PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 00, Number 0, Pages 000–000
S 0002-9939(XX)0000-0

COMPLEMENTARY COMPONENTS TO THE CUBIC PRINCIPAL HYPERBOLIC DOMAIN

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Abstract. We study the closure of the cubic Principal Hyperbolic Domain and its intersection $\mathcal{P}_\lambda$ with the slice $\mathcal{F}_\lambda$ of the space of all cubic polynomials with fixed point 0 defined by the multiplier λ at 0. We show that any bounded domain $\mathcal{W}$ of $\mathcal{F}_\lambda \setminus \mathcal{P}_\lambda$ consists of J-stable polynomials f with connected Julia sets $J(f)$ and is either of *Siegel capture* type (then $f \in \mathcal{W}$ has an invariant Siegel domain U around 0 and another Fatou domain V such that $f|_V$ is two-to-one and $f^k(V) = U$ for some $k > 0$) or of *queer* type (then a specially chosen critical point of $f \in \mathcal{W}$ belongs to $J(f)$, the set $J(f)$ has positive Lebesgue measure, and carries an invariant line field).

1. Introduction

In this paper, we study topological dynamics of complex cubic polynomials. We denote the *Julia set* of a polynomial f by $J(f)$ and the *filled Julia set* of f by $K(f)$. Let us recall classical facts about quadratic polynomials. The Mandelbrot set $\mathcal{M}_2$, perhaps the most well-known mathematical set outside of the mathematical community, can be defined as the set of all complex numbers c such that the sequence

$$c, \quad c^2 + c, \quad (c^2 + c)^2 + c, \dots$$

is bounded. The numbers c label polynomials $z^2 + c$. Every quadratic polynomial can be reduced to this form by an affine coordinate change.

By definition, $c \in \mathcal{M}_2$ if the orbit of 0 under $z \mapsto z^2 + c$ is bounded, i.e., $0 \in K(z^2 + c)$. Note that 0 is the only critical point of the polynomial $z^2 + c$ in $\mathbb{C}$. Generally, the behavior of critical orbits to a large extent determines the dynamics of other orbits. For example, by a classical theorem of Fatou and Julia, $c \in \mathcal{M}_2$ if and only if $K(z^2 + c)$ is connected. If $c \notin \mathcal{M}_2$, then the set $K(z^2 + c)$ is a Cantor set.

The central part of the Mandelbrot set, the so called *Principal Hyperbolic Domain* PHD_2, is bounded by a cardioid called the *Main Cardioid*. By definition, the Principal Hyperbolic Domain PHD_2 consists of all parameter values c such that the polynomial z^2+c is hyperbolic, and the set $K(z^2+c)$ is a Jordan disk (a polynomial of any degree is said to be *hyperbolic* if the orbits of all its critical points converge to attracting cycles). Equivalently, $c \in \mathrm{PHD}_2$ if and only if $z^2 + c$ has an attracting fixed point. The closure of PHD_2 consists of all parameter values c such that $z^2 + c$

Received by the editors November 11, 2014; revised November 30, 2015; second revision January 20, 2016.
2010 *Mathematics Subject Classification.* Primary 37F45; Secondary 37F10, 37F20, 37F50.
Key words and phrases. Complex dynamics; Julia set; polynomial-like maps; laminations.

has a non-repelling fixed point. As follows from the Douady–Hubbard parameter landing theorem [DH8485, Hub93], the Mandelbrot set itself can be thought of as the union of the main cardioid and *limbs* (connected components of $\mathcal{M}_2 \setminus \overline{\mathrm{PHD}}_2$) parameterized by reduced rational fractions $p/q \in (0,1)$.

This motivates our study of higher degree analogs of PHD_2 started in [BOPT14]. More precisely, complex numbers c are in one-to-one correspondence with affine conjugacy classes of quadratic polynomials (throughout we call affine conjugacy classes of polynomials *classes* of polynomials). Thus a natural higher-degree analog of the set $\mathcal{M}_2$ is the *degree d connectedness locus* $\mathcal{M}_d$ defined as the set of classes of degree d polynomials f, all of whose critical points do not escape, or, equivalently, whose Julia set $J(f)$ is connected. The *Principal Hyperbolic Domain* PHD_d of $\mathcal{M}_d$ is defined as the set of classes of hyperbolic degree d polynomials with Jordan curve Julia sets. Equivalently, the class $[f]$ of a degree d polynomial f belongs to PHD_d if all critical points of f are in the immediate attracting basin of the same attracting (or super-attracting) fixed point. In [BOPT14] we describe properties of cubic polynomials f such that $[f] \in \overline{\mathrm{PHD}}_d$; notice that Theorem 1.1 holds for any $d \geqslant 2$.

Theorem 1.1 ([BOPT14])**.** *If $[f] \in \overline{\mathrm{PHD}}_d$, then f has a non-repelling fixed point, no repelling periodic cutpoints in $J(f)$, and all its non-repelling periodic points, except at most one fixed point, have multiplier 1.*

Observe that, strictly speaking, in [BOPT14] we claim that all non-repelling periodic *cutpoints* in the Julia set $J(f)$, except perhaps one, have multiplier 1; however, literally repeating the same arguments one can prove the version of the results of [BOPT14] given by Theorem 1.1 (i.e., we can talk about all non-repelling periodic points of f, not only its periodic cutpoints). Theorem 1.1 motivates the following definition; notice that from now on in the paper we concentrate upon the cubic case (thus, unlike Theorem 1.1, Definition 1.2 deals with cubic polynomials).

