.4, Theorem 2.2 and Lemma 3.2 applied to a P-linear map for a representative P of A, the pattern A forces a pattern C of rotation number p/q which has no block structure over any pattern of rotation pair (p, q). If p/q is not equal to the rotation number of A then A itself has no block structure over any pattern of rotation pair (p, q) and we set $C = A$.

By Lemma 4.2, C forces a pattern B of rotation pair (p, q). We know already that C has no block structure over B. Since p, q are coprime, B is not a doubling. Therefore by Theorem 2.4 for every $m \geqslant 1$ the pattern C forces a pattern of rotation pair (mp, mq). Since A forces C, this completes the proof. □

Let us now investigate the case when p/q is the rotation number of A and an endpoint of $L(A)$. For any positive integer k we denote by $S(k)$ the set consisting of k and all numbers standing to the right of k in the Sharkovskiĭ ordering.

Theorem 4.4. *Let A be a non-passing pattern for $(f,\varphi) \in \mathcal{V}_n$. Assume that its rotation number p/q, where p, q are coprime, is an endpoint of $L(A)$. Then the set of all i for which A forces a pattern of rotation pair (ip, iq) is of the form $S(k)$ for some positive integer k.*

Proof. By Lemma 4.2, A forces a pattern B of rotation pair (p,q). If A forces a pattern C of rotation number p/q without a block structure over B then by Theorem 2.3 B forces C. By Lemma 4.2, C forces a pattern D of rotation pair (p,q). Since $B \neq C$, by Theorem 2.1 $B \neq D$, and therefore B has no block structure over D. Since p, q are coprime, D is not a doubling. Therefore by Theorem 2.4 for every $m \geqslant 1$ the pattern B (and thus the pattern A) forces a pattern of rotation pair (mp, mq), and the theorem holds with $k = 3$.

Assume now that A does not force any pattern of rotation number p/q without a block structure over B. Let P be a representative of A in a P-linear map f and let m be the period of B. By Theorem 2.3, any pattern C of rotation number p/q and period iq has a representative R in f contained in the union of convex hulls of blocks of P. If I is one of these convex hulls and $g = f^m|_I$ then g is a map from I to I satisfying the assumptions of Lemma 2.5 and such that $Q = R \cap I$ is a cycle of g of period i. Now the existence of k as in the statement of the theorem follows from the Sharkovskiĭ Theorem for g and Lemma 2.5. □

When we put together Theorems 4.3 and 4.4, we see that the set $\mathrm{rp}(A)$ for a non-passing pattern A has the same form as for a circle map of degree 1. We can visualize it easily when we use the modified rotation pairs, defined in the introduction. Using them we can restate Theorems 4.3 and 4.4 as follows.

Theorem 4.5. *Let A be a non-passing pattern for $(f,\varphi) \in \mathcal{V}_n$. Then there are patterns B_1 and B_2 forced by A, of modified rotation pairs (t_1, m_1) and (t_2, m_2), respectively, such that $\mathrm{mrp}(A) = [(t_1, m_1), (t_2, m_2)]$. Moreover, $m_i = 3$ unless the rotation number of A is t_i. In particular, if $t_1 \neq t_2$ then at least one of m_1, m_2 is equal to 3.*

The next two lemmas, dealing with n-od maps, are only loosely related to rotation numbers. Sometimes it is very important to know the periods of primitive patterns forced by a given pattern A. It turns out that there has to be at least one such pattern.

Lemma 4.6. *Each pattern forces a primitive pattern.*

Proof. Let Γ be the shortest loop in the graph given by a pattern A. If it passes twice through the same branch, there are arrows α and β in Γ ending on the same branch. Assume that β ends closer to a then α, and replace α with an arrow beginning at the same place as α, but ending where β ends. With this replacement we get a shorter loop, a contradiction. Thus, Γ gives us a primitive pattern. □

In fact, to find a loop corresponding to a primitive pattern forced by A it is enough to consider points closest to a on their respective branches.

We apply Lemma 4.6 to obtain the following result.

Lemma 4.7. *Assume that a pattern A forces no primitive pattern of period larger than* 2. *Then the set of periods of patterns forced by A is $S(k)$ for some k.*

Proof. Denote by m the period of A and by K the set of periods of patterns forced by A. Suppose first that A forces a pattern of period 1. If A has a block structure over a pattern of period 1 then all points of a representative of A lie on the same branch and then $K = S(k)$ for some k. Otherwise, by Theorem 2.4, A forces patterns of all periods, so $K = S(3)$.

