JULIA SETS OF EXPANDING POLYMODIALS

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Abstract. We continue studying branched covering maps of the plane with expanding properties, which we call *expanding polymodials*. They are analogous to piecewise expanding interval maps and have properties similar to those of complex polynomials (in particular, the Julia set and a lot of other notions from complex dynamics can be defined for expanding polymodials). In this paper we include the case when the Julia set is disconnected, study its topological properties and its Hausdorff dimension.

1. Introduction

A number of papers and even books (see [ALM] or [BC]) are devoted to one-dimensional dynamics, i.e. to studying continuous maps of one-dimensional (branched) manifolds (interval, circle, trees, graphs). However, with the exception of these maps, the dynamics of arbitrary continuous maps of manifolds is not extensively studied. This is quite understandable, because continuity puts little restriction on maps of spaces of dimension higher than 1, and to be able to achieve substantial results such restrictions should be introduced. These restrictions could be of smooth or topological nature, or both. The topological restrictions so far have been almost exclusively represented by the assumption that the map is a homeomorphism, or at least a local embedding. In other words, there are few studies of the dynamics of maps of manifolds with *singular points* (i.e. points at which the map is not a local homeomorphism) in dimension higher than 1, e.g. on the plane.

Of topological classes of such maps of the plane, branched covering maps are perhaps the most likely choice for attempts to develop a consistent theory of their dynamics. In fact, one class of such maps is studied in great detail for a lot of time by now. Of course we mean complex maps such as polynomials and rational functions. Clearly, it is only possible to study them, because on top of being branched covering maps they have an extremely powerful machinery of holomorphic functions behind them. A reasonable question then is whether other properties can serve as a replacement for the analytic properties of rational functions.

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The analogy with interval maps is instructive here. Namely, two most widely studied classes of continuous interval maps are smooth maps (loosely, they are similar to complex polynomials) and piecewise expanding maps. Thus, one can hope to define maps of the plane which would be similar to piecewise expanding interval maps and develop dynamics for them. In the paper [BCM] we suggest one such class, namely the *expanding polymodials*. Essentially, they are branched covering maps, which are locally expanding at all regular points and locally expanding within wedges at singular points. This paper is continuation of [BCM], where the main focus was on studying expanding polymodials for which the Julia set (which can be defined for them) is connected. Here however we concentrate on the case when the Julia set is disconnected, including its topological properties and estimates of its Hausdorff dimension.

Let us describe shortly what we do in this paper. In Section 2 we recall the definition of expanding polymodials and their basic properties from [BCM]. In Section 3 we introduce so-called *extrees* similar to *geometric coding trees* (see, e.g., [P], [PUZ], [PS]). In Section 4 we study topological entropy and invariant measures for arbitrary expanding polymodials. In Section 5 we study topological properties of the Julia set and show that if f is an expanding polymodial then there exists a dendrite containing its Julia set $J(f)$. This study is continued in Section 6 where we consider in detail non-trivial components of $J(f)$ and prove that such a component is always (pre)periodic and contains a singular point of f in its orbit. In Section 7 we provide estimates for the Hausdorff dimension of the Julia set (independently of whether it is connected or not). To do this, we have to impose some additional regularity assumptions on the map.

2. Definition and basic properties

Let us recall the definition of expanding polymodials and their basic properties from [BCM].

By $D_r(z)$ we will denote the closed disk of radius r centered at z. We say that a set A is *fully invariant* for f if $f^{-1}(A) = A$.

A map $f : \mathbb{C} \to \mathbb{C}$ is an *expanding polymodial* if there is a finite set $S = S(f) \subset \mathbb{C}$ and a constant $\alpha = \alpha(f) > 1$ such that:

(a) f is continuous, not one-to-one, and orientation preserving (meaning orientation preserving on every open set on which it is a homeomorphism on its image),
(b) every point of $\mathbb{C} \setminus S$ has an open neighborhood U such that if $z, w \in U$ then $|f(z) - f(w)| \geq \alpha|z - w|$,
(c) every $\zeta \in S$ has an open neighborhood U such that the closure of U can be covered by finitely many closed sets U_i such that if $z, w \in U_i$ then $|f(z) - f(w)| \geq \alpha|z - w|$,
(d) there is no nonempty fully invariant set contained in S.

We will always assume that S is the smallest set with property (b) and we will call its elements *singular points*. All other points will be *regular*. The images of singular points are *singular values*, all other points are *regular values*.

The simplest examples of quadratic expanding polymodials are *complex tent* maps (or shortly *c-tent* maps), given by the formula

$$f_c(z) = c\frac{z^2}{|z|} + 1,$$

where c is a complex number of absolute value larger than 1.

In [BCM] we introduce some basic notions and prove the following basic results about expanding polymodials.

Theorem 2.1. *An expanding polymodial can be extended to a map from the Riemann sphere to itself. Then it is an orientation preserving branched covering of degree larger than 1.*

Lemma 2.2. *The composition of two expanding polymodials, f and g, is an expanding polymodial, provided there is no fully invariant set contained in $S(f \circ g)$. In particular, an iterate of an expanding polymodial is an expanding polymodial.*

For a curve δ we will refer to a curve δ' such that $\delta = f \circ \delta'$, as a *lifting* of δ.

Lemma 2.3. *Let f be an expanding polymodial and $\alpha = \alpha(f)$. Let $\gamma : [a, b] \to \mathbb{C}$ be a curve of length M. Then the following holds.*

1. *For every lifting β of γ, the length of β is at most M/α.*
2. *If $f(z) = \gamma(a)$ then there is a lifting β of γ such that $z = \beta(a)$.*

It follows that if $z, w \in \mathbb{C}$, and n is a positive integer, then there exists a point $\zeta \in \mathbb{C}$ such that $f^n(\zeta) = w$ and $|f^i(z) - f^i(\zeta)| \le |f^n(z) - w|/\alpha^{n-i}$ for $0 \le i \le n$.

Lemma 2.4. *For any expanding polymodial f, if r is sufficiently large then the set $f^{-1}(D_r(0))$ is contained in the interior of $D_r(0)$.*

The Julia set $J(f)$ of an expanding polymodial f is the set of those points of $\mathbb{C}$ whose trajectories are bounded.

Theorem 2.5. *Let f be an expanding polynomial. Then:*

1. *$J(f)$ is compact, fully invariant, nonempty, has empty interior and has no isolated points;*
2. *$J(f)$ is connected if and only if it contains all singular points of f;*
3. *periodic points are dense in $J(f)$;*
4. *the complement of $J(f)$ is connected;*
5. *for any $z \in J(f)$ and $\varepsilon, r > 0$ there is N such that if $n > N$ then $D_r(0) \subset f^n(D_\varepsilon(z))$.*

3. External trees

Parallel to exrays (external rays) introduced in [BCM], we introduce here a similar tool, *geometric coding trees* (see, e.g., [P], [PUZ], [PS]). In order to unify terminology, we will rather call them *extrees*. They work more or less like exrays, with several modifications. One extree replaces not one exray, but a system of exrays. They work equally well in the cases of connected and disconnected Julia set. For a system of exrays, if $J(f)$ is connected then f restricted to its Julia set is a factor of the map $z \mapsto z^d$ of the unit circle onto itself. With the extree, this map is replaced by the full one-sided d-shift. However, on the Cantor set which is the phase space of this shift there is an order, preserved by the shift.

An additional advantage of extrees, as compared to the systems of exrays, is that it is easier to make the whole (open and infinite) tree embedded into $\mathbb{C}$ than to make exrays embedded and disjoint. We will start with lemmas that prepare this embedding. As always, we assume that f is a expanding polymodial of degree d. Similarly as in [BCM], we can take a circle C centered at 0, that bounds an open disk D such that $J(f) \cup S(f) \subset f^{-1}(D) \subset D$ (such D exists by Lemma 2.4). We denote by B the interior of $D \setminus f^{-1}(D)$. When we speak about curves, we will mean sometimes their parameterizations, and sometimes subsets of $\mathbb{C}$.

The first lemma is well known, but for completeness we include its simple proof.

Lemma 3.1. *If U is an open connected subset of $\mathbb{C}$ then every two points of U can be connected by a rectifiable curve.*

Proof. Let $z \in U$ and let V be the set of those points of U that can be connected to z by a rectifiable curve. Then clearly both V and $U \setminus V$ are open. Since U is connected and $z \in V$, we get $V = U$. ■

Lemma 3.2. *Let $z \in C$ and let $f^{-1}(z) = \{z_1, \dots, z_d\}$. Then there exist disjoint (except for the common endpoint z) rectifiable simple curves $\gamma_1, \dots, \gamma_d$ contained in B (except for the endpoints), such that γ_j joins z with z_j for $j = 1, \dots, d$.*

Proof. Take $\varepsilon > 0$ and let $\beta(t) = (1+t)z$ for $t \in [0, \varepsilon]$. If ε is sufficiently small, then the preimage of the curve β under f consists of disjoint rectifiable curves $\beta_1, \dots, \beta_d$ contained in B (except for their endpoints $z_1, \dots, z_d$), such that β_j joins z_j with some point $w_j \in B$ for $j = 1, \dots, d$.

The point z can be joined by a straight line segment β_1' contained in B (except for the endpoint z) with some point $w_1' \in B$. By Lemma 3.1, there is a rectifiable curve β_1'', joining w_1 with w_1', and contained in $B \setminus \bigcup_{j=2}^d \beta_j$. The concatenation of the curves β_1', β_1'' and β_1 joins z with z_1. It is rectifiable, and in order to make it simple we use the standard procedure of cutting off the loops. In such a way we get a desired curve γ_1.

Now we repeat the above construction with the index 1 replaced by 2 and the set $B \setminus \bigcup_{j=2}^d \beta_j$ replaced by $B \setminus \left(\bigcup_{j=3}^d \beta_j \cup \gamma_1\right)$. In such a way we get γ_2. We continue, and in order to construct γ_k we replace $B \setminus \bigcup_{j=2}^d \beta_j$ in the construction by the component of $B \setminus \left(\bigcup_{j=k+1}^d \beta_j \cup \bigcup_{j=1}^{k-1} \gamma_j\right)$ containing w_k. Therefore the curves $\gamma_1, \dots, \gamma_d$ are also pairwise disjoint (except for the common endpoint z). ■

Let us call a point $z \in \mathbb{C}$ *fully regular* if there are no $k, n \geq 0$ such that $f^k(z) \in f^n(S(f))$. There are only countably many points that are not fully regular.

