ENTROPY AND OVER-ROTATION NUMBERS FOR INTERVAL MAPS

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Abstract. Recently, over-rotation pairs and numbers for periodic orbits of interval maps have been introduced and studied. Here we find minimal entropy of periodic orbits with a given over-rotation number or pair.

1. Introduction

Combinatorial Dynamics deals with properties of dynamical systems having various types of periodic orbits (we will use a shorter name *cycles* for periodic orbits), see e.g. [ALM]. For interval maps, a natural notion of a type of a cycle is its *pattern*, that is the cyclic permutation that we get when we look at how the map acts on the points of the cycle, ordered from the left to the right.

Patterns are partially ordered by the *forcing relation*. A pattern A *forces* pattern B if every continuous map having a cycle of pattern A has a cycle of pattern B. Thus, if we know which patterns are forced by a given pattern A, we have enormous information about the structure of an interval map with a cycle of pattern A. Unfortunately, the forcing relation is rather complicated. Therefore it makes sense to consider notions weaker than pattern. This limits the information we get, but makes it easier to obtain it. One of such notions (the one that in fact started Combinatorial Dynamics, see [S]) is *period*. The Sharkovskiĭ Theorem gives the forcing relation among periods. This is a linear ordering, so the characterization of all the periods forced by the given one is simple. However, knowing only periods of cycles is much less than knowing their patterns.

In [BM2] we proposed a new notion of a type of a cycle, the *over-rotation pair*. If P is a cycle of f of period q, then the over-rotation pair of P is (p, q), where $2p$ is the number of points $x \in P$ such that $f(x) - x$ and $f^2(x) - f(x)$ have different signs (if f has only one fixed point a, then it is easy to check that p is equal to the number

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of points $x \in P$ such that $x > a$ and $f(x) < a$). The number p/q is called the *over-rotation number of* P. It turns out that the forcing relation among over-rotation pairs is also linear. On the other hand, one gets much more information from the over-rotation pair than from the period alone. Therefore we consider over-rotation pairs as an excellent compromise between patterns and periods, and continue their investigation in the present paper.

Note that cycles with the same pattern have the same over-rotation pairs. Thus, we can speak of over-rotation pairs and numbers of patterns.

One of the questions that arise immediately after introduction of a notion of a type of a cycle is how complicated the map has to be if it has a cycle of a given type. The complexity of a map is usually measured by its topological entropy. Thus, the question is: if we have an interval map with a cycle of given pattern, period, over-rotation pair, or over-rotation number, what is the minimal topological entropy of such a map?

For a pattern, the answer is simple (see [BGMY]). A map f is called *P-monotone* if it is monotone between consecutive (in space) elements of P and constant to the left of the leftmost and to the right of the rightmost element of P. All P-monotone maps have the same entropy h that can be easily computed. It is less than or equal to the entropy of any map with a cycle of the same pattern as P. Thus h is the required minimal entropy. We shall refer to it as the *entropy of a cycle* or the *entropy of a pattern*. The paper [BGMY] provides us also with the minimal entropy formula for a given period. Here we address the problem of finding minimal entropy for a given over-rotation pair or number. In fact, the main part of the problem is to find minimal entropy of patterns with a given over-rotation number. Then the result for over-rotation pairs follows easily (see Section 4).

A similar problem has been addressed for rotation numbers in [BM1]. *Rotation pairs* and *numbers* have been introduced in [B] (see also [BK]). They are defined in a way similar to over-rotation numbers, except that the first element of the pair measures how many points of the cycle are mapped to the left by the map. As noticed in [BM1], for *unimodal* maps (i.e. for maps with a unique turning point) rotation and over-rotation numbers of cycles coincide, and the same is true for rotation and over-rotation pairs. Clearly, the notion of a turning point makes sense for patterns, so one can talk of unimodal patterns even without referring to the corresponding maps; in any case it is obvious that rotation and over-rotation numbers (and also rotation and over-rotation pairs) of unimodal patterns are the same.

