ON ROTATION INTERVALS FOR INTERVAL MAPS

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Abstract. Following [B6-B8] we introduce *rotation numbers* and *intervals* for interval maps and prove some of their properties. In particular we study on what ω-limit sets the endpoints of the rotation intervals may be assumed. We also show that in piecewise-monotone case a theorem very close to that proven in [M2] for circle maps holds.

0. Introduction

One of the remarkable results in one-dimensional dynamics is the Sharkovskii theorem. To state it let us first introduce the *Sharkovskii ordering* for positive integers:

$$(*) \qquad 3 \succ_S 5 \succ_S 7 \succ_S \dots \succ_S 2\cdot 3 \succ_S 2\cdot 5 \succ_S 2\cdot 7 \succ_S \dots \succ_S 8 \succ_S 4 \succ_S 2 \succ_S 1$$

Denote by $Sh(k)$ the set of all integers m such that $k \succeq_S m$ and by $Sh(2^\infty)$ the set $\{1, 2, 4, 8, \dots\}$. Also denote by $P(\varphi)$ the set of periods of cycles of a map φ.

Theorem S[S]. *If $g : [0,1] \to [0,1]$ is continuous, $m \succ_S$ and $m \in P(g)$ then $n \in P(g)$ and so there exists $k \in \mathbb{N} \cup 2^\infty$ such that $P(g) = Sh(k)$.*

Theorem S characterizes sets of periods of interval maps. Similar result concerning circle maps of degree one is due to Misiurewicz. To state it we need some more definitions. The most important and historically the first among them is the notion of the *rotation number* introduced by Poincaré [P] for circle homeomorphisms. Newhouse, Palis and Takens [NPT] extended it onto circle degree one maps, introduced the notion of *rotation interval* and proved some properties of rotation intervals; their work was continued by Ito in [I]. We summarize the properties of rotation intervals proven in [NPT], [I] in Theorem INPT but first let us introduce necessary notations and definitions. Let $f : S^1 \to S^1$ be a map of degree 1, $\pi : \mathbb{R} \to S^1$ be the natural projection; let us fix a lifting F of f. If $x \in S^1, X \in \pi^{-1}x$ then we

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denote the set of all limit points of the sequence $\dfrac{F^n(X)}{n}$ by $I_F(x)$; the notation is correct since for a degree one map f the set $I_F(x)$ does not depend on the choice of $X \in \pi^{-1}x$. If $I_F(x) = \{\rho_F(x)\}$ is a one-point set then $\rho_F(x)$ is called the *rotation number* of x. Set $\bigcup_{x\in S^1} I_F(x) \equiv I_F$; the following theorem is proven in [NPT], [I] (properties of circle maps without cycles may also be deduced from [AK]).

Theorem INPT [I],[NPT]. (1) *I_F is a closed interval.*

(2) *If f has periodic points then the set of all rotation numbers of periodic points is dense in I_F; otherwise f is monotonically semiconjugate to an irrational rotation by angle α and $I_F = \{\alpha\}$.*

Since liftings of the same degree one circle map f differ by integers we may fix F and use f as a subscript from now on. The set I_f is called the *rotation set (interval)* of f. In fact the rotation set consists of all possible speeds with which points move to infinity under iterations of F; in particular if the circle map in question is the rotation then all points move to infinity with the same speed and the rotation set is a degenerate interval consisting of this speed only. If x is an f-periodic orbit of period q and X is its lifting then there exists a well-defined integer p such that $F^q(X) = X + p$. Denote a pair (p, q) by $rp(x)$ and call it the *rotation pair* of x; then $\rho_F(x) = \rho(x) = p/q$. Denote by $RP(f)$ the set of all rotation pairs of cycles of f. For real numbers $a \le b$ let $N(a, b) = \{(p, q) \in \mathbb{Z}^2 : p/q \in (a, b)\}$ (in particular $N(a, a) = \emptyset$). For $a \in \mathbb{R}$ and $l \in \mathbb{Z}^+ \cup \{2^\infty\} \cup \{0\}$ let $Q(a, l)$ be empty if a is irrational or $l = 0$; otherwise let it be $\{(ks, ns) : s \in Sh(l)\}$ where $a = k/n$ with k, n coprime (see [M2]). The following beautiful result related to Theorem INPT was obtained in [M2].

Theorem M[M2]. *For a continuous circle map f of degree 1 there exist $a, b \in \mathbb{R}, a \le b$ and $l, r \in \mathbb{Z}^+ \cup \{2^\infty\}$ such that $I_f = [a, b]$, $RP(f) = N(a, b) \cup Q(a, l) \cup Q(b, r)$.*

Let us give another well-known interpretation for the rotation numbers and sets (see, e.g., [MZ]). Namely, let the function $\phi_f : S^1 \longrightarrow \mathbb{R}$ be such that $\phi_f(z) = F(Z) - Z$ for some $Z \in \pi^{-1}z$; since $F(Z') - Z' = F(Z'') - Z''$ whenever $\pi(Z') = \pi(Z'')$ the function ϕ_f is well-defined and continuous. Then $I_f(z)$ is the set of limit points of the sequence $\frac{1}{n}\sum_{i=0}^{n-1} \phi_f(f^i z)$ and I_f is the union of all such sets taken over all points of the circle; the above stated theorems describe the properties of the sets $I_f(z)$ and I_f. Once the problem is stated this way it is easy to extend it (for one-dimensional maps it is done in [B6], see also [Z]). Indeed, given a map one can choose a function , consider Cesaro averages of this function along orbits of points and their limits (called *functional rotation sets* of points) and study the union of all these limits (called *functional rotation set* of the map); clearly in this broad form the question about properties of functional rotation sets may be asked for almost any maps and functions (see [B6]). One can hope that for some classes of maps functional rotation sets have nice structure for a large variety of functions. At the same time choosing specific functions (similarly to the circle case) one can probably obtain a lot of information about dynamics of the map.

In this paper we are mostly interested in interval maps; more precisely, we are investigating to what extent Theorem INPT and Theorem M may be generalized for interval maps with other functions playing the role of ϕ_f. In [B6] we give necessary definitions and state sufficient conditions for the analog of Theorem INPT to be true in case of circle or interval maps and bounded measurable functions. We are not working with measurable functions just for the sake of generality; the results of [B6] apply to a specific measurable function closely connected to the map (see [B6-B8] and this paper below). The necessity to deal with measurable functions contributes to somewhat lengthy definitions but in our view it pays off allowing to obtain results in rather general form and thus making them widely applicable. One of our main tools is the "spectral decomposition theorem" for one-dimensional maps ([B1-B3]). Also we would like to point out that related problems in symbolic dynamics are considered by K. Ziemian [Z].

Let us state the results of [B6] in a particular case which is in fact our focus in the present paper. Let $f : [0,1] \longrightarrow [0,1]$ be a continuous map, $L = \{x : fx < x\}$, $R = \{x : fx > x\}$ and $Fixf = \{x : fx = x\}$. Let the function $\xi_f = \xi$ be such that $\xi(x) = 1$ if $x \in L$, $\xi(x) = 0$ if $x \in R$ and $\xi(x) = 1/2$ if $x \in Fixf$ (we omit the subscript to simplify the notation). The function $\xi(x)$ certainly depends on the map f and to some extent characterizes the dynamics of f. In this sense the function ξ reminds of the function $\phi_f : S^1 \longrightarrow \mathbb{R}$ defined for the circle maps, and the analogy can be extended further as we are about to see. Indeed, let us apply the above described approach involving functional rotation sets to the function ξ. To begin with let us consider ξ-rotation sets for points. For a point x this is the set $I_f(x)$ of limit points of the sequence $\frac{1}{n}\sum_{i=0}^{n-1}\xi(f^i x)$. It is easy to see that $I_f(x)$ is closed and connected; moreover, $I_f(x) \subset [0,1]$.

In particular if x is a periodic point of period greater than 1 then $I_f(x)$ shows how big is the part of its orbit which consists of points mapped to the left. Let us for the moment assume also that there is a point, say, a such that for any point $z < a$ from the orbit of x we have $z < fz$ and for any point $y > a$ from the orbit of x we have $fy < a < y$ (periodic orbits of this kind turn out to be of major interest for us in what follows). Then intuitively speaking the number $I_f(x)$ shows with what speed the point x rotates around the point a under iterations of f (although this interpretation is not at all precise and we shall not give it precise meaning in any sense it helps to get some idea about the motivation behind the studying of the function ξ). For example, it is easy to check that any unimodal periodic orbit is of this kind (although there are certainly non-unimodal periodic orbits with the same properties); here by *unimodal* we mean periodic orbits such that the map on them has a single extremum which is maximum. An example of a non-unimodal periodic orbit of period 7 and rotation number 2/7 with these properties is given on Fig. 1.

Parallel to considering rotation sets of points one can consider limit sets of measures for these points; the limit sets of measures are closely connected to the functional rotation sets. Indeed, let $P_f(\mu) = \int_0^1 \xi\, d\mu$ for any measure μ and $V_f(x)$ be the set of all limit points of Cesaro averages of iterates of δ-measure concentrated at x. If P_f is continuous on $V_f(x)$ then $I_f(x) = P_f(V_f(x))$. Certainly if the function ξ were continuous then P_f would be continuous everywhere and so $I_f(x) = P_f(V_f(x))$ would be true for any x. Yet the function ξ has discontinuities; obviously the set D_f of discontinuities of ξ is a subset of $Fixf$. It is well-known that if μ is a measure and $\mu(D_f) = 0$ then P_f is continuous at μ. Thus we conclude that in order to guarantee that for the points we work with we have $I_f(x) = P_f(V_f(x))$ we may restrict ourselves onto points x (we call them *admissible*) such that for any measure $\nu \in V_f(x)$ we have $\nu(D_f) = 0$; then P_f is continuous on $V_f(x)$ and so $I_f(x) = P_f(V_f(x))$. In particular a point whose ω-limit set does not contain fixed points is admissible.

Figure 1: An example of a non-unimodal periodic orbit.

Denote the set of all admissible points by Ad_f. Note also that $V_f(x)$ is connected and closed [DGS].

We call $I_f(x)$ the *rotation set* of x; if $I_f(x) = \{\rho_f(x)\}$ is a one-point set then $\rho_f(x)$ is called the *rotation number* of x (not all points have rotation numbers but, for example, periodic points do). If $V_f(x)$ contains a single measure ν then we call $\rho_f(x) = P_f(\nu)$ the *rotation number* of the measure ν. Let $V_f = \bigcup_{x\in Ad_f} V_f(x)$. We call the set $P_f(V_f) = \bigcup_{x\in Ad_f} I_f(x) \equiv I_f$ the *rotation set* of f. In the definition we take the union only over the set Ad_f of all admissible points; it is worth mentioning that this is not that restrictive, for as we will see later in some important cases the rotation set defined as the union of $I_f(x)$ over all points (i.e. not only for admissible ones) simply coincide with the rotation set defined as above. Also, if $V_f(x) = \{\mu\}$ then the point x is called *generic* for the measure μ. If it is clear what map is considered we will omit subscript in notations for rotation numbers and the like. Although we are giving the definition for the particular function it is clear that a similar definition can be given for other functions and some of the results are in fact true for other functions as well ([B6]). Note also that in case of continuous functions the set of admissible points coincides with the entire manifold and therefore on the circle our functional rotation sets and classical ones are the same.

