EXPANDING POLYMODIALS

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Dedicated to Anatole Katok on the occassion of his 60th birthday

Abstract. We investigate what happens when we trade analyticity of complex polynomials for expansion properties, similarly as for piecewise expanding maps versus smooth maps in real dimension one. We get a class of maps that we call expanding polymodials. We investigate their properties, that turn out to be similar to those of complex polynomials. After introducing the Julia set and proving its basic properties, we concentrate on the case when the Julia set is connected. The other case, as well as the other aspects of the theory, will be addressed in forthcoming papers.

1. Introduction

Two classes of continuous interval maps that are considered most often are smooth maps and piecewise expanding maps. If we replace the interval by the complex plane, complex polynomials form a class that replaces smooth interval maps. However, we do not know of any attempts of finding a class that would replace piecewise expanding maps. This paper is intended to close this gap in the theory. Since our maps are hybrids between (complex) polynomials and polymodal (in fact, also unimodal) interval maps, we will use the word "polymodials".

Many proofs from the complex dynamics work because of the "miracles" that occur in the theory of holomorphic functions. Since our maps are not holomorphic, we have to compensate by making another strong assumption. Namely, we assume that they are kind of expanding. Thus, we will call our maps *expanding polymodials*. Sometimes expanding properties lead to simpler proofs or stronger results than in the complex dynamics, for instance we get local connectedness of the Julia sets. However, often we miss the holomorphic "miracles". We have to stress that close to singular (critical) points we can have only partial expanding properties, similarly to what happens close to the turning points of piecewise expanding interval maps. Thus, we will not get too strong results, like the existence of Markov partitions.

Expanding polymodials are in particular branched coverings of the Riemann sphere. Such maps have been considered, e.g. in [Pi], [Wh], but from a different perspective.

Since we are describing a large class of maps with many interesting properties and close ties to complex polynomials, the subject is vast. In this paper we only give basic definitions and constructions and prove simple properties, usually analogous to the properties of complex polynomials.

In Section 2 we give definitions and examples. In Section 3 we prove the basic properties of expanding polymodials and their Julia sets (defined here as the set of all points whose orbits are

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bounded). In Section 4 we define systems of external rays, similar to the canonical system of external rays for complex polynomials. Our external rays are much worse for two reasons. First, the construction of the classical external rays uses either the fact that the map is holomorphic or the fact that it is nicely conjugate to $z \mapsto z^d$ at the infinity. In our case we do not have those properties (the conjugacy exists, but is not nice). The second, more important, reason is that we need the rays to be (locally) rectifiable, and this is difficult to achieve with classical methods. Our construction applies to the case when the Julia set of an expanding polymodial is connected and allows us to prove that then it is locally connected.

In Section 5 we use the systems of external rays to produce invariant laminations, much like for complex polynomials. The main difference is that here we may get infinite gaps in laminations (that is, gaps with infinitely many sides). We also formulate several corollaries relating properties of expanding polymodials to known properties of laminations.

2. Definitions and examples

Formally the maps we are defining have nothing to do with the complex structure of the plane. However, quite often the complex notation is better, so we will treat the plane as the complex plane $\mathbb{C}$.

By $D_r(z)$ we will denote the closed disk of radius r centered at z. We say that a set A is *fully invariant* for f if $f^{-1}(A) = A$.

A map $f : \mathbb{C} \to \mathbb{C}$ is an *expanding polymodial* if there is a finite set $S = S(f) \subset \mathbb{C}$ and a constant $\alpha = \alpha(f) > 1$ such that:

(a) f is continuous, not one-to-one, and orientation preserving (meaning orientation preserving on every open set on which it is a homeomorphism on its image),
(b) every point of $\mathbb{C} \setminus S$ has an open neighborhood U such that if $z, w \in U$ then $|f(z) - f(w)| \geq \alpha|z - w|$,
(c) every $\zeta \in S$ has an open neighborhood U such that the closure of U can be covered by finitely many closed sets U_i such that if $z, w \in U_i$ then $|f(z) - f(w)| \geq \alpha|z - w|$,
(d) there is no nonempty fully invariant set contained in S.

Usually the sets U_i from (c) are sectors between two half-lines emerging from ζ, however this is not necessary.

We will always assume that S is the smallest set with property (b) and we will call its elements *singular points*. All other points will be *regular*. The images of singular points are *singular values*, all other points are *regular values*. We use the term “singular” rather than the more popular in this context “critical” in order to disassociate it from the idea that the derivative is zero.

Condition (d) rules out degenerate maps like $z \mapsto 2z^2/|z|$. The assumption that f is not one-to-one rules out homeomorphisms.

Instead of assuming that each application of f expands distances by factor at least α, we could have assumed that for every n the application of f^n expands distances by factor at least $\lambda\alpha^n$ for some $\lambda > 0$. However, this would result in making conditions defining expanding polymodials, as well as proofs, more complicated, without adding any new ideas. This modification can be always made if some interesting applications warrant it.

The simplest examples of quadratic expanding polymodials are *complex tent* maps (or shortly *c-tent* maps), given by the formula

$$f_c(z) = c\frac{z^2}{|z|} + 1,$$

where c is a complex number of absolute value larger than 1. If $-2 \leq c < -1$ is real, then for x real we have $f_c(x) = 1 - (-c)|x|$, that is f_c is the usual tent map of slope $-c$. Thus, c-tent maps are complex versions of tent maps in the same sense as quadratic polynomials are complex versions of logistic maps.

By replacing in the formula for c-tent maps $z^2/|z|$ by $z^n/|z|^{n-1}$ we get expanding polymodials of degree n. However, those maps have only one singular point. Examples of expanding polymodials of degree larger than 2 with more than one singular point can be obtained in the following way. We start from a polynomial P of degree n that has no fully invariant set consisting entirely of critical points. If z_0 is a critical point of P then we compose P in a small neighborhood of z_0 with a diffeomorphism (except at $P(z_0)$) φ that fixes the point $P(z_0)$, is supported on a small neighborhood of $P(z_0)$, and if $|P(z) - P(z_0)|$ is of order $|z - z_0|^k$ then $|\varphi(z) - P(z_0)|$ is of order $|z - P(z_0)|^{1/k}$. Then in a neighborhood of z_0 the quantity $|\varphi(P(z)) - P(z_0)|$ will be of order $|z - z_0|$ (that is, the limit at z_0 of the ratio will be finite and non-zero). We perform this construction for every critical point of P and then multiply the resulting function by complex constant with a sufficiently large absolute value. It is clear that the map obtained in such a way will be an expanding polymodial of the same degree as P and its singular points will be exactly the critical points of P.

We want to study the dynamics of our maps. To this end we define an analogue of the Julia set, which essentially is the non-wandering set in the case of expanding polymodials. This allows us to exploit similarities to complex polynomials, and especially quadratic polynomials. The simplest way to define the Julia set is to define the filled Julia set K as the set of the points with bounded orbits and the Julia set J as its boundary. However, we will see in the next section that they are equal, so we can just define J as the set of the points with bounded orbits.

3. Basic properties of expanding polymodials

We start with proving the simplest properties. In this section we assume that f is an expanding polymodial. In the proofs we will refer to the conditions (a)-(d) from the definition of expanding polymodials. By the *inverse image* of a set B under a map f we mean the set $f^{-1}(B)$ while a *preimage* means one point or one component of the inverse image.

Lemma 3.1. *If f is an expanding polymodial then the following properties hold.*

(1) *If z is a regular point then there is an open neighborhood U of z such that $f|_U$ is an orientation preserving homeomorphism onto its image.*
(2) *For every point $x \in \mathbb{C}$ the set $f^{-1}(x)$ is discrete; in particular, for any bounded set B the set $f^{-1}(x) \cap B$ is finite.*

Proof. (1) By (b), there is an open neighborhood V of z such that $f|_V$ is one-to-one. Take an open neighborhood U of z such that the closure of U is compact and contained in V. Then $f|_{\overline{U}}$ is a homeomorphism onto its image. It is orientation preserving by (a).

(2) Suppose that $f^{-1}(x)$ is not discrete. Then there is a sequence of points $y_i \in f^{-1}(x)$ convergent to some point $y \in \mathbb{C}$. This implies that f is not finite-to-one in any neighborhood of y, while by (b) and (c) every point has a neighborhood in which f is finite-to-one, a contradiction. Clearly this implies that if B is bounded then $f^{-1}(x) \cap B$ is finite. ∎

Lemma 3.2. *Any expanding polymodial is an open map.*

Proof. A map is called *light* if the inverse image of any point is totally disconnected. Let f be an expanding polymodial. By Lemma 3.1(2) the map f is light. It is proven in [BOT] that any light plane map with the totally disconnected set of singular values is open. Now, f has a finite set of singular points, so it has a finite set of singular values, and therefore it is open. ∎

From the definition of an expanding polymodial and Lemmas 3.1 and 3.2 we get immediately the next result.

