NEW ORDER FOR PERIODIC ORBITS OF INTERVAL MAPS

Alexander Blokh and Michał Misiurewicz

November 7, 1995

Abstract. We propose a new classification of periodic orbits of interval maps via over-rotation pairs. We prove for them a theorem similar to the Sharkovskiĭ Theorem.

1. Introduction

In the theory of discrete dynamical systems, periodic orbits (called also *cycles*) play a very important role. The problem of coexistence of various types of cycles for a given map admits particularly nice answers in dimension one. However, one has to decide what to understand by a "type" of a cycle. For interval maps, two choices have been widely adopted. One is to look only at the period of a cycle. Then the results are very strong. For instance, the following Sharkovskiĭ Theorem holds. To state it let us first introduce the *Sharkovskiĭ ordering* for the set $\mathbb{N}$ of positive integers:

$$3 \succ 5 \succ 7 \succ \dots \succ 2\cdot 3 \succ 2\cdot 5 \succ 2\cdot 7 \succ \dots 2^2\cdot 3 \succ 2^2\cdot 5 \succ 2^2\cdot 7 \succ \dots \succ 8 \succ 4 \succ 2 \succ 1$$

Denote by $\mathrm{Sh}(k)$ the set of all positive integers m such that $k \succ m$, together with k, and by $\mathrm{Sh}(2^\infty)$ the set $\{1, 2, 4, 8, \dots\}$. Denote also by $Per(f)$ the set of periods of cycles of a map f (by a period we mean the least period).

Theorem 1.1 ([S]). *If $f : [0, 1] \to [0, 1]$ is a continuous map, $m \succ n$ and $m \in \mathrm{Per}(f)$, then $n \in \mathrm{Per}(f)$. Therefore there exists $k \in \mathbb{N} \cup \{2^\infty\}$ such that $\mathrm{Per}(f) = \mathrm{Sh}(k)$. Conversely, if $k \in \mathbb{N}\cup\{2^\infty\}$ then there exists a continuous map $f : [0, 1] \to [0, 1]$ such that $\mathrm{Per}(f) = \mathrm{Sh}(k)$.*

Unfortunately, the classification of cycles by period only is very coarse. Another choice has quite opposite features. If we look at the permutations determined by the cycles then the classification is very fine, but the results are much weaker than for periods (see e.g. [ALM]).

1991 *Mathematics Subject Classification.* 58F03, 54H20.

Key words and phrases. Periodic orbits, rotation numbers, interval maps.

The second author was partially supported by the NSF grant DMS-9305899

Recently the third possible choice has been discovered ([B1]-[B3], see also [BK]). It gives better classification than just by periods, and on the other hand, it admits a full description of possible sets of types. Namely, one defines the *rotation pair* of a cycle as (p, q), where q is the period of the cycle and p is the number of its elements which are mapped to the left of themselves (for cycles of period 1 one can take $p = 1/2$, but it is better to exclude them from our considerations). Let us denote the rotation pair of a cycle P by $\mathrm{rp}(P)$ and the set of the rotation pairs of all cycles of a map f by $\mathrm{RP}(f)$. The number p/q is called the *rotation number* of the cycle P.

We introduce the following partial ordering among all pairs of integers (p, q) with $0 < p < q$. We will write $(p, q) \gtrdot (r, s)$ if either $1/2 \le r/s < p/q$, or $p/q < r/s \le 1/2$, or $p/q = r/s = m/n$ with m and n coprime and $p/m \succ r/m$ (notice that $p/m, r/m \in \mathbb{N}$).

Theorem 1.2 ([B1]). *If $f : [0, 1] \to [0, 1]$ is continuous, $(p, q) \gtrdot (r, s)$ and $(p, q) \in \mathrm{RP}(f)$ then $(r, s) \in \mathrm{RP}(f)$.*

This theorem makes it possible to give a full description of the sets of rotation pairs for continuous interval maps (as in Theorem 1.1, all theoretically possible sets really occur), see [B1]. This description is similar to the one for circle maps of degree one (see [M1]).

The aim of this paper is to introduce another notion of rotation pairs. Since the name *rotation pairs* is already reserved, we will call them *over-rotation pairs* (similarly to the rotation pairs they generate *over-rotation numbers*). Compared to rotation pairs, we gain in simplicity of both results and proofs. Moreover, the information we get from rotation and over-rotation pairs is in a general situation different.