Definition 1.2 ([BOPT14])**.** Let CU be the family of classes of cubic polynomials f with connected $J(f)$ such that f has a non-repelling fixed point, no repelling periodic cutpoints in $J(f)$, and all its non-repelling periodic points, except at most one fixed point, have multiplier 1. The family CU is called the *Main Cubioid.*

Let $\mathcal{F}$ be the space of polynomials

$$f_{\lambda,b}(z) = \lambda z + bz^2 + z^3, \quad \lambda \in \mathbb{C}, \quad b \in \mathbb{C}.$$

An affine change of variables reduces any cubic polynomial f to the form $f_{\lambda,b}$. Note that 0 is a fixed point for every polynomial in $\mathcal{F}$. The set of all polynomials $f \in \mathcal{F}$ such that 0 is non-repelling for f is denoted by $\mathcal{F}_{nr}$ (in other words, $\mathcal{F}_{nr}$ is the set of all polynomials $f_{\lambda,b}$ with $|\lambda| \leqslant 1$). Define the *λ-slice* $\mathcal{F}_\lambda$ of $\mathcal{F}$ as the space of all polynomials $g \in \mathcal{F}$ with $g'(0) = \lambda$. The space $\mathcal{F}$ maps onto the space of classes of all cubic polynomials with a fixed point of multiplier λ as a finite branched covering. This branched covering is equivalent to the map $b \mapsto a = b^2$, i.e., classes of polynomials $f_{\lambda,b} \in \mathcal{F}_\lambda$ are in one-to-one correspondence with the values of a. Thus, if we talk about, say, points $[f]$ of $\mathcal{M}_3$, then it suffices to take $f \in \mathcal{F}_\lambda$ for some λ. There is no loss of generality in that we consider only perturbations of f in $\mathcal{F}$.

Assume that $J(f)$ is connected. In [Lyu83, MSS83], the notion of *J-stability* was introduced for any holomorphic family of rational functions: a map is *J-stable* *with respect to a family of maps* if its Julia set admits an equivariant holomorphic motion over some neighborhood of the map in the given family. We say that $f \in \mathcal{F}_\lambda$ is *stable* if it is J-stable with respect to $\mathcal{F}_\lambda$ with $\lambda = f'(0)$, otherwise we say that f is *unstable*. The set $\mathcal{F}^{st}_\lambda$ of all stable polynomials $f \in \mathcal{F}_\lambda$ is an open subset of $\mathcal{F}_\lambda$. A component of $\mathcal{F}^{st}_\lambda$ is called a (λ-)*stable component*.

Definition 1.3. The *extended closure* $\overline{\mathrm{PHD}}_3^e$ *of the cubic Principal Hyperbolic Domain* $\overline{\mathrm{PHD}}_3$ is the union of $\overline{\mathrm{PHD}}_3$ and classes of all polynomials from all λ-stable components Λ with $|\lambda| \leqslant 1$ such that for all $b \in \mathrm{Bd}(\Lambda)$, we have $[f_{\lambda,b}] \in \overline{\mathrm{PHD}}_3$.

It turns out that properties of polynomials from $\overline{\mathrm{PHD}}_3$ listed in Theorem 1.1 are inherited by polynomials from the extended closure $\overline{\mathrm{PHD}}_3^e$.

Theorem 1.4 ([BOPT14]). *We have* $\overline{\mathrm{PHD}}_3^e \subset \mathrm{CU}$.

In [BOPT14b] we studied polynomials $f \in \mathcal{F}_{nr}$ that have connected quadratic-like Julia sets containing 0. Lemma 2.1, Theorem 2.3 and Corollary 4.1 are proven in [BOPT14b]; we include them in this paper for the sake of completeness.

Our aim is to continue to study properties of $\overline{\mathrm{PHD}}_3$. For a compact set $X \subset \mathbb{C}$, define the *topological hull* $\mathrm{TH}(X)$ of X as the union of X with all bounded components of $\mathbb{C} \setminus X$. We will write $\mathcal{P}_\lambda$ for the set of polynomials $f \in \mathcal{F}_\lambda$ such that $[f] \in \overline{\mathrm{PHD}}_3$. In our Main Theorem we describe the dynamics of polynomials f belonging to bounded components of the set $\mathrm{TH}(\mathcal{P}_\lambda) \setminus \mathcal{P}_\lambda$ where $|\lambda| \leqslant 1$.

Consider a cubic polynomial f with a non-repelling fixed point such that $[f]$ does not belong to $\overline{\mathrm{PHD}}_3$; we call such polynomials *potentially renormalizable*. Whenever we talk about a potentially renormalizable polynomial, we always assume that it has a non-repelling fixed point. In that case we may assume that $f \in \mathcal{F}_{nr}$ (i.e., we have $f(z) = f_{\lambda,b}(z) = \lambda z + bz^2 + z^3$ with $|\lambda| \leqslant 1$).