Lemma 3.3. *The composition of two expanding polymodials, f and g, is an expanding polymodial, provided there is no fully invariant set contained in $S(f \circ g)$. In particular, an iterate of an expanding polymodial is an expanding polymodial.*

Now we start to investigate some implications of the expanding properties of f from (b) and (c). For a curve δ we will refer to a curve δ' such that $\delta = f \circ \delta'$, as a *lifting* of δ.

Lemma 3.4. *Let f be an expanding polymodial and $\alpha = \alpha(f)$. Let $\gamma : [a, b] \to \mathbb{C}$ be a curve of length M. Then the following holds.*

(1) *For every lifting β of γ, the length of β is at most M/α.*
(2) *If $f(z) = \gamma(a)$ then there is a lifting β of γ such that $z = \beta(a)$.*

Proof. We have to estimate the length of β. We use the definition of the length of a curve. Let P be a finite subset of $[a, b]$, such that $a, b \in P$. For every point of $\beta([a, b])$ we choose its neighborhood from (b) or (c). From this cover we choose a finite subcover of $\beta([a, b])$ and a finer finite closed cover. Next we refine this closed cover by covering any closed set contained in a neighborhood from (c) by finite number of subsets, by intersecting it with the sets U_i from (c). Finally, we take the inverse image under β of this refinement. In such a way we get some finite closed cover $\mathcal{A}$ of $[a, b]$.

There exists a finite set $Q \subset [a, b]$, containing P, and such that every two consecutive points of Q belong to the same element of $\mathcal{A}$. We can construct Q from the left to the right, using an obvious "greedy" method. When we start from a point $x \in Q$, we find the closest to x from the right point y of P. Then we can find a set $A \in \mathcal{A}$ such that $x \in A$ and x is not equal to the rightmost point z of $A \cap [x, y]$, and choose z as the next point of Q. Clearly, the construction can be repeated for z, now necessarily with a different set B playing the role of A. Since $\mathcal{A}$ is finite, after a finite number of steps we get to the next point of P.

By the construction of $\mathcal{A}$, if p, q belong to the same element of $\mathcal{A}$ then $\|f(\beta(p)) - f(\beta(q))\| \geq \alpha\|\beta(p) - \beta(q)\|$. Since $f \circ \beta = \gamma$, we get that $\|\beta(p) - \beta(q)\|$ is not larger than the length of $\gamma|_{[p,q]}$ divided by α. Consequently, the length of the broken line joining the consecutive points $\beta(p)$, $p \in P$, is not larger than M/α. This is true for every P, so the length of the curve β is not larger than M/α.

(2) By the condition (b) of the definition of expanding polymodials, f restricted to $\mathbb{C} \setminus S$ is a local homeomorphism. Therefore there is no problem with lifting pieces of γ that are disjoint from $f(S)$. Observe that if $\gamma|_{(c,d)}$ is such maximal piece, each of its liftings δ has finite length and therefore the limits of δ at c and d exist. By continuity, the images of those limits under f are equal to $\gamma(c)$ and $\gamma(d)$ respectively. Thus, we can lift even $\gamma|_{[c,d]}$. We can do it with any prescribed $\delta(e)$ if $e \in (c, d)$ and $f(\delta(e)) = \gamma(e)$. We will show that we can do it even if $e = c$ (or $e = d$).

Suppose that we want to find a lifting δ of $\gamma|_{[c,d]}$ such that $\delta(c) = w$ and $f(w) = \gamma(c)$. Since f is open, we can choose y close to w, such that $f(y) = \gamma(e)$ for some $e \in (c, d)$ that is as close to c as we want. Then we take a lifting δ of $\gamma|_{[c,d]}$ such that $\delta(e) = y$. Then $\delta(c)$ will be as close to y (and therefore also to w) as we want and $f(\delta(c)) = \gamma(c) = f(w)$. However, by Lemma 3.1(2) the inverse image of every point is a discrete set, so we have $\delta(c) = w$.

In view of the above construction, if we try to choose β in the usual "piece by piece" method (that is, if we defined β already on $[a, c]$ for some c then we define it on $[c, d]$ for some $d > c$, etc.), the only problems arise if the set $A = \gamma^{-1}(f(S))$ is not discrete. However, we can deal easily also with this problem. By Lemma 3.1(2), f is finite-to-one on any bounded set. By already proven part (1) of the lemma, in our construction starting from a point z we will never leave the closed disk with radius M/α centered at z. The set $B = f^{-1}(f(S))$ intersected with this disk is finite. The problems start when β reaches a point of B. But then if the parameter $t \in [a, b]$ varies by less than α multiplied by the minimal distance between points of B (call this number ε), we shouldn't be able to get to any other point of B with β. Thus, if two points $s, t \in A$ ($s < t$) are less than ε apart, we simply set

$\beta(t) = \beta(s)$. In the intervals shorter than ε, disjoint form A and with endpoints belonging to A, we choose β using the method described earlier, and we see that such a piece of β is a loop. Thus, we can build β piece by piece, from a to b. ■

The next theorem shows that the map f is not only locally, but globally finite-to-one.

Theorem 3.5. *Let f be an expanding polymodial. Then f is a surjection, and there exists d such that all points have no more than d preimages under f.*

Proof. By Lemma 3.4 we can lift every rectifiable curve starting in a point of $f(\mathbb{C})$, so every point of $\mathbb{C}$ has a preimage under f. Therefore f is a surjection.

Let us consider a circle C bounding a closed disk D. Take a point $z \in C$ and join it by straight segments with all other points of C. Then lift those segments starting from some point $w \in f^{-1}(z)$. By Lemma 3.4 and the assumption that on $\mathbb{C} \setminus S$ the map f is a local homeomorphism, if the distance of w from S is larger than the diameter of C divided by α, we get a set D' such that $f|_{D'}$ is a local homeomorphism from D' onto D. Since D is simply connected, this local homeomorphism must be a homeomorphism, and in particular the component C' of $f^{-1}(C)$ containing w is homeomorphic to a circle.

Let us now assume that C does not pass through any point of $f(S)$. Let C' be a component of $f^{-1}(C)$. Since f restricted to this component is a local homeomorphism and maps C' onto C, the set C' is homeomorphic either to a circle or to the real line. Since any point has only finitely many preimages in any bounded region, in the latter case C' must be unbounded (all the time we are using Lemma 3.4; otherwise we could have "partial" inverse images). However, this contradicts the preceding paragraph. Therefore C' is homeomorphic to a circle. Moreover, the closure D' of the bounded component of $\mathbb{C} \setminus C'$ is homeomorphic to a disk, and since f is open, it is easy to see that $f(D') = D$. Moreover, there are no points of $f^{-1}(D)$ immediately outside C' (remember that f is a local homeomorphism on C'), so D' is a component of $f^{-1}(D)$.

Let us apply this in the following situation. Let E be a closed disk containing S in its interior. Then the set $E \cup f(E)$ is bounded, so it is contained in the interior of a closed disk D, bounded by a circle C. Clearly, C does not pass through any point of $f(S)$, so every component of $f^{-1}(C)$ is homeomorphic to a circle and bounds a component of $f^{-1}(D)$. Since f is open, any boundary point of $f^{-1}(D)$ belongs to $f^{-1}(C)$, so these are all components of $f^{-1}(D)$, unless $f^{-1}(D)$ is the whole plane. However, the latter is impossible, since f is a surjection. Let D', bounded by C' be the component containing E. Now we continuously enlarge the circle C. Let us denote a circle concentric with C and with radius t (larger than the radius of C) by C_t. This construction leads to the continuous "enlargement" of C' (remember that f there is a local homeomorphism), so we get the topological circles C'_t. The union of all C'_t most be equal to $\mathbb{C} \setminus D'$. Otherwise there would be a point $z \in \mathbb{C}$ which is the limit of a sequence of points $z_n \in C'_{t_n}$ with $t_n \to \infty$. This contradicts continuity of f.

In such a way we get a local homeomorphism $f|_{\mathbb{C}\setminus D'} : \mathbb{C} \setminus D' \to \mathbb{C} \setminus D$. Both $\mathbb{C} \setminus D'$ and $\mathbb{C} \setminus D$ are homeomorphic to an annulus, so this map has a finite degree d' (every point has exactly d' preimages). On the other hand, the number of preimages of every point in D is commonly bounded (in view of the conditions (b) and (c) and by the compactness of D) so the number of preimages of every point of $\mathbb{C}$ under f is commonly bounded. ■

By using Lemma 3.4 repeatedly, we get immediately the following lemma.