In Section 2 we present a definition of over-rotation pairs and discuss this notion from various points of view. In Section 3 we prove the theorem on classification of possible sets of over-rotation pairs. In Section 4 we compare briefly rotation and over-rotation numbers.

2. Definition; three points of view

Let $f : I \to I$ be a continuous map and let P be a cycle (a periodic orbit) of f of period $q > 1$. Let m be the number of points $x \in P$ such $f(x) - x$ and $f^2(x) - f(x)$ have different signs. Then the pair $(m/2, q)$ will be called the *over-rotation pair* of P and will be denoted by $\mathrm{orp}(P)$ and the number $m/(2q)$ will be called the *over-rotation number* of the cycle P. The set of the over-rotation pairs of all cycles of f will be denoted by $\mathrm{ORP}(f)$. Notice that the number m above is even, positive, and does not exceed $q/2$. Therefore in an over-rotation pair (p, q) both p and q are integers and $0 < p/q \le 1/2$.

2.1. First point of view: two dimensions.

We can think of the map f as an action being performed in two dimensions. The interval is being bent (perhaps in many places), stretched or contracted (differently in different places) and put back into itself. This point of view is widely adopted in the study of homeomorphisms of two-dimensional spaces, whether we look at a horseshoe ([H]), Hénon map ([BC]), or the inverse limit of an interval map ([Ba]). The first number in the over-rotation pair shows how many times $f(x)$ goes around x as we move along the orbit (how much the orbit is "twisted"). One can visualize this by imagining a point and its image connected by a piece of elastic string.

In a special case of a map with one fixed point, this picture can be replaced by a point connected by a string to the fixed point. Therefore in this case we count how many times the orbit of our periodic point goes around the fixed point.

In this model one point moves over (and under) another. This motivates our name (*over*-rotation) for the pair.

2.2. Second point of view: rotation theory.

In most of branches of mathematics one of the central problems is how to classify objects that are studied in that branch. Usually classifications make use of invariants, that is characteristics of objects that are the same for isomorphic objects. In the theory of topological dynamical systems the role of an isomorphism is played by a topological conjugacy, so we have to look for invariants of topological conjugacy (for interval maps we can require additionally that the conjugacy has to preserve orientation).

A good example of an invariant is the rotation set for a circle map of degree one. This notion admits far reaching generalizations (see e.g. [Z]). Namely, if X is a compact space, $f \in C(X,X)$ (a continuous map from X to X), $\varphi \in B(X,R)$ (a Borel real function on X), then we can look at the ergodic averages of φ. If they converge at a point x then we call the limit the *rotation number* of x. The set of all rotation numbers for a given map f is the *rotation set* of f. If φ depends on f in such a way that for conjugate systems the function φ is also transported by the same conjugacy, then the rotation set is an invariant.

Some variation of this approach is to look only at the periodic points. For these points rotation numbers always exist. In such a way we arrive at the following ideas.

Let I be a closed interval and let $\Phi : C(I,I) \to B(I,\mathbb{R})$ be an operator. If $\Phi(f)(x) = \Phi(g)(y)$ for all points x and y such that f restricted to the orbit of x is conjugate via an orientation preserving homeomorphism to g restricted to the orbit of y, then we call Φ *combinatorially defined*. In other words, Φ is combinatorially defined if $\Phi(f)(x)$ depends only on the combinatorics of the f-orbit of x: for every $f, g \in C(I,I)$ and $x, y \in I$, if the sign of $f^m(x) - f^k(x)$ is the same as the sign of $g^m(y) - g^k(y)$ for every $m, k \geq 0$ then $\Phi(f)(x) = \Phi(g)(y)$. If we fix n and replace in the above definition $m, k \geq 0$ by $m, k \in [0, n]$, we get Φ *n-combinatorially defined*. If it is n-combinatorially defined for some n, we call it *finitely combinatorially defined*.

Now, if x is a periodic point of f of period n, then $\rho_{f,\Phi}(x) = \dfrac{1}{n}\sum_{i=0}^{n-1} \Phi(f)(f^i x)$ is the *Φ-rotation number of x*, and $(\sum_{i=0}^{n-1} \Phi(f)(f^i x), n)$ is the *Φ-rotation pair of x*. If Φ is combinatorially defined then the sets of all Φ-rotation numbers and of all Φ-rotation pairs of periodic points of a map are invariant with respect to orientation preserving conjugacies.