We can now state the corollary of the results of [B6] for the above introduced rotation sets; this corollary is close to Theorem INPT. Note first that it makes sense to consider only those interval maps which have periodic non-fixed points; otherwise the dynamics of a map is trivial (any orbit converges to a fixed point) and also the rotation interval may be not well defined. So from now on let us assume that f is a continuous interval map with non-fixed periodic points.

Proposition 0.1[B6]. *$I_f \subset [0,1]$ is an interval, rotation numbers of periodic points are dense in I_f and for any $c \in I_f$ there is an admissible point x such that $V_f(x)$ contains a single measure and $\rho_f(x) = c$.*

The main difference between Theorem INPT and Proposition 0.1 (one may also say between the circle case and the interval one) is that in interval case I_f is not necessarily closed, and it is easy to suggest corresponding examples. Indeed, let $f : [0,1] \to [0,1]$ be a map constructed as follows: (1) all points $x_n = \dfrac{1}{n}$ are fixed; (2) all intervals $[\dfrac{1}{n+1}, \dfrac{1}{n}]$ are invariant; (3) if $g_n = f|[\dfrac{1}{n+1}, \dfrac{1}{n}]$ then $I_{g_i} \subsetneqq I_{g_{i+1}}$ for any i. Then it is easy to see that I_f is not closed because it is the union of all the sets I_{g_i}. However in an important particular case one can specify Proposition 0.1. Let $\mathcal{G}$ be the family of all interval maps g such that the following holds: (1) if z is a g-fixed point then there is a neighborhood (a,b) of z such that $z \notin int\, g(a,z)$ and $z \notin int\, g(z,b)$, (2) there are finitely many pairwise disjoint closed intervals such that g is monotone on each of them and all fixed points of g belong to their union (here as everywhere in the paper by "monotone" we mean "non-strictly monotone"). Note that the intervals may be chosen so that their endpoints are fixed points; also, we do not require that these intervals are non-degenerate. In particular, if g has finitely many fixed points which have the property (1) from above then $g \in \mathcal{G}$ since in this case the fixed points form the required family of intervals. Also, if g is piecewise-monotone then $g \in \mathcal{G}$. Indeed, one can divide $[0,1]$ into intervals of monotonicity, then choose on each of these intervals the leftmost and the rightmost fixed points (if any; also these points may coincide) and declare the interval in-between them one of the intervals we have to find; clearly the family of all such intervals is finite which proves that $g \in \mathcal{G}$.

Proposition 1.12. *Let $f \in \mathcal{G}$. Then the following holds.*

(1) *Either $I_f \subset (0,1)$ is closed, or $I_f = (0,b]$, $b < 1$, or $I_f = [a,1)$, $a > 0$, or $I_f = (0,1)$.*

(2) *If $a \in I_f$ is an endpoint of I_f then there is a measure μ such that $supp\, \mu$ contains no fixed points, $\rho(\mu) = a$ and $f|supp\, \mu$ is minimal and $I_f(x) = a$ for any $x \in supp\, \mu$.*

Actually facts similar to Proposition 0.1 hold for a variety of functions playing the role of ξ in the aforementioned construction (see [B6]). In the present paper however we deal mainly with the rotation numbers and sets "generated" by the function ξ in the above sense; the reason is that in this case additional results close to Theorem M can be obtained (see [B7]). Namely, for a non-fixed periodic point y of period $p(y)$ the number $l(y) = card\{orb(y) \cap L\}$ is well-defined; we call the pair $rp(y) = (l(y), p(y))$ the *rotation pair* of y and denote the set of all rotation pairs of periodic non-fixed points of f by $RP(f)$. For example, the rotation pair of any periodic orbit of period 2 is $(1,2)$. Also, the rotation pair of the periodic orbit on Fig. 1 is $(2,7)$. Clearly, $\rho_f(y) = \dfrac{l(y)}{p(y)}$. Let us introduce the following ordering among all pairs of positive integers (k,n) such that $k < n$:

1) if $k/l \in (1/2, p/q)$ then $(p,q) \gtrdot (k,l)$;

2) if $p/q \neq 1/2$ then $(p,q) \gtrdot (k,2k)$ for any k;

3) if $p/q = k/l = m/n$ where m, n are coprime then $(p, q) \gtrdot (k, l)$ if and only if $(p/m) \succ_S (k/m)$ (note that both (p/m) and (k/m) are integers).

Theorem 0.2 [B7]. *If $(p, q) \gtrdot (k, l)$ and $(p, q) \in RP(f)$ then $(k, l) \in RP(f)$.*

As an example of how Theorem 0.2 may be applied let us show how it implies a weak version of the Sharkovskii theorem usually stated as "Period 3 implies chaos" (see [LY]); in other words let us deduce from Theorem 0.2 the fact that if a interval map f has a periodic orbit of period 3 then it has periodic orbits of all periods. Indeed, there are only two possible types of periodic orbits of period 3: 1) $x = f^3x < fx < f^2x$; 2) $x = f^3x < f^2x < fx$. Let us begin assuming that f has a periodic point x of the first type. Then $rp(x) = (1, 3), \rho(x) = \frac{1}{3}$. Clearly, for any odd number $2k+1$ bigger than 3 we have $(1, 3) \gtrdot (k, 2k+1)$ since $\frac{1}{3} < \frac{k}{2k+1} \le \frac{1}{2}$; therefore the map f has a periodic orbit of period $2k + 1$. Moreover, for any even number $2m$ we have $(1, 3) \gtrdot (m, 2m)$ since $\frac{1}{3} < \frac{m}{2m} = \frac{1}{2}$; hence the map f has periodic points of all even periods. This finishes consideration of the case when the map f has period 3 orbits of the first type; the case when it has period 3 orbits of the second type is similar. Actually, these arguments may be easily extended to show how the Sharkovskii theorem may be deduced from Theorem 0.2 (see [B7]); note also that in fact Theorem 0.2 provides not only periods but also some additional information about orbits (namely, their rotation numbers and pairs).

It is easy to see that Theorem 0.2 implies the following

Corollary 0.3 [B7]. (1) *For a continuous interval map f with non-fixed periodic points there exist $0 \le a \le 1/2 \le b \le 1$ and $l, r \in \mathbb{Z}^+ \cup \{2^\infty\} \cup \{0\}$ such that $\operatorname{int} I_f = (a, b)$, $RP(f) = N(a, b) \cup Q(a, l) \cup Q(b, r)$, if $a < b = 1/2$ then $r = 3$, if $a = 1/2 < b$ then $l = 3$, if $a = b = 1/2$ then $r = l \ne 0$, if $a = 0$ then $l = 0$ and if $b = 1$ then $r = 0$.*

(2) [B8] *If a, b, l, r are numbers satisfying all the properties from the statement (1) then there is a continuous interval map f such that $RP(f) = N(a, b) \cup Q(a, l) \cup Q(b, r)$ and $\operatorname{int} I_f = (a, b)$.*

Corollary 0.3 characterizes all possible sets $RP(f)$ of rotation pairs which interval maps may have and establishes the connection between the sets $RP(f)$ and I_f similar to that established in Theorem M. There is however some difference between Theorem M and Corollary 0.3 (i.e. between interval and circle cases) dealing with this connection; the difference concerns sets $Q(a, l)$ and $Q(b, r)$ which in both cases consist of rotation pairs of periodic points with rotation numbers a and b respectively.

First of all the rotation set in the circle case is always a closed interval while in the interval case it is an interval which is not necessarily closed (we do not assume in Corollary 0.3 that the map belongs to $\mathcal{G}$). Secondly, in the circle case if a is rational then $Q(a, l)$ is never empty (the same takes place for b and $Q(b, r)$). However unlike in the circle case in the interval case $Q(a, l)$ may be empty even if a is rational; namely, due to the definition it happens if l is 0. Yet it turns out that in case of piecewise-monotone maps the connection between the sets $RP(f)$ and I_f is closer than in general and reminds that from Theorem M even as far as the sets $Q(a, l)$ and $Q(b, r)$ are concerned. Namely, for piecewise-monotone maps one can prove Theorem 2.2. The substance of this theorem is that except for the cases when $a = 0$ or $b = 1$ it rules out the aforementioned two differences between the statements of Theorem M and Corollary 0.3; i.e. rotation intervals of piecewise-monotone maps are closed in $(0, 1)$ and, moreover, if $a > 0$ is rational then $l > 0$ and so $Q(a, l) \neq \emptyset$ (the same holds for b and $Q(b, r)$).

Theorem 2.2. *Let f be a continuous piecewise-monotone interval map. Then the following cases are possible.*

(1) *There exist $0 < a \leq 1/2 \leq b < 1$ and $l, r \in \mathbb{Z}^+ \cup \{2^\infty\}$ such that $I_f = [a, b]$, $RP(f) = N(a, b) \cup Q(a, l) \cup Q(b, r)$.*

(2) *There exist $1/2 \leq b < 1$ and $r \in \mathbb{Z}^+ \cup \{2^\infty\}$ such that $I_f = (0, b]$, $RP(f) = N(a, b) \cup Q(b, r)$.*

(3) *There exist $a \leq 1/2$ and $l \in \mathbb{Z}^+ \cup \{2^\infty\}$ such that $I_f = [a, 1)$, $RP(f) = N(a, b) \cup Q(a, l)$.*

(4) *$I_f = (0, 1)$, $RP(f) = N(0, 1)$.*

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1. Preliminaries and properties of maps from $\mathcal{G}$

Throughout the paper we deal with continuous maps of the interval. First we prove the following

Proposition 1.1. *For any continuous f we have $0 \notin I_f$, $1 \notin I_f$.*

Proof. Indeed, otherwise by Proposition 0.1 there is an admissible point x such that $V_f(x) = \{\mu\}$ contains a single measure and, say, $\rho_f(x) = 0$. By the ergodic decomposition and the fact that $\xi \geq 0$ we may assume that μ is ergodic, $x \in \omega(x)$ is its typical point and $\omega(x) = supp\,\mu$ is the support of the measure μ (i.e. the smallest closed invariant set of μ-measure 1). Note that the definition of the support of a measure implies that if $supp\,\mu \cap U \neq \emptyset$ and U is open then $\mu(supp\,\mu \cap U) > 0$. Consider the set $A = supp\,\mu \cap L$ where $L = \{x : fx < x\}$. Then L is open and so if $A \neq \emptyset$ then $\mu(A) > 0$ which implies that $P_f(\mu) = \rho_f(x) > 0$ contradicting the assumption. So $A = \emptyset$ which means that for any point $z \in \omega(x)$ we have $fz \geq z$, thus $\omega(x) = \{y\}$ is a fixed point. Since x is admissible it may only happen when y belongs to an open interval of fixed points; but then $\rho_f(x) = 1/2$ which finally shows that $0 \notin I_f$. The same way we can show that $1 \notin I_f$. $\Box$

Let I_f be an interval with an enpoints $a \leq b$; we now study the case when $a \neq 0$ (or $b \neq 1$). To this end we need a few preliminary facts and definitions; whenever possible we shall try to state them in less generality in order to (hopefully) simplify the reading. Let $T : X \to X$ be a map of a compact infinite metric space (X, d) into itself. A dynamical system (X, T) is said to have the *specification property* [Bo] if for any $\varepsilon > 0$ there exists an integer $M = M(\varepsilon)$ such that for any $k > 1$, for any k points $x_1, x_2, \dots, x_k \in X$, for any integers $a_1 \leq b_1 < a_2 \leq b_2 < \dots < a_k \leq b_k$ with $a_i - b_{i-1} \geq M$, $2 \leq i \leq k$ and for any integer p with $p \geq M + b_k - a_1$ there exists a point $x \in X$ with $T^p x = x$ such that $d(T^n x, T^n x_i) \leq \varepsilon$ for $a_i \leq n \leq b_i, 1 \leq i \leq k$.