Take a fully regular point z that is not in the Julia set. Moreover, we assume that $|z|$ is so large, that z belongs to a circle C satisfying our assumptions. Using Lemma 3.2, we join z with its preimages by disjoint (except for the common endpoint z) rectifiable simple curves contained in B (except for the endpoints). It may happen that some of those curves pass through points that are on the forward trajectories of the singular points. Then we have to modify such a curve in such a way that it does not pass through those "bad" points. This can be done in a simple way, since there are finitely many singular points and every trajectory can pass through B only once. Therefore we can do it while preserving all properties of those curves, listed above. We call those (modified) curves *edges of the first generation.* The preimage under f^n of an edge of the first generation is the union of d^n disjoint curves (except for some curves having common endpoints), which we call *edges of the* $(n+1)$*-st generation.* The union of the edges of all generations is our *extree.*

If $k \neq n$ then $f^{-k}(B)$ is disjoint from $f^{-n}(B)$, so the edges of different generations may intersect only at the endpoints and only if the numbers of generations differ by 1. The valence of those points of intersection (which are the inverse images of z under the iterates of f; we call those points the *vertices* of the extree) is $d+1$. We have one incoming edge of generation n and d outcoming edges of generation $n+1$. Moreover, the cyclic order of the germs of those edges is the same as of their images. We can speak just about the order of the outcoming germs in the neighborhood of a vertex minus the germ of the incoming edge (as always, we use the counterclockwise direction, so if we visualize the incoming germ as coming from above and outcoming ones below the vertex, the order will be from the left to the right). The map f^n sends this vertex to z and is an orientation preserving homeomorphism in a neighborhood of the vertex. The image of the incoming edge comes from the complement of D; the images of the outcoming edges go into D, and their order is preserved by f^n. Note that we do not count z as a vertex, we will call it the *root* of the extree instead. Additionally we define the *generation* of a vertex w as the number n such that $f^n(w) = z$.

The readers that dislike the fact that we visualize the extree (at least locally) as growing downwards instead of upwards, can think of it as growing at the Antipodes. In fact, the situation is more complicated and our extrees grow in all directions. Figure 3.1 shows an extree for the c-tent map with $c = 1 + i$ (the singular point does not belong to the Julia set in this example).

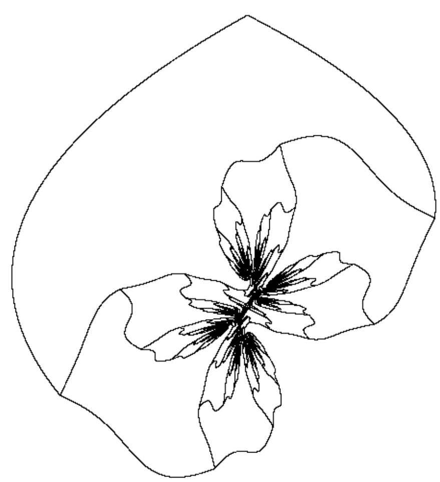

FIGURE 3.1. An extree

Let us call the curves (open at one endpoint) obtained by taking one edge from each generation and concatenating them, the *branches* of the extree. Of course in order to get a curve (which, by the definition, is continuous), the chosen branches from the generations n and $n+1$ have to have a common vertex. By the same argument as for the exrays (see [BCM]), the branches have to land, that is we can add the missing endpoint of a branch. Moreover, they land uniformly, that is the distance between any point of the edge of generation n in a branch and the endpoint of this branch can be bounded from above by a constant independent of the branch, and those constants converge (exponentially) to 0 as n goes to infinity. We will call the extree together with the endpoints a *closed extree*.

In view of what we already said, we can think about an extree not only as a subset of $\mathbb{C}$, but also as a continuous map of the *abstract d-tree* T_d into $\mathbb{C}$. This abstract d-tree is a subset of the unit square $[0,1]^2$. Let K_d be the Cantor set obtained by starting from the interval $[0,1]$ (the interval of the 0-th generation) and in each step dividing each interval into $2d-1$ intervals of equal length and removing every other one (that is, $d-1$ of them) in order to get d closed pairwise disjoint intervals of the next generation. If x is the midpoint of an interval I of the n-th generation and $x_1, \dots, x_d$ are the midpoints of the intervals of the $(n+1)$-st generation contained in I then the segments with endpoints $(x, 2^{-n})$ and $(x_j, 2^{-n-1})$ are edges of T_d of the n-th generation (for every I, j). The union of the edges of all generations is T_d (see Figure 3.2). The closure $\overline{T_d}$ of T_d is equal to $T_d \cup K_d$ (more precisely, $K_d \times \{0\}$ rather than K_d, but we will not distinguish between these two sets). We will call it the *closed abstract d-tree*. There is a natural ordering (from left to right) of the edges of a given generation.

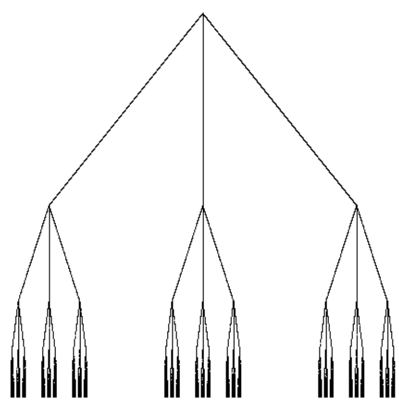

FIGURE 3.2. The abstract 3-tree

We can define a map φ_d from the union of the edges of generation at least 2 of T_d onto T_d, by mapping first d^n edges of generation $n+1$ affinely onto all edges of generation n, with the order preserved, then similarly next d^n edges, etc. Now we can think of the edges of the n-th generation as components of the inverse images under φ_d^{n-1} of the edges of the first generation. Thus, if we fix a homeomorphism ψ (preserving order) from the union of the edges of the first generation for T_d onto the similar union for an extree T, we can extend it to a homeomorphism $\psi : T_d \to T$ that preserves order of the edges in every generation and such that $f \circ \psi = \psi \circ \varphi_d$.

By continuity and exponential convergence, we can extend ψ to a continuous map between the closed trees. It does not have to be a homeomorphism. However, it maps K_d onto the set of landing points of T. We can also extend φ_d to the whole

$\overline{T_d}$. Note that restricted to K_d it will be conjugate to the full d-shift. Moreover, $\varphi_d|_{K_d}$ preserves order on K_d intersected with each interval of first generation. By continuity, we have $f \circ \psi = \psi \circ \varphi_d$ on the whole $\overline{T_d}$, and in particular on K_d.

Let us look closer at the map $\psi|_{K_d}$. We will call it a *Cantor limit map* and denote by L_C.

Let us draw some analogies between the systems of exrays and extrees. The proof of the following lemma is partially different than the proof of an analogous lemma from [BCM].

Lemma 3.3. *The set $L_C(K_d)$ is equal to $J(f)$.*

Proof. The proof that $L_C(K_d) \subset J(f)$ is the same as for exrays. Let us prove that $J(f) \subset L_C(K_d)$. Take a point $z \in J(f)$ and its neighborhood U. By Theorem 2.5 (5), there is n such that the root of the extree T belongs to $f^n(U)$. Therefore z belongs to the closure of T. Since the map $\psi : \overline{T_d} \to \overline{T}$ is continuous and $\overline{T_d}$ is compact, we get $z \in \psi(\overline{T_d})$. However, $\psi(T_d) = T \subset \mathbb{C} \setminus J(f)$, so $z \in \psi(K_d) = L_C(K_d)$. ■

One disadvantage of extrees compared to systems of exrays is that the limit Cantor map is not unique up to compositions with simple maps. If we deform the edges of the first generation in such a way that they move through $f^n(w)$ for some $w \in S(f)$ and $n > 0$, then there is a discontinuous change in the Cantor limit map. Moreover, the change in the order in which the points of $f^{-1}(z)$ are joined with the root z of the extree causes a change in the Cantor limit map. The latter change corresponds to the multiplication by a $(d-1)$-st root of 1 for the systems of exrays, but while this multiplication is continuous on the circle, the analogous map of K_d is discontinuous.

Observe that we can consider the Cantor set K_d to be a subset of a circle rather than an interval (we enlarge the interval and glue together its endpoints, leaving a gap in the Cantor set there). Let us look at the lamination defined on the set K_d analogously to the way we defined laminations on the circle in [BCM]. In order to visualize it, we consider K_d to be a subset of $\mathbb{S}^1$, as in the preceding paragraph. The proof that there are no linked leaves is the same simple argument as in [BCM] (or as for complex polynomials), since once two branches separate, they stay disjoint. Thus, we really have a lamination, although it is defined only on K_d. Moreover, this lamination is fully invariant under the shift on K_d. To distinguish it from the laminations considered in [BCM], we will call it a *Cantor lamination*.

Now, in the same way as for usual laminations, we see that there are at most countably many lamination gaps (they should not be confused with the gaps in the Cantor set). Thus, we get the following result.

Lemma 3.4. *For L_C, the preimages of all but countably many points have cardinality at most 2.*

4. Invariant measures and entropy

Let us note that by Lemma 2.4 and Theorem 2.5 $J(f)$ is the set of nonwandering points for a expanding polymodial f, so all probability invariant measures are concentrated on $J(f)$. If we consider f on the Riemann sphere, there is an additional nonwandering point at infinity, but that modifies the whole picture only in a trivial way. In any case, the topological entropy $h(f)$ of f is equal to $h(f|_{J(f)})$.

As for the complex polynomials, extrees provide a powerful tool for investigating invariant measures and entropy. In some respect the situation for expanding polymodials is better, in another respect worse. We do not have to worry about the branches that do not land. Moreover, it is easy to see that for every ergodic invariant measure on K_d with full support the Cantor limit map is at most two-to-one almost everywhere (by the way, similar arguments can be used for complex polynomials if we know that all branches land). Indeed, if a measure is not at most two-to-one almost everywhere, there is a gap (in the lamination sense, that is, the inverse image under L_C of a point, of cardinality larger than 2) of positive measure. Since the gaps are disjoint and the measure is invariant ergodic, the measure must be concentrated on the trajectory of a periodic gap. Since the measure has full support, this trajectory constitutes the whole K_d, and this contradicts Lemma 3.3 and Theorem 2.5.

On the other hand, there may be points of $J(f)$ whose preimage under the Cantor limit map are Cantor sets. Such phenomenon does not occur for complex polynomials.

We will use the same notation as in the preceding section. Moreover, we will use the standard notation of the form $\varphi_*(\kappa)$ for the image of a measure κ under a map φ (that is, $(\varphi_*(\kappa))(A) = \kappa(\varphi^{-1}(A))$ for any Borel set A).

Let us consider the measure μ for which the shift $\sigma : K_d \to K_d$ is the Bernoulli $(1/d, \dots, 1/d)$ shift. We will call it the *uniform measure.* It is the weak-* limit of the measures μ_n equidistributed on the set of all vertices of n-th generation of the closed abstract tree $\overline{T}$.

It is well known that if a map φ is at most n-to-1 almost everywhere then φ_* preserves the measure entropy (see, e.g., [PaT], Lemma IV.6). Thus, we get the following lemma.

Lemma 4.1. *If κ is an ergodic invariant probability measure on K_d, its support is K_d and $\nu = L_{C*}(\kappa)$ then $h_\kappa(\sigma) = h_\nu(f)$.*

A kind of converse also holds.