Let $\mathcal{A} = \bigcup_{|\lambda|<1} \mathcal{F}_\lambda$. For each $g \in \mathcal{A}$, let $A(g)$ be the basin of 0. Perturbing a potentially renormalizable polynomial $f \in \mathcal{F}_{nr}$ to a polynomial $g \in \mathcal{A}$, we see that $g|_{A(g)}$ is two-to-one (otherwise $[f] \in \overline{\mathrm{PHD}}_3$). We use this in Lemma 2.1 to show that a potentially renormalizable polynomial $f \in \mathcal{F}_{nr}$ has two distinct critical points. A critical point c of f is said to be *principal* if there is a neighborhood $\mathcal{U}$ of f in $\mathcal{F}$ and a holomorphic function $\omega_1 : \mathcal{U} \to \mathbb{C}$ defined on $\mathcal{U}$ such that $c = \omega_1(f)$, and, for every $g \in \mathcal{U} \cap \mathcal{A}$, the point $\omega_1(g)$ is the critical point of g contained in $A(g)$. By Theorem 2.3, if $f \in \mathcal{F}_{nr}$ is potentially renormalizable, then the point $\omega_1(f)$ is well-defined; let the other critical point of f be $\omega_2(f)$. It is easy to see that $\omega_1(f) \in K(f)$.

Definition 1.5. Let $\mathcal{W}$ be a bounded component of $\mathrm{TH}(\mathcal{P}_\lambda) \setminus \mathcal{P}_\lambda$, where $|\lambda| \leqslant 1$. Then $\mathcal{W}$ is said to be of *Siegel capture* type if any $f \in \mathcal{W}$ has an invariant Siegel domain U around 0 and another Fatou domain V such that $f|_V$ is two-to-one and $f^{\circ k}(V) = U$ for some $k > 0$. Also, $\mathcal{W}$ is said to be of *queer* type if the set $J(f)$ contains the critical point $\omega_2(f)$. In this case, it can be shown (Theorem 4.11) that $J(f)$ has positive Lebesgue measure, and carries an invariant line field.

Observe that polynomials from components of Siegel capture type and from components of queer type have connected Julia sets.

Main Theorem. *Let* $\mathcal{W}$ *be a bounded component of* $\mathrm{TH}(\mathcal{P}_\lambda) \setminus \mathcal{P}_\lambda$*, where* $|\lambda| \leqslant 1$*. Then any polynomial* $f \in \mathcal{W}$ *is stable and has neither repelling periodic cutpoints nor neutral periodic points distinct from* 0 *in* $K(f)$*. Moreover,* $\mathcal{W}$ *is either of* Siegel capture *type or of* queer *type.*

Notation and Preliminaries: we write $\overline{A}$ for the closure of a subset A of a topological space and $\mathrm{Bd}(A)$ for the boundary of A; the n-th iterate of a map f is denoted by $f^{\circ n}$. We let $\mathbb{C}$ stand for the complex plane, $\mathbb{C}^*$ for the Riemann sphere, $\mathbb{D}$ for the open unit disk in $\mathbb{C}$ centered at 0, and $\mathbb{S} = \mathrm{Bd}(\mathbb{D})$ for the unit circle. We identify the unit circle $\mathbb{S}$ with $\mathbb{R}/\mathbb{Z}$ and denote by $\overline{\beta\gamma}$ the chord with endpoints β, $\gamma \in \mathbb{S}$. The d-tupling map of the unit circle is denoted by σ_d. We will talk about principal sets of arbitrary continuous paths $\gamma : (0, \infty) \to \mathbb{C}$ such that $\lim_{t\to\infty} \gamma(t) = \infty$, not necessarily external rays. The principal set of γ is defined as $\bigcap_{\varepsilon>0} \overline{\gamma(0, \varepsilon)}$. We also assume knowledge of basic notions from complex dynamics, such as *Green function, dynamic rays* (*of specific argument*), *Böttcher coordinate, Fatou domain, repelling, attracting, neutral periodic points, parabolic, Siegel, Cremer periodic points, impressions, principal sets*, and the like (see, e.g., [McM94]).

2. Critical points of potentially renormalizable polynomials

Throughout Section 2, we consider a potentially renormalizable polynomial $f = f_{\lambda,b}$ with $|\lambda| \leqslant 1$. Recall that, if $g \in \mathcal{A}$ is close to f, then $f|_{A(g)}$ is two-to-one and contains exactly one critical point of g denoted by $\omega_1(g)$. Let $\omega_2(g)$ be the other critical point of g. Thus, maps $g \in \mathcal{A}$ close to f have two distinct critical points with very different properties. Consistently approximating f by polynomials $g \in \mathcal{A}$, we can distinguish between critical points of f as well.

Lemma 2.1 ([BOPT14b]). *The polynomial f has two distinct critical points.*

Proof. Assume that $\omega(f)$ is the only critical point of f (then it has multiplicity two). Let $\mathcal{C}$ be the space of all polynomials $g \in \mathcal{F}$ with a multiple critical point $\omega(g)$. This is an algebraic curve in $\mathcal{F}$ passing through f. The map taking $g \in \mathcal{C}$ to $g'(0)$ is a non-constant holomorphic function. Hence there are polynomials $g \in \mathcal{C}$ arbitrarily close to f, for which $|g'(0)| < 1$. The class of any such polynomial g belongs to PHD_3 as the immediate basin of 0 with respect to g must contain the multiple critical point $\omega(g)$, contradicting our assumption on f. □

We are ready to give the following definition.