Lemma 1.2 [B1-B3]. *Let $f : [0, 1] \to [0, 1]$ be continuous and mixing. Then f has the specification property.*

Clearly Lemma 1.2 implies the following

Corollary 1.3. *If $f : [0,1] \to [0,1]$ is mixing, $a_1, a_2, \dots, a_n \in [0,1]$ and $\varepsilon > 0$ then there is a periodic orbit P which visits an ε-neighborhood of a_i for every i.*

Now we state some results from [B7]. Say that a map f has the *right horseshoe* if there are points a, b, c such that $fc \le a = fa < b < c \le fb$ and the *left horseshoe* if there are points a, b, c such that $fc \ge a = fa > b > c \ge fb$. The importance of horseshoes for interval maps was first discovered in [MS], [M1]. Intuitively speaking the existence of horseshoes is to some extent equivalent to the richness of the dynamics of the map. The following lemma to some extent confirms this a bit vague statement.

Lemma 1.4[B7]. *If a map f has the right (resp. the left) horseshoe a, b, c then $(s,t) \in RP(f)$ for any (s,t) such that $0 < s/t \le 1/2$ (resp. $1 > s/t \ge 1/2$) and there is a periodic orbit Q with $rp(Q) = (s,t)$ lying completely to the right (resp. left) of a; in particular if a map f has both the right and the left horseshoe then $RP(f) = \{(s,t) : 0 < s < t\}$.*

Suppose that $S = \{x_1 < x_2 < \dots < x_q\}$ is a periodic orbit and there is l such that $fx_i > x_i$ for any $1 \le i \le l$ and $fx_j < x_j$ for any $l+1 \le j \le q$; then S is said to be *forcing a unique fixed point* and the family of all periodic orbits of all maps with these properties is denoted by $\mathcal{F}$. It is easy to give an example of a periodic orbit from $\mathcal{F}$; say, any periodic orbit of period 2 or 3 belongs to $\mathcal{F}$, and so does any unimodal periodic orbit. Also, the orbit on Fig. 1 belongs to $\mathcal{F}$. As shown in [B7] Lemma 1.4 together with some additional arguments implies the following

Lemma 1.5[B7]. *Let f have a periodic orbit $P \notin \mathcal{F}$. Then f has both the right and the left horseshoes, $I_f = (0,1)$ and $RP(f) = \{(s,t) : 0 < s < t\}$.*

One of our major focuses in this paper is the set $RP(f)$ of all rotation pairs of an interval map f. Lemma 1.5 fully describes this set in case when a map admits a periodic point not belonging to $\mathcal{F}$, so from now on we consider only interval maps whose periodic points belong to $\mathcal{F}$.

As it has already been mentioned there is a special kind of periodic orbits which plays an important role in our studying. In order to introduce these periodic orbits we first need to specialize Theorem 0.2 in a particular case. Due to this theorem if there is a periodic point P such that $\rho(P) = p/q$ where p, q are coprime then there will be a periodic orbit S such that $rp(S) = (p,q)$. The results of [B7] show that we will always be able to find this orbit S with some specific properties listed in the following

Lemma 1.6[B7]. *Let P be a periodic orbit, $\rho(P) = p/q$ where p, q are coprime. Then there is a periodic orbit S such that $rp(S) = (p,q)$. Moreover, we can find $S = \{x_1 < x_2 < \dots < x_q\} \in \mathcal{F}$ such that the following holds.*

(1) *There is a number l such that $fx_i > x_i$ for any $1 \le i \le l$ and $fx_j < x_j$ for any $l+1 \le j \le q$.*

(2) *If $A_0(S) = \{x \in S : x \le x_l, fx \le x_l\}$, $A_1(S) = \{x \in S : x \le x_l, fx \ge x_{l+1}\}$, $A_2(S) = \{x \in S : x_{l+1} \le x\}$ then:*

(a) *$fx \le x_l$ for any $x \in A_2(S)$;*

(b) *f is increasing on $A_0(S)$ and decreasing on $A_1(S) \cup A_2(S)$.*

(c) *$f^2x \le x$ for $x \in A_1(S)$ and $x \le f^2x$ for $x \in A_2(S)$ where $f^2x = x$ if and only if x is of period 2.*

In fact it is easy to see that (a) and (b) imply (c). We call a periodic orbit S with all the properties from Lemma 1.6 *characteristic* for p/q. It is easy to give examples of characteristic periodic orbits; for instance, the periodic orbit on Fig. 1 is characteristic.

Let us call a finite sequence of points $\{f^i x, f^{i+1}x, \dots, f^j x\}$ a *time segment* of the orbit of x or simply a *time segment*. Later we will use the fact that by Lemma 1.6 a characteristic orbit S may be divided into a few alternating time segments of the following two types.

(1) A *time segment of spiral type* (in short a *spiral*) is the sequence of points $z, fz, \dots, f^{2k}z$ from S such that $f^{2k}z < f^{2k-2}z < \dots < z \le x_l < x_{l+1} \le fz < f^3 z < \dots < f^{2k-1}z$, maximal by inclusion among all time segments in S with these properties.

(2) A *time segment of shift type* is the sequence of points $y < fy < \dots < \dots < f^s y \le x_l$ from S, maximal by inclusion among all time segments in S with these properties.

In order to simplify understanding let us discuss the division of a few particular characteristic orbits into time segments. First let us consider a unimodal orbit of period 3, i.e. the orbit of a point x such that $x = f^3x < fx < f^2x$. Then $x < fx$ is the only time segment of shift type in the orbit and $f^3x < fx < f^2x$ is the only spiral in the orbit. Now let us consider the orbit on Fig. 1, i.e. the orbit Q of a periodic point x such that $x = f^7x < fx < f^5x < f^2x < f^3x < f^4x < f^6x$ (although it is easy to follow our arguments on the formal level the picture is also helpful here). It is easy to see that the points x_l and x_{l+1} which have to divide the orbit into the set of points mapped to the left and the set of points mapped to the right are in this case points f^3x and f^4x. The only spiral in the orbit is $x = f^7x < f^5x < f^3x < f^4x < f^6x$; the only time segment of shift type is $x < fx < f^2x < f^3x$.

We include the spatial structure of a spiral in its definition; however in fact it is not that necessary since any maximal time segment in a characteristic orbit in which each point is mapped to the other side of the interval $[x_l, x_{l+1}]$ has this structure. Indeed, by Lemma 1.6(b).(2) when the point is mapped back to the same side of this interval by f^2 it finds itself farther away from $[x_l, x_{l+1}]$. Now, let $\{z, fz, \dots, f^r z\}$ be a maximal time segment of spiral type. If $z \ge x_{l+1}$ then one can take a point $\zeta \in S$ which is mapped into z; by Lemma 1.6(b).(2) $\zeta \le x_l$ and $f^2\zeta < \zeta$, so ζ can be safely added to the existing time segment which will remain a spiral contradicting the maximality of $\{z, fz, \dots, f^r z\}$. So one can assume that $z \le x_l$. Now, if $f^r z \ge x_{l+1}$ then one can add $f^{r+1}z$ to the time segment in question and it is just as easy to check that the time segment remains a spiral; the same contradiction now implies that $f^r z \le x_l$ finishing the verification (clearly if $z \le x_l$ and $f^r z \le x_l$ then r is even).

Finally let us state some technical facts established in [B7]. Consider a periodic orbit $P \in \mathcal{F}$; we may assume that $P = \{x_1 < \dots < x_q\}$ and there is a number l such that $fx_i > x_i$ for any $1 \le i \le l$ and $fx_j < x_j$ for any $l+1 \le j \le q$. Then obviously there is a fixed point $z \in (x_l, x_{l+1})$ (may be even more than one). Pick up such a fixed point z and call it a *singled out* fixed point for P. Now, let us call a non-degenerate interval I *admissible* if one of its endpoints is z and the other one belongs to P.

One can imagine a piece of rubber rope nailed down at the fixed point z whose second endpoint, say, x belongs to P; this rope covers exactly one admissible interval. Let us see what happens if one applies the map f to the endpoints of this piece of rope; this way we explain the appearance of important for us in the future *chains of intervals* or simply *chains*. The point z is fixed so we made no mistake securing it. Yet w moves into fw, so now the piece of rope covers another admissible interval; the fact that the rope is made of rubber is essential here. Let us allow at this point in time shrinking of the rope (obviously the nailed down point z cannot move in any case) so that in its new position it still covers an admissible interval; it corresponds to the moving the other endpoint of the piece of rope closer to z into another point from P keeping it on the same side with respect to z. After that one can again apply f to the endpoints of the piece of rope and go on with the process.

A sequence of admissible intervals appearing in this process is called a *chain*. More precisely, a sequence of admissible intervals $I_0 = [y_0, z], I_1 = [y_1, z], \dots$ is called a *chain (of intervals)* if $I_{j+1} \subset [z, fy_j]$ for all $j \geq 0$. If a chain of intervals is periodic we call it a *loop (of intervals)*; sometimes we also call a finite chain of intervals $I_0 = [y_0, z], \dots, I_k = [y_k, z]$ a *loop (of intervals)* if in addition to the usual properties of chains $I_0 \subset [z, fy_k]$. Let us discuss elementary properties of chains. First of all it is easy to see that if $I_0, \dots, I_{k-1}$ is a loop then $k > 1$ since the image of an admissible interval cannot contain this interval. Also, we do not require that intervals in a chain all are distinct; the same is true for loops.

Let ϕ be a function defined on the family of all admissible intervals such that $\phi([x, z]) = 0$ if $x < z$ and $\phi([z, x]) = 1$ if $z < x$. For any loop $\bar{\alpha} = \{I_0, \dots, I_{k-1}\}$ let us call the pair of numbers (p, k) the *rotation pair* of $\bar{\alpha}$ where $p = \sum_{j=0}^{k-1} \phi(I_j)$; also let us call the number $\rho(\bar{\alpha}) = p/k$ the *rotation number* of $\bar{\alpha}$. We finish this series of definitions with the following one: a sequence $\{y_1, \dots, y_l\}$ is called *non-repetitive* if it cannot be represented as several repetitions of a smaller sequence.

Lemma 1.7[B7]. *Let $\bar{\alpha} = \{I_0, \dots, I_{k-1}\}$ be a loop. Then there are the following possibilities.*

(1) *Let k be even, $\phi(I_j) = 0$ if j is even and $\phi(I_j) = 1$ if j is odd. Then f has a point x of period 2.*

(2) *Let the first possibility fail. Then there is a periodic point $x \in I_0$ such that $x \neq z, f^j x \in I_j (0 \leq j \leq k-1), f^k x = x$ and so $\rho(x) = \rho(\bar{\alpha})$. Moreover, if the sequence of numbers $\{\phi(I_0), \dots, \phi(I_{k-1})\}$ is non-repetitive then $rp(x) = rp(\bar{\alpha})$.*

Any point x with the properties from Lemma 1.7 is said to be *generated* by $\bar{\alpha}$. There is also another obvious connection between periodic orbits forcing a unique fixed point and loops which works the other way round. Namely, if $x \in \mathcal{F}$ is a periodic point of period k and z is the singled out fixed point for $orb\, x$ then $[x, z], [fx, z], \dots, [f^{k-1}x, z]$ is a loop of intervals. We denote it by $\bar{\alpha}(orb\, x) = \bar{\alpha}(x)$ and say that the loop $\bar{\alpha}(orb\, x)$ is *generated* by x (or $orb\, x$). Note that if a loop of intervals is generated by a periodic oorbit then it is non-repetitive.