Lemma 4.2. *Let ν be an ergodic invariant probability measure on $J(f)$ of positive entropy for f. Then there is an ergodic invariant probability measure κ on K_d such that $\nu = L_{C*}(\kappa)$ and $h_\kappa(\sigma) = h_\nu(f)$.*

Proof. Let $B \subset K_d$ be the set of all points $x \in K_d$ such that $x \le y$ for every $y \in K_d$ such that $L_C(y) = L_C(x)$ (remember that $K_d \subset [0,1]$). Then the map $\tau = (L_C|_B)^{-1}$ is Borel and maps $J(f)$ onto B in a one-to-one way. Therefore the measure $\hat{\kappa} = \tau_*(\nu)$ is a probability measure on K_d and $L_{C*}(\hat{\kappa}) = \nu$. Now we get an invariant measure $\tilde{\kappa}$ from $\hat{\kappa}$ in a usual way, that is as the weak-* limit of a convergent subsequence of the sequence $\left((1/n)\sum_{k=0}^{n-1} \sigma_*^k(\hat{\kappa})\right)_{n=1}^{\infty}$. This measure is invariant for σ. Since $L_{C*}(\hat{\kappa}) = \nu$, the measure ν is invariant for f and $f \circ L_C = L_C \circ \sigma$, we have also $L_{C*}(\sigma_*^k(\hat{\kappa})) = \nu$ for all k. Therefore we get $L_{C*}(\tilde{\kappa}) = \nu$.

Look at the ergodic decomposition of $\tilde{\kappa}$. There is a measure ξ on the set of all ergodic invariant probability measures on K_d such that $\tilde{\kappa}$ is equal to the integral of the identity function with respect to $d\xi$. Then ν is equal to the integral of L_{C*}

with respect to $d\xi$. However, ν is ergodic, so the image under L_{C*} of ξ-almost every measure is ν. In particular, there exists an ergodic invariant probability measure κ on K_d such that $L_{C*}(\kappa) = \nu$.

By Lemma 3.4, for L_C the preimages of all but countably many points have cardinality at most 2. Since ν has positive entropy, $\nu(A) = 0$ for every countable set A. Thus, L_C is at most 2-to-1 κ-almost everywhere, so by the lemma from [PaT] cited earlier, $h_\kappa(\sigma) = h_\nu(f)$. ■

Let us stress that in Lemma 4.1 we cannot replace the assumption that the support of κ is equal to the whole K_d by an assumption on the entropy of κ. In the situation described later in Example 5.3 (a c-tent map with $c \in (-2, -1)$ real with 0 periodic), if the period of 0 is n then $L_C^{-1}(0)$ is mapped to itself by σ^n in the 2-to-1 way. One can perform coding by looking whether the branches reach 0 from the upper or lower half-plane, and see that σ^n restricted to $L_C^{-1}(0)$ is conjugated to the full 2-shift. Therefore there is a lot of ergodic measures of positive entropy for σ whose images under L_{C*} are equal to the measure concentrated on the f-orbit of 0.

Set $\nu = L_{C*}(\mu)$. By Lemma 4.1, $h_\nu(f) = \log d$.

Theorem 4.3. *The topological entropy of f is $\log d$. The measure ν is the unique measure of maximal entropy for f.*

Proof. Since $f|_{J(f)}$ is a factor of the full d-shift, the topological entropy of f is not larger than the topological entropy of the d-shift, that is, $\log d$. On the other hand, by Lemma 4.1, $h_\nu(f) = h_\mu(\sigma) = \log d$, so by the Variational Principle $h(f) = \log d$ and ν is a measure of maximal entropy.

Suppose that ξ is an ergodic measure of maximal entropy for f. By Lemma 4.2, there exists an invariant measure κ on K_d such that $L_{C*}(\kappa) = \xi$ and $h_\kappa(\sigma) = \log d$. Since μ is the only measure of maximal entropy for σ, we have $\kappa = \mu$, and therefore $\xi = \nu$. Thus, ν is the only ergodic measure of maximal entropy for f, and therefore the only (not necessarily ergodic) measure of maximal entropy for f. ■

The measure ν can be obtained, as always in such highly regular cases, as the limit of measures on preimages of a point under higher and higher iterates of f (cf. [L] for the rational case).

Theorem 4.4. *Let $z \in \mathbb{C}$ be a fully regular point. Let ν_n be the probability measure equidistributed on $f^{-n}(z)$. Then ν is the weak-* limit of the sequence $(\nu_n)_{n=1}^{\infty}$.*

Proof. Assume first that $z \notin J(f)$. Then there is k such that $|f^k(z)|$ is so large, that there exists an extree T with the root $f^k(z)$. Let T_d be the abstract d-tree and $\psi : \overline{T_d} \to \overline{T}$ the map constructed in the preceding section. Denote $x = \psi^{-1}(z)$. Let μ_n be the probability measure equidistributed on $\varphi_d^{-n}(x)$, where φ_d is also the map from the preceding section. Clearly the measures μ_n converge in the weak-* topology to μ and $\psi_*(\mu_n) = \nu_n$. Therefore the measures ν_n converge in the weak-* topology to ν.

Now, if $z \in J(f)$, then we choose a fully regular point $w \notin J(f)$ with sufficiently large modulus and construct an extree T with the root at w. The Cantor limit map sends K_d onto the whole Julia set, so there is $x \in K_d$ such that $L_C(x) = z$. Let μ_n be the probability measure equidistributed on $\sigma^{-n}(x)$. Clearly the measures μ_n

converge in the weak-* topology to μ. In order to complete the proof as in the first case, we have to show that $L_{C*}(\mu_n) = \nu_n$. The only obstacle to this could be that two elements of $\sigma^{-n}(x)$ are mapped to the same point by L_C. However, that would mean that two different branches of T landing at $z = L_C(x)$ are mapped to the same branch by f^n, so there is a singular point on the trajectory of $L_C(x)$. Since z is also on this trajectory and is fully regular, we get a contradiction. ∎

5. Julia sets and dendrites

In this section we prove the following theorem.

Theorem 5.1. *If f is an expanding polymodial then there exists a dendrite containing its Julia set $J(f)$.*

As we mentioned in Section 1, it allows us to make a number of conclusions concerning topological properties of $J(f)$ and dynamical properties of $f|_{J(f)}$.

We will prove Theorem 5.1 by constructing the dendrite in steps. Recall that by a preimage of a connected set we mean a component of the full inverse image of the set.

Consider the following object. A set $D \subset \mathbb{C}$ is homeomorphic to a closed disk. In its interior there are disjoint sets $D_1, \dots, D_k$, each of them also homeomorphic to a closed disk. On the boundary of D we choose a point p and on the boundary of each D_j a point p_j. Then we connect p with each p_j with simple curves γ_j (we treat them as subsets of $\mathbb{C}$). Those curves are disjoint, except the common endpoint p, and are disjoint from the union of sets D_j, again except the corresponding endpoints (see Figure 1). We will call the sets $D, D_1, \dots, D_k$ *bubbles* and the set

$$B = \operatorname{int}(D) \setminus \left(\bigcup_{j=1}^{k} D_j \cup \bigcup_{j=1}^{k} \gamma_j \right)$$

a *bubbler*. The bubbler B has the *outer boundary*, which is the boundary of D, and the *inner boundary*, which is the union of the boundaries of the bubbles D_j and the curves γ_j. We will refer to D as the *base bubble* of B, to $D_1, \dots, D_k$ as *secondary bubbles* of B, to $p, p_1, \dots, p_k$ as *special points* of $D, D_1, \dots, D_k$ respectively, and to $\gamma_1, \dots, \gamma_k$ as *joining curves* of B.

A foliation of a bubbler will be called *standard* if each leaf has one end on the inner boundary and the other end on the outer boundary, and the bubbler with this foliation is homeomorphic to an open disk with the foliation by the horizontal segments (see Figure 5.2). A foliation will be called *rectifiable* if each leaf is rectifiable and their lengths are commonly bounded.

Lemma 5.2. *A bubbler with rectifiable boundary admits a rectifiable standard foliation.*

Proof. A disk foliated by horizontal segments is homeomorphic with the upper half-plane $H \subset \widehat{\mathbb{C}}$ foliated by semicircles centered at 0. Denote the latter foliation by $\mathcal{F}$. Consider the Möbius map M given by the formula $M(z) = (z - i)/(z + i)$. Then $M(\infty) = 1$, $M(-1) = i$, $M(0) = -1$, $M(1) = -i$ and $M(i) = 0$. Therefore M maps H onto the unit disk $\mathbb{D}$, and the image of the unit semicircle is the vertical

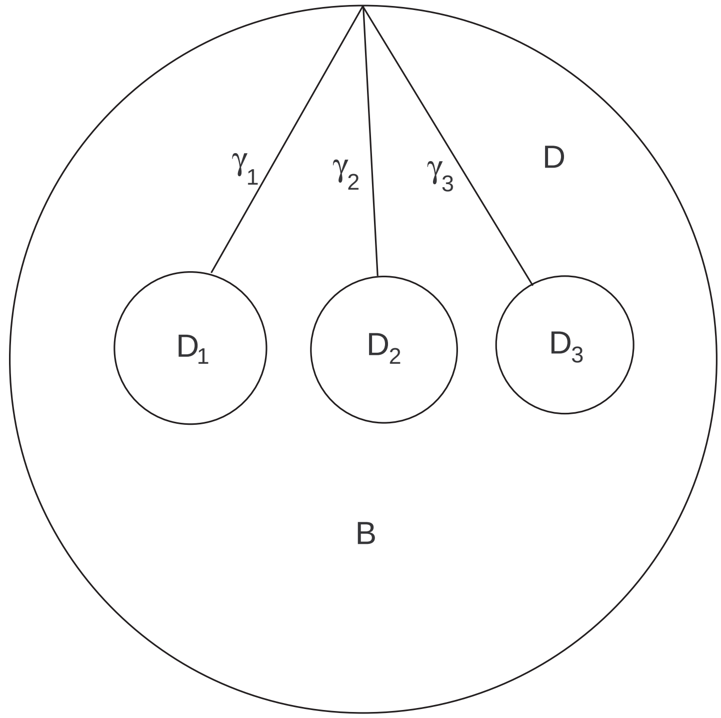

FIGURE 5.1. A bubbler

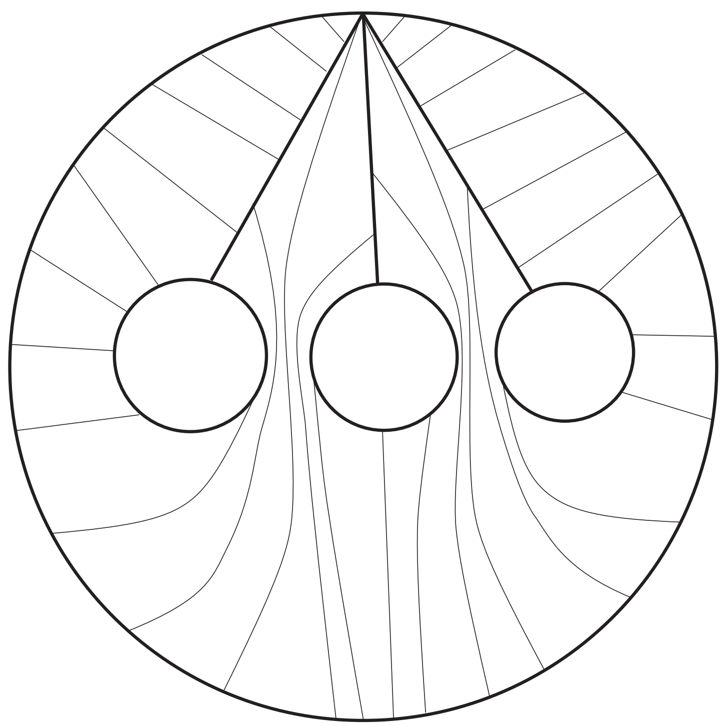

FIGURE 5.2. A standard foliation of a bubbler

diameter of $\mathbb{D}$. Two special points of the boundary of H that are not endpoints of any leaves of the foliation are 0 and ∞, and their images under M are -1 and 1.