Lemma 3.6. *If f is an expanding polymodial, $z, w \in \mathbb{C}$, and n is a positive integer, then there exists a point $\zeta \in \mathbb{C}$ such that $f^n(\zeta) = w$ and $|f^i(z) - f^i(\zeta)| \le |f^n(z) - w|/\alpha^{n-i}$ for $0 \le i \le n$.*

It will be convenient to refer to any point ζ satisfying Lemma 3.6 as a *pull back* of w along the orbit segment $z, f(z), \dots, f^n(z)$.

Now we have tools to prove the next properties of expanding polymodials. They basically mean that the infinity is attracting.

Lemma 3.7. *For any expanding polymodial f there is a constant $s \geq 0$ such that for every $z \in \mathbb{C}$ we have $|f(z)| \geq \alpha(|z| - s)$. Moreover, if r is sufficiently large then the set $f^{-1}(D_r(0))$ is contained in the interior of $D_r(0)$.*

Proof. Set $s = \max\{|w| : f(w) = 0\}$. Take $z \in \mathbb{C}$. Let ζ be a pull back of 0 along $z, f(z)$. Then $f(\zeta) = 0$ (and therefore $|\zeta| \leq s$) and $|z - \zeta| \leq |f(z)|/\alpha$. Thus, $|f(z)| \geq \alpha(|z| - s)$.

Now take $r > s\alpha/(\alpha - 1)$ and assume that $|f(z)| \leq r$. Then $|z| \leq s + r/\alpha < r$, so $f^{-1}(D_r(0))$ is contained in the interior of $D_r(0)$. ■

Lemma 3.8. *For any expanding polymodial f there is a number $r > 0$ such that if $|f^k(z)| > r$ for some k then the sequence $(|f^n(z)|)_{n=0}^{\infty}$ grows at least exponentially.*

Proof. By Lemma 3.7, there is $s \geq 0$ such that $|f(z)| \geq \alpha(|z| - s)$ for every $z \in \mathbb{C}$. Take $r > s\alpha/(\alpha - 1)$; then $\alpha(1 - s/r) > 1$. If $|z| \geq r$ then we get $|f(z)|/|z| \geq \alpha(1 - s/r)$. ■

Sometimes it is better to look at an expanding polymodial as a map of the Riemann sphere onto itself.

Theorem 3.9. *An expanding polymodial can be extended to map from the Riemann sphere to itself. Then it is an orientation preserving branched covering of degree larger than 1.*

Proof. Let f be an expanding polymodial. By Lemma 3.7, there is $s \geq 0$ such that $|f(z)| \geq \alpha(|z| - s)$ for every $z \in \mathbb{C}$. Thus, as z goes to infinity, so does $f(z)$. This means that if we set additionally $f(\infty) = \infty$ then f will become a continuous map of the Riemann sphere into itself.

If U is an open neighborhood of the infinity, its complement is a compact subset of the plane, so the image of this complement is a compact set missing the infinity. Therefore $f(U)$ is a neighborhood of the infinity. Together with Lemma 3.2 it shows that f is open on the whole Riemann sphere. By this, Theorem 3.5 and Stoilow's Theorem ([S], [Wh]) it follows that f is an orientation preserving branched covering of degree larger than 1. ■

In the rest of the paper we will denote the degree of the expanding polynomial f by d. Thus, every regular value has d preimages under f.

For an expanding polymodial f we define its *Julia set* $J(f)$ as the set of those points of the plane whose orbits are bounded. Figure 3.1 shows the Julia set for the c-tent map with $c = -0.0989715585843 + 1.1674545084468i$ (for approximately this value of c the singular point 0 is periodic of period 4).

Lemma 3.10. *The Julia set of an expanding polymodial is compact, fully invariant and nonempty.*

Proof. Let f be an expanding polymodial. By Lemmas 3.8 and 3.7, there is $r > 0$ such that $J(f)$ is equal to the intersection of the sets $f^{-n}(D_r(0))$, $n = 0, 1, \dots$, which form a descending sequence of nonempty compact sets. Therefore $J(f)$ is compact and nonempty. By the definition, for any $z \in \mathbb{C}$ we have $z \in J(f)$ if and only if $f(z) \in J(f)$, so $J(f)$ is fully invariant. ■

Lemma 3.11. *The Julia set of an expanding polymodial has empty interior.*

Proof. Let f be an expanding polymodial. Take $z \in J(f)$ and $w \notin J(f)$. By Lemma 3.6, for any n there exists a pull back ζ of w along $z, \dots, f^n(z)$. Then $f^n(\zeta) = w$ and $|z - \zeta| \leq |f^n(z) - w|/\alpha^n \leq M/\alpha^n$, where $M = \sup_{\xi \in J(f)} |\xi - w|$ is finite by Lemma 3.10. Again by Lemma 3.10, $\zeta \notin J(f)$.

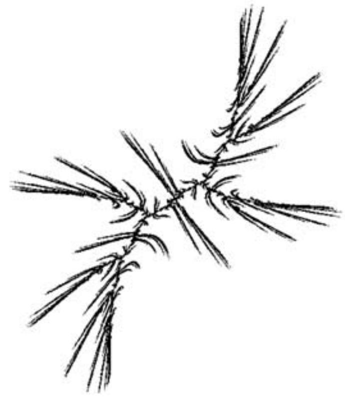

FIGURE 3.1. A Julia set

Thus, arbitrarily close to z we can find points that are not in $J(f)$, so z is not an interior point of $J(f)$. ∎

Lemma 3.12. *The complement of the Julia set of a expanding polymodial is connected.*

Proof. Let f be an expanding polymodial. Suppose that $F = \mathbb{C} \setminus J(f)$ is not connected. Then it has a bounded component U. The image of any component of F is contained in a component of F. If $f^n(U)$ is contained in a bounded component of F for every n then the orbit of every point of U is bounded, a contradiction. Therefore there is a bounded component V of F such that $f(V)$ is contained in the unbounded component W of F. Since the closure of V is compact, $f(V)$ is a bounded set, and therefore it cannot be equal to W. The set W is arcwise connected; indeed, if there are two different components of its arcwise connectivity, then because they are open and disjoint we get a contradiction with the fact that W is connected. Consider a curve from a point of $f(V)$ to infinity contained in W. It will have to exit $f(V)$ at a point $z \in W$ that belongs to the boundary of $f(V)$. On the other hand, since f is an open map, the point z belongs to the image of the boundary of V, which is in turn contained in $J(f)$, and so $z \in J(f) \cap W$, a contradiction. ∎

Lemma 3.13. *Let f be an expanding polymodial. For any $z \in J(f)$ and $\varepsilon, r > 0$ there is N such that if $n > N$ then $D_r(0) \subset f^n(D_\varepsilon(z))$.*

Proof. Let $w \in D_r(0)$ and let n be a positive integer. By Lemma 3.6, there exists a pull back ζ of w along $z, \dots, f^n(z)$. Then $f^n(\zeta) = w$ and $|z - \zeta| \le |f^n(z) - w|/\alpha^n$. Since $J(f)$ is compact and invariant, there is a constant M, independent of n, such that $|f^n(z)| < M$. Thus, $|z-\zeta| < (M+r)/\alpha^n$, so if $n > \log((M+r)/\varepsilon)/\log\alpha$ then $\zeta \in D_\varepsilon(z)$. ∎

Lemma 3.14. *The Julia set of an expanding polymodial has no isolated points.*

Proof. Suppose first that $J(f)$ is finite. Since it is fully invariant and nonempty, by (d) there exists a regular point $z \in J(f)$. If $f(z)$ is a regular value, it has at least 2 preimages, so there is $w \ne z$ such that $f(w) = f(z)$. If it is a singular value, there is $w \in S$ such that $f(w) = f(z)$. Since z is a regular point, also $w \ne z$. In both cases, since $J(f)$ is fully invariant, $w \in J(f)$. One of the points z, w (call it w_0) is not periodic. Since f is a branched covering, it is a surjection, so there are points w_j, $j = 1, 2, \dots$, such that $f(w_j) = w_{j-1}$ for $j = 1, 2, \dots$. Since w_0 is not periodic, $w_j \ne w_k$ if $j \ne k$. Since $J(f)$ is fully invariant, they all belong to $J(f)$, a contradiction. This proves that $J(f)$ is infinite.