The main result of this paper is that among patterns of a given over-rotation number minimal entropy is attained on a unimodal one. This, together with the results of [BM1], gives the formula for the minimal entropy. Namely, according to [BM1], the minimal entropy of a cycle with rotation number ϱ is attained on a unimodal cycle and is equal to $\log \lambda(\varrho)$, where $\lambda(\varrho)$ is the unique root of the equation

$$\sum_{\varrho<k/n<1-\varrho} t^{-n} = 1$$

(the summation is over all pairs of natural numbers k, n such that $\varrho < k/n < 1-\varrho$). Thus, the minimal entropy of a cycle with over-rotation number ϱ is also $\log \lambda(\varrho)$.

The paper is organized as follows. In Section 2 we introduce the simplest patterns with a given over-rotation number, *over-twist patterns* and prove their properties that will be useful later. In Section 3, we provide necessary estimates of the entropy of over-twist patterns. In Section 4 we obtain the formula for minimal entropy of patterns with a given over-rotation number and with a given over-rotation pair.

2. Over-twist patterns

If a pattern A forces a pattern B then $h(A) \geq h(B)$. Therefore when we are looking for patterns with minimal entropy among those with over-rotation number ϱ, it is enough to consider those that do not force any other pattern of over-rotation number ϱ. Similarly to twist patterns for rotation numbers, we call them *over-twist* patterns (and accordingly we will speak of *over-twist cycles*). By [BM2], if $p/q < p'/q' \leq 1/2$ then any pattern of over-rotation number p/q forces a pattern of over-rotation number p'/q'. Since the forcing relation is antisymmetric, we see that an over-twist pattern cannot force a pattern of a smaller over-rotation number. In particular, if f is P-monotone and f has a cycle of a smaller over-rotation number than P, then P cannot be over-twist.

Restricting our attention to over-twist patterns allows us to use in the proofs specific properties of those patterns. The rest of this section is devoted to establishing those properties.

Let P be a cycle of a P-monotone map f. As in [BM2] we call P (and its pattern) *convergent* if there are no $x, y \in P$ such that $f(x) < x < y < f(y)$. This is equivalent to f having only one fixed point. We will always denote this fixed point by a. Those points $x \in P$ for which x and $f(x)$ lie on the same side of a are called *green*, and those for which x and $f(x)$ lie on the opposite sides of a are called *black*. The cycle P (and its pattern) is called *green* if it is convergent and f is increasing on the set of green points and decreasing on the set of black points of P.

For a map f with a unique fixed point a it is useful to look at *admissible loops of intervals* $\alpha = J_0 \to J_1 \to \cdots \to J_{q-1} \to J_0$, where $\to$ means f*-covering* (that is, $J \to K$ if $K \subset f(J)$) and every interval J_i has a as one of its endpoints. If p is the number of arrows $J \to K$ in α such that J is to the right of a and K is to the left of a, then we say that the over-rotation number of α is p/q. A cycle Q (other than $\{a\}$) for which there is a point $x \in Q$ with $f^q(x) = x$ and $f^i(x) \in J_i$ for $i = 0, 1, \ldots, q-1$, is *associated* to α. Clearly, its over-rotation number is the same as that of α. By [B] (cf. Lemma 3.4 of [BM2]), if the over-rotation number of α is not $1/2$ then there exists a cycle associated to α.

Let us denote by J_y the interval with endpoints a and y. If P is a cycle of f of period q and $x \in P$ then we call the loop $J_x \to J_{f(x)} \to \cdots \to J_{f^{q-1}(x)} \to J_x$ the *fundamental admissible loop* of P.

To avoid problems with the pattern of period 1 (what is its over-rotation number?; is it green?) and because by [BM2] the only over-twist pattern of rotation number $1/2$ is that of period 2, we consider over-twist patterns only for over-rotation numbers different from $1/2$ (without repeating it explicitly). Anyway, the minimal entropy of patterns with over-rotation number $1/2$ is clearly equal to 0.

Theorem 2.1. *Over-twist patterns are green.*

Proof. Let A be an over-twist pattern. There is a map f and a cycle P of pattern A such that f is P-monotone and there is no other cycle of pattern A (such f is called *P-adjusted*, see e.g. [MN], [ALM]). By [BM2] if A is *divergent* (not convergent) then it forces a pattern of a smaller over-rotation number than that of A, a contradiction. Therefore A is convergent, so f has a unique fixed point (a, as always).