Let B be a bubbler with rectifiable boundary. This boundary is locally connected, so if $\varphi : \mathbb{D} \to B$, where $\mathbb{D}$ is the unit disk, is a Riemann map, it can be extended to the closure of $\mathbb{D}$ as a continuous map (we will call it also φ). We may choose φ in such a way that the lower half of the unit circle is mapped by φ to the outer boundary of B, and the upper half is mapped to the inner boundary of B (and the points -1 and 1 are both mapped into the special point p of D). Then the image $\mathcal{G}$ of the foliation $\mathcal{F}$ under $\varphi \circ M$ is a standard foliation of the bubbler B. Any leaf of $\mathcal{G}$ is the image of the vertical diameter of $\mathbb{D}$ under the map $z \mapsto \varphi(M(t \cdot M^{-1}(z)))$ for some positive constant t. This map is a composition of a Riemann map φ with a Möbius map preserving $\mathbb{D}$, so it is also a Riemann map. By Lemma 10.5 of [Po], the length of the image of a diameter of $\mathbb{D}$ under a Riemann map is bounded by a universal constant times the length of the boundary of B. Thus, the foliation $\mathcal{G}$ is rectifiable. ■

Now we assume that f is an expanding polymodial. We are really interested in the case when the trajectory of at least one singular point escapes to infinity, however, we do not have to make this assumption. We start by taking a circle C, centered at 0, of a very large radius. The radius should be so large that all singular points and singular values of f are inside C, and the inverse image of C is inside C. Moreover, we choose this radius in such a way that C is disjoint from the trajectories of all singular points of f. Let D be the closed disk bounded by C.

Now we construct our first bubbler, with the base bubble D. We start by choosing a sufficiently large integer n. Since C is disjoint from the trajectories of all singular points, all components of $f^{-n}(D)$ will be homeomorphic to closed disks, bounded by the components of $f^{-n}(C)$. The closures of those disks will be our secondary bubbles. The number n should be so large that

(1) $f^{-n}(D)$ contains only those singular points of f that are in $J(f)$,
(2) if two singular points of f are in the same bubble, they are in the same component of $J(f)$.

Note that we are talking about the singular points of f, not of f^n. Clearly, (1) holds if n is large enough. Since the intersection of the sets $f^{-n}(D)$ over all n is equal to $J(f)$, for sufficiently large n (2) also holds. Thus, we fix n so large that (1) and (2) hold, and we get bubbles $D_1, \dots, D_k$. Now we pick special points $p, p_1, \dots, p_k$ on the boundaries of the bubbles $D, D_1, \dots, D_k$ respectively, and join them by rectifiable curves in such a way that we get a bubbler. We call this bubbler B. Observe that the boundary C of D is a circle, and the boundaries of $D_1, \dots, D_k$ are contained in its inverse image under f^n, so they are also rectifiable (see Lemma 2.3). Therefore the boundary of B is rectifiable.

In the second step we construct k bubblers $B_1, \dots, B_k$, whose base bubbles are $D_1, \dots, D_k$. Fix $m \in \{1, \dots, k\}$. Observe that $f|_{D_m}$ is a branched covering map onto $f(D_m)$, the set $f(D_m)$ is homeomorphic to a closed disk and contains some of the sets $D_1, \dots, D_k$ in its interior. We have

$$f^{-1}\left(\bigcup_{j=1}^{k} D_j\right) = f^{-(n+1)}(D),$$

so for every j the components of $f^{-1}(D_j)$ are homeomorphic to closed disks and are contained in the union of the interiors of the bubbles $D_1, \dots, D_k$. We take those components of $f^{-1}\left(\bigcup_{j=1}^k D_j\right)$ that are contained in the base bubble D_m as secondary bubbles of B_m. Now we have to choose special points. For D_m we already have such point, namely p_m. If E is a secondary bubble of B_m, then $f(E) = D_j$ for some j. We choose as the special point of E one of the preimages of p_j. Then we join special points by rectifiable curves in such a way that we get a bubbler (see Figure 5.3). Again, the bubbler B_m that we get, has rectifiable boundary.

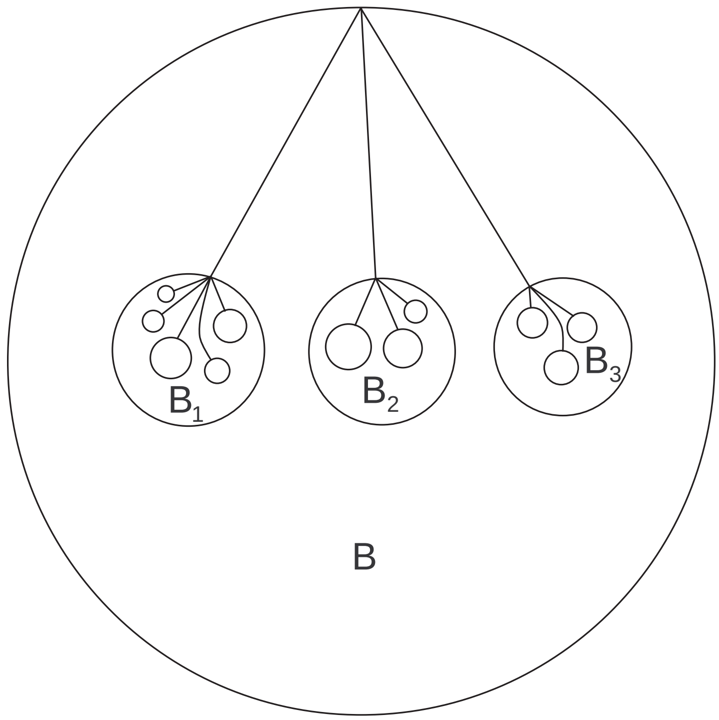

FIGURE 5.3. Bubblers B and B_j

According to Lemma 5.2, for bubblers $B, B_1, \dots, B_k$ there exist rectifiable standard foliations $\mathcal{F}, \mathcal{F}_1, \dots, \mathcal{F}_k$ respectively. Since they come together nicely on the boundaries of the bubbles D_j, we get a rectifiable foliation of the union of the bubblers $B, B_1, \dots, B_k$ and the boundaries of $D_1, \dots, D_k$. The leaves of the foliation connect the points of C with points of the interior boundaries of $B, B_1, \dots, B_k$, except that the leaves do not connect points of C with points of the boundaries of $D_1, \dots, D_k$, but rather "pass through" the latter boundaries.

The third step of the construction is to pull back the bubblers and the foliation constructed in the second step. A preimage of a bubbler is not necessarily a bubbler. Therefore we have to define a *prebubbler*. The difference in the definition of a bubbler and prebubbler is that we allow more than one special point (but finitely many of them) on the boundary of a bubble, whether the base or secondary one. Each joining curve joins a special point of the base bubble with a special point of a secondary bubble. A special point of the base bubble may be an endpoint of several joining curves, while a special point of a secondary bubble is an endpoint of exactly one joining curve. Moreover, we require that the union of the secondary bubbles and the joining curves (including their endpoints) is connected and simply connected. Of course, a bubbler is a prebubbler. Figure 5.4 shows a prebubbler mapped to a bubbler by a branched covering map of degree 2. Inside the central secondary bubble of the prebubbler B_L there is a singular point of f.

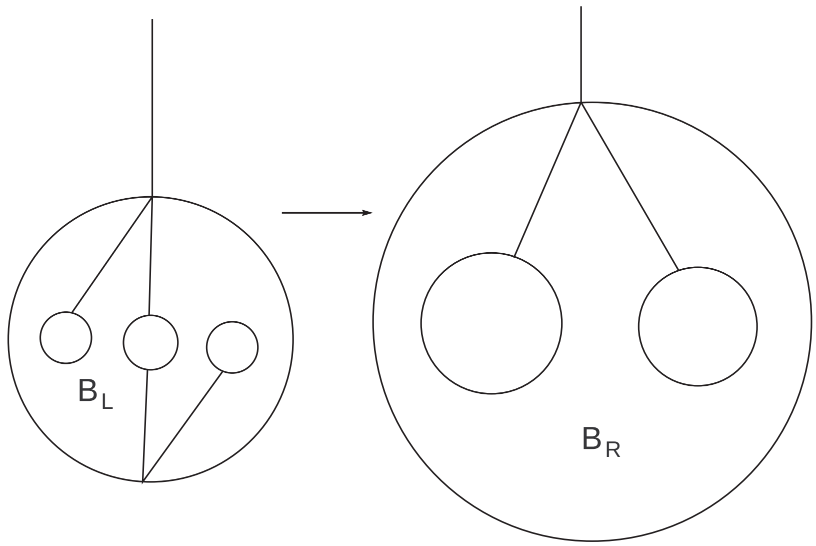

FIGURE 5.4. A prebubbler mapped onto a bubbler

Another feature visible in Figure 5.4 is a piece of a joining curve β from a bubbler (or prebubbler) of the preceding generation. Here by the generation of a bubble D' we mean a number m such that $f^{m+n-1}(D') = D$ (in particular, the bubbles $D_1, \ldots, D_k$ constructed in the second step of the construction are of the first generation). We assume that the bubbler B_R has been constructed in the second step, so this piece comes from the first step. The prebubbler B_L is its preimage and the map has degree 2. Therefore there are 2 preimages of β ending at this prebubbler. However, the base bubble of B_L is a secondary bubble of the bubbler constructed in the second step. Therefore the joining curve, whose piece is visible there, is a curve constructed in the second step, and is not a preimage of β. At further stages the picture would be different (recall that in the third step of the construction we simply pull back the bubblers). Hence if B_R were constructed in the third step, then we would see two pieces of the joining curves attached to B_L and they would be preimages of β.