Before we pass on to the proof of Proposition 1.12 we need to state some results from [B1-B3]. The invariant probability measure concentrated on a periodic orbit (and thus equidistributed) is called a *CO-measure* [DGS]; denote the set of all such measures $CO(f)$ and the CO-measure concentrated on a periodic orbit S by $\nu(S)$. Theorem 1.8 is a simplified version of the result obtained in [B1,B3] for interval maps and in [B2] for graph maps.

Theorem 1.8[B1-B3]. *Let $f : X \to X$ be a continuous circle map with $Per\, f \neq$*

$\emptyset$ or non-strictly periodic map, μ be an invariant probability measure. Then the following statements are equivalent:

(1) *there exists x such that $\mu(\omega(x)) = 1$;*

(2) *there exists a generic point for μ;*

(3) *μ can be approximated arbitrary well by a CO-measure.*

In fact Theorem 1.8 follows from the spectral decomposition constructed in [B1-B3]; we will state here two more results from [B1-B3] which are parts of the spectral decomposition, but first let us briefly describe how limit sets are classified in [B1-B3]. An interval I is called *periodic (of period k)* or *k-periodic* if $J, \dots, f^{k-1}J$ are pairwise disjoint and $f^kJ \subset J$; the set $M = \bigcup_{i=0}^{k-1} f^iJ \equiv orb\, J$ is then called a *cycle of intervals* (we write also $per(J) = per(M) = k$). A map restricted on a cycle of intervals is called *non-strictly periodic*. Fix an infinite set $\omega(x)$ and consider the family $\mathcal{A}$ of all cycles of intervals $orb\, I$ such that $\omega(x) \subset orb\, I$. There are two possibilities.

1) *Periods of sets $orb\, I \in \mathcal{A}$ are not bounded.* Then there exists a nested family of cycles of intervals containing $\omega(x)$ with periods tending to infinity. This allows to semiconjugate $f|\omega(x)$ to a transitive translation in a compact group and implies many properties of $f|\omega(x)$; in particular $\omega(x)$ cannot contain periodic points.

2) *Periods of sets $orb\, I \in \mathcal{A}$ are bounded.* Then there exists a minimal cycle of intervals $orb\, J \in \mathcal{A}$. It is easy to see that all points $y \in \omega(x)$ have the following property: if U is a neighborhood of y in $orb\, J$ then $\overline{orb\, U} = orb\, J$ (otherwise $\overline{orb\, U}$ generates a cycle of intervals $orb\, K$ such that $\omega(x) \subset orb\, K \subsetneqq orb\, J$ which is a contradiction). The idea is to consider all the points $z \in orb\, J$ with this property. They form a set B which turns out to be a maximal by inclusion limit set with some important properties.

The limit sets of the second type are described in Proposition 1.9. To state it we need more definitions. Let $\xi : K \longrightarrow K$ and $\xi' : K' \longrightarrow K'$ be non-strictly periodic, K and K' be homeomorphic. Let $\phi : K \longrightarrow K'$ be a monotone semiconjugacy between ξ and ξ' and $F \subset K$ be a ξ-invariant closed set such that $\phi(F) = K'$, for any $x \in K'$ we have $int\, \phi^{-1}(x) \cap F = \emptyset$ and so $\phi^{-1}(x) \cap F \subset \partial\phi^{-1}(x)$, $1 \le card\{\phi^{-1}(x) \cap F\} \le 2$. Then we say that ϕ *almost conjugates* $\xi|F$ to ξ'. Finally let M be a cycle of intervals; consider a set $\{x \in M$: for any relative neighborhood U of x in M we have $\overline{orb\, U} = M\}$; it is easy to see that this is a closed invariant set. It is called *a basic set* and denoted by $B(M, f)$ provided it is infinite; if $B(M, f)$ exists we say that M *generates* a basic set.

Proposition 1.9[B1-B3]. *Let $B = B(M, f)$ be a basic set. Then $f|M$ is surjective and there exist a transitive map $g : M' \longrightarrow M'$, M' homeomorphic to M, and a monotone map $\phi : M \longrightarrow M'$ such that ϕ almost conjugates $f|B$ to g. Furthermore, the following holds:*

(a) *$B \subset \overline{Per\, f}$ is a perfect set;*

(b) *$f|B$ is transitive and if $\omega(z) \supset B$ then $\omega(z) = B$ ($\equiv B$ is a maximal limit set);*

(c) *$h(f|B) > 0$;*

(d) *if $x \in B$ is not isolated in B from a side T then for any T-semi-neighborhood $U \subset M$ of x we have $\overline{orb\, U} = M$.*

The following proposition follows immediately from [B1-B3].

Proposition 1.10[B1-B3]. *Suppose that $\omega(x)$ contains a periodic orbit of period n but does not coincide with it. Then there is a unique basic set $B(M) \supset \omega(x)$ and the period of M is not bigger than n.*

We will also need the following

Proposition 1.11. *If $f : [0,1] \to [0,1]$ is a transitive map and there are two fixed points y and z then the following holds:*
1) *if there is a point $x \in (y,z)$ such that $x < fx$ then $0 \in \overline{I_f}$;*
2) *if there is a point $x \in (y,z)$ such that $x > fx$ then $1 \in \overline{I_f}$.*

Proof. Suppose first that there are open disjoint intervals $U < V < W$ such that either
1) $fx > x$ if $x \in U$, $fx < x$ if $x \in V$ and $fx > x$ if $x \in W$ or
2) $fx < x$ if $x \in U$, $fx > x$ if $x \in V$ and $fx < x$ if $x \in W$.

By Corollary 1.3 there is a periodic orbit which visits U, V and W; clearly this orbit does not force a unique fixed point. Therefore by Lemma 1.5 the map f has the rotation interval $I_f = (0,1)$ and we are done. So from now on we may assume that there are no intervals U, V, W with the properties from above; this only can happen if there is a fixed point $z \in (0,1)$ such that for any $x < z$ we have $fx \ge x$ and for any $x > z$ we have $fx \le x$. Consider this situation in more details assuming that the first case from the proposition takes place (one can deal with the second case similarly). Then we may also assume that there is another f-fixed point $y < z$. Consider the set $f[y,z]$; clearly there is a point $\zeta \in (z,1)$ such that $f[y,z] = [y,\zeta]$ (ζ has to lie strictly to the right of z since otherwise $[y,z] \neq [0,1]$ is contradicting transitivity). Let us show that $y \in f[z,\zeta]$. Indeed, otherwise the interval $[z,\zeta] \cup f[z,\zeta]$ is invariant and at the same time it does not contain y so it does not coincide with the entire $[0,1]$ which again contradicts transitivity. Thus one can find a point $x \in (y,z)$ so that $fx > z$ and $f^2x = y$; clearly it shows that f admits a right horseshoe and so by Lemma 1.4 $0 \in \overline{I_f}$ completing the proof. □

Let us consider as an example of how Proposition 1.11 may be applied the case when M is a periodic interval of period 1 which generates a basic set. The existence of a monotone semiconjugacy between $f|M$ and a transitive map $g : [0,1] \to [0,1]$ implies that $I_f \supset I_g$. Thus if g has at least two fixed points then either $0 \in \overline{I_f}$ or $1 \in \overline{I_f}$; we will make use of this observation later on.

We can now prove Proposition 1.12 which is the central result of this section.

Proposition 1.12. *Let $f \in \mathcal{G}$. Then the following holds.*

(1) *Either $I_f \subset (0,1)$ is closed, or $I_f = (0,b]$, $b < 1$, or $I_f = [a,1)$, $a > 0$, or $I_f = (0,1)$.*

(2) *If $a \in I_f$ is an endpoint of I_f then there is a measure μ such that $\operatorname{supp}\mu$ contains no fixed points, $\rho(\mu) = a$ and $f|\operatorname{supp}\mu$ is minimal and $I_f(x) = a$ for any $x \in \operatorname{supp}\mu$.*

Proof. Due to Propositions 0.1 and 1.1 in order to prove the first statement it is enough to show that if $a > 0$ is the left endpoint of $\overline{I_f}$ then $a \in I_f$. By Corollary 0.3 $a \le 1/2$, and if $a = 1/2$ then Proposition 1.12 follows from Corollary 0.3. So we may assume that $a < 1/2$. If there is at least one f-periodic orbit which does not belong to $\mathcal{F}$ then by Lemma 1.5 $I_f = (0,1)$. Thus if P is an f-periodic orbit then $P \in \mathcal{F}$; in other words $P = \{x_1 < x_2 < \dots < x_k\}$ and for some l we have that $fx_i > x_i$ if $i \le l$ and $fx_i < x_i$ if $i > l$. Also let $\mathcal{I} = \{I_1, \dots, I_k\}$ be the intervals containing all fixed points of f whose endpoints are fixed points such that f is monotone on each of them; the existence of these intervals follows from the definition of $\mathcal{G}$. Let $\mathbf{J} = \bigcup_{i=1}^{r} J_i$. The set $\mathbf{J} \cap [x_l, x_{l+1}] = C(P)$ is a finite union of intervals from $\mathcal{J}$ since neither x_l nor x_{l+1} belong to $\mathbf{J}$. Let $z(P)$ and $z'(P)$ be the leftmost and the rightmost points of $C(P)$ respectively; then both are fixed, and we consider $z(P)$ as the singled out point for P. As an example let us consider the case when there is a unique f-fixed point, say, ζ (for instance it may happen if the map f is unimodal); then the family $\mathcal{F}$ consists of a single degenerate interval $[\zeta, \zeta]$ and obviously $z(P) = \zeta$.

If there is a periodic point x such that $\rho_f(x) = a$ then we have nothing to prove. So we assume that such a periodic point does not exist. Let $1/2 > a_0 > a_1 > \dots$ be a sequence of numbers such that $\lim_{i\to\infty} a_i = a$. By Corollary 0.3 there are periodic orbits of rotation numbers smaller than or equal to a_i. Let the smallest among their periods be q_i and the smallest rotation number of a point of period q_i be $p_i/q_i \le a_i$; we may assume that a periodic orbit S_i is characteristic for p_i/q_i. Consider the smallest number $j > i$ such that $q_j > q_i$ and $a_j < p_i/q_i$, find a number p_j and a periodic orbit S_j and then continue. Also, we can get rid of all the numbers $a_{i+1}, \dots, a_{j-1}$ and renumber a_j into a_{i+1}. Therefore finally we will have a sequence of numbers $1/2 > a_0 \ge p_0/q_0 > a_1 \ge p_1/q_1 \dots$ and periodic orbits S_i characteristic for p_i/q_i where q_i is the smallest period of a periodic point with the rotation number less than or equal to a_i and $\dfrac{p_i}{q_i}$ is the smallest rotation number of a periodic point of period q_i. Furthermore, there are only finitely many possible sets $C(S_i)$ (all these sets are finite unions of intervals taken from a finite family of intervals $\mathcal{F}$), so considering a subsequence we may assume that $C(S_i) = \bigcup_{j=1}^{t} J'_j$ for any i (here all J'_j belong to $\mathcal{J}$). Let $z(S_i) = z, z'(S_i) = z'$.