Clearly there is a lot of combinatorially defined operators. We are interested mainly in those for which the sets of all rotation numbers and of all rotation pairs are relatively easy to compute. There is a situation when such computation seems to be both easy and natural. Namely, if the function $\varphi = \Phi(f)$ is integer-valued (or cohomologous to an integer-valued function ψ, i.e. $\varphi = \psi + \xi - \xi \circ f$ for a bounded function ξ), then both components p and q of the Φ-rotation pair (p, q) of a periodic point x are integers and the Φ-rotation number p/q of x is rational. The property which simplifies the computation of the set of Φ-rotation pairs is the following "completeness" property: if $p/q < r/s < k/m$ and f has periodic points with Φ-rotation numbers p/q and k/m then it has also a periodic point with Φ-rotation number r/s and period s (that is, with Φ-rotation pair (r, s)). Clearly "completeness" implies that the closure of the set of all Φ-rotation numbers of periodic points is connected. Thus, it is important to find combinatorially defined operators that produce sets of rotation pairs with this completeness property.

Rotation numbers mentioned in Section 1 are produced by such operator. It is defined by setting $\Phi_r(f)(x)$ equal to 0 if $f(x) > x$, to 1/2 if $f(x) = x$, and to 1 if $f(x) < x$. It is 1-combinatorially defined. The operator that produces over-rotation numbers is given by $\Phi_{or}(f)(x)$ equal to 1/2 if $(f^2(x)-f(x))(f(x)-x) \leq 0$ and 0 otherwise. It is 2-combinatorially defined. In the case when f has a unique fixed point a, the function $\Phi_{or}(f)$ is cohomologous to the function ψ such that $\psi(x)$ is 1 if $x > a$ and $f(x) < a$, 1/2 if $x \geq a$ and $f(x) = a$, and 0 otherwise (take $\xi(x)$ equal to 1/2 if $x \geq a$ and 0 otherwise).

Hence, proving completeness of the sets of over-rotation pairs is a step in the realization of our program of looking for useful combinatorially defined operators.

2.3. Third point of view: forcing.

According to the general scheme (see e.g. [M2]), we consider the set of all cycles of $f \in C(I, I)$ with various equivalence relations. We get various equivalence classes, determined for instance by period, rotation pair, over-rotation pair, or the cyclic permutation when the points of a cycle are numbered from left to right (in this case we will speak of a *pattern* of a cycle; it is called a *cycle* in [MN] and an *oriented pattern* in [ALM]). For each relation, we say that an equivalence class A *forces* an equivalence class B if every continuous interval map with a representative of A (that is, a cycle belonging to A) has a representative of B. In such a way we get forcing among periods, rotation pairs, over-rotation pairs, and patterns. Then the question of coexistence of various types of cycles can be addressed via investigation of those forcing relations.

Forcing among periods gives us the Sharkovskiĭ ordering. This ordering is linear, so the situation is fairly simple. However, the equivalence classes are very large. For patterns, the equivalence classes are decently small, but forcing gives only a partial ordering, difficult to investigate. For rotation and over-rotation pairs the equivalence classes are not as large as for periods. Forcing is given for them by the relation $\gg$ (this is proved for rotation pairs in [B1]; for over-rotation pairs we prove it in this paper). The difference is that whereas for the rotation pairs forcing is only a partial ordering, for the over-rotation pairs it is a linear one. Thus, using over-rotation pairs we get a situation that is basically not more complicated than the situation for periods, but we dig much deeper into the structure of cycles.

Clearly, cycles with the same pattern have the same over-rotation pairs. Therefore we can speak of over-rotation pairs and over-rotation numbers of patterns.

3. Classification theorem for sets of over-rotation pairs

We want to state our main theorem in a form similar to Theorem 1.1. For this we need notation similar to $\mathrm{Sh}(n)$. Let $\mathbb{M}$ be the set consisting of 0, 1/2, all irrational numbers between 0 and 1/2, and all pairs (α, n), where α is a rational number from $(0, 1/2]$ and $n \in \mathbb{N} \cup \{2^\infty\}$. Then for $\eta \in \mathbb{M}$ the set $\mathrm{Ovr}(\eta)$ is equal to the following. If η is an irrational number, 0, or 1/2, then $\mathrm{Ovr}(\eta)$ is the set of all over-rotation pairs (p, q) with $\eta < p/q \le 1/2$. If $\eta = (r/s, n)$ with r,s coprime, then $\mathrm{Ovr}(\eta)$ is the union of the set of all over-rotation pairs (p, q) with $r/s < p/q \le 1/2$ and the set of all over-rotation pairs (mr, ms) with $m \in \mathrm{Sh}(n)$. Notice that in the latter case, if $n \neq 2^\infty$, then $\mathrm{Ovr}(\eta)$ is equal to the set of all over-rotation pairs (p, q) with $(nr, ns) \gtrdot (p, q)$, plus (nr, ns) itself. Notice also that $\mathrm{Ovr}(1/2)$ is the empty set (this is necessary since we excluded the fixed points).