Now by Lemma 3.13 and full invariance of $J(f)$, every open set intersecting $J(f)$ contains an infinite subset of $J(f)$. ∎

Lemma 3.15. *Periodic points are dense in the Julia set of an expanding polymodial.*

Proof. Let f be an expanding polymodial. Denote by M the diameter of $J(f)$. Then $|z - f^n(z)| \leq M$ for every $z \in J(f)$ and $n > 0$. Take a point $z_0 \in J(f)$ and fix $n > 0$. Using inductively Lemma 3.6 (that is, pulling back the point z_{k-1} along $z_{k-1}, \dots, f^n(z_{k-1})$) we get points z_k, $k = 1, 2, \dots$, such that $f^n(z_k) = z_{k-1}$ and $|z_k - z_{k-1}| \leq M/\alpha^{kn}$ for $k = 1, 2, \dots$. Then the sequence $(z_k)_{k=0}^{\infty}$ converges to some point w and $|w - z| \leq \sum_{k=1}^{\infty} \alpha^{-kn} = 1/(\alpha^n - 1)$. By taking n arbitrarily large we get w arbitrarily close to z. By continuity of f^n we have $f^n(w) = w$. ∎

4. External rays

As follows from Section 3, studying dynamics of expanding polymodials boils down to studying it on their Julia sets. This is of course analogous to the case of complex polynomials. In that case, a major tool is the so-called *external rays*. They are used in order to foliate the basin of infinity, but are actually applied in a neighborhood of the Julia set to understand better the dynamics on it and to introduce an invariant lamination connected with this dynamics. Complex polynomials are behaving very regularly close to the infinity, and thus for them external rays are constructed starting from the infinity.

In the case of expanding polymodials we do not have much information (other than topological) about the behavior close to the infinity, nor is it necessary to construct the rays from infinity – after all, we are interested in the Julia set. Thus we modify the construction to suit our needs. We start our construction at some circle with a big, but finite, radius. We then take advantage of metric properties of expanding polymodials and construct *rectifiable* rays which allows us to show that rays are landing in a continuous fashion. The price we pay for this is that the external rays are not determined uniquely for general expanding polymodials. However, this turns out to be not so crucial for some major conclusions. To avoid misinterpretation by the readers used to work with the classical external rays, we will call our external rays *exrays*.

In order to be as close as possible to the picture that we know for polynomials, we put some effort into making our exrays pairwise disjoint. This also simplifies some of the proofs later.

Let f be an expanding polymodial whose all singular points belong to the Julia set. The Julia set of f is contained in the disk $D_r(0)$ for some r. Fix $R > r$ such that $f(D_r(0))$ is contained in the interior of $D_R(0)$. Moreover, we assume (according to Lemma 3.7) that $f^{-1}(D_R(0)) \subset D_R(0)$. Let C be the circle of radius R centered at 0.

By Theorem 3.9, f extended to the Riemann sphere is a branched covering. Moreover, the point at infinity is attracting. Therefore, if γ is a simple closed curve bounding a region D containing all singular values, then $f^{-1}(\gamma)$ is a simple closed curve bounding $f^{-1}(D)$. In particular, $f^{-1}(C)$ is homeomorphic to a circle and is contained in $D_R(0)$. On the other hand, it is disjoint from $D_r(0)$, so it is contained in the annulus centered at 0 with radii r and R. Moreover, $f^{-1}(C)$ bounds $f^{-1}(D_R(0))$, and since $D_R(0)$ contains $J(f)$, so does $f^{-1}(D_R(0))$.

We choose an orientation preserving homeomorphism $\varphi : C \to f^{-1}(C)$ so that the map $f \circ \varphi : C \to C$ is conjugate to the map $z \mapsto z^d$ of the unit circle $\mathbb{S}^1$ to itself. In fact, we can choose φ in such a way that $f \circ \varphi$ is equal to any prescribed orientation preserving map $\xi : C \to C$ of degree d. Just go to the universal covers of C and $f^{-1}(C)$ and compose the inverse of the lifting of f with the lifting of ξ. The resulting map is a lifting of an orientation preserving homeomorphism from C to $f^{-1}(C)$, and we choose this homeomorphism as φ.

In the situation described above the following lemma holds.

Lemma 4.1. *Let A be the closure of the region bounded by C from outside and by $f^{-1}(C)$ from inside. Then there exists a homeomorphism $E : C \times [0, 1] \to A$ such that:*

1. *$E(z, 0) = z$ for $z \in C$,*
2. *$E(z, 1) = \varphi(z)$ for $z \in C$,*
3. *the curves γ_z, given by $\gamma_z(t) = E(z, t)$, have lengths bounded by a constant independent of $z \in C$.*

Proof. The main problem is that while we know that $f^{-1}(C)$ is rectifiable, in general it is not smooth, or regular in any other sense. The main step of the construction is to reach $f^{-1}(C)$ with a continuous family of disjoint rectifiable curves. Then the modification of the construction, so that in particular (1) and (2) are satisfied, can be achieved via standard methods, that we will describe later in the proof.

We start by applying the inversion ι (that is, $\iota(z) = 1/z$) to A (see Figure 4.1). Then we denote by B the region bounded by $\iota(f^{-1}(C))$. This region is homeomorphic to a disk. Therefore there exists an analytic univalent map ϱ from the unit disk onto B, such that $\varrho(0) = 0$. Since the curve $f^{-1}(C)$ is rectifiable, so is $\iota(f^{-1}(C))$, and thus, by Lemma 10.5 of [Po], the lengths of the images under ϱ of all radii of the unit disk are commonly bounded. Since ϱ is analytic and $\varrho'(0) \neq 0$, the image under ϱ of the circle C_ε, centered at 0 and with a sufficiently small radius ε, has the property that its distinct points have distinct arguments (to see it, observe that the position vector and the tangent vector at any point of $\varrho(C_\varepsilon)$ are never parallel).

Now we apply ι ($= \iota^{-1}$) back. In such a way we get a homeomorphism $\iota \circ \varrho$ of the closed annulus A_1, centered at 0 and with radii ε and 1, onto the closure A_2 of the region bounded by $f^{-1}(C)$ from inside and $(\iota \circ \varrho)(C_\varepsilon)$ from outside. This region contains $A \setminus f^{-1}(C)$ and any two distinct points of its outer boundary have distinct arguments. Let us call the segments with the constant argument extending from one component of the boundary of A_1 to the other one "radii" (of A_1). Then the images under $\iota \circ \varrho$ of the "radii" of A_1 have commonly bounded lengths. Our main goal is achieved. However, the images of the "radii" of A_1 under $\iota \circ \varrho$ may intersect C in a very complicated fashion. Moreover, (1) and (2) are not satisfied. Therefore we need to amend the construction.

To this end, we choose a circle C_1, centered at 0, of radius smaller than the radius of C and a circle C_2 (also centered at 0) of yet smaller radius, but still so large, that the open disk bounded by C_2 contains $f^{-1}(C)$ (see Figure 4.2). Then we compose $\iota \circ \varrho$ with the continuous map τ that is the identity on the disk bounded by C_2 and maps the pieces of the "radii" of A_2 affinely to the "radii" of the annulus bounded by C_1 and C_2. In other words, τ contracts the "radii" of the annulus bounded by $(\iota \circ \varrho)(C_\varepsilon)$ and C_2 to the "radii" of the annulus bounded by C_1 and C_2. Then $\tau \circ \iota \circ \varrho$ is a homeomorphism of A_1 onto the closure A_3 of the region bounded by $f^{-1}(C)$ from inside and by C_1 from outside. Still the images of the "radii" of A_1 under this homeomorphism have commonly bounded lengths. By precomposing this map with a homeomorphism from $C \times [1/2, 1]$ to A_1 that preserves the "radii", we get E restricted to $C \times [1/2, 1]$. Clearly, we can do it in such a way that (2) is satisfied. We extend it to the whole $C \times [0, 1]$, so that (1) is also satisfied, by taking as the missing parts of the curves γ_z parts of the spirals (the argument is an affine function of the modulus). By the construction, (3) is also satisfied. ∎

The curves γ_z are the initial segments of exrays. Now we continue exrays by taking inverse images of those initial segments. We extend γ_z by attaching to it the preimage of another γ_{z_1} which "starts" where γ_z "ends", then attach a preimage under f^2 of yet another γ_{z_2}, and so on. That is, exrays are curves $\gamma : [0, \infty) \to \mathbb{C}$ such that for any nonnegative integer n and $t \in [0, 1]$ we have $f^n(\gamma(n + t)) = \gamma_z(t)$ for some z depending on n, but not on t (of course γ is continuous).