Thus, we can assume that A forces no primitive patterns of period other than 2. Then any pattern B forced by A has the same property. However, by Lemma 4.6 such B has to force some primitive pattern, so it forces a primitive pattern of period 2.

Suppose that m is even and A has a block structure over a primitive pattern of period 2. If P is a representative of A in a P-linear map f then for f^2 P decomposes into 2 cycles, each of them contained in one branch. Therefore K consists of elements of some set $S(l)$ multiplied by 2, and of 1 (from $\{a\}$). This set is equal to $S(2l)$, and the proof is complete in this case.

Assume now that m is even and A does not have a block structure over a primitive pattern of period 2. Then by Theorem 2.4 A forces patterns of all even periods (and the pattern of $\{a\}$ of period 1). Thus, K contains $S(6)$. If $K = S(6)$, we are done; otherwise A forces a pattern of odd period larger than 1.

The last case we have to consider is when A forces a pattern of odd period larger than 1 (this contains the case of m odd). Let s be the smallest such period, and let B be a pattern of period s forced by A. Consider a P-linear map f where P is a cycle of pattern B. Take a point $x \in P$. Let Γ be the loop $x \to f(x) \to f^2(x) \to \cdots \to f^s(x) = x$ and let γ be the loop which by Lemma 3.2(2) corresponds to a primitive pattern of period 2. Then a concatenation of Γ and γ is a loop of length $s + 2$ for which by Lemma 3.2(1) there exists an associated periodic point $y \neq a$ such that $f^{s+2}(y) = y$. Let C be the pattern of the orbit of y. By Theorem 2.2, B forces C. Therefore the period of C, which is a divisor of $s + 2$, can be only $s + 2$. If C forces a pattern B' of period s, we repeat the above construction with b replaced by B' and get a pattern C' of period $s + 2$ forced by B', etc. We claim that after finitely many such steps we get a pattern D of period $s + 2$ which does not force any pattern of period $s + 2$. Indeed, otherwise we get an infinite sequence of patterns $B, C, B', C', B'', C'', \ldots$ in which every pattern forces the next one and their periods are $s, s + 2, s, s + 2, s, s + 2, \ldots$. By Theorem 2.1 those patterns are all distinct, and since there are only finitely many patterns of a given period, we get a contradiction.

Thus, B forces a pattern D of period $s+2$ which does not force any pattern of odd period less than $s + 2$ but larger than 1. Now this construction can be repeated and by induction we get that B (and therefore A) forces patterns of all odd periods larger than or equal to s. Hence, $K = S(s)$. □

A map from $\mathcal{U}_n$ that has no primitive cycles of period larger than 2 behaves very similar to a map from $\mathcal{U}_2$ (or $\mathcal{U}_1$), that is an interval map. Thus, Lemma 4.7 is basically the

Sharkovskiĭ Theorem for those maps. The essential step in the proof is to show that a pattern of odd period larger than 1 forces patterns of all larger odd periods. We did it by looking at the loops in the graph given by this pattern. Alternatively, one can use rotation numbers, in the way very similar to the proof of the Sharkovskiĭ Theorem from [7,8]. An example of how this could be done is given in the following proposition.

Proposition 4.8. *The following properties hold.*

(1) *If a pattern A for $(f,\varphi)\in\mathcal{V}_n$ has rotation pair (r,s) and forces a pattern of rotation pair (r',s') such that (r,s) and (r',s') are Farey neighbors then the set* $\mathrm{mrp}(A)$ *contains* $[(r/s,1),(r'/s',3)]$ *if* $r/s<r'/s'$, *or* $[(r'/s',3),(r/s,1)]$ *if* $r'/s'<r/s$.

(2) *If A is a triod pattern of period $2k+1$ forcing no primitive pattern of period other than 2 then the set of modified rotation pairs of patterns forced by A contains either* $[(k/(2k+1),1),(1/2,3)]$ *or* $[(1/2,3),((k+1)/(2k+1),1)]$, *and hence A forces patterns of all periods from $S(2k+1)$.*

Proof. (1) Assume that $r/s<r'/s'$; the proof for $r'/s'<r/s$ is similar. By Proposition 3.3(2), for any pair (u,t) of positive integers with $r/s<u/t\leqslant r'/s'$ there is a loop with rotation pair (u,t). If u,t are coprime, then the corresponding cycle of f has rotation pair (u,t). Now Theorem 2.4 implies that $[(r/s,1),(r'/s',3)]\subset\mathrm{mrp}(A)$.