Theorem 3.1. *If $f : [0, 1] \to [0, 1]$ is a continuous map, $(p, q) \gtrdot (r, s)$ and $(p, q) \in \mathrm{ORP}(f)$ then $(r, s) \in \mathrm{ORP}(f)$. Therefore there exists $\eta \in \mathbb{M}$ such that $\mathrm{ORP}(f) = \mathrm{Ovr}(\eta)$. Conversely, if $\eta \in \mathbb{M}$ then there exists a continuous map $f : [0, 1] \to [0, 1]$ such that $\mathrm{ORP}(f) = \mathrm{Ovr}(\eta)$.*

We devote the rest of this section to the proof of this theorem.

Recall that if (p, q) is an over-rotation pair then we have $0 < p/q \le 1/2$. Therefore the ordering $\gtrdot$ on the set of all over-rotation pairs is a linear ordering and the sets $\mathrm{Ovr}(a)$ are exactly the sets that with each pair (p, q) contain all pairs (r, s) such that $(p, q) \gtrdot (r, s)$. Hence "therefore" in the statement of the theorem is justified.

We divide the proof of Theorem 3.1 into 6 steps.

(1) If f has a cycle with points $x < y$ such that $f(x) < x$ and $f(y) > y$ then $\mathrm{ORP}(f) = \mathrm{Ovr}(0)$.
(2) If f has a cycle with over-rotation number α and $\beta \in [\alpha, 1/2]$ is rational then f has a cycle with over-rotation number β.
(3) If p, q are coprime and f has a cycle with over-rotation number p/q then f has a cycle with over-rotation pair (p, q).
(4) If f has a cycle with over-rotation number α and $\alpha < r/s \le 1/2$ then f has a cycle with over-rotation pair (r, s).
(5) If p, q are coprime, $n \succ m$, and f has a cycle with over-rotation pair (np, nq) then it has a cycle with over-rotation pair (mp, mq).
(6) For every $\eta \in \mathbb{M}$ there is $f \in C(I, I)$ with $\mathrm{ORP}(f) = \mathrm{Ovr}(\eta)$.

We shall call a cycle (and the pattern it represents) *divergent* if it has points $x < y$ such that $f(x) < x$ and $f(y) > y$ (as in (1)). A cycle (pattern) that is not divergent will be of course called *convergent*.

We shall use in the proofs the standard technique of loops of intervals. An interval J *f-covers* an interval K (we write then $J \to K$) if $K \subset f(J)$. If we have a *loop* of intervals $J_0 \to J_1 \to \cdots \to J_{n-1} \to J_0$ then there is a periodic point x such that $f^i(x) \in J_i$ for $i = 0, 1, \dots, n-1$ and $f^n(x) = x$. We will say that the orbit of x is *associated* to the loop. Any piece $J_i \to \cdots \to J_j$ (or $J_j \to \cdots \to J_0 \to \cdots \to J_i$) of the loop will be called a *block*.

Lemma 3.2. *If f has a divergent cycle then* $\mathrm{ORP}(f) = \mathrm{Ovr}(0)$.