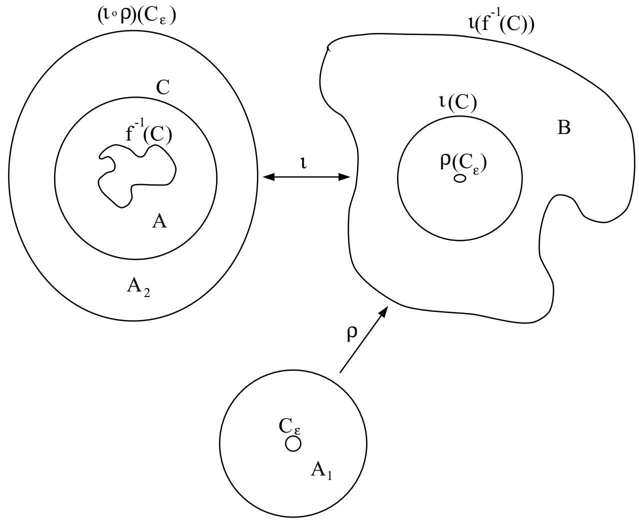

FIGURE 4.1. The maps ι and ϱ

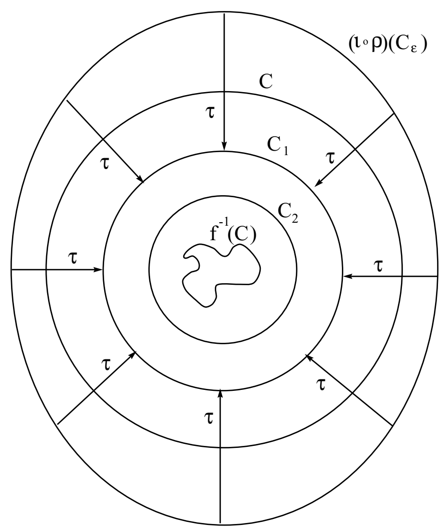

FIGURE 4.2. Modifications

To describe it more formally, consider the map $f \circ \varphi : C \to C$. By the construction it is conjugate to the map $z \mapsto z^d$ on the unit circle $\mathbb{S}^1$. That is, there exists an orientation preserving homeomorphism $\psi : \mathbb{S}^1 \to C$ such that for $z \in \mathbb{S}^1$

$$\psi(z^d) = f(\varphi(\psi(z))). \tag{4.1}$$

Note that (for a given φ) this ψ is unique up to the multiplication of z by the $(d-1)$-st roots of 1 because it is defined once the value of ψ at 0 is chosen.

Now for $t \in [0, 1]$ we define $G : \mathbb{S}^1 \times [0, 1] \to A$ (where A is as in Lemma 4.1) by

$$G(z, t) = E(\psi(z), t). \tag{4.2}$$

Next we extend it inductively for all $t \in [0, \infty)$ by the formula

$$f(G(z, t)) = G(z^d, t - 1). \tag{4.3}$$

To check that this is possible, we have to establish consistency, that is to show that (4.3) holds (with the definition (4.2)) for $t = 1$. We have by (4.1), (4.2), and Lemma 4.1 (1) and (2)

$$f(G(z, 1)) = f(E(\psi(z), 1)) = f(\varphi(\psi(z))) = \psi(z^d) = E(\psi(z^d), 0) = G(z^d, 0),$$

so, indeed, our definition of G is correct.

The map $G : \mathbb{S}^1 \times [0, \infty) \to \mathbb{C}$ that we get in such a way is a homeomorphism onto its image if restricted to any set $\mathbb{S}^1 \times [n, n+1]$. Moreover, the images of those sets are pairwise disjoint, except the boundaries. Thus, the whole G is a homeomorphism onto its image. For any $z \in \mathbb{S}^1$ the map $G(z, \cdot)$ is the exray starting at the point $\psi(z)$, with the initial segment $\gamma_{\psi(z)}$. We will refer to G as a *system of exrays*.

Theorem 4.2. *All exrays land uniformly. In other words, the map* $G : \mathbb{S}^1 \times [0, \infty) \to \mathbb{C}$ *can be extended to a continuous map on* $\mathbb{S}^1 \times [0, \infty]$.

Proof. By the construction, the lengths of the initial segments of exrays are commonly bounded. By Lemma 3.4, the lengths of their n-th preimages are uniformly bounded by constants that decrease exponentially with n. Therefore all exrays land. Moreover, for a given ε there is N such that for every exray γ the length of $\gamma|_{[N,\infty)}$ is less than $\varepsilon/3$. A sufficiently small change of z causes the change of $\gamma(N)$ by less than $\varepsilon/3$, so it causes the change $\lim_{t\to\infty} \gamma(t)$ by less than ε. ∎

By the above theorem, the *limit map* $L : \mathbb{S}^1 \to \mathbb{C}$, given by $L(z) = G(z, \infty)$, is continuous, and by (4.3),

$$f(L(z)) = L(z^d) \tag{4.4}$$

for $z \in \mathbb{S}^1$. In the next lemma we will prove that $L(\mathbb{S}^1)$ is the Julia set of f. Together with (4.4) it implies that L semiconjugates the map $z \mapsto z^d$ with $f|_{J(f)}$.

Lemma 4.3. *We have* $G(\mathbb{S}^1 \times [0, \infty)) = D_R(0) \setminus J(f)$ *and* $L(\mathbb{S}^1) = J(f)$.

Proof. Since $J(f)$ is fully invariant and $G(\mathbb{S}^1 \times [0, 1])$ is disjoint from it, so is $G(\mathbb{S}^1 \times [0, \infty))$. Moreover, $G(\mathbb{S}^1 \times [0, 1]) \subset D_R(0)$ and $f^{-1}(D_R(0)) \subset D_R(0)$. Therefore $G(\mathbb{S}^1 \times [0, \infty)) \subset D_R(0) \setminus J(f)$. On the other hand, for any point $z \in D_R(0) \setminus J(f)$ there is n such that $|f^n(z)| > R$. Let n be the smallest such number. Then $f^{n-1}(z)$ lies in the closure of the region bounded by C and $f^{-1}(C)$, so it belongs to $G(\mathbb{S}^1 \times [0, 1])$. Therefore $z \in G(\mathbb{S}^1 \times [0, \infty))$. This proves that $G(\mathbb{S}^1 \times [0, \infty)) = D_R(0) \setminus J(f)$.

By Lemma 3.11, $J(f)$ is contained in the closure of $D_R(0) \setminus J(f)$. The set $G(\mathbb{S}^1 \times [0, \infty])$ is compact as the image of a compact set under a continuous map, contains $D_R(0) \setminus J(f)$, and thus it contains $J(f)$. Therefore $J(f) \subset L(\mathbb{S}^1)$. On the other hand, for an exray γ we have $\gamma(t) \in f^{-n}(D_R(0))$ if $t > n$. Thus, the landing point w of γ belongs to the intersection of the sets $f^{-n}(D_R(0))$. By our choice of R, we have $f^{-1}(D_R(0)) \subset D_R(0)$, so this intersection is equal to $J(f)$. Thus $w \in J(f)$. This proves that $L(\mathbb{S}^1) = J(f)$. ∎

A plane continuum which is locally connected (and hence arcwise connected, see e.g. [Ku]), does not separate the plane and has empty interior, is called a *dendrite*. Our results yield the following corollary.

Corollary 4.4. *If all singular points of an expanding polymodial f belong to $J(f)$, then $J(f)$ is a dendrite.*

The proof of the kind of converse to the above corollary is simple.

Lemma 4.5. *If for an expanding polymodial f its Julia set is connected, then all its singular points belong to $J(f)$.*

Proof. Suppose that $c \notin J(f)$ is a singular point of f. Then the trajectory of c escapes to the infinity. Since $\mathbb{C} \setminus J(f)$ is open and connected, the point $f(c)$ can be connected with ∞ by a simple curve not passing through any other singular point and disjoint from $J(f)$. We can also find a simple curve β, disjoint from γ (except the endpoint), not passing through any other singular point, and joining $f(c)$ with a point of $J(f)$. If f is k-to-1 in a neighborhood of c, then the inverse images of γ and β will contain k curves each. All of them start from c, every other one goes to ∞, and every other one to some point of $J(f)$ (here we use the fact that $J(f)$ is fully invariant). Moreover, those curves are pairwise disjoint, except at c. This means that there are points of $J(f)$ in each of k sets into which the Riemann sphere is divided by $f^{-1}(\gamma)$. Since $f^{-1}(\gamma)$ is disjoint from $J(f)$, this shows that $J(f)$ is disconnected. ■

While there are many different systems of exrays, the limit map L is basically unique. More precisely, it is unique up to the composition with the rotations by $(d-1)$-st roots of 1. In order to see that we will make an additional construction, interesting by itself. While it can be considered a version of the Caratheodory theory of prime ends (see, e.g., [Ma]), we have a slightly different approach, that produces simple proofs in our special case.