The idea is to continue the construction from now on by simply pulling back the prebubblers created in the preceding step. To justify this method we need the following lemma.

Lemma 5.3. *A preimage of a prebubbler constructed in the construction above in the step $n \geq 2$ is a prebubbler.*

Proof. As we have already established, preimages of bubbles are bubbles. Thus, if we take a preimage B of a prebubbler, we get a base bubble B_b, some secondary bubbles inside, and some preimages of joining curves. The boundaries of bubbles are disjoint from $J(f)$. If there are no singular points in the base bubble then f restricted to it is a homeomorphism and there is nothing to prove. Otherwise, by the assumptions (1) and (2) that we made on D and n, all singular points contained in the base bubble B_b belong to the same component of $J(f)$. Therefore, they belong to the same secondary bubble B_s. This means that f restricted to $B_b \setminus B_s$ is a local homeomorphism of some degree m. We can make a slit from the bubble $f(B_s)$ to the boundary of $f(B_b)$ that cuts through the interior of the prebubbler $f(B)$ (see Figure 5.5). Then the preimages of this slit cut B into m sectors, and f restricted to each sector is a homeomorphism. Thus, the union of the secondary bubbles and joining curves of B consists of m copies of those objects for $f(B)$, disjoint except for their common bubble B_s. Therefore B is a prebubbler. ■

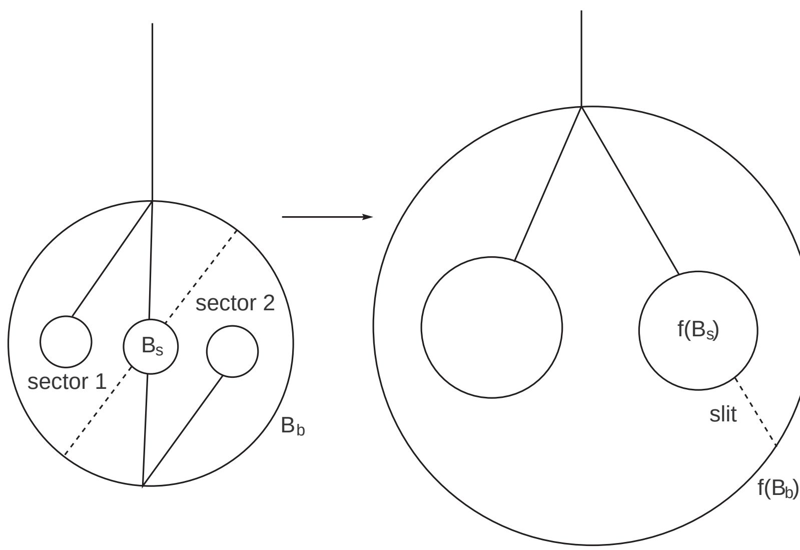

FIGURE 5.5. A slit and sectors

Thus, in the third step we take preimages of bubblers constructed in the second step, then preimages of those prebubblers, etc. We also take preimages of the corresponding foliations. Note that in any prebubbler the leaves start at the outer boundary and end either at a joining curve or at a secondary bubble. However, that secondary bubble is a primary bubble of the next generation, and so the latter leaves are then joined with the leaves of the foliation of the next generation. In other words, the leaves which end at a secondary bubble are extended by the leaves of the next generation.

In such a way we get two objects. The first one is the union K of all joining curves (including endpoints) and $J(f)$. The second one is the foliation $\mathcal{H}$ obtained by joining all foliations together. Since $J(f) = \bigcap_{j=1}^{\infty} f^{-j}(D)$, $\mathcal{H}$ foliates $\mathrm{int}(D) \setminus K$. All leaves of $\mathcal{H}$ begin at the circle C. Since the foliations $\mathcal{F}, \mathcal{F}_1, \dots, \mathcal{F}_k$ are rectifiable and the lengths of the leaves decay exponentially from one generation to the next, the foliation $\mathcal{H}$ is rectifiable too. Therefore all leaves land. There are two possibilities. Either a leaf lands at a point of a joining curve, or it is extended through all generations of bubbles. In the latter case, since the intersection of $f^{-j}(D)$ over all j is equal to $J(f)$, a leaf lands at a point of $J(f)$. In any case, a leaf lands at a point of K.

Lemma 5.4. *The set K is a dendrite.*

Proof. Let K_m be the union of the joining curves up to generation m and the bubbles of generation m. Of course each K_m is compact.

Clearly, K_1 is connected and simply connected. We get K_{m+1} from K_m by replacing the bubbles of generation m by the union of joining curves and bubbles of generation $m+1$. In each bubble of generation m this union is connected and simply connected. Moreover, it joins all special points on the boundary of this bubble. Therefore, if K_m is connected and simply connected, so is K_{m+1}. Hence, by induction, all sets K_m are connected and simply connected.

As the intersection of a descending family of compact connected simply connected sets, K is also a compact, connected and simply connected. It is the union of $J(f)$, which has empty interior, and countably many curves. Therefore it has empty interior.

It remains to show that K is locally connected. For this, we consider a map ψ that assigns to each point of C the landing point of the leaf of foliation $\mathcal{H}$ that begins at this point. Additionally, since the special point p of D has no leaf beginning at it, we set $\psi(p) = p$.

We show that ψ is continuous. Clearly, it is continuous at p. If a leaf starting at z lands on a joining curve, all nearby leaves land close to $\psi(z)$ on a joining curve (of the same generation, possibly plus or minus 1). Thus, ψ is continuous at z. If a leaf starting at z lands at a point of $J(f)$, then for every m there is a point of this leaf that belongs to the interior a bubble of m-th generation. Then all nearby leaves also have such points. Since the distance between such point and the landing point of the leaf decays exponentially with m, this shows that ψ is continuous at z also in this case. Hence, ψ is continuous on the whole circle C.

We know already that $\psi(C) \subset K$. We will show that $\psi(C) = K$. All points of the joining curves are landing points of leaves of the foliation $\mathcal{H}$. If $z \in J(f)$ then in any neighborhood of z there are points of bubbles of arbitrarily large generation. Those points belong to leaves of $\mathcal{H}$ and are arbitrarily close to the landing points of those leaves. Therefore $\psi(C)$ is dense in $J(f)$. Since ψ is continuous and C is compact, $J(f) \subset \psi(C)$, and hence $\psi(C) = K$. Thus, K is locally connected as a continuous image of a circle. ■

Now Theorem 5.1 follows from Lemma 5.4.

6. Nondegenerate subcontinua of the Julia set

We consider the situation when the Julia set $J(f)$ of an expanding polymodial f is disconnected, but not all singular points lie in the basin of the infinity. However, formally everything in this section applies to all expanding polymodials.

We will start with an immediate corollary to Theorem 5.1. We will call a continuum *nondegenerate* if it consists of more than one point.

Corollary 6.1. *The Julia set of an expanding polymodial has at most countably many nondegenerate components. Each of them is a dendrite. For every $\varepsilon > 0$ any family of pairwise disjoint subcontinua of $J(f)$ of diameter larger than ε is finite. In particular, there are only finitely many components of $J(f)$ of diameter larger than ε.*

We will say that a continuum A is *wandering* if the sets $f^n(A)$, $n = 0, 1, 2, \ldots$, are pairwise disjoint. Note that according to our definition, A can be wandering even if it is attracted to an attracting periodic orbit. Also, we will say that a continuum B is *periodic* (of period k) if $B, \dots, f^{k-1}(B)$ are pairwise disjoint, while $f^k(B) \subset B$. In this case by $\mathrm{orb}(B)$ we will denote the union $\bigcup_{i=0}^{k-1} f^i(B)$.

Theorem 6.2. *There are no wandering nondegenerate subcontinua of the Julia set $J(f)$ of an expanding polymodial f. If I is a periodic nondegenerate subcontinuum of $J(f)$ then the set $\mathrm{orb}(I)$ contains a singular point of f and $f|_{\mathrm{orb}(I)}$ is not 1-to-1.*

Since f is open and $J(f)$ is fully invariant, f maps components of $J(f)$ onto components of $J(f)$. Therefore from Theorem 6.2 we get immediately the following corollary.

Corollary 6.3. *If a component I of $J(f)$ is nondegenerate then there is $n \geq 0$ such that $f^n(I)$ is a periodic component of $J(f)$ and contains a singular point of f.*

Now we will proceed towards proving Theorem 6.2. By $\mathrm{diam}(I)$ we will denote the diameter of a set I. Recall that $\alpha(f)$ is the expansion constant of f (from the definition of an expanding polymodial). If a, b are points of a dendrite I then we write $[a, b]_I$ for the smallest connected set containing a, b and contained in I. If $a = b$, it is of course $\{a\}$, and if $a \neq b$ then it is an arc joining a with b.

Lemma 6.4. *Assume that I is a dendrite and none of the singular points of f belongs to I. Assume also that the convex hull of $f(I)$ contains at most one singular value of f. Then*

$$\mathrm{diam}(f(I)) \geq \frac{\alpha(f)}{2} \mathrm{diam}(I). \tag{6.1}$$

Proof. Assume first that the convex hull of $f(I)$ does not contain a singular value of f. We join $f(x)$ with $f(y)$ by a straight line segment γ (we can think of it as a curve). Then the lifting of γ starting at x ends at y. By Lemma 2.3 its length is at most $|f(x) - f(y)|/\alpha(f)$. This proves that $\mathrm{diam}(f(I)) \geq \alpha(f) \,\mathrm{diam}(I)$, so (6.1) holds in this case.

Assume now that the convex hull of $f(I)$ contains one singular value c of f. Let r be the infimum of the lengths of curves joining $f(x)$ with $f(y)$ in $\mathbb{C} \setminus \{c\}$ and homotopic to $[f(x), f(y)]_{f(I)}$ in $\mathbb{C} \setminus \{c\}$. For every $s > r$ there is such curve γ of length smaller than s. The lifting of γ starting at x ends at y. By Lemma 2.3 its length is smaller than $s/\alpha(f)$. This proves that $r \geq \alpha(f)|x - y|$. Since the curves used to define r can have self-intersections, we have $r \leq |f(x) - c| + |f(y) - c|$. Since the diameters of $f(I)$ and its convex hull are the same, we have $|f(x) - c| \leq \mathrm{diam}(f(I))$ and $|f(y) - c| \leq \mathrm{diam}(f(I))$. Therefore $2\,\mathrm{diam}(f(I)) \geq \alpha(f)|x - y|$. This proves (6.1) in this case. ∎

Let us now consider a more abstract situation. A point a of a dendrite I is said to be an *endpoint* of I if $I \setminus \{a\}$ is connected, and a *cutpoint* of I otherwise.