Proof. It is proved in [LMPY] (the proof of Lemma 3.1; see also [B1]) that if a cycle P is divergent then there are points $x < y < z$ of P such that $f(x) < x$, $f(y) \geq z$ and $f(z) \leq x$. Then there are fixed points a, b of f such that $x < a < y < b < z$ and $f(t) > t$ for every $t \in (a, b)$. Moreover, there is a point $c \in (a, y)$ such that $f(c) = b$. Set $J = [a, c]$, $K_1 = [c, y]$, $K_2 = [y, b]$, and $L = [b, z]$. Then the interval J f-covers J, K_1, K_2, the interval K_1 f-covers L, the interval K_2 f-covers L, and the interval L f-covers J, K_1, K_2. For an over-rotation pair (p, q) we take the periodic orbit Q associated to the loop $J \to J \to \cdots \to J \to K_1 \to L \to K_2 \to L \to K_2 \to L \to \cdots \to K_2 \to L \to J$ with $q - 2p$ J's followed by a block $K_1 \to L$ and $p - 1$ blocks $K_2 \to L$. This loop passes only once through K_1. The only points of K_1 that belong to other intervals of the loop are c and y, and they clearly do not belong to Q. Therefore the period of Q is equal to the length of the loop, that is q. The points of Q that are mapped to the right and then to the left or vice versa are obtained by starting the loop at K_1, K_2 and L. There are $2p$ of them, so the over-rotation pair of Q is (p, q), as desired. Hence, f has cycles with all possible over-rotation pairs. ■

Remark 3.3. Notice that in the proof above we did not really use the fact that the points x, y, z belong to the same periodic orbit. The same proof works for instance if $x < y < z$, $f(x) = f(z) = x$ and $f(y) = z$ (that is, f has a 2-horseshoe see e.g. [MN]). In particular, for the tent map f (given by $f(x) = 1 - |2x - 1|$) we get $\mathrm{ORP}(f) = \mathrm{Ovr}(0)$. □

The properties (2)-(5) are really about forcing of patterns. To check whether pattern A forces pattern B it is enough to consider a P-linear map f (linear between points of P and constant to the left and to the right of the smallest interval containing P), where P has pattern A. If A is divergent then by Lemma 3.2 properties (2)-(5) hold. Therefore we may assume that A is convergent. In such a case f has a unique fixed point. Thus, when proving (2)-(5) we may assume that $f \in \mathcal{U}$, where $\mathcal{U}$ is the family of all maps from $C(I, I)$ having a unique fixed point (we will always denote this fixed point by a). On the other hand, every cycle of $f \in \mathcal{U}$ is convergent, so once we assume that $f \in \mathcal{U}$, we do not have to assume that P is convergent.

As we already noticed (at the end of Subsection 2.2), if $f \in \mathcal{U}$ then the first element in the rotation pair of a cycle Q is the number of points $x \in Q$ such that $f(x) < a < x$.

If $f \in \mathcal{U}$ then we will use loops made of intervals of a special type. We will call an interval with one of the endpoints equal to a *admissible* (since one of the endpoints of the word "admissible" is "a"). A loop consisting of admissible intervals will be also called *admissible*. Clearly, loops can be concatenated, provided they have a common interval. We will say that a loop has *over-rotation pair* (p, q) if it has length q and there are p blocks $J \to K$ with J to the right of a and K to the left of a. The ratio p/q will be called the *over-rotation number* of the loop.

Lemma 3.4. *Let $f \in \mathcal{U}$. Then for every admissible loop there is a cycle of length at least* 2 *with the same over-rotation number as the loop.*

Proof. It is proved in [B1] that either there is a cycle P of period at least 2 associated to the loop or the intervals in the loop lie alternately to the left and to the right of a and then f has a cycle Q of period 2. In the first case the over-rotation numbers of the loop and the cycle P coincide. In the second case they are both $1/2$, so they also coincide. ■

Let $f \in \mathcal{U}$ and let P be a cycle of period $n \geq 2$ of f. If we denote by J_z the interval with endpoints a and z then for $x \in P$, $J_x \to J_{f(x)} \to \cdots \to J_{f^{n-1}(x)} \to J_x$ is an admissible loop. We will call it the *fundamental admissible loop of P*.

Notice that to determine the over-rotation number of a loop we look only at the blocks of length 2 in the loop (one can think about them as the arrows in the loop). Therefore if we concatenate loops of over-rotation pairs (p, q) and (r, s) then we get a loop of over-rotation pair $(p + r, q + s)$.