Let α be the fundamental admissible loop of P. Suppose that f is not increasing on the set of green points of P or not decreasing on the set of black points of P. Then (by the symmetry) we may assume that there are points $x, y \in P$ such that $x < y < a$ and either $f(y) < f(x) < a$ or $a < f(x) < f(y)$. We can modify α by replacing an interval J_x by J_y. Then a new loop β is admissible and has the same over-rotation number as α. Since we assume that this number is not $1/2$, f has a cycle Q of the same over-rotation number, associated to β.

Let us show that $Q \neq P$. Suppose that P is associated to β. Let z be the point of P closest to a from the right. Since $z \neq x$, the same interval J_z appears in α and β. Moreover, z is the unique point of P belonging to J_z. Therefore for i such that $x = f^i(z)$, by starting at J_z and following the orbit of z and the loop β for i steps we get $x \in J_y$. This cannot be the case, since $x < y < a$. Hence P is not associated to β, so $Q \neq P$.

Since f is P-adjusted, the pattern of Q is different from A. Thus, A forces a different pattern with the same over-rotation number, so it is not over-twist, a contradiction. This proves that f is increasing on the set of green points of P and decreasing on the set of black points of P, that is P (and therefore A) is green. ■

3. Estimates of entropy

For the estimates of topological entropy from below we will use the standard technique (see e.g. [BGMY]). Let $f : I \to I$ be a continuous interval map. If J_i $(i = 1, \dots, k)$ are intervals with pairwise disjoint interiors then we look at the transition matrix M, that is the $n \times n$ matrix (m_{ij}) such that $m_{ij} = 1$ if J_i f-covers J_j and $m_{ij} = 0$ otherwise. Then $h(f) \geq \log r(M)$, where $r(M)$ is the spectral radius of M.

Now we derive some estimates for the entropy of non-unimodal over-twist patterns.

Lemma 3.1. *Let f be a P-monotone map for a convergent cycle P such that both the leftmost and the rightmost points of P are green. Then $h(f) \geq \log 2$.*

Proof. Denote the leftmost and the rightmost points of P by e_l and e_r respectively. Then there are points $c_l, c_r \in P$ such that $f(c_l) = e_r$ and $f(c_r) = e_l$. Since P is convergent, we have $e_l < c_l < a < c_r < e_r$. Moreover, $f(e_l) < a$ and $f(e_r) > a$. Therefore both intervals $[e_l, c_l]$ and $[c_l, a]$ f-cover intervals $[a, c_r]$ and $[c_r, e_r]$, and both intervals $[a, c_r]$ and $[c_r, e_r]$ f-cover intervals $[e_l, c_l]$ and $[c_l, a]$. The corresponding transition matrix has 2 as an eigenvalue ($\langle 1, 1, 1, 1\rangle$ is an eigenvector). Therefore $h(f) \geq \log 2$. ∎

Lemma 3.2. *If A is a non-unimodal green pattern then $h(A) \geq \log \sqrt{3}$.*

Proof. Let P be a cycle of non-unimodal green pattern A, and let f be a P-monotone map. Look at the distribution (in space) of the green and black points of P. Unimodality means that there is one block of green points and one block of black points. Since P is non-unimodal, this is not the case. Therefore either both the leftmost and the rightmost points of P are green or there is a green point between black ones. In the first case we get by Lemma 3.1 $h(A) = h(f) \geq \log 2 > \log \sqrt{3}$.