Definition 2.2. A critical point c of f is said to be *principal* if there is a neighborhood $\mathcal{U}$ of f in $\mathcal{F}$ and a holomorphic function $\omega_1 : \mathcal{U} \to \mathbb{C}$ defined on this neighborhood such that $c = \omega_1(f)$, and, for every $g \in \mathcal{U} \cap \mathcal{A}$, the point $\omega_1(g)$ is the critical point of g contained in $A(g)$.

Now let us prove Theorem 2.3.

Theorem 2.3 ([BOPT14b]). *There exists a unique principal critical point of f.*

Proof. By Lemma 2.1, the two critical points of f are different. Then there are two holomorphic functions, ω_1 and ω_2, defined on a convex neighborhood $\mathcal{U}$ of f in $\mathcal{F}$, such that $\omega_1(g)$ and $\omega_2(g)$ are the critical points of g for all $g \in \mathcal{U}$. Suppose that neither $\omega_1(f)$, nor $\omega_2(f)$ is principal. Then, arbitrarily close to f, there are cubic polynomials g_1 and $g_2 \in \mathcal{A}$ with $\omega_2(g_1) \notin A(g_1)$ and $\omega_1(g_2) \notin A(g_2)$. Since $A(g_i)$ contains a critical point for $i = 1, 2$, we must have that $\omega_i(g_i) \in A(g_i)$.

The set $\mathcal{A}$ is convex. Therefore, the intersection $\mathcal{U} \cap \mathcal{A}$ is also convex, hence connected. Let $\mathcal{O}_i$, $i = 1$, 2, be the subset of $\mathcal{U} \cap \mathcal{A}$ consisting of all polynomials g with $\omega_i(g) \in A(g)$. By the preceding paragraph, $g_1 \in \mathcal{O}_1$ and $g_2 \in \mathcal{O}_2$. We claim that $\mathcal{O}_i$ is open. Indeed, if $g \in \mathcal{O}_i$, then there exists a Jordan disk $U \subset A(g)$ with $g(U)$ compactly contained in U, and $\omega_i(g) \in U$. If $\tilde{g} \in \mathcal{U} \cap \mathcal{A}$ is sufficiently close to

g, then $\tilde{g}(U)$ is still compactly contained in U, and $\omega_i(\tilde{g})$ is still in U, by continuity. It follows that $U \subset A(\tilde{g})$, in particular, $\omega_i(\tilde{g}) \in A(\tilde{g})$. Thus, $\mathcal{O}_i$ is open. Since $\mathcal{O}_1, \mathcal{O}_2$ are open and non-empty, the set $\mathcal{U} \cap \mathcal{A}$ is connected, and

$$\mathcal{U} \cap \mathcal{A} = \mathcal{O}_1 \cup \mathcal{O}_2,$$

the intersection $\mathcal{O}_1 \cap \mathcal{O}_2$ is nonempty. Note that $\mathcal{O}_1 \cap \mathcal{O}_2$ consists of polynomials, whose classes are in PHD_3. Since $\mathcal{U}$ can be chosen arbitrarily small, it follows that f can be approximated by maps $g \in \mathcal{A}$ with $[g] \in \mathrm{PHD}_3$, a contradiction.

The existence of a principal critical point of f is thus proved. The uniqueness follows immediately from our assumption on f. □

Denote by $\omega_1(f)$ the principal critical point of f. Obviously, $\omega_1(f) \in K(f)$. Let $\omega_2(f)$ be the other critical point of f. For $g_{\lambda,b}(z) = \lambda z + bz^2 + z^3$ with $|\lambda| \leqslant 1$ (i.e., for $g \in \mathcal{F}_{nr}$) sufficiently close to f, the point $\omega_1(g)$ is a holomorphic function of g.

3. Parabolic dynamics

Suppose that 0 is a parabolic point of f with rotation number p/q, i.e., we have $f'(0) = \exp(2\pi i p/q)$. It follows that $f^{\circ q}(z) = z + az^{m+1} + O(z^{m+2})$ for small z and some non-zero coefficient a, where $m = q$ or $2q$. A *repelling vector* is defined as a vector $v \in \mathbb{C}$ such that av^m is a positive real number. Repelling vectors define m straight rays originating at 0. These rays divide the plane into m open *attracting sectors*. Let S be an attracting sector, and D a small round disk centered at 0. The map $z \mapsto z^{-m}$ is defined on $S \cap D$ and takes $S \cap D$ into a subset of the plane containing the half-plane $\mathrm{Re}(w\overline{a}) < -M$ for some big $M > 0$. Moreover, the map $z \mapsto z^{-m}$ conjugates $f^{\circ q}|_{S\cap D}$ with a map F asymptotic to $w - ma$ as $w \to \infty$. If M is big enough, then F takes the half-plane $\mathrm{Re}(w\overline{a}) < -M$ into itself. An *attracting petal* P of f at 0 is defined as the closure of the pullback of this half-plane to S. An attracting petal depends on the choice of an attracting sector S and on the choice of the number M. The following properties of attracting petals are immediate:

1. any attracting petal P is a compact subset of the plane such that $tP \subset P$ for $t \in [0, 1]$;
2. if P is an attracting petal, then the map $f^{\circ q} : P \to \mathbb{C}$ is univalent, and we have $f^{\circ q}(P) \subset P$;
3. the set $f(P)$ lies in some attracting petal of f;

In what follows, given $f \in \mathcal{F}$ and small $\varepsilon > 0$, we define $g_{f,\varepsilon} \in \mathcal{F}$ as the cubic polynomial affinely conjugate to $(1 - \varepsilon)f$.

Lemma 3.1. *Let P be an attracting petal of f. If $\varepsilon > 0$ is sufficiently small, then P is contained in $A(g_{f,\varepsilon})$.*

Proof. Set $g = g_{f,\varepsilon}$. Let us show that *every* attracting petal $\widetilde{P}$ of f is contained in $A((1 - \delta)f)$ for every $0 < \delta < 1$. Assume that there are attracting petals $\widetilde{P}_0 = \widetilde{P}$, $\widetilde{P}_1$, $\dots$, $\widetilde{P}_{q-1}$ with $f(\widetilde{P}_i) \subset \widetilde{P}_{i+p \pmod q}$. It follows from property (1) of attracting petals that the map $(1 - \delta)f$ takes $\widetilde{P}_i$ to a subset of $\widetilde{P}_{i+q \pmod q}$. Hence, $\widetilde{P} \subset A((1 - \delta)f)$.

A conjugacy between $(1 - \varepsilon)f$ and g is given by the map $z \mapsto (1 - \varepsilon)^{1/2} z$, and we may choose ε so small that the set $(1 - \varepsilon)^{-1/2} P$ is contained in some (slightly bigger) attracting petal $\widetilde{P}$ of f. By the previous paragraph, $(1 - \varepsilon)^{-1/2} P \subset A((1 - \varepsilon)f)$, hence $P \subset A(g)$. □

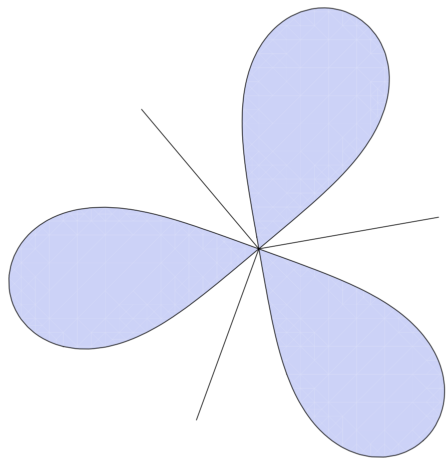

FIGURE 1. Attracting petals and repelling directions for the polynomial $f(z) = e^{2\pi i/3}z + z^2 + z^3$.

By a *parabolic domain at* 0, we mean a Fatou component of f containing some attracting petal.

Corollary 3.2. *Suppose that $f \in \mathcal{F}_\lambda$, $|\lambda| \leqslant 1$, is potentially renormalizable, and Ω is a parabolic domain of f at 0. For every compact set $D \subset \Omega$ and every sufficiently small $\varepsilon > 0$, we have $D \subset A(g_{f,\varepsilon})$.*

Proof. Let p/q be the rotation number, and let $P \subset \Omega$ be an attracting petal. We may assume that D is a Jordan disk. Replacing D with a bigger Jordan disk if necessary, we may assume that $D \cap P \neq \varnothing$. By compactness of D, there exists a positive integer m with the property $f^{\circ qm}(D) \subset P$.

Let $\varepsilon > 0$ be a small real number, and set $g = g_{f,\varepsilon} \in \mathcal{F}$. By Lemma 3.1, we have $P \subset A(g)$. We have $g^{\circ qm}(D) \subset P$ provided that ε is small enough. It follows that D is contained in some pullback of $A(g)$. Since $D \cap P \neq \varnothing$, this pullback must coincide with $A(g)$. □

Corollary 3.3 identifies $\omega_1(f)$ in the attracting and parabolic cases.

Corollary 3.3. *Let $f \in \mathcal{F}_{nr}$ be potentially renormalizable. If 0 is a parabolic (resp., attracting) fixed point of f, and c is a critical point of f belonging to a parabolic (resp., the attracting) domain Ω of f at 0, then $c = \omega_1(f)$. Thus, such Ω is unique.*

Proof. We may assume that 0 is parabolic. Let D be a small disk around c contained in Ω. By Corollary 3.2, if $\varepsilon > 0$ is small enough and $g = g_{f,\varepsilon}$, then $D \subset A(g)$. Let c_g be the critical point of g close to c. If ε is small enough, then $c_g \in D$. Therefore, $c_g \in A(g)$ and $c = \omega_1(f)$ by definition of the principal critical point. □

The proof of Corollary 3.4 is left to the reader; notice that the claim about the Julia set of a polynomial g being locally connected follows from [DH8485].