Lemma 3.5. *If $f \in \mathcal{U}$ has a cycle with over-rotation number α and $\beta \in [\alpha, 1/2]$ is rational then f has a cycle with over-rotation number β.*

Prof. Let P be a cycle of $f \in \mathcal{U}$ with over-rotation number α and let $\beta \in [\alpha, 1/2]$ be rational. Denote the period of P by n. Then there are non-negative integers r, s such that $(rn\alpha + s)/(rn + 2s) = \beta$. Let x and y be the points of P closest to a from the left and right respectively. Since x is mapped to the right and y to the left, $J_x \to J_y \to J_x$ is an admissible loop. The concatenation of s copies of this loop with r copies of the fundamental admissible loop of P (it exists since J_x appears in both loops) is a loop of over-rotation number β. By Lemma 3.4 f has a cycle of over-rotation number β. ■

Lemma 3.6. *Let p, q be coprime. If $f \in \mathcal{U}$ has a cycle with over-rotation number p/q, then f has a cycle with over-rotation pair (p, q).*

Proof. Let $f \in \mathcal{U}$ and let P be a cycle of the smallest period among the cycles of f with over-rotation number p/q. The over-rotation pair of P is (mp, mq) for some $m \geq 1$. Suppose that $m > 1$.

Consider the fundamental loop of P. This time we compute the first element of the over-rotation pair adding $1/2$ for every arrow that starts and ends on opposite sides of a. Look at such sums for q consecutive arrows of the loop. If we move with our block by one arrow along the loop then this sum can change at most by $1/2$. Since the average sum over such blocks is p, this means that there is a block over which that sum is exactly p. This block starts with some interval J_x and ends with J_y. When we compute the sum over the block, we add $1/2$ each time we move across a. Since this sum is an integer, x and y must lie on the same side of a. Therefore either $J_x \subset J_y$ or $J_y \subset J_x$. Hence, either our block forms a loop of over-rotation pair (p, q) or its complement to the fundamental loop of P forms a loop of over-rotation pair $((m-1)p, (m-1)q)$. This contradicts minimality of period of P. Thus, $m = 1$, so the over-rotation pair of P is (p, q). ■

We say that a cycle P of period $n > 1$ is a *reduction* of a cycle Q of period kn with $k > 1$ (cf. [MN]) if for the Q-linear map f there are pairwise disjoint intervals $K_1, K_2, \dots, K_n$ such that $f(K_i) \subset K_{i+1}$ for $i < n$, $f(K_n) \subset K_1$ and $P \cup Q \subset \bigcup_{i=1}^{n} K_i$. Notice that since K_i are pairwise disjoint and P has period n, there is one element of P and k elements of Q in each K_i. The direction in which the points of Q are mapped is the same as for the corresponding points of P. Therefore if P has over-rotation pair (p,n) then Q has over-rotation pair (kp,kn). Hence the over-rotation numbers of P and Q are equal.

We will call a cycle that has no reduction *irreducible*. Clearly, irreducibility of a cycle depends only on its pattern, so we can speak of *irreducible patterns*.

A cycle has a *division* (see [LMPY]) if either it has period 2 or it has a reduction of period 2. Otherwise a cycle has *no division*. The same terminology is used for patterns. The points along a cycle with a division are mapped alternately to the left and right (as in the proof of Lemma 3.2). Clearly, a pattern with a division has over-rotation number 1/2.

Lemma 3.7. *If $f \in \mathcal{U}$ has a cycle P with over-rotation number α and $\alpha < r/s \le 1/2$ then f has a cycle with over-rotation pair (r,s). The same is true if $\alpha = r/s \le 1/2$ and P is irreducible, unless the over-rotation pair of P is (p,q) with p,q coprime.*

Proof. Let P be a cycle of $f \in \mathcal{U}$ of over-rotation number α and pattern A, and let $\alpha \le r/s$.

Assume first that $r/s < 1/2$. By Lemmas 3.5 and 3.6 (and since for the P-linear map all cycles of this map represent patterns forced by A, see e.g. [ALM]) A forces a pattern B with over-rotation pair (p,q), where $p/q = r/s$ and p,q are coprime. If $\alpha < p/q$, B is not a reduction of A and $B \ne A$. This is also true if P (and therefore A) is irreducible and the over-rotation pair of P is not (p,q). Since p,q are coprime, B is irreducible. Since $p/q < 1/2$, the period of B is larger than 2. Hence the assumptions of Theorem 9.12 of [MN] are satisfied. By this theorem, for every n the pattern A forces a pattern C of period nq such that B is a reduction of C. For $n = s/q$ the pattern C has over-rotation pair (r,s). Hence, there is a representative of C in f and it has over-rotation pair (r,s).