Consider the set $\mathcal{H}$ of all rectifiable curves beginning at a point of $J(f)$ and otherwise disjoint from $J(f)$. In $\mathcal{H}$ we introduce a pseudometric ϱ. Namely, if $\beta, \gamma \in \mathcal{H}$, then $\varrho(\beta, \gamma)$ is the infimum of the lengths of curves δ, disjoint from $J(f)$, whose some initial part coincides with some initial part of β and some initial part of the curve δ^{-1}, obtained from δ by reversing time, coincides with some initial part of γ. Clearly, ϱ is a pseudometric, that is, it is symmetric and satisfies the triangle inequality.

There are distinct elements of $\mathcal{H}$ whose distance ϱ is zero, and therefore ϱ is not a metric. We will refer to such curves as *equivalent*. Now we may factor $\mathcal{H}$ by this equivalence relation and we get a space that we will denote by $\mathcal{G}$. In this space ϱ is a metric.

If G is a system of exrays then every exray $G(z, \cdot)$ can be treated as an element of $\mathcal{H}$ and therefore we get a natural map $M : \mathbb{S}^1 \to \mathcal{G}$. If two elements of $\mathcal{H}$ are equivalent, they have the same landing point. Therefore we can speak about the "landing point" map $\Lambda : \mathcal{G} \to J(f)$. With this notation, we have $L = \Lambda \circ M$. Let us also recall that C is the big circle used in the construction of G.

Lemma 4.6. *The maps M and Λ are continuous.*

Proof. Let $z, w \in \mathbb{S}^1$, take the curves $G(z, \cdot), G(w, \cdot) : [0, \infty] \to \mathbb{C}$, and consider them as elements of $\mathcal{H}$. Then choose a positive integer n and define a curve γ as follows. It starts at $L(z)$, then follows $G(z, t)$, as t runs from ∞ to n (that is, backwards), then follows the shorter part of $f^{-n}(C)$ to $G(w, n)$ (call this part β_n), and finally follows $G(w, t)$ as t runs from n to ∞. It is a curve of the type that is used to define ϱ. Thus, $\varrho(G(z, \cdot), G(w, \cdot))$ is not larger than the sum of the lengths of $G(z, [n, \infty])$, $G(w, [n, \infty])$, and β_n. The first two numbers decrease exponentially with n (this estimate is uniform in z, w). The same is true for the length of β_n, as long as f^n is 1-to-1 on β (the image is contained in C). However, by (4.3), we know that f^n is 1-to-1 on β_n if the difference of arguments of z and w is smaller than $2\pi/d^n$. Thus, as the distance between z and w goes to 0, we can take larger and larger n, and $\varrho(G(z, \cdot), G(w, \cdot))$ also goes to 0. This proves that M is continuous.

The map Λ is Lipschitz continuous with constant 1 by the definition. ■

Remark 4.7. From the above proof it follows that M, and therefore also L, is Hölder continuous with the exponent $\log\alpha(f)/\log d$. $\square$

Proposition 4.8. *The map M is a homeomorphism from $\mathbb{S}^1$ onto $\mathcal{G}$.*

Proof. Since M is continuous and $\mathbb{S}^1$ is compact, it is enough to prove that M is a bijection. We start by proving that it is 1-to-1.

Let $z, w \in \mathbb{S}^1$, $z \neq w$, and suppose that $M(z) = M(w)$. Then $L(z) = L(w)$. Let A and B be the two arcs of $\mathbb{S}^1$ joining z and w. There is n such that the images of both A and B under the n-th iterate of the map $\zeta \mapsto \zeta^d$ are equal to the whole $\mathbb{S}^1$. If $L(A) = \{L(z)\}$ or $L(B) = \{L(z)\}$ then by (4.4) the whole circle $\mathbb{S}^1$ would be mapped by L to one point. This contradicts Lemma 4.3. Therefore there are points $u \in A$ and $v \in B$ such that $L(u) \neq L(z)$ and $L(v) \neq L(z)$. For $\zeta \in \mathbb{S}^1$, denote the exray $G(\zeta, [0, \infty))$ (treated as a subset of $\mathbb{C}$) by K_ζ. The exrays K_u and K_v, together with $J(f)$, divide the disk $D_R(0)$ (used to construct G) into two sets, one of them containing K_z, the other one containing K_w. Therefore any curve joining a point of K_z with a point of K_w and disjoint from $J(f)$ has to intersect either K_u or K_v. Thus, $\varrho(G(z,\cdot), G(w,\cdot))$ cannot be smaller than twice the distance of $L(z)$ from either K_v or K_u. Both of those distances are positive, so $\varrho(G(z,\cdot), G(w,\cdot)) > 0$, a contradiction. This proves that M is 1-to-1.

Now we show that M is onto. Take a curve $\gamma \in \mathcal{H}$. If we start its parametrization from 0 (that is, $\gamma(0) \in J(f)$), we can find a sequence $t_n \to 0$ such that $\gamma(t_n) \in f^{-n}(C)$. Let $\beta_n = G(z_n, \cdot)$ be the external ray passing through $\gamma(t_n)$. Then, by the definition of ϱ and Theorem 4.2, $\varrho(\beta_n, \gamma) \to 0$ as $n \to \infty$. From the sequence (z_n) we can choose a subsequence convergent to some $z \in \mathbb{S}^1$. Since M is continuous, the curves $G(z,\cdot)$ and γ are equivalent. This shows that M is onto. $\blacksquare$

Theorem 4.9. *Let G_1, G_2 be systems of exrays for an expanding polymodial f of degree d and let $L_1, L_2 : \mathbb{S}^1 \to \mathbb{C}$ be the corresponding limit maps. Then there is λ such that $\lambda^{d-1} = 1$ and $L_1(z) = L_2(\lambda z)$ for all $z \in \mathbb{S}^1$.*

Proof. Let $M_1, M_2 : \mathbb{S}^1 \to \mathcal{G}$ be the maps corresponding to G_1, G_2 respectively. Let z_1, z_2 be such elements of $\mathbb{S}^1$ that $M_1(z_1) = M_2(z_2)$. Let γ_i be the exray $G_i(z_i,\cdot)$ for $i = 1, 2$. Then γ_1 and γ_2 are equivalent. We want to show that $f \circ \gamma_1$ and $f \circ \gamma_2$ are also equivalent. Since we even do not know whether an image under f of an arbitrary rectifiable curve is rectifiable, we cannot do it directly from the definition.

Let us take a large t and then $z_3 \in \mathbb{S}^1$ such that the exray $\gamma_3 = G_1(z_3,\cdot)$ passes through $G_2(z_2, t)$. By making t sufficiently large, we can make $\varrho(\gamma_3, \gamma_2)$ as small as we wish. Since $M_1(z_1) = M_2(z_2)$ and $\varrho(\gamma_1, \gamma_2)$, we can make in this way $|z_3 - z_1|$ as small as we wish. Since we know that the images under f of small pieces of the curves γ_i close to $J(f)$ have still small length, we can make $\varrho(f\circ\gamma_3, f\circ\gamma_2)$ as small as we wish. Finally, M_1 is continuous, so $\varrho(f\circ\gamma_3, f\circ\gamma_1)$ is also arbitrarily small. This proves that $\varrho(f\circ\gamma_1, f\circ\gamma_2) = 0$.

In view of (4.3), what we proved means that there is a map $F : \mathcal{G} \to \mathcal{G}$ such that $F(M_i(z)) = M_i(z^d)$ for $i = 1, 2$. Thus, $M_2^{-1} \circ M_1$ conjugates $z \mapsto z^d$ with itself. It is also easy to see that $M_2^{-1} \circ M_1$ preserves the orientation. Therefore there exists λ such that $\lambda^{d-1} = 1$ and $M_2^{-1}(M_1(z)) = \lambda z$ for all $z \in \mathbb{S}^1$. Thus, $M_1(z) = M_2(\lambda z)$, and since $L_i = \Lambda \circ M_i$, we get $L_1(z) = L_2(\lambda z)$, for all $z \in \mathbb{S}^1$. $\blacksquare$

Remark 4.10. It follows from Theorem 4.9 that there are $d-1$ possible limit maps. If from a system of exrays G we get one of them, L, it is easy to modify G to get all others. Namely, with the same C and φ, if the initial segments of exrays go around $f(C)$ one more time (this is easy to realize by the methods from the proof of Lemma 4.1), this causes the change of the point of intersection of each exray with $f^n(C)$ by d^{-n+1} of the full turn for $n = 2, 3, \ldots$. Therefore the shift of the landing point is by $1/(d-1)$ of the full turn. $\square$

5. Laminations

In this section we will again assume that the expanding polymodial f has all singular points in the Julia set. A major tool in studying polynomial maps on their Julia sets is a specific equivalence relation on the unit circle connected with the polynomial and called a *lamination*. Mimicking the construction of laminations for complex polynomials, we construct a lamination for an expanding polymodial f in this section of the paper. We start with the definitions.