Let us consider the second case. There is $k \geq 1$ and consecutive (in space) points $y_0 < y_1 < \cdots < y_k < y_{k+1}$ of P such that y_0 and y_{k+1} are black and $y_1, \dots, y_k$ are green. Since the points of P closest to a are black, all points y_i lie on the same side of a. By symmetry, we can assume that they are to the left of a. Set $z = f(y_{k+1})$. Since the cycle P is green, the right endpoint of the interval $f([y_{k+1}, a])$ is z. Moreover, all the points $y_1, \dots, y_k$ are green, so the right endpoint of $f([y_1, a])$ is also z. The interval $[y_1, z]$ is not invariant (only a part of P belongs to it), so the left endpoint of $f([y_1, z])$ (which is the same as the left endpoint of $f([a, z])$) is to the left of y_1. The closest point of P to y_1 from the left is y_0. Therefore $f([a, z])$ contains $[y_0, a]$. Thus, $[a, z]$ f-covers all three intervals $[y_0, y_k]$, $[y_k, y_{k+1}]$, $[y_{k+1}, a]$. On the other hand, $f(y_k) < a < z = f(y_{k+1}) < f(y_0)$, so each of those three intervals f-covers $[a, z]$. Thus, each of those three intervals f^2-covers each one, so $h(f^2) \geq \log 3$. Therefore $h(A) = h(f) \geq \log \sqrt{3}$. ∎

Lemma 3.3. *Let f be a P-monotone map for an over-twist cycle P of over-rotation number $\varrho < 1/3$. Then there is no point $x \in P$ with either $x < f^3(x) < f(x) < a < f^2(x)$ or $x > f^3(x) > f(x) > a > f^2(x)$.*

Proof. By symmetry, it is enough to consider only the case $x < f^3(x) < f(x) < a < f^2(x)$. Let α be the fundamental admissible loop of P. Consider the loop β obtained by deleting from α the intervals J_x, $J_{f(x)}$ and $J_{f^2(x)}$. Since $x < f^3(x) < a$, β is an admissible loop. If the period of P is q and its over-rotation number is p/q then the over-rotation number of β is $(p-1)/(q-3)$. Since $p/q < 1/3$, we have $(p-1)/(q-3) < p/q$. Therefore f has a cycle of over-rotation number smaller than the over-rotation number of P. Since f is P-monotone, it means that P cannot be over-twist. ∎

Lemma 3.4. *Let $f : I \to I$ be a continuous interval map. Assume that there are points $u, v, w \in I$ such that $u < v < f(v) < w$, $f(u) \geq w$, $f^2(v) \geq w$, and $f(w) \leq u$ (or $u > v > f(v) > w$, $f(u) \leq w$, $f^2(v) \leq w$, and $f(w) \geq u$). Then $h(f) \geq \log 2$.*

Proof. By symmetry it is enough to consider only the first case. Both intervals $[u, v]$ and $[v, f(v)]$ f-cover $[f(v), w]$, while $[f(v), w]$ f-covers all three intervals $[u, v]$, $[v, f(v)]$, and $[f(v), w]$. The characteristic polynomial of the corresponding transition matrix is $\lambda^3 - \lambda^2 - 2\lambda = \lambda(\lambda + 1)(\lambda - 2)$. Therefore the spectral radius of the matrix is 2, so $h(f) \geq \log 2$. ■

Lemma 3.5. *If A is a non-unimodal over-twist pattern with over-rotation number smaller than $1/3$ then $h(A) \geq \log 2$.*

Proof. Suppose that the over-rotation number of a non-unimodal over-twist pattern A is smaller than $1/3$, but the entropy of A is smaller than $\log 2$. Let P be a cycle of pattern A, and let f be a P-monotone map. Then $h(f) < \log 2$. Denote the leftmost and the rightmost points of P by e_g and e_b (without saying which one is which). Since we agreed to forget of over-twist patterns of over-rotation number $1/2$, we know that the period of P is larger than 2. By Theorem 2.1, P is green. Hence, if both points e_g and e_b are black, they form a period 2 cycle, a contradiction. If they are both green then $h(f) \geq \log 2$ by Lemma 3.1, also a contradiction. Therefore one of them is green and one black. We may assume (as the reader could have guessed) that e_g is green and e_b is black. Notice that $f(e_b) = e_g$.

In the following part of the proof the reader is strongly advised to make drawings indicating the ordering of the points involved and where they are mapped by f. Such interactive reading can be much more useful than just looking at the static drawings that we could provide.