Corollary 3.4. *Suppose that $f \in \mathcal{F}_{nr}$ is potentially renormalizable, 0 is parabolic and, for some k, the point $f^{\circ k}(\omega_2(f))$ belongs to a parabolic domain at 0. Then the maps $g = g_{f,\varepsilon}$ converge to f as $\varepsilon \to 0$, have locally connected Julia sets and are such that $\omega_1(g) \in A(g)$, $\omega_2(g) \notin A(g)$ and $g^{\circ k}(\omega_2(g)) \in A(g)$.*

4. Bounded components of $\mathcal{F}_\lambda \setminus \mathcal{P}_\lambda$ must be of Siegel capture type or of queer type

We need the notion of an active critical point introduced by McMullen in [McM00]. Set $i = 1$ or 2, and take $f \in \mathcal{F}_\lambda$. The critical point $\omega_i(f)$ is *active* if, for every neighborhood $\mathcal{U}$ of f in $\mathcal{F}_\lambda$, the sequence of the mappings $g \mapsto g^{\circ n}(\omega_i(g))$ fails to be normal in $\mathcal{U}$. If the critical point $\omega_i(f)$ is not active, then it is said to be *passive*.

Corollary 4.1 ([BOPT14b]). *Let λ be a non-repelling multiplier. Every bounded component $\mathcal{W}$ of $\mathcal{F}_\lambda \setminus \mathcal{P}_\lambda$ consists of stable maps.*

Proof. By [MSS83], to prove that $f \in \mathcal{W}$ is stable, it suffices to show that both critical points of f are passive. Note that, if $g \in \mathrm{Bd}(\mathcal{W})$, then the g-orbits of $\omega_1(g)$ and of $\omega_2(g)$ are bounded uniformly with respect to g. By the maximum principle, the f-orbits of $\omega_1(f)$ and $\omega_2(f)$ are uniformly bounded for all $f \in \mathcal{W}$, which implies normality. Thus both critical points are passive, and the corollary is proved. □

We want to improve the description of the dynamics of maps in a bounded component of $\mathcal{F}_\lambda \setminus \mathcal{P}_\lambda$ given in [BOPT14] (see Theorem 1.4). Let $f \in \mathcal{F}_\lambda, |\lambda| \leqslant 1$, belong to a bounded component $\mathcal{W}_f$ of $\mathcal{F}_\lambda \setminus \mathcal{P}_\lambda$. By Corollary 4.1, the map f is stable. A priori, $\mathcal{W}_f$ (and f) can be classified into five types:

Disjoint type: the critical point $\omega_2(f)$ lies in a periodic attracting basin but not in $A(f)$.

Attracting capture type: we have $|\lambda| < 1$, and $\omega_2(f)$ is *eventually* mapped to $A(f)$.

Parabolic capture type: λ is a root of unity, and $\omega_2(f)$ is eventually mapped to a periodic parabolic Fatou domain at 0.

Siegel capture type: the critical point $\omega_2(f)$ is eventually mapped to the Siegel disk around 0.

Queer type: we have $\omega_2(f) \in J(f)$.

In this section, we prove that bounded components of $\mathcal{F}_\lambda \setminus \mathcal{P}_\lambda$ of the first three types do not exist. We use [BFMOT12], where fixed and periodic points of various maps of plane continua were studied.

Definition 4.2. A *dendrite* is a locally connected continuum containing no Jordan curves. If $g : D \to D$ is a self-mapping of a dendrite D, a periodic point a is said to be *weakly repelling* is there exists an arc $I \subset D$ with endpoint a and a number k such that $g^{\circ k}(a) = a$ and, for any $x \in I \setminus \{a\}$, the point x separates the points a and $g^{\circ k}(x)$ in D.

Theorem 4.3 (Theorem 7.2.6 [BFMOT12]). *Suppose that $g : D \to D$ is a self-mapping of a dendrite D such that all its periodic points are weakly repelling. Then g has infinitely many periodic cutpoints.*

Lemma 4.4 has a topological nature.

Lemma 4.4. *If a polynomial F has locally connected Julia set with pairwise disjoint closures of its Fatou domains, then $J(F)$ contains infinitely many periodic repelling cutpoints unless $J(F)$ is a Jordan curve.*

Proof. Assume that $J(F)$ is not a Jordan curve. Define a map ψ which collapses Fatou domains of F to points and creates a dendrite D out of $J(F)$ with the map $h : D \to D$ induced by F; thus, $\psi \circ F = h \circ \psi$ and F is semiconjugate to h (if $J(F)$

is already a dendrite, then ψ can be taken as the identity map). Then all periodic points of h are weakly repelling. Indeed, the corresponding property is immediate if an h-periodic point is the ψ-image of an F-periodic point. Otherwise let U be a periodic Fatou domain of F. Observe that, by the assumption, U cannot have a critical point, say, c in $\mathrm{Bd}(U)$ since if it does, then F will have two Fatou domains, whose closures contain c, a contradiction.