(2) Let f be a P-linear map where P is cycle of pattern A. Let n be the largest in the Sharkovskiĭ ordering period of a pattern forced by A and let B be a pattern of period n forced by A. Then $n\leqslant 2k+1$ is odd. Let G be the graph associated with A and let Γ be the loop in G corresponding to B. Since B does not force a primitive pattern of period 1, all green arrows in Γ point towards a. Hence, after removing them from Γ we get a loop Γ', corresponding to a non-passing pattern C of period $m\leqslant n$. By Lemma 4.6, C forces a primitive pattern which must be of period 2.

Now, if C has rotation number $r/m\neq 1/2$ then the conclusion follows from Theorem 4.5. Assume that $r/m=1/2$. We show that the number of arrows removed from Γ was 1. Indeed, the case $n=3$ is straightforward, while if $n>3$ then after removing 2 arrows we would get a loop corresponding to a pattern of period dividing $n-2$. It cannot be of period 1 by assumptions and has therefore an odd period u such that $3\leqslant u<n$, a contradiction. Since at least one arrow was removed, it was exactly one, so the rotation pair of B is $((n-1)/2,n)$. This is a Farey neighbor of $(1,2)$, so the conclusion follows now from (1). □

5. Application to the triod maps

Now we are able to explain an intriguing phenomenon, noticed over 10 years ago, about the coexistence of periods for maps from $\mathcal{U}_3$. This theorem (see [1]) is very similar to the Sharkovskiĭ Theorem.

If we have an ordering of natural numbers (or its subset), we call a *tail* any non-empty subset that with any number k contains all numbers smaller than k in this ordering. Thus, the sets $S(k)$, $k\in\mathbb{N}'$ are exactly the tails of the Sharkovskiĭ ordering.

In [1] two other orderings were introduced, namely a *green ordering*

$$5 \succ 8 \succ 4 \succ 11 \succ 14 \succ 7 \succ 17 \succ 20 \succ 10 \succ 23 \succ 26 \succ 13 \succ \cdots$$

followed by the Sharkovskiĭ ordering multiplied by 3 (that is, numbers of the form $3k$ ordered according to the Sharkovskiĭ ordering of k's) and 1 at the end; and a *red ordering*

$$7 \succ 10 \succ 5 \succ 13 \succ 16 \succ 8 \succ 19 \succ 22 \succ 11 \succ 25 \succ 28 \succ 14 \succ \cdots$$

followed by the Sharkovskiĭ ordering multiplied by 3 and 1 at the end.

Theorem 5.1 [1]. *If $f \in \mathcal{U}_3$ then the set of periods of cycles of f is equal to the union of some tails of the Sharkovskiĭ, green and red orderings.*

As we mentioned in the introduction, this theorem was later generalized by Baldwin [4] to all continuous maps of the triod.

The intriguing phenomenon, mentioned at the beginning of this section, is a striking similarity of the tails of the green and red orderings to the sets of denominators of numbers from an interval whose one endpoint is $1/3$. Observe, that similar phenomenon relating the tails of Sharkovskiĭ ordering and the denominators of numbers from an interval whose one endpoint is $1/2$ has already been explained in [8] by means of rotation theory for interval maps. Also, let us remind the reader that similar phenomenon was discovered for maps of n-od (see, e.g., [3]).

Lemma 5.2. *Any tail of the green* (*respectively, red*) *ordering containing a number larger than* 1 *and not divisible by* 3 *is equal to the set of periods of all modified rotation pairs from the set* $[(a, m), (1/3, 3)]$ (*respectively,* $[(1/3, 3), (a, m)]$) *for some modified rotation pair* (a, m) *with* $0 < a < 1/3$ (*respectively,* $1/3 < a < 1/2$). *Conversely, any set of the above form is a tail of the green* (*respectively, red*) *ordering.*