Let c be the point of P that is mapped by f to e_b. It is also black and lies on the same side of a as e_g. Let us move backwards along the preimages of c belonging to P until we first meet a green point lying between c and e_b (denote it by d). The existence of such a point follows from the assumption that P is not unimodal. Indeed, suppose that there are no green points between c and e_b. Then since there are no black points between e_g and c (because P is a green cycle) we conclude that P is unimodal, a contradiction. By symmetry we may assume that $d < a$. Then by the choice of d we get that $f(d)$ is black, so $d < f(d) < a < f^2(d)$. Again by the choice of d, if $f^2(d)$ is green then it lies between e_g and c (and thus to the right of c, since d lies between e_b and c). Then by Lemma 3.4 (with $u = e_b$, $v = d$, $w = c$) we get $h(f) \geq \log 2$, a contradiction. Therefore $f^2(d)$ is black, so $f^3(d) < a$. If $f^3(d) > f(d)$ then the interval $[f(d), f^2(d)]$ is invariant, which is impossible. If $d < f^3(d) < f(d)$ then we get a contradiction by Lemma 3.3. It cannot also happen that $f^3(d) = d$ since then P would be of period 3 and over-rotation number $1/3$. Hence $f^3(d) < d$, so we get $f^3(d) < d < f(d) < a < f^2(d)$. Once more by the choice of d, if $f^3(d)$ is green then it lies between e_g and c (and thus to the left of c, since d lies between e_b and c). Then by Lemma 3.4 (with $u = c$, $v = d$, $w = f^2(d)$) we get $h(f) \geq \log 2$, a contradiction. Therefore $f^3(d)$

is black. Since both points $f^3(d)$ and $f(d)$ are black and $f^3(d) < f(d)$, we get $f^2(d) < f^4(d)$. Hence, $f^3(d) < d < f(d) < a < f^2(d) < f^4(d)$. Using Lemma 3.4 for the last time (with $u = f^3(d)$, $v = d$, $w = f^2(d)$), we get as always $h(f) \geq \log 2$, a contradiction. This completes the proof. ■

4. Patterns of minimal entropy

By [BM2], there are unimodal patterns of all over-rotation rational numbers from $(0, 1/2)$. Since the forcing relation restricted to the unimodal patterns with a turning point of the same kind (maximum or minimum) is a linear ordering, there is a unique over-twist unimodal pattern for each over-rotation number (up to symmetry). Therefore (if we identify patterns that differ by symmetry or think only of the unimodal maps that are increasing to the left of the turning point) we can speak of *the* over-twist unimodal pattern of a given over-rotation number.

Theorem 4.1. *Any non-unimodal over-twist pattern has entropy larger than the unimodal over-twist pattern of the same over-rotation number.*

Proof. Let A be a non-unimodal over-twist pattern of over-rotation number $\varrho \neq 1/2$ and let B be the unimodal over-twist pattern of the same over-rotation number. If $\varrho \geq 1/3$ then $h(B) \leq h(C)$, where C is the unimodal pattern of period 3. A simple computation yields $h(C) = \log((1 + \sqrt{5})/2)$. On the other hand, by Lemma 3.2, $h(A) \geq \log \sqrt{3}$. Since $\sqrt{3} > (1 + \sqrt{5})/2$, we get $h(A) > h(B)$. If $\varrho < 1/3$ then by Lemma 3.5, $h(A) \geq \log 2$, whereas $h(B) < \log 2$, so again $h(A) > h(B)$. ■

Let $\lambda(\varrho)$ be the number defined in the introduction. Then we get the following corollary to Theorem 4.1.

Theorem 4.2. *Minimal topological entropy of patterns of over-rotation number ϱ is $\log \lambda(\varrho)$.*

Now we consider the problem of minimal entropy of a pattern with a given over-rotation pair. In order to deal with all cases we introduce the notation λ_n for the largest zero of the polynomial $t^n - 2t^{n-2} - 1$ for $n \geq 3$ odd and $\lambda_1 = 0$. With this notation, the minimal entropy of all patterns of period $n2^k$, where n is odd, is $(1/2^k) \log \lambda_n$ (see [BGMY]).