Let us study spirals in S_i. Let $x, fx, \dots, f^{2k}x$ be one; then by the definition $f^{2k}x < \dots < f^2x < x < z < fx < \dots < f^{2k-1}x$. By the properties of characteristic orbits and the maximality of spirals if $v \in S_i$ is the point such that $fv = x$ then $v < x$; also let $u \in S_i$ be such that $fu = v$. Let m be the biggest among $0, 1, \dots, k$ number such that $v < f^{2m}x$. Fix n such that $\dfrac{n}{2n+1} > a_0$ and show that $m \le n$. Indeed, in the loop $\bar{\alpha}(S_i)$ there is a subsequence $[v, z], [x, z], \dots, [f^{2m-1}x, z]$; if we omit this subsequence from the loop and start the rest from $[f^{2m}x, z]$ we will be left with $\bar{\beta} = \{[f^{2m}x, z], [f^{2m+1}x, z], \dots, [u, z]\}$. Let us show that this is a loop. Indeed, it is enough to see that $[fu, z] = [v, z] \supset [f^{2m}x, z]$ which follows immediately, for by the choice of m we have $v < f^{2m}x < z$. Clearly the rotation pair $rp(\bar{\beta})$ is $(p_i - m, q_i - 2m - 1)$. Assume that $m > n$; then $\dfrac{m}{2m+1} > a_0 > a_i \ge \dfrac{p_i}{q_i}$ which implies that $\dfrac{p_i - m}{q_i - 2m - 1} < \dfrac{p_i}{q_i}$. By Lemma 1.7 the loop $\bar{\beta}$ generates a periodic orbit R of period $r \le q_i - 2m - 1 < q_i$ and the rotation number $\dfrac{p_i - m}{q_i - 2m - 1} < \dfrac{p_i}{q_i}$ which contradicts the properties of the sequence $a_0 \ge p_0/q_0 > a_1 \ge p_1/q_1 \dots$ established in the previous paragraph. So $m \le n$.

Furthermore, let w be the closest from the left to z point of $A_0(S_i)$ (i.e. the closest to z point w from S_i such that $w < z$ and $fw < z$). Let us show that there is at least one point among points $x, fx, \dots, f^{2n+2}x$ which lies to the left of z and is farther away from z then w. Consider two cases. If $m < k$ then by the choice of m we see that $f^{2m+2}x < v$; at the same time $v < z$, $fv < z$ (i.e. $v \in A_0(S_i)$) and so by the choice of w we have $v \le w$. Since $m \le n$ we get the required. Now, if $m = k$ then by the maximality of a spiral $f^{2k}x \in A_0(S_i)$ and again the required statement holds because $m \le n$. Let us show now that w cannot lie arbitrary close to z. Indeed, $fw < z$ and at the same time if y is the closest from the left to z point of S_i then $fy > z$ by the properties of characteristic orbits. So, $z \in \mathit{int}\, f[w, y]$. On the other hand $f \in \mathcal{G}$ and so there is $\varepsilon > 0$ such that $z \notin \mathit{int}\, f[z - \varepsilon, z]$. Therefore $w < z - \varepsilon$. Note also that if $z < z'$ then choosing ε to be small enough we may assume $z \notin \mathit{int}\, f[z' + \varepsilon, z']$ (let us remind the reader that z' is the rightmost point of the union of intervals $C(P)$). If however $z = z'$ we still may make the same assumption because of the properties of maps from $\mathcal{G}$. So in any case we may assume that $z \notin \mathit{int}\, f[z' + \varepsilon, z']$.

Now we can show that there is a neighborhood U of $[z, z']$ such that $S_i \cap U = \emptyset$ for large i. Indeed, notice that by the assumption $S_i \cap [z, z'] = \emptyset$ for any i. Let U be a neighborhood of $[z, z']$ such that if $U' = U \cap ((0, z) \cup (z', 1))$ and $W = (z - \varepsilon, 1)$ then $U' \subset W, fU' \subset W, \dots, f^{n+2}U' \subset W$. Note that the point $w \in S_i$ defined in the previous paragraph does not belong to W since $w < z - \varepsilon$. Consider the spiral $y, fy, \dots, f^{2l}y$ in S_i which starts at the closest from the left to z point y of S_i. Then by the properties of characteristic orbits fy is the closest from the right to z' point of S_i. At the same time by what was shown in the previous paragraph among points $y, fy, \dots, f^{n+2}y$ there is a point lying farther away from z then w and so definitely not belonging to W. The choice of U shows now that if $y \in U$ or $fy \in U$ then this is impossible. Therefore, $y \notin U$ and $fy \notin U$, or, in other words, S_i is disjoint from U.

Considering a subsequence we may assume that $\nu(S_i) \to \mu$. By Theorem 1.8 there is a point x' such that $\mathit{supp}\, \mu \subset \omega(x')$ and moreover μ has a generic point x''. Let us show that $\mathit{supp}\, \mu$ contains no fixed points and so $x'' \in Ad_f$ is an admissible point. First we prove that there is no fixed point in $[z', 1] \cap \mathit{supp}\, \mu$. Indeed, let $\zeta > z'$ be a fixed point; choose $\delta < \varepsilon$ so that if any two points are δ-close then their f-images are ε-close. Now, if a point $x \in S_i$ is δ-close to ζ then fx must be ε-close to $f\zeta = \zeta$ which is impossible for by the properties of characteristic orbits $fx < z$ and hence by the properties of S_i we have $fx < z - \varepsilon$. In other words we see that orbits S_i do not come even δ-close to the point ζ, therefore there are no fixed points in the set $\mathit{supp}\, \mu \cap [z', 1]$.

Suppose that $\zeta \in \mathit{supp}\, \mu \cap [0, z)$ is a fixed point and show that then $0 \in \overline{I_f}$ which is a contradiction. First let us show that $S_i \cap [\zeta, z] \ne \emptyset$. To this end it is enough to show that if y_i is the closest from the left to z point of S_i then $\zeta < y_i$ for large i, and indeed by the properties of maps from $\mathcal{G}$ and because $C(S_i) = \bigcup_{j=1}^{t} J'_j$ for any i we see that there are no fixed points in (y_i, z). Let us show now that $\mathit{supp}\, \mu \ne \{\zeta\}$. Indeed, by the properties of characteristic orbits and since $\nu(S_i)$-measure of $(z', 1]$ is $\dfrac{p_i}{q_i} > a > 0$ we have that $\mu(z', 1] \ge a$. At the same time $\zeta < z$ which proves that $\mathit{supp}\, \mu \ne \{\zeta\}$; therefore the ω-limit set $\omega(x')$ which contains $\mathit{supp}\, \mu$ contains a fixed point ζ but does not coincide with $\{\zeta\}$. Thus Proposition 1.10 implies that there is a basic set $B = B(M)$ containing $\omega(x')$ (and so containing ζ), and M is of period 1 because $\zeta \in B$ is a fixed point; also, as we have just seen $\mu(z', 1] \ge a > 0$ and so $[\zeta, z'] \subset M$.

Let ϕ be a monotone semiconjugacy between $f|M$ and a transitive map $g : [0, 1] \to [0, 1]$. Then $\phi(z') = \alpha$ is a g-fixed point and $\phi(\zeta) = \beta$ is a g-fixed point. W
e may assume that none of the points from S_i is mapped by ϕ into a g-fixed point

Indeed, otherwise the semiconjugacy implies that the whole orbit S_i is mapped into the same fixed point; since $\zeta \in supp\,\mu$ and at the same time there are points of S_i to the right of z' we see that $\alpha = \beta$ is this fixed point. Then an f-invariant interval $\phi^{-1}(\alpha) = [b, d]$ contains $[\zeta, z']$ and also all the orbits S_i. Therefore the way we define μ implies that $supp\,\mu \subset B \cap [b, d] = \{b, d\}$ where $b \leq \zeta$ and $d \geq z'$. We know that $\nu(S_i)$-measure of $(z', 1]$ converges to a and at the same time $\nu(S_i)$-measure of the neighborhood U of $[z, z']$ is 0 since U is disjoint from S_i for any i. Hence $z' < d$ and $\mu(d) = a < 1/2$. Together with $supp\,\mu \subset \{b, d\}$ it implies that d is a fixed point belonging to $[z', 1]$ which was shown to be impossible.

Thus all points of the orbits S_i are mapped by ϕ into points which are not fixed for the map g. Clearly it implies that $\alpha > \beta$ because if $\alpha = \beta$ then an f-invariant interval $\phi^{-1}(\alpha)$ contains all the orbits S_i since $S_i \cap [\zeta, z] \neq \emptyset$ for any i and all sets S_i are mapped by ϕ into a g-fixed point α which is a contradiction. All points of S_i lying between ζ and z' are mapped by f to the right; hence their ϕ-images are mapped to the right by g. Now by Lemma 1.11 $0 \in \overline{I_g} \subset \overline{I_f}$ which is a contradiction showing that $supp\,\mu$ contains no fixed points. Hence the above chosen generic for μ point x'' is admissible. Moreover, all periodic non-fixed points are admissible too, so we conclude that $\rho(\mu) = \lim_{i\to\infty} \rho(S_i) = \lim_{i\to\infty} \dfrac{p_i}{q_i} = a$. This proves the first statement of Proposition 1.12.

Let us pass on to the second statement; we will use the construction and notation from the proof of the first one. If periods q_i of orbits S_i do not grow then clearly we may assume that S_i converge to a periodic orbit Q with the rotation number $\rho(Q) = a$; since we suppose that such a periodic orbit does not exist (if it exists the proposition is trivial) we may assume that q_i grow to infinity. We saw that all the points from $supp\,\mu$ are admissible. Let $K \subset supp\,\mu$ be a minimal set and prove that for any invariant measure ν such that $supp\,\nu = K$ we have $\rho(\nu) = a$; since at least one such measure exists we get the required. Indeed, suppose that $\rho(\nu) > a$. Choose a generic for ν point $x \in K, x < z$. Clearly there is a big number s such that $x < f^s x < z$ and if r is the number of those of points $x, fx, \ldots, f^{s-1}x$ which lie to the right of z then $r/s > a$ (here we use the fact that $\rho(\nu) > a$). Working with a subsequence we may assume that there is a sequence of points $x_i \in S_i$, $\lim_{i\to\infty} x_i = x$. Thus for large i we have $x_i < f^s x_i$. Omitting the subsequence $[x_i, z], \ldots, [f^{s-1}x_i, z]$ from the loop $\bar{\alpha}(S_i)$ we get a new chain of intervals $\bar{\beta}$ which turns out to be a loop because $x_i < f^s x_i$. Now if i is so large that $\dfrac{p_i}{q_i} < \dfrac{r}{s}$ then $\rho(\bar{\beta}) = \dfrac{p_i - r}{q_i - s} < \dfrac{p_i}{q_i}$ and by Lemma 1.7 there is a periodic point y of rotation number $\rho(y) = \rho(\bar{\beta}) = \dfrac{p_i - r}{q_i - s}$ of period smaller than or equal to $q_i - s$ which contradicts the choice of the sequence $1/2 > a_0 \geq p_0/q_0 > a_1 \geq p_1/q_1 \ldots$. This completes the proof of Proposition 1.12. $\square$

Propositions 0.1 and 1.12 allow to give an alternative and simpler definition of the rotation set for maps from $\mathcal{G}$; in particular this definition is applicable in case of piecewise monotone interval maps.

An alternative definition of the rotation interval for maps from $\mathcal{G}$. Let $f \in \mathcal{G}$. Then the closure in $(0, 1)$ of the set of all rotation numbers of f-periodic points is the *rotation set* of f. In particular if rotation numbers of f-periodic points are bounded away from 0 and 1 then the rotation set of f is the closure of the set of all rotation numbers of periodic points of f in the usual sense.