We claim that there exists a periodic cutpoint x of $J(F)$ with $x \in \mathrm{Bd}(U)$. Indeed, we may assume that U is F-invariant. Take a component K of $J(F) \setminus \overline{U}$ (it exists since $J(F) \neq \mathrm{Bd}(U)$). Then $\overline{K} \cap \overline{U} = \{y\}$ for some point y. If y is not (pre)periodic, then there exists N such that for any $n \geqslant N$ any component of $J(F) \setminus \{F^{\circ n}(y)\}$ except for one component containing U contains no critical points. This implies that any such component T is wandering. However, T must contain pullbacks of U, a contradiction. Hence y is (pre)periodic, and some image x of y is a desired periodic cutpoint of $J(F)$ of period, say, m. Since U is invariant, the combinatorial rotation number of $F^{\circ m}$ at x is zero. This and the fact that x is repelling in $J(F)$ implies that the point $\psi(\overline{U})$ is a weakly repelling periodic point of h.

Thus, any periodic point of h is weakly repelling. By Theorem 4.3, the map h has infinitely many periodic cutpoints in $\psi(J(F))$. By construction, this implies that F has infinitely many periodic repelling cutpoints (recall that F has only finitely many periodic Fatou domains). □

Corollary 4.5 now easily follows.

Corollary 4.5. *Let $g \in \mathcal{A}$ be such that $g|_{A(g)}$ is two-to-one and $g^{\circ k}(\omega_2(g)) \in A(g)$. Then g has locally connected Julia set with pairwise disjoint closures of its Fatou domains. Thus, $J(g)$ contains infinitely many periodic repelling cutpoints.*

Proof. By [DH8485], the set $J(g)$ is locally connected. Suppose that two Fatou domains U, V are such that $\overline{U} \cap \overline{V} \neq \varnothing$. Since $A(g)$ is the only periodic Fatou domain of g, both U and V must eventually map to $A(g)$. Hence there exists a unique number m such that $g^{\circ m}(U) \neq g^{\circ m}(V)$ while $g^{\circ m+1}(U) = g^{\circ m+1}(V)$. This implies that the common point of $g^{\circ m}(\overline{U})$ and $g^{\circ m}(\overline{V})$ is critical, a contradiction. Now Lemma 4.4 implies the remaining claims of this lemma. □

To prove that bounded components of $\mathcal{F}_\lambda \setminus \mathcal{P}_\lambda$ cannot be of disjoint type or attracting (parabolic) capture types, we use perturbations as a tool. Thus, we need a few general facts about perturbations. Lemma 4.6 goes back to Douady and Hubbard [DH8485]. Recall that a *smooth (external) ray* of a polynomial P, for which $J(P)$ is not necessarily connected, is defined as a homeomorphic image of $\mathbb{R}$ in $\mathbb{C} \setminus K(P)$ accumulating only in $\{\infty\} \cup J(P)$, avoiding pre-critical points of P, and tangent to the gradient of the Green function of P. An external ray of argument $\theta \in \mathbb{R}/\mathbb{Z}$ is denoted by $R_P(\theta)$.

Lemma 4.6 (Lemma B.1 [GM93])**.** *Let f be a polynomial, and z be a repelling periodic point of f. If a smooth* periodic *ray $R_f(\theta)$ lands at z, then, for every polynomial g sufficiently close to f, the ray $R_g(\theta)$ is also smooth, lands at a repelling periodic point w close to z, and w depends holomorphically on g.*

Lemma 4.7 now easily follows.

Lemma 4.7. *Suppose that $h_n \to h$ is an infinite sequence of polynomials of degree d, and $\{\alpha, \beta\}$ is a pair of periodic arguments such that the external rays $R_{h_n}(\alpha)$,*

$R_{h_n}(\beta)$ land at the same repelling periodic point x_n of h_n. If $R_h(\alpha)$, $R_h(\beta)$ are smooth but do not land at the same periodic point of h, then one of these two rays must land at a parabolic point of h.

Proof. We may assume that x_n converge to an h-periodic point x. If both rays $R_h(\alpha)$, $R_h(\beta)$ land at distinct *repelling* periodic points, then, by Lemma 4.6, we get a contradiction with the fact that $R_{h_n}(\alpha)$, $R_{h_n}(\beta)$ land at x_n and $x_n \to x$. Since α and β are periodic, by the Snail Lemma, $R_h(\alpha)$ and $R_h(\beta)$ cannot land at a Cremer point. Hence one of the rays $R_h(\alpha)$, $R_h(\beta)$ must land at a parabolic periodic point. □

The next lemma deals with much more specific perturbations.

Lemma 4.8. *Let $f \in \mathcal{F}_{nr}$ be potentially renormalizable such that the point 0 is parabolic. Suppose that $f^{\circ k}(\omega_2(f))$ belongs to a parabolic Fatou domain at 0 for some $k > 0$. Then there are infinitely many repelling periodic cutpoints in $J(f)$.*

Proof. By Corollary 3.4, we can choose a sequence $\varepsilon_n \to 0$ so that the maps $g_n = g_{f,\varepsilon_n}$ have the following properties:

(1) each g_n has an attracting fixed point 0 and a critical point $d_n \approx \omega_2(f)$ such that $d_n \notin A(g_n)$ and $g_n^{\circ k}(d_n) \in A(g_n)$;
(2) the maps g_n are pairwise topologically conjugate on their Julia sets.