Proof. Let us prove this for the green ordering, the proof for the red one is similar. First notice that the green ordering is given by the periods of the following modified rotation pairs:

$$\left(\tfrac{1}{5}, 1\right), \left(\tfrac{1}{4}, 2\right), \left(\tfrac{1}{4}, 1\right), \left(\tfrac{3}{11}, 1\right), \left(\tfrac{2}{7}, 2\right), \left(\tfrac{2}{7}, 1\right), \left(\tfrac{5}{17}, 1\right), \ldots.$$

Moreover, we have

$$\tfrac{1}{5} < \tfrac{1}{4} < \tfrac{3}{11} < \tfrac{2}{7} < \tfrac{5}{17} < \cdots < \tfrac{1}{3}.$$

Odd terms of this sequence of fractions are of the form $(2k-1)/(6k-1)$, and the even ones are of the form $k/(3k+1)$. Thus, we have to prove that there is no number of the form $j/(6k-1)$ in $((2k-1)/(6k-1), 1/3)$ and no number of the form $j/2(3k+1)$ in $(k/(3k+1), 1/3)$. These statements are equivalent to the nonexistence of an integer j satisfying $6k-2 < 3j < 6k-1$ and $6k < 3j < 6k+2$ respectively, which is obvious. □

As shown in [1], green and red orderings are given by so called green and red patterns. Let us list some important properties of these patterns. Suppose that a green or red pattern A

has representative P in a map $f \in \mathcal{U}_3$. Then P has points on all three branches of the triod. There is an ordering of these branches, b_0, b_1, b_2 such that for the point p_i of P, closest to a on the branch b_i we have $f(p_i) \in b_{i+1}$, where the addition in the subscripts is modulo 3. Let us use the displacement φ such that the transition from b_i to b_{i+j} $(j = 0, 1, 2)$ corresponds to the displacement $j/3$. Then we get a loop of length 3 and rotation number $1/3$, so A forces a pattern of period 3 and rotation number $1/3$.

With this displacement, arrows of oriented graphs of green patterns have displacements only 0 and $1/3$, and of red patterns only $1/3$ and $2/3$. Thus, they are all non-passing. Now Theorem 4.5 and Lemma 5.2 explain the described phenomenon. Presence of the Sharkovskiĭ ordering multiplied by 3 at the end of the green and red orderings corresponds to the fact that the rotation numbers of green and red patterns is not $1/3$, so we get these periods from Theorem 4.5. Moreover, we get period 1 at the end, since a is a fixed point.

We know by Lemma 4.6 that the collection of primitive patterns forced by A is non-empty. It turns out that it is an important characteristic of A.

Theorem 5.3. *For triod maps fixing a, if a pattern forces primitive patterns of at least two different periods then it forces patterns of all periods* (*except perhaps* 1).

Before we prove this theorem, we consider how to use most effectively the rotation theory in the case when our pattern A forces a primitive pattern C of period 3. Let P be a representative of A in a P-linear map $f \in \mathcal{U}_3$. Then f has a cycle R of pattern C. This cycle has one point on each branch of the triod. There is an ordering of these branches, b_0, b_1, b_2 such that for the point r_i of R on the branch b_i we have $f(p_i) \in b_{i+1}$. As before, we will use the displacement φ such that the transition from b_i to b_{i+j} $(j = 0, 1, 2)$ corresponds to the displacement $j/3$. Then R has rotation number $1/3$.

Lemma 5.4. *In the above situation, if a pattern A does not force a primitive pattern of period* 1, *then the graph given by A contains a transitive non-passing subgraph containing loops corresponding to C and all patterns of maximal rotation number forced by A* (*that is, the right endpoint of $L(A)$*).

Proof. Let f and P be as above. Since A does not force a primitive pattern of period 1, all arrows in the graph G given by P which begin and end on the same branch, point towards a. Hence, if we remove these arrows, we get a graph G' which is transitive (we can pass from any vertex to any vertex along the loop corresponding to P with arrows of displacement 0 removed). This graph has only displacements $1/3$ and $2/3$, so it is non-passing.