In fact techniques developed in the proof of Proposition 0.12 allow us to show that sometimes considering all points (i.e. not only admissible ones) does not add anything to the rotation set of a map. Namely, one can prove the following

Proposition 1.13. *Suppose a is the only f-fixed point and there is an $\epsilon > 0$ such that $f[a-\epsilon, a] \subset [a, 1], f[a, a+\epsilon] \subset [0, a]$. Then the union $\bigcup_x I_f(x)$ coincides with the rotation set I_f of f.*

Proof. In the definition of the rotation set the union of sets $I_f(x)$ is taken over all admissible points. So it is enough to show that even for a non-admissible point x we have $I_f(x) \subset I_f$. Assume otherwise; then without loss of generality we may assume that $\alpha \in I_f(x) \setminus I_f, \alpha \neq \frac{1}{2}$. Since x is not admissible then $a \in \omega(x)$. Let us show that for any δ neither $f[a-\delta, a]$ nor $f[a, a+\delta]$ is degenerate. Indeed, otherwise we may assume that $f[a-\delta, a] = \{a\}$. Since $I_f(x) \neq \frac{1}{2}$ it implies that the orbit of x does not enter $[a-\delta, a]$ which by the continuity implies that the orbit of x does not enter $[a, a+\delta]$ either contradicting the fact that $a \in \omega(x)$. Now, if there are no preimages of c in $[0, a)$ then let $c' = 0$, if there is the closest from the left to a preimage of a then denote it by c', and if there is a sequence of preimages of a approaching a from the left choose a preimage of a in $[a-\epsilon, a)$ and denote it by c'. Similarly we find a preimage c'' of a to the right of a. Denote $[c', c'']$ by J; the choice of c', c'' implies that $f[a, c''] \leq a \leq f[c', a]$. Then since $I_f(x) \neq \frac{1}{2}$ we see that the orbit of x enters and leaves J infinitely many times.

Consider a nested sequence of chains of intervals $\bar{\beta} = \{[x, a], [fx, a], \dots, \}$ and construct a new sequence of chains of intervals as follows. Mark those pairs of times e_k, l_k when the orbit of x enters J and leaves it (in other words, $f^{e_k - 1}x \notin J, f^{e_k} \in J, \dots, f^{l_k}x \in J, f^{l_k+1} \notin J$). If $e_k \geq l_k - 1$ we will not change $\bar{\beta}$ at this place at all. If $e_k < l_k - 1$ then there are two cases.

1) $f^{e_k-1}x$ and $f^{l_k}x$ lie to the same side of a.

Then obviously $f^{l_k}x$ is closer to a than $f^{e_k-1}x$. In this case omit from $\bar{\beta}$ all the intervals $[f^i x, a], e_k - 1 \leq i \leq l_k - 1$ so that the corresponding part of the new sequence of intervals is $\dots, [f^{e_k-2}x, a], [f^{l_k}, a], \dots$. Clearly the new sequence of intervals is still a chain of intervals.

2) $f^{e_k-1}x$ and $f^{l_k}x$ lie to the distinct sides of a.

Then the properties of f at a imply that $f^{l_k-1}x$ is closer to a than $f^{e_k-1}x$. Similarly to the first case let us omit from $\bar{\beta}$ all the intervals $[f^i x, a], e_k - 1 \leq i \leq l_k - 2$ so that the corresponding part of the new sequence of intervals is $\dots, [f^{e_k-2}x, a], [f^{l_k-1}x, a], \dots$. Again, the new sequence of intervals is a chain of intervals.

In the end we will get a new chain of intervals $\bar{\gamma} = \{[f^{n_k}x, a] = I_k\}$ such that there are no three subsequent intervals in $\bar{\gamma}$ which would belong to J. The fact that this is a chain of intervals means that $[f^{n_k+1}x, a] \supset I_{k+1}$ for any k. Let us now change this chain of intervals once again as follows: if the number k is such that $f^{n_k}x \notin J$ then replace I_k by $[f^{n_k}, c']$ if $f^{n_k}x$ lies to the left of a and by $[c'', f^{n_k}x]$ if it lies to the right of a. Denote the resulting sequence of intervals by $I'_0, I'_1, \dots$. By the construction $fI'_k \supset I'_{k+1}$, all intervals in the new sequence lie either to the left or to the right of a and there are no three subsequent intervals in it which intersect

J. As usual in one-dimensional dynamics it allows to find a nested sequence of closed intervals $R_0 \supset R_1 \supset \dots$ such that for any $0 \le i \le k-1$ we have $f^i R_k \subset I'_k$ and also $f^k R_k = I'_k$.

Let $z \in \cap R_k$. There is a subsequence $\{m_k\}$ such that $\frac{1}{m_k} \sum_{i=0}^{m_k-1} \xi(f^i z) \to b$ where b is farther away from $\frac{1}{2}$ than α but lies to the same side of $\frac{1}{2}$ as α. Indeed, refining the chain of intervals $\bar\beta$ in order to get $\bar\gamma$ we were in fact omitting pieces of $\bar\beta$ within which intervals to the left and to the right of a were alternating; getting rid of these pieces could only move limits of the partial sums $\frac{1}{m} \sum_{i=0}^{m_k-1} \xi(f^i z)$ farther away from $\frac{1}{2}$. At the same time by the properties of the sequence $I'_0, I'_1, \dots$ there are no three consecutive iterates of z belonging to J and so $a \notin \omega(z)$. Therefore z is admissible and $I_f(z) \subset I_f$ which implies that $\alpha \in [b, \frac{1}{2}] \subset I_f$ completing the proof.

2. Rotation intervals for piecewise-monotone maps

For the sake of completeness let us begin this section with the sketch of the proof of Corollary 0.3 which as we have already mentioned in Introduction follows from Theorem 0.2; the analysis we are about to make will also explain what problem has to be solved in order to pass on from Corollary 0.3 to Theorem 2.2.

Note first that the rotation pair $(1, 2)$ is the $\gtrdot$-weakest and so always $(1, 2) \in RP(f)$ (let us remind the reader that we consider only the interval maps with non-fixed periodic points). Thus $1/2 \in I_f$ for any f. Assume that $I_f = \{1/2\}$. Then by Theorem 0.2 there exists a number l such that $RP(f) = Q(1/2, l)$. Now let us assume that $I_f \neq \{1/2\}$ and $int\, I_f = (a, b)$. Again by Theorem 0.2 we may conclude that $RP(f) \supset N(a, b)$. A more difficult problem concerns the endpoints of I_f, i.e. a and b, although in some cases obvious reasons show that the situation is similar to that of Theorem M. Indeed, if a is irrational then there are no periodic points x such that $\rho(x) \le a$; the similar fact holds for b. Taking into account that $Q(c, l) = \emptyset$ for an irrational c and any l we conclude that if both a, b are irrational then $RP(f) = Q(a, 3) \cup N(a, b) \cup Q(b, 3) = N(a, b)$. Moreover, if $a \notin I_f$ then again there are no periodic points x such that $\rho(x) \le a$, and the similar fact is true for b. Thus taking into account that $Q(c, 0) = \emptyset$ for any c we see that if $I_f = (a, b)$ then $RP(f) = Q(a, 0) \cup N(a, b) \cup Q(b, 0) = N(a, b)$. Summarizing this analysis we get Corollary 0.3.

Moreover, the analysis helps in dealing with Theorem 2.2 and in fact allows to state a problem closely connected to this theorem. Indeed, consider the following

Problem 1. *Let I_f be an interval containing its endpoint $a \neq 1/2$, a be rational. Is there a periodic point x such that $\rho_f(x) = a$? What are classes of maps for which such a periodic point exists?*

Suppose the answer to the first question is affirmative for a particular map f. Then the parts of Theorem 2.2 concerning interior points of the rotation interval I_f and the periodic points having the corresponding rotation numbers follow from Corollary 0.3; the parts concerning endpoints of the rotation interval will follow from Theorem 0.2 and the affirmative (for the map f) answer to the first question of Problem 1. Therefore if one can answer the second question of Problem 1, i.e.

find a class of maps for which the periodic point from this problem exists then Theorem 2.2 holds for this class of maps.

The major result of the rest of the paper is that a periodic point mentioned in Problem 1 exists for any piecewise-monotone map and so the arguments from above apply proving for piecewise-monotone maps Theorem 2.2. From now on we consider only piecewise-monotone maps. For any such map f a *lap* is its interval of monotonicity I such that any other interval of monotonicity $J \supset I$ has the same f-image (in other words a lap is an interval on which f is monotone with the maximal image). Clearly if the map f has some *flat spots* (i.e. intervals on which f is a constant) then for any maximal image there may well be infinitely many laps which differ from one another by pieces of flat spots. Let $l(f)$ be the number of laps of f which have distinct images; clearly $l(f)$ is well-defined. In what follows we need some well-known facts which we state next.

The following are slightly modified definitions from [BCMM]. Let $n \geq 2$. The *horseshoe map* $H : [0,1] \to [0,1]$ of type n^+ is defined as follows: for $x \in [i/n, (i+1)/n] (i = 0, \dots, n-1)$ let $H(x) = n(x-i/n)$ if i is even and $1-n(x-i/n)$ if i is odd. Then H is continuous, maps each of the n laps $I_1 = [0, 1/n], \dots, I_n = [(n-1)/n, 1]$ linearly onto $[0,1]$ and is increasing on the first lap (see Fig. 2 where example of a horseshoe of type 4^+ is given). The horseshoe map of type n^- is defined similarly but it is decreasing on the first lap. In either case we denote by $l(H)$ the number of laps of f. Now, let $P \subset [0,1]$ be finite. For $i = 0, \dots, n\,(n = l(H))$, define $d_i = \min\{|p - i/n| : p \in P\}$ and let $\bar{p}_i$ be a point in P such that $|\bar{p}_i - i/n| = d_i$. We say that P *fits* H if $n = 2$ or $n \geq 3$ and $d_i + d_{i+1} \leq 1/n\,(i = 1, \dots, n-2)$. In either case, the P-*truncation* or simply *truncation* H_P of H, defined by $H_P(x) = H(\bar{p}_i)$ if $|x - i/n| \leq d_i (i = 0, \dots, n)$ and $H(x)$ otherwise, is well-defined (see Fig. 2 which contains an example of the graph of a trunctaion; the graph of the truncation wherever it is different from that of the horseshoe is shown in dashed lines).

The importance of truncations of horseshoes (simply *truncations* in what follows) follows from the fact that any piecewise-monotone map f may be modeled by a truncation H in the sense that H has the same number of intervals of monotonicity as f and exhibits the same limit behavior. This actually follows from Milnor-Thurston kneading theory [MT] and may be stated in terms of kneadings; the result itself has been relied upon in literature (see [BCMM], [BC], [MN]) and definitely belongs to "folklore knowledge", so one can state it without proof. Since in this paper we deal with periodic orbits, their rotation pairs and numbers, we give the following weak version of the result which shows that to prove Theorem 2.2 it is enough to work with truncations.