Lemma 6.5. *Let I be a dendrite and $g : I \to I$ a continuous 1-to-1 map. Assume that there are no wandering nondegenerate subcontinua of I. Then for every cutpoint p of I there is $m \geq 1$ such that p belongs to an arc consisting of fixed points of g^m. Moreover, g is onto (so it is a homeomorphism).*

Proof. Observe first that if $g^k(a) = a$ and $g^n(b) = b$ then g^{kn} maps $[a, b]_I$ homeomorphically onto itself. Then in every maximal subinterval of $[a, b]_I$ without fixed points of g^{kn} there is a wandering subinterval. Therefore there are no such intervals, and this means that g^{kn} restricted to $[a, b]_I$ is the identity.

We will show that periodic points are dense in I. If they are not, their complement (which is invariant) contains nondegenerate components. If we take a continuum contained in such component, it cannot be wandering, so this component has to be periodic or preperiodic. This shows that there is a periodic component. Its closure M is a subcontinuum of I mapped to itself by some iterate $h = g^i$ of g. By the property proved in the preceding paragraph, h has at most one periodic point in M, and this point is an endpoint of M. On the other hand, it is well known that any dendrite has a fixed point property, so h has a unique fixed point $a \in M$.

We claim that h maps some arc $[a, y]_M$ into itself. Choose any point $b \in M$, $b \neq a$. The intersection of the arcs $[a, b]_M$ and $[a, h(b)]_M$ is an arc $[a, c]_M$ for some $c \in M$, $c \neq a$. Set $x = h^{-1}(c)$, then $x \in [a, b]_M$. We have $x \in [c, b]_M$ or $x \in [a, c]_M$. In the first case $h([a, x]_M) \subset [a, x]_M$. In the second case $[a, x]_M \subset h([a, x]_M) \subset h^2([a, x]_M) \subset \dots$ and all those sets are arcs. Therefore their union is an arc with one endpoint a and h maps this arc onto itself. This proves our claim. As before, we conclude that $[a, y]_M$ consists of fixed points of h. In particular, we found in M periodic points of g other than a, a contradiction. This shows that periodic points of g are dense in I.

Now the statement of the lemma follows from the observation from the beginning of the proof and the density of periodic points. ■

The reader can easily construct a dendrite homeomorphism without wandering subcontinua for which the map on the set of endpoints is an adding machine.

Proof of Theorem 6.2. Suppose that I is a wandering nondegenerate subcontinuum of the Julia set of an expanding polymodial f. By Corollary 6.1, I is a dendrite and the diameters of $f^n(I)$ converge to 0. Choose an integer m such that $(\alpha(f))^m > 2$. By Lemma 2.2, the map $g = f^m$ is an expanding polymodial. Its constant of expansion $\alpha(g)$ is at least $(\alpha(f))^m$, so it is greater than 2. By the definition of the Julia set we have $J(g) = J(f)$, and hence I is a wandering nondegenerate subcontinuum of $J(g)$.

If n is sufficiently large then $g^n(I)$ does not contain any singular point of g and $\mathrm{diam}(g^{n+1}(I))$ is so small that the convex hull of $g^{n+1}(I)$ contains at most one singular value of g. Therefore by Lemma 6.4, $\mathrm{diam}(g^{n+1}(I)) > \mathrm{diam}(g^n(I))$. This contradicts the fact that the diameters of $g^n(I)$ converge to 0. Therefore there are no wandering nondegenerate subcontinua of $J(f)$.

Assume now that I is a periodic nondegenerate subcontinuum of $J(f)$ of period k. If $f^k|_I$ is a 1-to-1 map then by Lemma 6.5 there exist non-trivial arcs in I on which some iterate of f is the identity. This is impossible because the iterates of f are also expanding polymodials. Therefore $f^k|_I$ (and hence $f|_{\mathrm{orb}\, I}$) is not 1-to-1, and hence a singular point of f has to be contained in $\mathrm{orb}(I)$. ■

7. Hausdorff dimension of the Julia set

So far we have been studying topological properties of the Julia sets of expanding polymodials. This was possible in particular because their expanding properties are strong enough to allow us to obtain qualitative results like Theorem 5.1 or Theorem 6.2. However, these properties are too weak to obtain more quantitative characteristics of the Julia set such as its Hausdorff dimension, except the trivial statement, that it is in $[0,2]$. While we will give an example that it can be 0, we do not have an example that it can be 2. We conjecture that it cannot. However, we even do not know whether the Lebesgue measure of the Julia set of an expanding polynomial f has to be zero.

More precisely, the difficulties in estimating the Hausdorff dimension of the Julia set are due to the fact that while f is expanding, it does not have to be regular in any reasonable sense. The shape of an image or a preimage of a ball is unpredictable. Thus, in order to get better estimates of $\mathrm{HD}(J(f))$ (by $\mathrm{HD}(\cdot)$ we denote the Hausdorff dimension of a set), we have to make additional assumptions. To get an estimate from below, we need only to assume that f is Lipschitz continuous. To get an estimate from above, we have to assume that it is sufficiently smooth. Since it is not our aim to look for the weakest possible assumptions, we will settle for C^2 (apart from the singular points).

Let $|U|$ denote the diameter of a set U, and define the diameter of a collection of sets $\{U_i\}$ by $|\{U_i\}| = \sup_i |U_i|$. A collection $\{U_i\}$ is said to be a δ-cover of a set A if $A \subset \bigcup_i U_i$ and $|\{U_i\}| < \delta$. We recall the definition of the Hausdorff measure and dimension of A. For $s \geq 0$ and $\delta > 0$, we let

$$\mathcal{H}^s_\delta(A) = \inf\left\{\sum_{i=1}^{\infty} |U_i|^s : \{U_i\}_{i=1}^{\infty} \text{ is a } \delta\text{-cover of } A\right\}.$$

The s-dimensional Hausdorff measure of A is

$$\mathcal{H}^s(A) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(A),$$

and the Hausdorff dimension of A is the number

$$\mathrm{HD}(A) = \sup\{s : \mathcal{H}^s(A) = \infty\} = \inf\{s : \mathcal{H}^s(A) = 0\}.$$

To begin with, let us consider the estimate from below. The main result of [DZG] and [M] is that the Hausdorff dimension of a set is larger than or equal to the topological entropy of the map on this set divided by the logarithm of the Lipschitz constant of this map. From this and from Theorem 4.3 we get immediately the following theorem.

Theorem 7.1. *If an expanding polymodial f of degree d restricted to its Julia set is Lipschitz continuous with the Lipschitz constant L then $HD(J(f)) \geq \log d / \log L$.*

Now we give the example we promised.

Example 7.2. We will construct an expanding polymodial f with $HD(J(f)) = 0$. It will be similar to a c-tent map. More precisely, it will be of the form

$$f(z) = \varphi(|z|)\frac{z^2}{|z|^2} - 2$$

for some strictly increasing real valued function φ on $[0, \infty)$ such that φ^{-1} is Lipschitz continuous with a constant less than 1, $\varphi(0) = 0$ and $\varphi(1) = 3$.

Since φ^{-1} is Lipschitz continuous with a constant less than 1, φ is expanding, and therefore f is an expanding polymodial. If x is real and nonnegative, then $f(x) = \varphi(x) - 2$, and if x is negative then $f(x) = f(-x)$. Since $f(0) = -2$ and $f(1) = f(-1) = 1$, the interval $[-1, 1]$ is mapped by f onto $[-2, 1]$. All points of $[-2, -1)$ are mapped to $(1, \infty)$, which is contained in the basin of attraction of the infinity. Let A be the set of all points of $[-1, 1]$, whose trajectories are contained in $[-1, 1]$. It is equal to the intersection of unions of intervals, exactly as in the classical construction of the Cantor set, except that the lengths of those intervals depend on the function φ and thus may (and in our case will) be different. As long as the function φ satisfies the conditions listed in the preceding paragraph, the maximal length of the intervals obtained on each step converges exponentially to 0. Thus, A is homeomorphic to the Cantor set (in particular, A is closed). Since the degree of f is 2 and every point of A has 2 preimages in A, the set A is fully invariant. Fix a point $b \in A$; by Lemma 3.13 of [BCM] every point of $J(f)$ is the limit of a sequence of preimages of b under the iterates of f, and since A is fully invariant, $J(f)$ is contained in $\overline{A} = A$. All points of A have bounded trajectories, so they belong to $J(f)$. Thus, $J(f) = A$. Observe that the same arguments would apply to any fully invariant compact set on which the map is 2-to-1.

Define the set A_n as the set of all points x such that $f^i(x) \in [-1, 1]$ for $i = 0, 1, \dots, n$. The idea of the construction is to simultaneously build a sequence of collections of intervals A_n and define the map f on them (and eventually on the entire plane), so that the Hausdorff dimension of A is 0. To this end we need to look closely at the sets A_n. Observe that the set A_n is the union of 2^n intervals. Thus, $A_0 = [-1, 1]$, A_1 is the union of 2 intervals $[-1, a]$ and $[a, 1]$ for some $0 < a < 1$, and in general A_n is the set of 2^n closed intervals. The set $A_n \setminus A_{n+1}$ is also the union of 2^n intervals (each of them contained in the corresponding interval of A_n). We will call those intervals *gaps of n-th generation*; in other words, each component I of the set A_n contains exactly one gap of n-th generation, and if we remove this gap we will obtain the two components of A_{n+1} contained in I.

Each gap of n-th generation must be mapped by f to a gap of $(n-1)$-st generation. We will require this map to be affine. Since f is continuous, if a sequence of sets A_n is given, this requirement will define f on the gap of the 0-th generation, then on the gaps of the 1-st generation, and so on. Moreover, if the sets A_n are chosen so that their intersection is homeomorphic to the Cantor set, then eventually it will define the map on the union of all gaps, and hence by continuity on the entire interval $[-1, 1]$. Of course not every choice of sets A_n leads to an expanding polymodial. In order for f to be of the right form, A has to be symmetric with respect to 0 and the ratio of the lengths of gaps of $(n-1)$-st generation to the gaps of n-th generation should be at least some $\alpha > 1$.

In order to simplify the construction, we make all gaps of n-th generation of the same length, and all components of each set A_n of the same length too (although, clearly, the length of a component of A_n will be larger than the length of a gap of n-th generation). The gap of the 0-th generation will have length 2/3, and the components $[-1, -1/3]$ and $[1/3, 1]$ of A_1 will have also lengths 2/3, so the gaps of the first generation are shorter. Then we make the lengths of the components of A_n and the lengths of the gaps of n-th generation decrease very rapidly with n. Thus the conditions from the preceding paragraph will be satisfied. We make φ

also affine on $[0,1/3]$ and $[1,\infty)$. The slope of φ on $[0,1/3]$ is 3, and we make the slope of φ on $[1,\infty)$ also 3. This completes the definition of f, except that we have to specify the lengths of the components of A_n.