Assume now that $r/s = 1/2$. Then $s = 2r$, so s is even. If $\alpha < 1/2$ then P has no division. If $\alpha = 1/2$, P is irreducible and its rotation pair is not $(1,2)$ then also P has no division. Theorem 2.3 of [LMPY] yields that if f has a periodic orbit with no division then it has a periodic point of period odd and larger than 1. Hence, f has a cycle of an odd period $k > 1$. Since $k \succ s$, f has a cycle of period s. In particular, f has a cycle Q of period s and of pattern that does not force any pattern of period odd and larger than 1. Then, again by the same theorem of [LMPY], Q has a division. Therefore the over-rotation number of Q is 1/2, so the rotation pair of Q is (r,s). ■

Lemma 3.8. *Let p,q be coprime, let $n \succ m$, and let $f \in \mathcal{U}$ have a cycle with over-rotation pair (np,nq). Then f has a cycle with over-rotation pair (mp,mq).*

Proof. Let P be a cycle of f of over-rotation number (np,nq) and pattern A. We may assume that $m \ne n$, so in particular $n > 1$. We may also assume that f is P-linear. If P is irreducible then by Lemma 3.7 f has a cycle of over-rotation pair (mp,mq).

Assume that P is reducible. Then P has an irreducible reduction Q. The over-rotation pair of Q is (kp,kq) for some k. If $k > 1$ then again by Lemma 3.7 f has a cycle of over-rotation pair (mp,mq).

Assume that $k = 1$. Then look at f^q restricted to the interval K_1 from the definition of reduction. This interval is f^q-invariant and $f^q|_{K_1}$ has a cycle $P \cap K_1$ of period n. By

Theorem 1.1 it has a cycle of period m. Then the f-orbit of any point of this cycle is a cycle of f of over-rotation pair (mp, mq). ∎

Now we are ready to prove the first part of Theorem 3.1. As we noticed, because of Lemma 3.2 we may assume that $f \in \mathcal{U}$. Suppose that $(p, q) \gtrdot (r, s)$ and $(p, q) \in \mathrm{ORP}(f)$. If $p/q < r/s$ then f has a cycle of rotation pair (r, s) by Lemma 3.7. If $p/q = r/s$, we get the same conclusion by Lemma 3.8.

To prove the second part of Theorem 3.1, we use the family of truncated tent maps, as in the proof of the second part of the Sharkovskiĭ Theorem in [ALM]. If f_1 is the tent map (see Remark 3.3) then $f_t(x) = \min(f_1(x), t)$ defines the tent map truncated at level t.

Lemma 3.9. *For every $\eta \in \mathbb{M}$ there is $t \in [0, 1]$ such that* $\mathrm{ORP}(f_t) = \mathrm{Ovr}(\eta)$.

Proof. Fix $\eta \in \mathbb{M}$. By Remark 3.3 $\mathrm{Ovr}(\eta) \subset \mathrm{ORP}(f_1)$. For each $\zeta \in \mathrm{Ovr}(\eta)$ there are finitely many cycles of f_1 with over-rotation pair ζ. Therefore there exists the smallest t_ζ such that f_{t_ζ} still has at least one of them. Let t be the supremum of t_ζ over all $\zeta \in \mathrm{Ovr}(\eta)$. Clearly, $\mathrm{Ovr}(\eta) \subset \mathrm{ORP}(f_t)$. Suppose that f_t has a cycle P with rotation pair $\zeta \notin \mathrm{Ovr}(\eta)$. Then P is also a cycle of f_v, where $v = \max P$ is the rightmost point of P (notice that $v \le t$). From the definition of the ordering $\gtrdot$ it follows that there exists $\vartheta \neq \zeta$ such that $\zeta \gtrdot \vartheta$ and $\mathrm{Ovr}(\eta) \subset \mathrm{Ovr}(p/q, n)$, where $\vartheta = (np, nq)$ and p, q are coprime. We know already that this implies that f_v has a cycle Q with over-rotation pair ϑ. Hence if $u = \max(Q)$ then $u < v \le t$ (the inequality between u and v is strict because $Q \neq P$) and $\mathrm{Ovr}(\eta) \subset \mathrm{Ovr}(p/q, n) = \mathrm{ORP}(f_u)$. This contradicts the definition of t. Hence, $\mathrm{ORP}(f_t) = \mathrm{Ovr}(\eta)$. ∎

This completes the proof of Theorem 3.1.