To complete the proof we have to show that C and all patterns of maximal rotation number forced by A correspond to loops in G whose arrows have no displacement 0. For C it is obvious, since all arrows of that loop have displacement $1/3$. If a loop Γ in G has an arrow with displacement 0 then this arrow can be removed from the loop and since this arrow pointed towards a, what is left is also a loop. It has the same total displacement as Γ, but smaller length, so it has larger rotation number than Γ. Therefore Γ cannot correspond to a pattern with maximal rotation number among patterns forced by A. This completes the proof. □

Remark 5.5. Let us consider the displacement such that the transition from b_i to b_{i+j} ($j = 0, 1, 2$) corresponds to the displacement $j/3$. Then the loop corresponding to any primitive cycle of period 2 has displacements $1/3$ and $2/3$ (and so this cycle has rotation number $1/2$). Thus, if we assume in Lemma 5.4 that C is a primitive pattern of period 2 instead of 3, its proof will still work.

Proof of Theorem 5.3. If a pattern A forces a primitive pattern B of period 1 and a primitive pattern C of period 2 or 3, then it has no block structure over B, and by Theorem 2.4 it forces patterns of all periods.

Assume now that A forces primitive patterns of periods 2 and 3. By a similar argument as above, A forces patterns of all even periods. Let b be the right endpoint of $L(A)$. By Lemma 5.4 there is a transitive non-passing subgraph G' of the graph G given by A with $L(G')$ containing $[1/3, b]$. Then $b \geqslant 1/2$ since A forces a primitive pattern of period 2 and all such patterns have rotation number $1/2$. For every odd $k > 1$ we have $1/3 \leq ((k-1)/2)/k < 1/2$ and the integers $(k-1)/2, k$ are coprime. Hence by Proposition 3.3 there is a loop in G' with rotation pair $((k-1)/2), k)$. This loop corresponds to some pattern of period k forced by A.

6. More about triod maps

Actually, for the triod maps we can prove more than in the preceding section. In Lemma 5.4 we considered a situation when a pattern A forces a primitive pattern of period 3, but does not force a primitive pattern of period 1. Then we used rotation theory to get patterns forced by A with rotation number larger than $1/3$. Now we replace period 1 by period 2 and look at the other side of $1/3$.

Thus, as in Lemma 5.4, we assume that a pattern A forces a primitive pattern C of period 3; P is a cycle of pattern A of a P-linear map $f \in \mathcal{U}_3$; f has a cycle R of pattern C; the displacement is chosen in the same way as there. Thus, possible displacements are $0, 1/3, 2/3$ and all arrows corresponding to R have displacement $1/3$.

When talking about the arrows in the graph G given by P, referring constantly to the displacements is cumbersome, so as in other papers (e.g., [1]) we will color-code them, and for the sake of making pictures (by the reader), we adopt the convention that the branches are numbered counterclockwise. Thus, arrows with displacement $1/3$ will be *black*. They lead to the next branch. The arrows with displacement 0 are *green*, and they lead to the same branch. They come in two varieties, *inbound* and *outbound*, depending on whether the beginning is further from a than the end or not. The arrows ending where they began will be counted as outbound. The arrows with the displacement $2/3$ are *red* and they lead to the previous branch (we draw them in the clockwise direction, although the displacement is $2/3$, not $-1/3$).

We make an additional assumption, namely that A does not force a primitive pattern of period 2. This has an immediate consequence.

Lemma 6.1. *In the above situation, there are no points $x, y \in P$ and arrows: black from x to x' and red from y to y' such that $x' \geqslant y$ and $y' \geqslant x$.*

Proof. If there were such points and arrows, then there would be arrows from x to y and from y to x. Since x and y lie on different branches, this would imply that there is a primitive cycle of period 2, a contradiction. □

A loop consisting of black arrows will be called a *black loop*. Similarly, we will speak of *black paths*. A point $x \in P$ will be called *black recurrent* if there is a black loop passing through it. By our assumptions, there are black recurrent points in P. For example, by Lemma 3.2(2) for any cycle of f there exists a loop associated with it; then the loop associated with the cycle R is black and hence all points of P in this loop are black recurrent.

Lemma 6.2. *If a point $x \in P$ is black recurrent then there is a black loop of length* 3 *passing through it.*

Proof. Look at the branch to which x belongs. The black loop to which x belongs passes through finitely many points on this branch. If we write these points in the order of their appearance in the black loop then there must be two consequent points z, y such that $y \geqslant x \geqslant z$. In other words, there is a black path of length 3 beginning at z and ending at y. We can replace the beginning of the first arrow in this path by x and the end of the last arrow also by x, and we get a black loop of length 3 passing through x. □

We will say that two black recurrent points are *black equivalent* if there is a black loop passing through both of them. Clearly, black equivalence is an equivalence relation. We will call a graph H a *supergraph* if it consists of a black loop of length 3 and a green arrow from a vertex of this loop to itself.