Two pairs (z_1, z_2) and (w_1, w_2) of points of $\mathbb{S}^1$ are called *linked* if the chord joining z_1 with z_2 intersects the chord joining w_1 with w_2 inside the unit disk and *unlinked* otherwise. It is natural to say that a family of pairwise unlinked chords "laminates" the circle. Extending this, we can say, that basically, a lamination is a closed equivalence relation on the circle such that the convex hulls of equivalence classes "laminate" the open disk. More precisely, we define a lamination as follows (we follow [D], [McM] and [BL1, BL2], cf. [Th]).

An equivalence relation ℓ on the unit circle $\mathbb{S}^1$ will be called a *lamination* if it satisfies the following properties (for clarity, we will use notation $z \sim_\ell w$ rather than $z\ell w$):

(L1) ℓ is *closed*: the graph of ℓ is a closed set in $\mathbb{S}^1 \times \mathbb{S}^1$;
(L2) ℓ is *unlinked*: if $z_1 \sim_\ell z_2$ and $w_1 \sim_\ell w_2$, but $z_1 \not\sim_\ell w_1$, then the pairs (z_1, z_2) and (w_1, w_2) are unlinked;
(L3) each equivalence class of ℓ is totally disconnected.

Note that a closed subset of a circle is totally disconnected if and only if it has empty interior. If we fix a lamination then we will refer to its equivalence classes simply as *classes*. A class that consists of exactly two points is called a *leaf*, while a class that consists of at least three points is called a *gap* (cf. [Th]).

Geometrically, if a class is a leaf, we connect its elements by a chord; if it is a gap, we draw only chords connecting consecutive (in the circular order) points. In such a way, in the latter case we get a polygon or a generalized polygon with infinitely many sides (if the equivalence class is uncountable, there will be points of this class not belonging to any side, so it is a very generalized polygon). While the names "leaf" and "gap" come from the geometric representation of a lamination given above, the reader has to remember that they really refer to subsets of $\mathbb{S}^1$. Our definitions are closer to [D], [McM] than the original geometric definitions of Thurston [Th].

Fix an integer $d > 1$ and denote the map $z \mapsto z^d$ of $\mathbb{S}^1$ onto itself by σ_d. For a subset $A \subset \mathbb{S}^1$ and $z, w \in A$ we will say that w *follows* z in A if the arc from w to z in the counterclockwise direction does not contain any other point of A. We will call a lamination ℓ *invariant* (or σ_d*-invariant*) if

(L4) for every class A the set $\sigma_d(A)$ is also a class;
(L5) if A is a class and w follows z in A then $\sigma_d(w)$ follows $\sigma_d(z)$ in $\sigma_d(A)$.

Property (L5) is called "consecutive preserving" in [K2] and is equivalent to "covering with positive orientation" of [BL1, BL2] (cf. [Th]).

The limit map L (there are $d - 1$ of them, so we choose one), defined in the preceding section, gives us a lamination $\ell = \ell(f)$, similarly as for the complex polynomials. That is, we define an equivalence relation ℓ in $\mathbb{S}^1$ by setting $z \sim_\ell w$ if and only if $L(z) = L(w)$. Note that by Theorem 4.9 this equivalence relation is independent of the choice of the limit map L. By (4.4), the map L semiconjugates σ_d with $f|_{J(f)}$. Therefore the Julia set $J(f)$ is homeomorphic to the quotient space $\mathbb{S}^1/\ell$.

Theorem 5.1. *For an expanding polymodial f with all singular points in its Julia set, the relation $\ell(f)$ is an invariant lamination.*

Proof. Since our relation is given by a continuous map, it is automatically closed. Therefore (L1) holds.

To prove that (L2) holds, assume that $L(z_1) = L(z_2) \neq L(w_1) = L(w_2)$ and show that the pairs (z_1, z_2) and (w_1, w_2) are not linked. We will use the same notation as in the preceding section. Additionally, we will denote by $\mathcal{E}(z)$ the exray (in a geometrical sense) corresponding to z, that is the set $G(\{z\} \times [0, \infty))$. If (z_1, z_2) and (w_1, w_2) are linked, then the sets $\mathcal{E}(z_1) \cup \mathcal{E}(z_2) \cup \{L(z_1)\}$ and $\mathcal{E}(w_1) \cup \mathcal{E}(w_2) \cup \{L(w_1)\}$ must have a nonempty intersection. Since different exrays are disjoint and they are disjoint from $J(f)$, while $L(z_1), L(w_1) \in J(f)$, this is impossible.

If (L3) does not hold then there is a whole arc of points of $\mathbb{S}^1$ with the same image under L. The image of such an arc under some iterate of σ_d would be the whole circle, so by (4.4) the set $L(\mathbb{S}^1)$ would consist of one point. This contradicts Lemma 4.3. Therefore (L3) holds.

Now we prove (L4) and (L5). Since we know by Theorem 3.9 that f is a branched covering, every singular point z has its *order*, that is a number $n > 1$ such that in some small neighborhood of z the map f acts (in a possibly distinct maps around z and $f(z)$) like $\zeta \mapsto \zeta^n$. If z is a regular point then the same is true for $n = 1$, so we may say that the order of z is 1.

By (4.4), if $z \sim_\ell w$ then $\sigma_d(z) \sim_\ell \sigma_d(w)$. Hence, the image of a class is contained in a class.

Let us take $z \in J(f)$ and look at the exrays landing at z and $f(z)$. Let n be the order of z. Assume first that only finitely many exrays land at z and $f(z)$. If we go around z in the counterclockwise direction, we see consecutively exrays $\mathcal{E}(z_1), \mathcal{E}(z_2), \dots, \mathcal{E}(z_k)$ landing at z, and around $f(z)$ exrays $\mathcal{E}(w_1), \mathcal{E}(w_2), \dots, \mathcal{E}(w_j)$ landing at $f(z)$ (we may assume that $\sigma_d(z_1) = w_1$). Since in the small neighborhoods of z and $f(z)$ the map f looks like $\zeta \mapsto \zeta^n$, and locally images and preimages of exrays are exrays (by (4.3)), we have $k = nj$ and $\mathcal{E}(z_1)$ is mapped to $\mathcal{E}(w_1)$, $\mathcal{E}(z_2)$ to $\mathcal{E}(w_2)$, ..., $\mathcal{E}(z_{j+1})$ (if $n > 1$) again to $\mathcal{E}(w_1)$, etc. This proves simultaneously (L4) and (L5) in this case. If infinitely many rays land at z or $f(z)$, the same must be true for the other point, and by the similar arguments (the map f looks like $\zeta \mapsto \zeta^n$, and locally images and preimages of exrays are exrays) (L4) and (L5) hold. ■

Thus, we see that ℓ is an invariant lamination. In fact, the situation is very similar as for the complex polynomials. The main difference is that the gaps may be infinite.

Let us define the *valence* of a point $z \in J(f)$ as the number of components of $J(f) \setminus \{z\}$. It may happen that the connected Julia set of an expanding polymodial has a point of infinite valence.

Example 5.2. Let f be the c-tent map with $c \in (-2, -1)$ real and such that the singular point 0 is periodic. A concrete example is $c = -(\sqrt{5} + 1)/2$, when 0 is periodic of period 3. Then $[1/(c + 1), -1/(c + 1)]$ is the maximal invariant interval on the real line. It is contained in the Julia set of f. Since $1/(c + 1) < 1 < -1/(c + 1)$, a neighborhood of the singular value 1 on the real line is contained in $J(f)$. Since $J(f)$ is fully invariant, the inverse image of this neighborhood is also contained in $J(f)$. However, the inverse image of this neighborhood is homeomorphic to a tree containing 0, and 0 has valence 4 in this tree. Now we take next n inverse images of this tree, where n is the period of 0, and get a tree contained in $J(f)$, containing 0, and in which 0 has valence 8. By continuing this process we get trees contained in $J(f)$, containing 0, and in which 0 has valence 2^k for $k = 4, 5, 6, \dots$. Since $J(f)$ is simply connected, this means that the valence of 0 in $J(f)$ is infinite. In Figure 5.1 we show the Julia set of the c-tent map with $c = -(\sqrt{5} + 1)/2$ in a small neighborhood (of diameter approximately $1/250$) of 0. □

Let us study the points of infinite valence in more detail. To do so we need the following result obtained for laminations in [K1] and independently in [BL1, BL2]. By a *preperiodic class* we mean a class whose orbit under σ_d is finite; so this notion includes also periodic classes.