Proposition 2.1. *Let f be a piecewise-monotone map. Then there is a truncation H such that $l(f) = l(H), RP(f) = RP(H), I_f = I_H$.*

In order to make use of Proposition 2.1 we need to study general properties of truncations. As it follows from the definition if H is a truncation then there may be some (no more than $l(H) - 1$ though) *flat spots*, i.e. intervals on which H is a constant; moreover, f has distinct directions of monotonicity on both sides of any flat spot. Thus there are spots-maxima and spots-minima. Clearly the orbits of all points from a spot are the same. Now, if I is an interval such that $f^n I \subset I$ (e.g., periodic interval of period n) then $f^n|I$ may have some flat spots; on the other hand, on any interval $J \subset I$ on which f does not have flat spots and is monotone it is in fact linear with the slope bigger than of equal to 2. Let us show that there

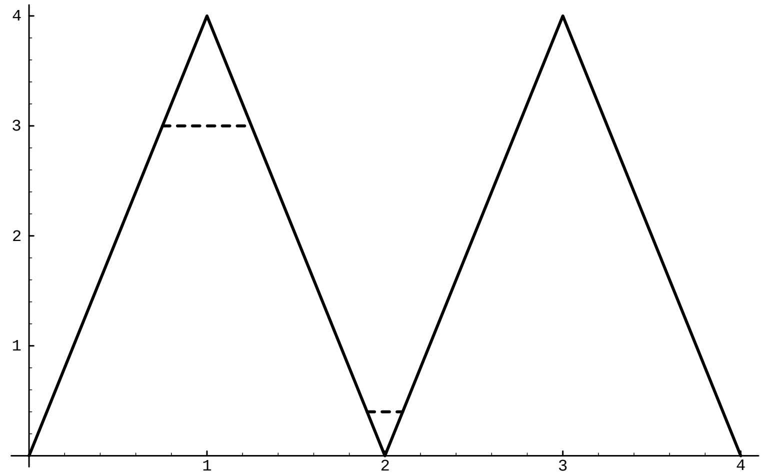

Figure 2: A horseshoe and its truncation.

are preimages of periodic points in $int\, I$. Indeed, if not then we may assume that $I = [a, b]$ and $f^n|int\, I$ moves all the points to the right. Thus b is f^n-fixed. If f^n has a flat spot which ends up at b then f^n maps points of this flat spot into b which is a contradiction to the assumption that there are no preimages of periodic points in $int\, I$. At the same time if b is not an endpoint of a flat spot then points from a small neighborhood of b are mapped away from b by f^n (for the slope is bigger than 1) which is a contradiction to the fact that $f^n|int\, I$ moves all the points to the right. This proves the following

Proposition 2.2. *If H is a truncation and I is an interval such that $H^n(I) \subset I$ then I contains preimages of periodic points in its interior.*

It is easy to see that Proposition 2.2 implies the following

Proposition 2.3. *Let H be a truncation, x be a point which is never mapped into a flat spot of f. Then x can be approximated from either side by preimages of periodic points of H and so these preimages are dense outside the set of preimages of flat spots of f.*

Proof. Let U be a compact semi-neighborhood of x. Let us consider the orbit of U. We claim that there are iterates $n < m$ such that $f^nU \cap f^mU \neq \emptyset$. First, no iterate of I is degenerate since x is never mapped into flat spots. Now, if infinitely many iterates of U intersect flat spots of H or its extrema which do not belong to flat spots then the fact that there are only finitely many flat spots of H and the above mentioned extrema implies that some of the iterates of U are not disjoint. So we may assume that f^kU intersects neither flat spots nor the above mentioned extrema for big k. But then for any big k we have $\lambda(f^{k+1}U) \geq 2\lambda(f^k U)$ where $\lambda(G)$ is the Lebesgue measure of G which is clearly impossible. Thus there are $n < m$ such that $f^n U \cap f^m U \neq \emptyset$. Consider the set $\bigcup_{i=0}^{\infty} f^{(m-n)i}(f^n U)$; obviously this is an f^{m-n}-invariant interval with the closure, say, K. By Proposition 2.2 there are preimages of periodic points in K; so there are preimages of periodic points in U which is the required. $\square$

Any flat spot of a truncation assumes either locally maximal or locally minimal value of f; then we call it *spot-maximum* or *spot-minimum* respectively. Suppose that H' and H'' are two truncations of the horseshoe map of the same type. Suppose also that flat spots of H'' contain those of H'; then if $I' \subset I''$ are maximal flat spots of H', H'' respectively then $H'(I') > H''(I'')$, and if $I' \subset I''$ are minimal flat spots of H', H'' then $H'(I') < H''(I'')$. We say then that H' *forces* H''. The term is justified since for example all periodic points of H'' are certainly periodic points of H', so in particular $RP(H'') \subset RP(H')$. In the rest of the paper talking about *close* maps we mean close in C^0-topology maps. Also, if a truncation H has the property that all its local extrema are eventually mapped into periodic points (we call these local extrema *eventually periodic*) say that H is a *Markov truncation.*

Proposition 2.4. *Let H be a truncation. Then there is an arbitrary close to H Markov truncation H' with $l(H) = l(H')$ which forces H.*

Proof. The definition of a truncation implies that some of local extremal values of a map may be assumed on its flat spots. The only local extremal values which are not assumed on flat spots are $0, 1$. Let us show that if every flat spot of a truncation H' is eventually periodic then H' is Markov. Indeed, it is enough to show that 0 and 1 are eventually periodic. If none of them belongs to a flat spot it follows immediately from the definition of a horseshoe. If, say, 0 belongs to a flat spot and 1 does not then 0 is eventually periodic by the assumption about H' and 1 is either mapped into 0 (and so is eventually periodic too) or is a fixed point. Finally if both 0 and 1 belong to flat spots then they both are eventually periodic due to the assumption about H'.

Thus it is enough to show that there is a truncation H' with all its flat spots eventually periodic which is arbitrary close to H and forces H. To this end we will step by step change H on its flat spots getting H' in the end of this process. First let us agree not to change the map on all its eventually periodic flat spots. Now, let I be such a flat spot that $H^k(I), k > 0$, is disjoint from any flat spot. Then by Proposition 2.3 there is an eventually periodic point in any semi-neighborhood of $H(I)$. If I is a spot-maximum then let us pick up an eventually periodic point y in a small right semi-neighborhood of $H(I)$; if I is a spot-minimum let us pick up an eventually periodic point y in a small left semi-neighborhood of $H(I)$. Clearly one can now change H only on I and get a new truncation H' which forces H, has the flat spot I' which belongs to I and is such that $H'(I') = y$. Clearly this decreases the number of non-eventually periodic flat spots of H at least by one. Therefore if we keep doing this we will in finitely many steps make all the flat spots of H eventually periodic which completes the proof. $\square$

Markov truncations and in fact all maps of the interval with eventually periodic local extrema (we will call these maps *Markov* too) have been studied for years; in particular it is very well known that their properties are strongly connected with those of a Markov subshift of a finite type which can be constructed for any Markov map (see, e.g. [MN] or [ALM]). Let us describe one of the ways it can be done. Let f be a Markov map of the interval, $0 = c_0 < c_1 < \dots < 1 = c_n$ be a set of eventually periodic points such that $f|[c_i, c_{i+1}]$ is monotone for any i. Let $C'' = \{c''_0, c''_1, \dots, c''_m$ be the union of all finite orbits of points $c_0, \dots, c_n$. For any $[c''_i, c''_{i+1}]$ let $[\alpha_i, \beta_i]$ be the convex hall of the intersection $Fix\, f \cap [c''_i, c''_{i+1}]$; now let us add α_i and β_i to C'' unless they already belong to C''. Finally we get a set C such that: 1) $f(C) \subset C$; 2) if $c' < c''$ are the adjacent points from C then $f|[c', c'']$ is monotone; 3) if $c' < c''$ are adjacent points from C and there is a fixed point in $[c', c'']$ then c', c'' are fixed points too.

Consider the partition $\mathcal{I}$ of $[0, 1]$ into intervals generated by the set C. Then for any interval $I \in \mathcal{I}$ its image fI coincides with the union of a few intervals from the same partition $\mathcal{I}$. Let us consider an oriented graph whose vertices are intervals of $\mathcal{I}$ and whose arrows connect two vertices I' and I'' if and only if $fI' \supset I''$; this graph generates a subshift of finite type $\sigma : A_f \longrightarrow A_f$. The properties of this subshift are related to those of f. In particular the problem of finding the sets $RP(f)$ and I_f may be restated in terms of $\sigma : A_f \longrightarrow A_f$. Before we do this let us remind the reader that we consider only maps with at least some periodic non-fixed points (otherwise the dynamics of the map is trivial and very well known). Moreover, if $I_f = \{1/2\}$ then the description of the set $RP(f)$ follows immediately from Corollary 0.3, so we assume that $I_f \neq \{1/2\}$. Let us assign a number $\chi(I)$ to any of intervals from $\mathcal{I}$. Namely, if both endpoints of I are fixed then let $\chi(I) = 1/2$. Otherwise there are no fixed points in $int\, I$ and so the direction in which f moves points on I is well-defined; if points get mapped to the left let $\chi(I) = 1$, otherwise let $\chi(I) = 0$.

Let us consider for $\sigma : A_f \longrightarrow A_f$ the same construction which led to the definition of a rotation number and a rotation pair; similar approach is due to K. Ziemian [Z] who studies rotation numbers for arbitrary Markov subshifts of finite type. Namely, if $\mathbf{k} = \{I_0, \dots\} \in A_f$ then let $\chi(\mathbf{k}) = \chi(I_0)$. Now, for any $\mathbf{k} \in A_f$ one can consider its χ-rotation set $I_\sigma(\mathbf{k})$ exactly like it has been done before for interval maps. The function χ is obviously continuous on A_f so all points are admissible in our sense and it is natural to call the union of sets $I_\sigma(\mathbf{k})$ over all $\mathbf{k} \in A_f$ the *χ-rotation set* (or simply *rotation set*) of $\sigma|A_f$. In particular for an n-periodic point $\mathbf{k}$ of σ_f one can define its *rotation pair* $rp(\mathbf{k}) = (p, n)$ where $p = \sum_{i=0}^{n-1} \chi(\sigma^i \mathbf{k})$, and the *rotation number* $\rho(\mathbf{k}) = p/n$ of $\mathbf{k}$. Notice that if $n > 1$ then the construction implies that $\chi(\sigma^i \mathbf{k}) \neq 1/2$ for any i.

There is a well-known connection (see, e.g., [MN]) between the dynamics of $\sigma|A_f$ and that of f, in particular between their periodic orbits, which is obtained if we code the orbits on the interval by the elements of the partition $\mathcal{I}$. We will state a weak version of the corresponding folklore result in terms of rotation numbers of periodic orbits.

Proposition 2.5. *The sets of rotation numbers of periodic non-fixed points of f and σ are the same.*

The following result from [Z] is an important tool for us; we state it in the above described situation but in fact it holds for much broader defined rotation numbers.

Proposition Z [Z]. *Let B be the closure of the set of all rotation numbers of periodic points of A_f. Then for any rational number $r \in B$ there is a periodic point $x \in A_f$.*

We are ready now to give the affirmative answer to the first question of Problem 1 for piecewise-monotone maps. Note first that by Proposition 1.12 either $I_f \subset (0,1)$ is closed, or $I_f = (0, b]$, $b < 1$, or $I_f = [a, 1)$, $a > 0$, or $I_f = (0, 1)$. In the next Proposition 2.6 we consider the question of existence of a periodic orbit with the rotation number which is equal to an endpoint of I_f in these cases. Clearly the question makes sense only if the endpoint of I_f is neither 0 nor 1; also, without loss of generality we consider only the left endpoint of I_f. Moreover, Proposition 2.6 also contains for piecewise-monotone maps the inverse statement to that of Lemma 1.6 showing that if 0 or 1 belong to the rotation set of a piecewise-monotone map then the map has a horseshoe.