Lemma 6.3. *Either all points of P are black equivalent or G contains a supergraph.*

Proof. Let B be an equivalence class of the black equivalence relation. Denote by x_0, x_1, x_2 the elements of B furthest from a on consecutive branches and by y_0, y_1, y_2 the elements of B closest to a on those branches. Take $j \in \{0, 1, 2\}$. There is a black arrow $x' \to x_{j+1}$, where $x' \in B$. Then $x_j \geqslant x'$, so there is a black arrow $x_j \to x_{j+1}$. Similarly, there is a black arrow $y_j \to y'$ for some $y' \in B$. Then $y' \geqslant y_{j+1}$, so there is a black arrow $y_j \to y_{j+1}$. Thus, B is bounded from outside and inside by black loops of length 3. All points of P between these loops belong to B (if, for instance, $x_0 > z > y_0$ then there are black arrows $x_2 \to z \to y_1$).

No black arrow beginning inside or on the outer loop can end outside the outer loop, since then the end of this arrow would belong to B. By Lemma 6.1, no red arrow beginning inside or on the outer loop can end outside the outer loop. Thus, if there is a point of P outside the outer loop, there is a green arrow beginning inside or on the outer loop and ending outside the outer loop, and hence there is a point on the outer loop and a green arrow beginning and ending at it.

Similarly, no black arrow ending outside or on the inner loop can begin inside the inner loop, since then the beginning of this arrow would belong to B. By Lemma 6.1, no red arrow ending outside or on the inner loop can begin inside the inner loop. Thus, if there is a point of P inside the inner loop, then there is a green arrow beginning inside the inner loop and ending outside or on the inner loop, and hence there is a point on the inner loop and a green arrow beginning and ending at it. This completes the proof. □

Lemma 6.4. *The graph G contains either a supergraph or a transitive non-passing subgraph containing loops corresponding to C and all patterns of minimal rotation number forced by A (that is, the left endpoint of $L(A)$).*

Proof. If not all points of P are black equivalent then by Lemma 6.3 G contains a supergraph. Assume that all points of G are black equivalent. Let H be a subgraph obtained from G by removing all red arrows. Clearly, it is transitive and it contains a loop corresponding to C. Let Γ be a loop in G corresponding to a pattern of minimal rotation number forced by A. Suppose that there is a red arrow $x \to y$ in Γ. By Lemma 6.2, there is a black loop $x \to x_1 \to x_2 \to x$. By Lemma 6.1, $x_2 > y$, and thus the arrow $x \to y$ in Γ can be replaced by two black arrows $x \to x_1 \to y$. This gives a loop with the same total displacement but longer than Γ, that is a loop with a smaller rotation number than Γ, a contradiction. Therefore Γ is also a loop in H. This completes the proof. □

Note that a supergraph is transitive and non-passing. Moreover, if it is present then by Theorem 2.4 A forces patterns of rotation pairs $(0, q)$ for all q. Thus, in the same way as we obtained Theorems 4.3 and 4.4 (restated as Theorem 4.5), we get immediately from Lemmas 5.4 and 6.4 the following result. We denote by $M^+(\alpha)$ and $M^-(\alpha)$ the set of all (modified rotation) pairs (t, m) with $t \geqslant \alpha$ and $t \leqslant \alpha$, respectively.

Theorem 6.5. *Let A be a pattern for $(f, \varphi) \in \mathcal{U}_3$. Assume that there is the ordering of the branches b_0, b_1, b_2 of the triod such that the transition from b_i to b_{i+j} $(j = 0, 1, 2)$ is $j/3$ (the addition in the subscripts is modulo 3). Assume also that A forces a primitive pattern of rotation pair $(1, 3)$.*

(1) *If A does not force a primitive pattern of period 1 then there is a pattern B_2 forced by A, of modified rotation pair (t_2, m_2), and a positive integer m_0, such that* $\operatorname{mrp}(A) \cap M^+ = [(1/3, m_0), (t_2, m_2)]$. *Moreover, $m_i = 3$ unless the rotation number of A is t_i (here $t_0 = 1/3$).*

(2) *If A does not force a primitive pattern of period 2 then there is a pattern B_1 forced by A, of modified rotation pair (t_1, m_1), and a positive integer m_0, such that* $\operatorname{mrp}(A) \cap M^- = [(t_1, m_1), (1/3, m_0)]$. *Moreover, $m_i = 3$ unless the rotation number of A is t_i (here $t_0 = 1/3$).*

Thus, if A does not force primitive patterns of periods 1 and 2, we get the same result as in Theorem 4.5. We conjecture that appropriate versions of rotation theory can be constructed for patterns on the n-od forcing primitive patterns of only one period.