When we take the cover of A by the components of A_n, we have 2^n of them, and the length of each of them is some number b_n. To get $\mathrm{HD}(A)=0$, it is enough to have $(\log(2^n))/(-\log b_n)\to 0$ as $n\to\infty$. This is satisfied if we set $b_0=2$ and $b_n=(4/3)2^{-n^2}$ for $n\geq 1$. Then in particular $b_1=2/3$ and $2b_2=1/6<2/3$, as it should be. For $n\geq 1$ the gaps of n-th generation have lengths

$$\frac{4}{3}\left(2^{-n^2}-2\cdot 2^{-(n+1)^2}\right)=\frac{4}{3}\cdot 2^{-n^2}\left(1-4^{-n}\right).$$

The ratio of two consecutive numbers of this form is equal to

$$\frac{\frac{4}{3}\cdot 2^{-n^2}(1-4^{-n})}{\frac{4}{3}\cdot 2^{-(n+1)^2}(1-4^{-n-1})}=2\cdot\frac{4^n-1}{1-4^{-n-1}}>6,$$

and the length of the gap of 0-th generation is $2/3$, while the gaps of the first generation have length $1/2$. This gives the stretching factor $\alpha=4/3>1$. Thus, f is an expanding polymodial with the Julia set of Hausdorff dimension 0. $\square$

Now we pass on to the upper estimate on the Hausdorff dimension of the Julia set of an expanding polymodial. Unlike our lower estimates, they require much more information about the map. Moreover, it may seem that the proofs are more difficult. However, to some extent this is because for the lower estimates we have the results of [DZG] and [M], which allow us to get these estimates almost for free (modulo Theorem 4.3 which is already proven), while before we can obtain the upper estimates we need to develop certain tools. Let us start by describing the information we want to know about an expanding polymodial whose Julia set we study.

Let $f:\mathbb{C}\to\mathbb{C}$ be an expanding polymodial of degree d with singular set $S(f)$. We assume that $f|_{\mathbb{C}\setminus S(f)}$ is of class C^2 and Lipschitz continuous. We denote the Lipschitz constant of f by L and its minimal expansion rate by α. The minimal Jacobian is denoted by β. Instead of considering the set $S(f)$, we will rather use the set $\tilde{S}$, which is the union of $S(f)$ and all periodic orbits of the points of $S(f)$.

At every point $z\in\mathbb{C}$ we have its *local degree*, that is a number p such that f is p-to-1 in a neighborhood of z. Of course, this number is 1 at any regular point of f and larger than 1 at any singular point. If z is a periodic point of period n of f, we define the *local periodic degree* at z as the n-th root of the product of local degrees at points of the orbit of z. Finally, the *periodic degree* of f is the maximum of local periodic degrees at all periodic points. We will denote it by d_p. Note that $d_p=1$ if and only if f has no periodic singular points.

Theorem 7.3. *The Hausdorff dimension of the Julia set J of an expanding polymodial is less than or equal to*

$$\max\left(\frac{\log\frac{dL^2}{\beta}}{\log L},\frac{\log d_p}{\log\alpha}\right).$$

Note that if f has no periodic singular points then $\log d_p = 0$, so the estimate is simply $(\log \frac{dL^2}{\beta})/(\log L)$. Observe also that if $d \geq \beta$ then $(\log \frac{dL^2}{\beta})/(\log L) \geq 2$ and the estimate from Theorem 7.3 holds automatically. Similarly, it holds automatically if $d_p \geq \alpha^2$. On the other hand, if $d < \beta$ and $d_p < \alpha^2$ then, by Theorem 7.3, $\mathrm{HD}(J) < 2$.

We will prove this theorem in steps.

Upper bounds on $\mathrm{HD}(J)$ can be obtained by producing arbitrarily small δ-covers $\{U_i\}$ of J and considering $\sum_{i=1}^{\infty} |U_i|^s$. Given a set U of diameter δ, the connected components of $f^{-n}(U)$ may be chopped up and covered with balls of diameter $\delta' < \delta$. In this way, a δ'-cover of J may be produced from an existing δ-cover. When the sets U are closed disks, and the preimage sets comprising $f^{-n}(U)$ are particularly regular, this method is commonly referred to as "cutting up ellipses", see e.g. [F]. We use a version of this here.

By $D(z,r)$ we will denote the closed ball centered at z with radius r. For a set V and a positive integer n we will call a point z (V,n)-*good* if none of the points $z, f(z), f^2(z), \ldots, f^{n-1}(z)$ belongs to V. Otherwise of course it will be called (V,n)-*bad.*

Lemma 7.4. *Let V be a neighborhood of $\tilde{S} \cup \{\infty\}$, and n be a positive integer. Then there exists $\varepsilon > 0$ such that if $r \leq \varepsilon$ and z is (V,n)-good then the component of $f^{-n}(D(f^n(z),r))$ containing z is contained in a rectangle of side lengths s and t, where $\frac{2r}{L^n} \leq s \leq t$, and $st \leq \frac{8r^2}{\beta^n}$.*

Proof. Since f is Lipschitz continuous, if δ is sufficiently small then there is a closed neighborhood W of $\tilde{S} \cup \{\infty\}$ such that whenever z is (V,n)-good, all sets $A, f(A), f^2(A), \ldots, f^{n-1}(A)$, where $A = D(z,\delta)$, are disjoint from W. The map f restricted to $\mathbb{C} \setminus W$ is a local diffeomorphism of class C^2. Therefore there is $\varepsilon > 0$ such that under our assumptions on z there is a branch g of f^{-n} mapping $D(f^n(z),\varepsilon)$ diffeomorphically onto a neighborhood of z. This g is of class C^2, and thus if ε is sufficiently small and $r \leq \varepsilon$ then $g(D(f^n(z),r))$ is contained in an ellipse which we get when we replace g by its linear approximation, enlarged homothetically by a factor $\sqrt{2}$. This ellipse is contained in turn in a rectangle of side lengths $s \leq t$, where $s/2$ and $t/2$ are the semiaxes of the ellipse. The area of the ellipse is at most $2\pi r^2/\beta^n$, so because the area st of the rectangle is by the factor $4/\pi$ greater than the area of the ellipse, we conclude that $st \leq 8r^2/\beta^n$. The estimate $s \geq 2r/L^n$ follows immediately from the Lipschitz property of f. ■

Lemma 7.5. *Under the assumptions of the preceding lemma, if a point w has k_g (V,n)-good preimages and k_b (V,n)-bad preimages under f^{-n}, then $f^{-n}(D(w,r))$ can be covered by $8k_g(L^2/\beta)^n$ closed disks of radius $\sqrt{2}r/L^n$ and k_b closed disks of radius r/α^n.*

Proof. Consider a point $x \in f^{-n}(D(w,r))$. If the component of $f^{-n}(D(w,r))$ to which x belongs, contains also a (V,n)-good point of $f^{-n}(w)$ then, by Lemma 7.4, x belongs to a rectangle form that lemma. There are k_g such rectangles. If x does not belong to a component of $f^{-n}(D(w,r))$ containing a (V,n)-good point of $f^{-n}(w)$ then, by Lemma 3.4 of [BCM], it lies in a closed disk of radius r/α^n centered at a (V,n)-bad point of $f^{-n}(w)$. There are k_b such points.

Thus, it remains to prove that a rectangle from Lemma 7.4 can be covered by $8(L^2/\beta)^n$ closed disks of radius $\sqrt{2}r/L^n$. To do this, we first cover the rectangle by squares of side length $q = 2r/L^n$. We can do this with lm squares, where l, m are integers, if $s/q \le l$ and $t/q \le m$. By Lemma 7.4, $1 \le s/q \le t/q$, so there exist integers l, m such that $s/q \le l < 2s/q$ and $t/q \le m < 2t/q$. Then, again by Lemma 7.4, $lm \le 4st/q^2 \le 8L^{2n}/\beta^n$. As the last step we cover each square by a closed disk of radius $\sqrt{2}r/L^n$. ∎

Now we fix n, V, ε, r as in Lemma 7.4 and a point $z_0 \in \mathbb{C}$, and define a *disk tree* T by induction. The vertices of our tree will be disks, and they will be divided into generations. There will be only one disk of the first generation, namely $D(z_0, r)$. If we have a disk A of k-th generation, then we apply Lemma 7.5, where w is the center of A and r is replaced by the radius of A. The new disks obtained in such a way will belong to $(k+1)$-st generation and will be joined with A by edges of T. Thus, the new disks cover f^n-preimages of the disks of the previous generation. In order to have the radii of the disks decrease to 0 as the generation number goes to infinity (and to be sure that they all are smaller than ε), we assume that n is so large that $\sqrt{2}/L^n < 1$. For every vertex v of generation $m > 1$, we will denote by $\varphi(v)$ the unique vertex of generation $m-1$ which is joined with v by an edge. We will then consider φ as a map, so we will speak of $\varphi^k(v)$ and $\varphi^{-1}(v)$.

We will call a positive integer n *admissible* if $L^n > \sqrt{2}$, periods of all periodic singular points of f divide n, and if for any two points $z, w \in \tilde{S}$ with $f^m(z) = w$ we can find a number $m_1 < n$ such that $f^{m_1}(z) = w$.

Recall that V is a neighborhood of $\tilde{S} \cup \infty$. We will be considering the set V of a special form. Namely, we choose a sufficiently large number R and a small number δ, and define V as the union of the open disks of radius δ centered at the points of $\tilde{S}$ and the complement of the closed disk of radius R centered at 0. The union of the disks of radius δ centered at the periodic points of $\tilde{S}$ will be denoted by V'. Clearly, $V' \subset V$.

We will paint the vertices of the disk tree T into 3 colors. If the center of v (remember that v is a disk) is (V, n)-good then v will be painted *green*. If the center z of v is (V, n)-bad, and z belongs to V', we will paint v *red*. If the center z of v is (V, n)-bad and z does not belong to V', we will paint v *yellow*.

Lemma 7.6. *For given positive integers n, K, with n admissible, there exist V of the above form and $\varepsilon > 0$ such that the following properties hold.*

(1) *If a vertex v of generation m is not green and $\varphi(v)$ is green then the vertices $\varphi^j(v)$ are green for $j = 1, 2, \ldots, \min(K, m-1)$.*
(2) *If vertices v and $\varphi(v)$ are yellow then $\varphi^2(v)$ is not yellow.*
(3) *If a vertex v is red then the vertex $\varphi(v)$ is not yellow.*
(4) *If a vertex v is red then at most d_p^n of the elements of $\varphi^{-1}(v)$ are red.*

Proof. Consider δ and R that define V. If we take δ sufficiently small then it takes more than Kn iterates of f to get from a 2δ-neighborhood of a point $z \in \tilde{S}$ to a 2δ-neighborhood of a point $w \in \tilde{S}$, unless $f^k(z) = w$ (but then either $k \le n$ or both z and w belong to the same periodic orbit). Also, we can take R so big that first Kn images of 2δ-neighborhoods of points of $\tilde{S}$ are contained in the disk of radius R centered at 0.