Theorem 5.3. *For a σ_d-invariant lamination any non-preperiodic class has no more than 2^d elements.*

Theorem 5.3 implies that any infinite class must be preperiodic. Let us study periodic infinite classes of ℓ. We need the following well-known folklore lemma.

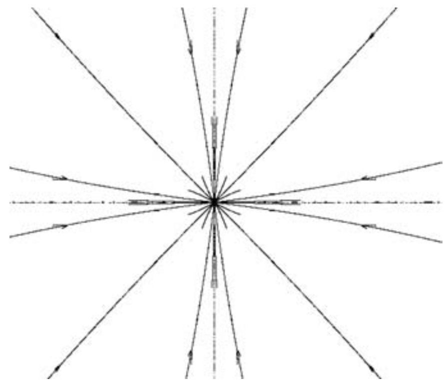

FIGURE 5.1. Point of infinite valence

Lemma 5.4. *Let g be a homeomorphism of a compact metric space K onto itself, for which there exist $\beta > 1$ and $\varepsilon > 0$ such that for every two points x, y with $d(x, y) \leq \varepsilon$ we have $d(g(x), g(y)) \geq \beta d(x, y)$. Then K is finite.*

Proof. The space K is compact, so the inverse homeomorphism g^{-1} is uniformly continuous. Therefore there is $\delta > 0$ such that if $d(x, y) < \delta$ then we have $d(g^{-1}(x), g^{-1}(y)) < \varepsilon$, so $d(g^{-1}(x), g^{-1}(y)) \leq d(x, y)/\beta$. There is a finite set $E \subset K$ which is δ-spanning, that is, for every $x \in K$ there is $y \in E$ such that $d(x, y) \leq \delta$. Then for every n the set $g^{-n}(E)$ has the same cardinality as E and is (δ/β^n)-spanning. If $K \neq E$, then there is a subset E_1 of K of larger cardinality than E. If γ is the minimal distance between distinct elements of E_1 then we can take n such that $2\delta/\beta^n < \gamma$. This gives us a contradiction, since each ball centered at an element of $g^{-n}(E)$ with radius δ/β^n contains at most one element of E_1. This proves that K is finite. ∎

Let us apply Lemma 5.4 to our laminations.

Proposition 5.5. *Let A be an infinite periodic class of period m of the lamination ℓ. Then $\sigma_d^m|_A$ is k-to-1 for some $k > 1$.*

Proof. Observe that σ_d-invariant laminations are also invariant for σ_d^m and for σ_d^m the class A is fixed. By (L4), $\sigma_d^m(A) = A$, and by (L5), $\sigma_d^m|_A$ is k-to-1 for some $k \geq 1$. It remains to show that $k > 1$.

Suppose that $k = 1$. Since $\sigma_d^m|_A$ is continuous and onto A, and A is compact, $\sigma_d^m|_A$ is a homeomorphism. Clearly, σ_d^m extends the distance between two sufficiently close points by the factor of d^m. Thus, by Lemma 5.4, A is finite, a contradiction. ∎

Observe that the above proposition applies not only to our lamination ℓ, but to any σ_d-invariant lamination. Let us now go back to the study of the lamination ℓ generated by the limit map L.

Proposition 5.6. *For $z \in J(f)$ the valence of z is equal to the number of elements of $L^{-1}(z)$.*

Proof. Assume first that $L^{-1}(z)$ has n elements for some finite n. The set $L^{-1}(z)$ divides $\mathbb{S}^1$ into n arcs $A_1, \dots, A_n$. The union of exrays landing at z, together with z itself, divides the disk $D_R(0)$ into corresponding n parts $B_1, \dots, B_n$. Exrays corresponding to the elements of A_i are contained in B_i and are landing there. Therefore there are points of $J(f)$ in each of the sets B_i, and consequently, the valence of Z in $J(f)$ is at least n. If the set $L^{-1}(z)$ is infinite, then the same argument, together with (L3), show that the valence of z is infinite. Thus, in all cases, the valence of z is larger than or equal to the number of elements of $L^{-1}(z)$.

Suppose now that the valence of z is strictly larger than the number of elements of $L^{-1}(z)$. Then $L^{-1}(z)$ is finite. As before, let $A_1, \dots, A_n$ be the arcs into which $L^{-1}(z)$ divides $\mathbb{S}^1$. Then there is i such that A_i is the union of several, but more than one, nonempty sets of the form $L^{-1}(B) \cap A_i$, where B is a component of $J(f) \setminus \{z\}$. Those sets are pairwise disjoint and open (since the components of $J(f) \setminus \{z\}$ are open and L is continuous). This means that A_i is not connected, a contradiction. This completes the proof. ■

Remark 5.7. Both "valence" and "number of elements" recognize only one infinity. However, in Example 5.3 the set of components of $J(f) \setminus \{0\}$ is countable, while $L^{-1}(0)$ is homeomorphic to the Cantor set. □

Now Example 5.2 and Proposition 5.6 show that there are expanding polymodials with laminations having infinite gaps.

Consider the image of the Lebesgue measure on $[0, 1]$ under the map $t \mapsto e^{2\pi i t}$. We will refer to it as the Lebesgue measure on $\mathbb{S}^1$. Of course this measure is invariant and ergodic with respect to the map σ_d.

Proposition 5.8. *Let μ be an ergodic probability measure on $\mathbb{S}^1$, invariant with respect to the map σ_d, and with the support equal to the whole circle. Then the union of all gaps of ℓ has measure μ zero. In particular, the Lebesgue measure of the union of all gaps of ℓ is zero.*

Proof. In the geometrical representation of ℓ a gap corresponds to a subset of the unit disk with nonempty interior, and two distinct gaps are disjoint. Therefore there are at most countably many gaps. Thus, it is enough to prove that if A is a gap then $\mu(A) = 0$.

Suppose that A is a gap of positive measure. Let A_n ($n = 0, 1, 2, \dots$) be the image of A under the n-th iterate of σ_d. Then $\mu(A_0) \le \mu(A_1) \le \mu(A_2) \le \dots$. If the sets A_n are pairwise disjoint, the measure of their union is infinite. Therefore they are not pairwise disjoint. Since ℓ is invariant, for some n we get $A_{n+k} = A_n$ with $k > 0$. Then the set $B = \bigcup_{j=n}^{n+k-1} A_j$ is closed, invariant and has positive measure. Since μ is ergodic, it has full measure. Therefore the support of μ is contained in B, so $B = \mathbb{S}^1$. This contradicts (L3). ■

Let us now briefly mention several properties of the dynamical system $f|_{J(f)}$ (with f being an expanding polymodial) which follow from certain known facts about topological dynamics of laminations. In doing so we rely upon the fact that $f|_{J(f)}$ is conjugate to the factor of σ_d on the quotient space $\mathbb{S}^1/\ell$. By [BL1, BL2], this factor map has no wandering continua (a set A is said to be *(φ-)wandering* if $\varphi^n(A) \cap \varphi^m(A) = \emptyset$ for any two non-negative integers $n \ne m$). Thus, the following theorem holds.

Theorem 5.9. *If f is an expanding polymodial whose singular points belong to its Julia set $J(f)$ then $f|_{J(f)}$ has no wandering continua.*

Let us now discuss the so-called *backward stability* for expanding polymodials. Out interest in this property is motivated by the fact that it is important for the dynamics in general and has some nice consequences (e.g. in studying Milnor attractors, [BM]). The definition is analogous to the Lyapunov stability but applies to backward orbits under a given map rather than to forward orbits (see [L], [BM], [BO]). Namely, let T be a metric continuum. We say that $f : T \to T$ is *backward stable* if for any δ there exists ε[1] such that for any continuum K with $\mathrm{diam}(K) \le \varepsilon$, any $n \ge 0$ and any component M of $f^{-n}(K)$, $\mathrm{diam}(M) \le \delta$ (such components are called *pull-backs*). By [BO], the factor map of σ_d on the quotient space $\mathbb{S}^1/\ell$ of an invariant lamination ℓ is backward stable. We conclude that the following holds.

Theorem 5.10. *If f is an expanding polymodial then $f|_{J(f)}$ is backward stable.*

[1] After all, it is *backward* stability

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