Theorem 4.3. *Minimal topological entropy of patterns of over-rotation pair (p, q) is $\log \lambda(p/q)$ if $p/q < 1/2$ and $(1/2^k) \log \lambda_n$ if $p/q = 1/2$ and $q = n2^k$ with n odd.*

Proof. Let us deal first with the case of over-rotation numbers smaller than $1/2$. Look at the over-rotation pair (nk, nm) with k, m coprime and $k/m < 1/2$. Let A be the unimodal pattern of over-rotation number k/m and entropy $\log \lambda(\varrho)$. We claim that there exists a unimodal pattern B of period nm that is an *extension* of A (see e.g. [ALM]). This means that if Q is a cycle with pattern B then the points of Q come in m blocks of consecutive (in space) points, each block is mapped onto a block, if we collapse each block into a point then we get a cycle with pattern A, and the map is monotone on all blocks except perhaps one. When we look at the first return map on one block, we get some pattern C and then we say that B is an extension of A by C. Notice that we can construct A from B and C. If both A and C are unimodal, we can choose the non-monotone block to be the one that becomes the turning point after collapsing, and then B will be also unimodal. This proves our claim.

In such a situation we have $h(B) = \max(h(A), (1/m)h(C))$ (see e.g. [ALM]). Since $k/m < 1/2$, we have $h(A) > (1/2)\log 2$, whereas $m > 2$ and $h(C) < \log 2$, so $(1/m)h(C) < (1/2)\log 2$. Therefore $h(B) = h(A) = \log \lambda(k/m)$. Together with Theorem 4.2 this proves our theorem in the first case.

Instead of trying to understand the above construction, the readers familiar with the theory of unimodal maps (see e.g. [CE]) may notice that all this is well known. In particular, the kneading sequence of B is the asterisk product of the kneading sequences of A and C.

Now, if $p/q = 1/2$ then q is even. By [BGMY] (see also [ALM]), minimal possible entropy of a pattern of period q is $(1/2^k)\log \lambda_n$, where $q = n2^k$ with n odd. The cycles with this entropy have *division*, that is if we follow the orbit then the points lie alternately to the left and to the right of the fixed point. Therefore their over-rotation number is $1/2$. This proves the theorem in the second case. ■

Remark 4.4. For every rotation pair there is a *unimodal* pattern realizing minimal entropy. □

Remark 4.5. Since for unimodal maps rotation and over-rotation pairs coincide, Theorem 4.3 is valid also for rotation pairs instead of over-rotation ones. □

References

[ALM] L. Alsedà, J. Llibre and M. Misiurewicz, *Combinatorial Dynamics and Entropy in Dimension One*, Advanced Series in Nonlinear Dynamics, vol. 5, World Scientific, Singapore, 1993.

[BGMY] L. Block, J. Guckenheimer, M. Misiurewicz, and L.-S. Young, *Periodic points and topological entropy of one dimensional maps*, Global Theory of Dynamical Systems, Lecture Notes in Math., vol. 819, Springer, Berlin, 1980, pp. 18–34.

[B] A. Blokh, *Rotation numbers, twists and a Sharkovskii-Misiurewicz-type ordering for patterns on the interval*, Ergod. Th. & Dynam. Sys. **15** (1995), 1–14.

[BM1] A. Blokh, M. Misiurewicz, *Entropy of twist interval maps*, MSRI preprint #41-94 (1994).

[BM2] A. Blokh, M. Misiurewicz, *New order for periodic orbits of interval maps*, preprint (1995).

[BK] J. Bobok and M. Kuchta, *X-minimal orbits for maps on the interval*, preprint.

[CE] P. Collet and J.-P. Eckmann, *Iterated Maps on the Interval as Dynamical Systems*, Progr. in Phys. **1**, Birkhäuser, Boston, 1980.

[MN] M. Misiurewicz and Z. Nitecki, *Combinatorial Patterns for Maps of the Interval*, Mem. Amer. Math. Soc. **456** (1990).

[S] A. N. Sharkovskiĭ, *Co-existence of the Cycles of a Continuous Mapping of the Line into Itself*, Ukrain. Math. Jour. **16** (1964), 61–71.

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