If we now choose a sufficiently small ε, we see that the first three properties are satisfied. Observe that a point which is sufficiently close to a point of $\tilde{S}$ will stay in the disk of radius R centered at 0 for a long time. Moreover, once a trajectory leaves a very small neighborhood of an element z of $\tilde{S}$, it takes a long time for it to get to a small neighborhood of some element w of $\tilde{S}$, unless $f^j(z) = w$ for some j. It cannot get back close to the trajectory of z (if δ is sufficiently small), because f is an expanding polynomial. Since n is admissible, this proves (1) and (2) (two yellow vertices in a row may occur, since when cutting a trajectory into pieces of length n, we may cut it at a place after leaving a neighborhood of z and before arriving to a neighborhood of w). To see that (3) is satisfied, note that once a trajectory finally leaves a small neighborhood of a periodic orbit of a singular point, it still stays close to this periodic orbit, where there are no other singular points.

To count the maximal possible number of red elements of $\varphi^{-1}(v)$, we have to know how many elements of $f^{-n}(z)$ is close to z, if z is very close to a periodic point w of $\tilde{S}$. If the period is k then n is a multiple of k (since it is admissible), so the number of the elements of $f^{-n}(z)$ is close to z is at most the local periodic degree at w to the power k to the power n/k, that is, the local periodic degree at w to the power n. This proves (4). ■

We know now which vertex is of which color, but there still may be different shades. We define K shades of green and 2 shades of yellow. Assume first that $1 \le k \le K-1$. If v is a green vertex of T, vertices $\varphi^j(v)$ are defined and green for $j = 1, 2, \dots, k-1$, and the vertex $\varphi^k(v)$ is defined, but it is not green, then we say that the *shade* of v is k. The shade of the rest of green vertices is K. If v is yellow and $\varphi(v)$ is defined and yellow, then the shade of v is 2. The shade of all other yellow vertices is 1.

With this notation, we can summarize our knowledge of the tree T in the following directed graph $\mathcal{G}$.

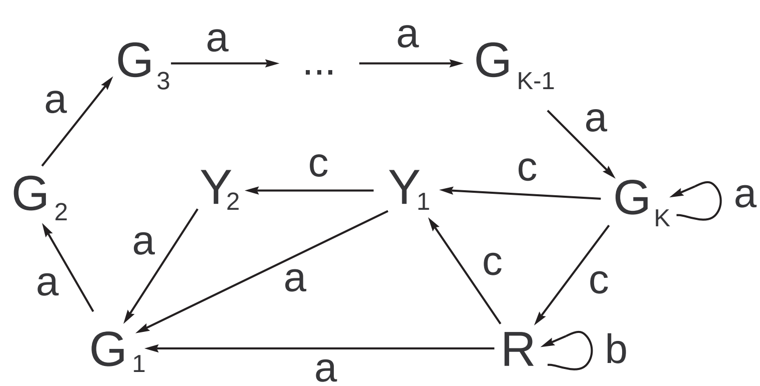

FIGURE 7.1. The graph $\mathcal{G}$ for T

The notation for vertices is obvious, for example, G_j means green, shade j. An arrow from A to B means that it may happen that v has color/shade B and $\varphi(v)$ color/shade A (note that the arrows go in the opposite direction than φ). The absence of an arrow means that this cannot happen (by Lemma 7.6 or the shading rules). Let us explain now the letters a, b, c by the arrows. A letter corresponding to an arrow $A \to B$ indicates an upper bound on the number of elements of $\varphi^{-1}(v)$

of color/shade B if v is of color/shade A, and the ratio of the radii of those elements to the radius of v. By Lemmas 7.5 and 7.6 (4), those numbers and ratios are the following.

For a: number $8(dL^2/\beta)^n$, ratio $\sqrt{2}/L^n$.

For b: number d_p^n, ratio $1/\alpha^n$.

For c: number d^n, ratio $1/\alpha^n$.

Now we are in a position to complete the proof of Theorem 7.3. The estimate in this theorem is the real number s such that the pressure of the subshift of finite type with the graph $\mathcal{G}$, for the observable (potential) that is the logarithm of the (number)$\cdot$(ratio)^s above, is zero. Such method is standard for computing the Hausdorff dimension (see, e.g. [Pe]), except that in our case we perform kind of coding and the observable is not that simple.

Proof of Theorem 7.3. Denote the radius of the vertex of T of generation 1 by r. For every vertex v of generation m we define its *color m-code* as the sequence of vertices of the graph $\mathcal{G}$ corresponding to the color/shade of $\varphi^{m-1}(v), \varphi^{m-2}(v), \dots, \varphi(v), v$. In other words, the color m-code of v is a path of length $m-1$ in $\mathcal{G}$. From this path we can easily read off an upper bound for the number of vertices of generation m with this color m-code and their radii. Namely, the bound for the number is the product of the numbers corresponding to the letters a, b, c by the arrows of the path, and the radius is the product of the corresponding ratios, times r.

Fix $s > 0$ and set $s_a = 8(dL^2/\beta)^n(\sqrt{2}/L^n)^s$, $s_b = d_p^n(1/\alpha^n)^s$ and $s_c = d^n(1/\alpha^n)^s$. We will say that s_a, s_b, s_c are the *quantities associated* to arrows in $\mathcal{G}$ marked a, b, c respectively. The product of quantities associated to the arrows of a given path will be called the quantity associated to this path.

Now, the sum of the diameters to the power s of all vertices of generation m is not larger than the sum over all paths of length $m-1$ of quantities associated to those paths (call this number $N(m)$) times $(2r)^s$. In every path in $\mathcal{G}$ there can be at most 3 arrows marked c in a row, and every block of arrows marked c is followed by a block of arrows marked a of length at least K. Therefore if we assume that $s_a < 1$ and $s_b < 1$, if K is so large that $s_c^3 s_a^K < 1$ then there are constants $\lambda > 0$ and $\mu \in (0,1)$ such that for every m and a path of length $m-1$, the quantity associated to this path is less than $\lambda\mu^m$.

Our plan is to show that $N(m) \to 0$ as $m \to \infty$. As we will see later, this means that the s-dimensional Hausdorff measure of J is equal to 0, thus implying that $\mathrm{HD}(J) \le s$. In order to show that $\lim_{m\to\infty} N(m) = 0$ if K is sufficiently large, it remains to prove that the number of paths of length $m-1$ is not larger than $\lambda_1\mu_1^m$ for some constants $\lambda_1 > 0$ and $\mu_1 \in (0, 1/\mu)$. This means that the topological entropy of the subshift of finite type given by the graph $\mathcal{G}$ is smaller than $-\log\mu$ (for sufficiently large K). We can compute the entropy of this subshift using the rome method (see [BGMY], [ALM]). The vertices R, G_K of $\mathcal{G}$ form a rome. There are paths of length $K+2, K+1, 1$ from G_K to itself, a path of length 1 from G_K to R, paths of lengths $K+2, K+1, K$ from R to G_K, and a path of length 1 from R to itself. Therefore the determinant that we have to compute in the rome method is equal to

$$(x^{-(K+2)} + x^{-(K+1)} + x^{-1} - 1)(x^{-1} - 1) - (x^{-(K+2)} + x^{-(K+1)} + x^{-K})(x^{-1})$$
$$= (1 - x^{-1})^2 - x^{-(K+2)}(2x + 1).$$

Topological entropy of the subshift is equal to the logarithm of the largest zero of this function (call it Φ_K). We have $\Phi_K(1) < 0$, while if $\nu > 1$ then, if K is sufficiently large, $\Phi_K(x) > 0$ for all $x > \nu$. This means that the limit of the largest zero of Φ_K is 1 as $K \to \infty$. This completes the proof that if K is sufficiently large then $\lim_{m\to\infty} N(m) = 0$.

Now we use the standard method of estimating the Hausdorff dimension of J. We cover it by a finite number of balls with radii less than the number ε specified earlier. The computation made above shows that if $s_a < 1$ and $s_b < 1$ (we take K sufficiently large) then the s-dimensional Hausdorff measure of J is zero. Thus, in this case, $\mathrm{HD}(J) \le s$. Therefore $\mathrm{HD}(J)$ is less than or equal to the minimal number s for which $\max(s_a, s_b) = 1$. We have $s_a = 1$ for

$$s = \frac{n \log \frac{dL^2}{\beta} + \log 8}{n \log L - \log \sqrt{2}}$$

and $s_b = 1$ for $s = (\log d_p)/(\log \alpha)$. Letting n go to infinity, we get the desired estimate of $\mathrm{HD}(J)$. ■

Let us apply Theorem 7.3 to our main examples of expanding polymodials, that is, to tent maps. Then $L = 2|c|$, $\beta = 2|c|^2$ and $d = 2$. Thus,

$$\frac{\log \frac{dL^2}{\beta}}{\log L} = \frac{2 \log 2}{\log 2 + \log |c|} < 2.$$

Thus, together with Theorem 7.1, we get the following corollary.

Corollary 7.7. *If f is a c-tent map then the Hausdorff dimension of its Julia set is positive. If additionally 0 is not periodic for f then this dimension is less than 2.*

For c-tent maps with 0 periodic of period k, we have $d_p = \sqrt[k]{2}$ and $\alpha = |c|$, so in Theorem 7.3 we have to take also into account the number

$$\frac{\log d_p}{\log \alpha} = \frac{\log 2}{k \log |c|}.$$

Unfortunately this number can be larger than 2. For instance, 0 is periodic of period 7 for c equal approximately $0.11248827878329313 + 0.99741901747024\,23i$. Then $|c|$ is approximately 1.00605043758868500, so $(\log 2)/(7 \log |c|)$ is over 16. We can improve estimates by a more careful study of the preimages of balls that are close to the periodic orbit of 0, but it seems that with $|c|$ so close to 1 with relatively small period 7, some new ideas would be necessary to get an upper estimate of the Hausdorff dimension less than 2.

Of course we can improve the estimate from Theorem 7.3 also for general expanding polymodials if there are no periodic singular points. For instance, we can apply it to f^n instead of f, or take into account that large stretching can occur in a completely different place than small Jacobian. This should lead to a "pressure" formula (see e.g. [Pe]). If the image of the unit ball under the derivative of f at

z has semiaxes $L(z) \geq \alpha(z)$, then the Hausdorff dimension should be less than or equal to the value of s for which the pressure for the logarithm of the function

$$\Phi_s(z) = \frac{L(z)}{\alpha(z)} \left(\frac{1}{L(z)} \right)^s$$

is equal to 0. Perhaps some assumptions on regularity at singular points (for instance, that the limit of Φ_s exists at every singular point) should be assumed.

However, for c-tent maps such a formula would yield the same estimate as Theorem 7.3. The real questions are whether this formula not only gives an upper estimate, but the actual value of $\mathrm{HD}(J)$, and whether it is valid also in the case when there are periodic singular points.

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