If A forces a primitive pattern of period 2, but not of period 1, we apply Remark 5.5 and get the following result.

Theorem 6.6. *Let A be a pattern for $(f,\varphi)\in\mathcal{U}_3$. Assume that there is the ordering of the branches b_0, b_1, b_2 of the triod such that the transition from b_i to b_{i+j} $(j=0,1,2)$ is $j/3$ (the addition in the subscripts is modulo 3). Assume also that A forces a primitive pattern of period 2, but does not force a primitive pattern of period 1. Then there is a pattern B_2 forced by A, of modified rotation pair (t_2, m_2), and a positive integer m_0, such that* $\mathrm{mrp}(A)\cap M^+=[(1/2,m_0),(t_2,m_2)]$. *Moreover, $m_i=3$ unless the rotation number of A is t_i (here $t_0=1/3$).*

Our tools allow us to obtain a new independent proof of Theorem 5.1.

New proof of Theorem 5.1. Clearly, it is enough to show that periods of cycles forced by a pattern A form the union of tails of the Sharkovskiĭ, green and red orderings.

If A does not force a primitive pattern of period 3, this follows from Lemma 4.7. Assume that A forces a pattern of period 3. If A forces a primitive pattern of some other period, then it forces patterns of all periods by Theorem 5.3 (we get period 1 from $\{a\}$). Otherwise, by Theorem 6.5 rotation theory applies to A and by Lemma 5.2 the set of periods of patterns forced by A is the union of tails of green and red orderings. □

Of course, there is a converse to Theorem 5.1, as in case of Theorems 1.1, 1.2 and 1.3. That is, for every union of tails of the Sharkovskiĭ, green and red orderings, there is a map $f\in\mathcal{U}_3$ with this set of periods. However, this is only a matter of constructing simple examples (see [1]).

References

[1] L. Alsedà, J. Llibre, M. Misiurewicz, Periodic orbits of maps of Y, Trans. Amer. Math. Soc. 313 (1989) 475–538.

[2] L. Alsedà, J. Llibre, M. Misiurewicz, Combinatorial Dynamics and Entropy in Dimension One, Adv. Ser. in Nonlinear Dynamics, Vol. 5, World Scientific, Singapore, 1993.

[3] L. Alsedà, J. Moreno, Linear orderings and the full periodicity kernel for the n-star, Math. Anal. Appl. 180 (1993) 599–616.

[4] S. Baldwin, An extension of Sharkovskii theorem to the n-od, Ergodic Theory Dynamical Systems 11 (1991) 249–271.

[5] L. Block, J. Guckenheimer, M. Misiurewicz, L.-S. Young, Periodic points and topological entropy of one-dimensional maps, in: Lecture Notes in Math., Vol. 819, Springer, Berlin, 1980, pp. 18–34.

[6] A. Blokh, Rotation numbers, twists and a Sharkovskii–Misiurewicz-type ordering for patterns on the interval, Ergodic Theory Dynamical Systems 15 (1995) 1–14.

[7] A. Blokh, Functional rotation numbers for one dimensional maps, Trans. Amer. Math. Soc. 347 (1995) 499–513.
[8] A. Blokh, M. Misiurewicz, New order for periodic orbits of interval maps, Ergodic Theory Dynamical Systems 17 (1997) 565–574.
[9] M. Misiurewicz, Periodic points of maps of degree one of a circle, Ergodic Theory Dynamical Systems 2 (1982) 221–227.
[10] M. Misiurewicz, Z. Nitecki, Combinatorial patterns for maps of the interval, Mem. Amer. Math. Soc. 456 (1990).
[11] A.N. Sharkovskiĭ, Co-existence of the cycles of a continuous mapping of the line into itself, Ukrain. Mat. Zh. 16 (1964) 61–71.
[12] K. Ziemian, Rotation sets for subshifts of finite type, Fund. Math. 146, pp. 189–201.