Proposition 2.6. *Let f be a piecewise-monotone map. Then the following holds.*

(1) *Let $I_f = [a, b] \subset (0, 1)$ or $I_f = [a, 1)$. If a is rational then there is a periodic point x such that $\rho_f(x) = a$.*

(2) *Let $I_f = (0, b], b < 1$ or $I_f = (0, 1)$. Then f has the right horseshoe.*

Proof. 1) Let $a = p/n$ where p, n are coprime; by Theorem 0.2 we may assume that $a \neq \dfrac{1}{2}$ and so $n > 2$. By Proposition 2.1 there exists a truncation H with the same number m of laps as the map f itself such that $RP(f) = RP(H), I_f = I_H$. Therefore we may assume from the very beginning that $f = H$. Now, making use of Proposition 2.4 we can find a sequence of Markov truncations $H_i \to H$ each of which forces H; H_i converge to H in C^0-topology. Thus as we noticed before Proposition 2.4 $RP(H_i) \supset RP(H)$. For every H_i let us consider the corresponding subshift of finite type $\sigma_i : A_i \longrightarrow A_i$ and the closure B_i of the set of rotation numbers of all its periodic points of periods greater than 1. By Proposition 2.5 B_i coincides with the closure of the set of rotation numbers of all H_i-periodic points of periods greater than 1 which contains a by Proposition 0.1, the assumptions made in case (1) and the choice of H_i. Now Proposition Z implies that there is a σ-periodic point of the rotation number a which by Proposition 2.5 implies that there is an H_i-periodic point x_i of rotation number a; moreover by Theorem 0.2 we may assume that $rp_{H_i}(x_i) = (p, n)$ so that in particular all points x_i are of period $n > 2$.

We may also assume that $x_i \to x$. Let us prove that x is in fact an H-periodic point and $rp_H(x) = (p, n)$; this will mean that x is the required point. Indeed, $H_i \to H$ in C^0-topology. Clearly it is enough to show that there is an $\epsilon > 0$ such that the minimal distance between points from $orb\, x_i$ is greater than ϵ. It is easy to see that to prove this it is enough to show that for some $\delta > 0$ the diameter of $orb\, x_i$ is greater than δ. Indeed, let such δ exist and yet the minimal distance between points from $orb\, x_i$ is not bounded away from 0. Then we may assume that there is a number r such that $|H_i^r(x_i) - x_i| \to 0$ as $i \to \infty$ for some r which does not depend on i. The fact that p and n are coprime implies that among n pairs of points $\{x_i, H_i^r(x_i)\}, \{H_i x_i, H_i^{r+1}(x_i)\}, \dots, \{H_i^{n-1} x_i, H_i^{n-1+r} x_i\}$ there is at least one pair such that the two points in it are mapped by H_i into different directions, which means that the interval between them contains an H_i-fixed point. At the same time by the continuity the distance between the points in this pair is very small, and so they are very close to the fixed point; again by the continuity we see that the whole orbit of x_i is close to this fixed point and so the diameter of $orb\, x_i$ is also very small contradicting the assumption.

So let G be a horseshoe. Choose $\delta > 0$ so that for any turning point $d \in (0, 1)$ of G we have $G[d - \delta, d + \delta] \cap [d - \delta, d + \delta] = \emptyset$. Now, let F be a truncation of G, Q be an

F-periodic orbit of period $k > 2$, J be the smallest interval containing Q; assume that the length of J is less than δ and show that it leads to the contradiction. Since k is bigger than 2 we conclude that F has at least one extremum in $int\, J$, and the definition of a truncation implies that so does G. Denote a turning point of G which belongs to $int\, J$ by c; clearly $c \neq 0, 1$ and by the assumption $J \subset [c-\delta, c+\delta]$. Since J contains a periodic orbit of period k then J does not belong to a flat spot of F. Therefore by the definition of a truncation $F(J) \subset G(J)$. At the same time $J \subset [c - \delta, c + \delta]$ and so by the choice of δ we have $G(J) \cap J = \emptyset$ implying that $F(J) \cap J = \emptyset$ which is impossible since J contains a periodic orbit of F. The contradiction completes the proof of the first statement of Proposition 2.6.

2) Assume that $I_f = (0, b], b < 1$ or $I_f = (0, 1)$. If there is an f-periodic orbit which forces more than one fixed point (see the definition in Section 1) then by Lemma 1.5 f has both the left and the right horseshoes. So we may assume that all periodic orbits of f force a unique fixed point. By Proposition 0.1 there is a sequence of periodic points y_i such that $\rho(y_i) \to 0$. Moreover, by Lemma 1.6 we may choose all periodic orbits of y_i to be characteristic. Clearly we may assume that periods of y_i grow to infinity. Also, the fact that $\rho(y_i) \to 0$ implies that there are longer and longer time segments in orbits of y_i such that all the points from the beginning to the end of the time segment are mapped to the right by f. We may assume that $y_i < f(y_i) < \dots < f^{i^2}(y_i)$ for any i. Clearly there is a time segment of the length i within $\{y_i, f(y_i), \dots, f^{i^2}(y_i)\}$ such that the distance between its leftmost and rightmost points is no more than $\frac{1}{i}$; choosing the corresponding points z_i from the orbits of y_i we may assume that in fact for any i we have $z_i < f(z_i) < \dots < f^i(z_i) \le z_i + \frac{1}{q} i$ and also that $z_i \to z$. Clearly $f(z) = z$.

Let us consider possible kinds of local behavior at z. Since f is piecewise monotone then there are only few different types of such behavior; namely, if we choose small left semi-neighborhood U_l and right semi-neighborhood U_r of z then $f(U_l)$ lies either to the right or to the left of z and the same holds for U_r. We may assume that z_i approach z from the right. Indeed, suppose otherwise. If $f(U_l)$ lies to the left of z then clearly none of the points $z_i \in U_l$ is periodic which contradicts the choice of z_i. So $f(U_l)$ lies to the right of z; replacing z_i by $f(z_i)$ we can find the required sequence of periodic points approaching z from the right, thus we may assume it from the very beginning. Then $f(U_r)$ has to lie to the right of z because $z < z_i < f(z_i)$. Let us show that if there is a point $\zeta \in orb\, z_i$ such that $\zeta < z$ then f has the right horseshoe. First let us choose a fixed point z' closest from the left to z_i; then f maps all the points in $(z', z_i]$ strictly to the right. Obviously $z \le z'$, so there is the smallest non-negative j such that $f^j(z_i) > z'$ and $f^{j+1}(z_i) < z'$; clearly $j > 0$ and $f^j(z_i) > z_i$ since otherwise $f^j(z_i) \in (z', z_i)$ while by the choice of z' all the points from (z', z_i) are mapped to the right. Now, let k be the smallest non-negative number such that $f^k(z_i) \ge f^j(z_i)$. Then by the choice of k we have $z' < f^{k-1}(z_i) < f^j(z_i)$, so that in the end we have $f(f^j(z_i)) \le z' = f(z') < f^{k-1}(z_i) < f^j(z_i) \le f(f^{k-1}(z_i))$ which shows that f has the right horseshoe. Hence we may assume that all the orbits of z_i lie to the right of z. Denote the leftmost point of the orbit of z_i by α_i and the rightmost point of the same orbit by β_i.

Let us prove that diameters of the orbits of z_i cannot converge to 0. Indeed, clearly there is at least one turning point of f between the leftmost and the right-most points of $orb\, z_i$ for any i. If diameters of the orbits of z_i converge to 0 then we can refine a sequence of $\{z_i\}$ so that intervals $[\alpha_i, \beta_i]$ approach z and are pairwise disjoint; since every such interval contains a turning point we get a contradiction with the fact that f is piecewise monotone. Let us show now that the fact that diameters of the orbits of z_i do not converge to 0 implies that f has the right horseshoe. Indeed, let $\epsilon > 0$ be such that $diam\,(orb\, z_i) > \epsilon$. All the orbits of z_i are characteristic; thus if z_i'' is the rightmost point of the orbit of z_i then $f(z_i'')$ is the leftmost point of this orbit, and we may assume that z_i'' converge to some point $z'' \geq z + \epsilon$ and $f(z_i'')$ converge to z. Every point z_i'' has the preimage z_i''' in the orbit of z_i such that $z < z_i''' < f(z_i''') = z_i''$ and we may assume that z_i''' converge to some point $z''' \in (z, z'')$ (the fact that $z < z'' < z'''$ easily follows from the continuity arguments). Finally we have $f(z''') = z = fz < z'' < f(z'') = z'''$ which means that f has the right horseshoe completing the proof. $\square$

As an example let us consider *unimodal* maps. For the sake of simplicity and without loss of generality we will assume that there is a point $c \in (0,1)$ such that $f|[0,c]$ is increasing, $f|[c,1]$ is decreasing, $f(c) = 1, f(1) = 0$ and there is no more than one f-fixed point in $[0,c]$. By the unimodality there are no two consecutive iterates mapped to the left in any orbit. Therefore all rotation numbers of periodic orbits are less than or equal to $\frac{1}{2}$; moreover, if we consider the rotation set $I_f(x)$ of an arbitrary point x then as usual $I_f(x)$ is an interval and because of the same reason we see that $I_f(x) \leq \frac{1}{2}$. In any case by Theorem 0.2 the right endpoint of the rotation interval of f is $\frac{1}{2}$; moreover, the union $\bigcup_x I_f(x) = T_f$ in this case contains $\frac{1}{2}$ and lies to the left of $\frac{1}{2}$.

Consider a few possibilities. If there is a fixed point $d \in [0,c]$ then f has the right horseshoe since $0 = f(1) < d = f(d) < c < f(c) = 1$, so the rotation interval I_f of f is $(0, \frac{1}{2}]$. It is easy to see that in this case we have $T_f = [0, \frac{1}{2}]$. Indeed, there is a point x whose orbits stays for longer and longer periods of time in small neighborhoods of d to the right of d; the definition of the set $I_f(x)$ now implies that $I_f(x) = \{0\}$ and so $T_f = [0, \frac{1}{2}]$. If there is no fixed point in $[0,c]$ then f does not have the right horseshoe and the rotation interval $I_f = [\mu, \frac{1}{2}], \mu > 0$ of f coincides with the usual closure of the rotation numbers of f-periodic points. Note that by Theorem 0.2 for any rational number from the interior of the rotation interval there are infinitely many periodic points of different periods and this rotation number, so if I_f is not degenerate then for any finite set of periodic orbits A the rotation interval coincides with the closure in $(0,1)$ of the set of rotation numbers of periodic points from $Per\, f \setminus A$. Moreover, if a unimodal map f has a degenerate rotation interval I_f then by Theorem 0.2 $I_f = \{\frac{1}{2}\}$ and if f has periodic points of infinitely many periods then again the rotation interval coincides with the closure in $(0,1)$ of the set of rotation numbers of periodic points from $Per\, f \setminus A$. Relying upon proposition 1.13 in the proof of its second statement we thus obtain the following

Corollary 2.7. *Let f be a unimodal map with periodic points of infinitely many periods, $A \subset Per\, f$ be a finite set of periodic points. Then the following holds.*

(1) *If there is a fixed point in $[0,c]$ then* $\bigcup I_f(x) = I_f = (0, \frac{1}{2}]$*; moreover, in*

this case $\bigcup_x I_f(x) = T_f = [0, \frac{1}{2}]$ *coincides with the closure of the set of all rotation numbers of periodic points from* $Per\, f \setminus A$.

(2) *If there is no fixed point in* $[0, c]$ *then* $I_f = T_f$ *coincides with the closure of the set of all rotation numbers of periodic points from* $Per\, f \setminus A$.

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