4. Comparison with rotation numbers

We start by an example showing that in a general situation rotation numbers and over-rotation numbers are completely different. We speak of numbers rather than of pairs, since the second component of a pair – the period – remains the same. We can also speak of the rotation and over-rotation sets, defined as the closure of the set of the rotation (respectively over-rotation) numbers of all cycles of the map. By the completeness theorems, those sets are intervals.

Let $I = [1, 5]$ and let f be the P-linear map, where $P = \{1, 2, 3, 4, 5\}$ and $f(1) = 2,\ f(2) = 3,\ f(3) = 5,\ f(4) = 1,\ f(5) = 4$. One can easily check that the over-rotation number of P is $1/5$, whereas the rotation number of P is $2/5$. Moreover, the over-rotation set of f is $[1/5, 1/2]$, but the rotation set of f is $[1/4, 2/3]$.

Nevertheless, there is some connection between over-rotation sets and rotation sets. It is proven in [B1] that every pattern of rotation number α forces a convergent pattern of the same rotation number and such that if P is its representative in the P-linear map $f \in \mathcal{U}$ then f is decreasing on one side of a. For such a pattern it is easy to check that its over-rotation number is $\min(\alpha, 1 - \alpha)$ (in particular, the rotation and over-rotation numbers in unimodal maps coincide).

This has two consequences. The first is that if the over-rotation interval of a map is $[\alpha, 1/2]$ then its rotation set is contained in $[\alpha, 1-\alpha]$. In particular, one can observe that in our example $[1/4, 2/3] \subset [1/5, 4/5]$. This illustrates the fact that the pieces of information about the map contained in the sets $\mathrm{RP}(f)$ and $\mathrm{ORP}(f)$ complement each other. The second consequence is that one can deduce the completeness theorem for rotation pairs of [B1] from our Theorem 3.1.

References

[ALM] L. Alsedà, J. Llibre and M. Misiurewicz, *Combinatorial Dynamics and Entropy in Dimension One*, Advanced Series in Nonlinear Dynamics, vol. 5, World Scientific, Singapore, 1993.

[Ba] M. Barge, *Horseshoe maps and inverse limits*, Pacific J. Math. **121** (1986), 29–39.

[BC] M. Benedicks and L. Carleson, *The dynamics of the Hénon map*, Ann. Math. **133** (1991), 73–169.

[BGMY] L. Block, J. Guckenheimer, M. Misiurewicz, and L.-S. Young, *Periodic points and topological entropy of one dimensional maps*, Global Theory of Dynamical Systems, Lecture Notes in Math., vol. 819, Springer, Berlin, 1980, pp. 18–34.

[B1] A. Blokh, *Rotation Numbers, Twists and a Sharkovskii-Misiurewicz-type Ordering for Patterns on the Interval*, Ergod. Th. & Dynam. Sys. (to appear).

[B2] ———, *Functional Rotation Numbers for One-Dimensional Maps*, Trans. Amer. Math. Soc. **347** (1995), 499–514.

[B3] ———, *On Rotation Intervals for Interval Maps*, Nonlineraity **7** (1994), 1395–1417.

[BK] J. Bobok and M. Kuchta, *X-minimal orbits for maps on the interval*, (preprint).

[H] T. Hall, *Fat one-dimensional representatives of pseudo-Anosov isotopy classes with minimal periodic orbit structure*, Nonlinearity **7** (1994), 367–384.

[M1] M. Misiurewicz, *Periodic points of maps of degree one of a circle*, Ergod. Th. & Dynam. Sys. **2** (1982), 221–227.

[M2] M. Misiurewicz, *Formalism for studying periodic orbits of one dimensional maps*, European Conference on Iteration Theory (ECIT 87), World Scientific, Singapore, 1989, pp. 1–7.

[MN] M. Misiurewicz and Z. Nitecki, *Combinatorial Patterns for Maps of the Interval*, Mem. Amer. Math. Soc. **456** (1990).

[S] A. N. Sharkovskiĭ, *Co-existence of the Cycles of a Continuous Mapping of the Line into Itself*, Ukrain. Math. Jour. **16** (1964), 61–71.

[Z] K. Ziemian, *Rotation sets for subshifts of finite type*, Fundam. Math. **146** (1995), 189–201.

Department of Mathematics, University of Alabama in Birmingham, University Station, Birmingham, AL 35294-2060

E-mail address: ablokh@math.uab.edu

Department of Mathematical Sciences, IUPUI, 402 N. Blackford Street, Indianapolis, IN 46202-3216

E-mail address: mmisiure@math.iupui.edu