The second requirement can be fulfilled because for any sequence g_n satisfying (1), the topological dynamics of $g_n : J(g_n) \to J(g_n)$ can be of finitely many types (observe that there are finitely many Thurston invariant laminations modeling polynomials with property (1)).

Now, by Corollary 4.5, there are infinitely many pairs of periodic angles α_i, β_i such that the external rays $R_{g_n}(\alpha_i), R_{g_n}(\beta_i)$ land at the same periodic repelling cutpoint $x_i(g_n)$ of $J(g_n)$. By Lemma 4.7, this implies that, for infinitely many subscripts i, the points $x_i(g_n)$ converge to a repelling periodic cutpoint $x_i(f)$ of $J(f)$, and all these points are distinct. □

We will need the following lemma.

Lemma 4.9. *If $\mathcal{V}$ is a stable domain in $\mathcal{F}_\lambda$, then $f \in \mathcal{V}$ does not have a periodic neutral point other than 0.*

Proof. If f is stable with a neutral periodic point $x = x(f) \neq 0$, then it follows that all maps $g \in \mathcal{V}$ have a neutral periodic point $x(g) \neq 0$ of the same period. Clearly, this implies that the multiplier of $x(g)$ for $g \in \mathcal{V}$ must be constant, a contradiction. □

We are ready to prove Lemma 4.10.

Lemma 4.10. *Let $\mathcal{W}$ be a component of $\mathrm{TH}(\mathcal{P}_\lambda)\setminus\mathcal{P}_\lambda$. Then any polynomial $f \in \mathcal{W}$ has connected Julia set $J(f)$ and has neither repelling periodic cutpoints nor non-repelling periodic points distinct from 0. In particular, $\mathcal{W}$ is either of Siegel capture type or of queer type.*

Proof. Assume that $f \in \mathcal{W}$. Since the critical points of polynomials from $\mathrm{Bd}(\mathcal{W})$ are non-escaping, it follows that the critical points of f are also non-escaping. Hence $J(f)$ is connected. Consider now two possibilities. Recall that, by Corollary 4.1, all maps in $\mathcal{W}$ are stable.

First, it may happen that $\mathcal{W}$ is a stable component. Then by Definition 1.3 where the extended closure $\overline{\mathrm{PHD}}_3^e$ of the cubic Principal Hyperbolic Domain $\overline{\mathrm{PHD}}_3$ is defined it follows that the polynomial f belongs to $\overline{\mathrm{PHD}}_3^e$. Hence, by Theorem 1.1, the polynomial f cannot have attracting periodic points distinct from 0 or repelling periodic cutpoints distinct from 0 of its Julia set. On the other hand, by Lemma 4.9, the polynomial f cannot have neutral periodic points distinct from 0. This completes the proof of the lemma in the case when $\mathcal{W}$ is a stable component.

Second, suppose that $\mathcal{W}$ is not a stable component. Then there is a unique stable component $\mathcal{V}$ in $\mathcal{F}_\lambda$ such that $\mathcal{V} \supsetneqq \mathcal{W}$. It follows that there is a polynomial $g \in \mathrm{Bd}(\mathcal{W}) \cap \mathcal{V}$; in other words, there is a polynomial $g \in \mathcal{V}$ such that $[g] \in \overline{\mathrm{PHD}}_3$, and the maps $f|_{J(f)}$ and $g|_{J(g)}$ are quasi-symmetrically conjugate.

Suppose that f has a repelling periodic cutpoint x of its Julia set. Clearly, $x \neq 0$. The corresponding periodic point y of g cannot be repelling by Theorem 1.1. Since $f|_{J(f)}$ and $g|_{J(g)}$ are conjugate, it follows that the only possibility for $y \neq 0$ is that y is a parabolic periodic point. However this contradicts Lemma 4.9. Hence f has no repelling periodic cutpoints of its Julia set. Moreover, by Lemma 4.9, the polynomial f has no neutral periodic points distinct from 0.

Finally, suppose that f has an attracting periodic point distinct from 0. Then the fact that f and g are quasi-conformally conjugate on their Julia sets implies that g has either (1) an attracting periodic Fatou domain U not containing 0 or (2) a parabolic periodic Fatou domain with a parabolic point $z \neq 0$ on its boundary. Since $[g] \in \overline{\mathrm{PHD}}_3$, case (1) is impossible by Theorem 1.1. On the other hand, case (2) is impossible by Lemma 4.9. This completes the proof of the first claims of the lemma.

Let us use this to show that $\mathcal{W}$ is either of Siegel capture type or of queer type. Indeed, if $f \in \mathcal{W}$ is of disjoint type, then f has an attracting periodic point distinct from 0, which is impossible by the above. If f is of attracting capture type, then, by Corollary 4.5, the set $J(f)$ contains infinitely many periodic repelling cutpoints, again a contradiction with the above. Finally, if f is of parabolic capture type, then, by Lemma 4.8, we see that f has infinitely many periodic repelling cutpoints, and we obtain the same contradiction with the above. Therefore, $\mathcal{W}$ is either of Siegel capture type or of queer type. □

The following theorem describes components of queer type.

Theorem 4.11. *Let $\mathcal{W}$ be a bounded stable component of $\mathcal{F}_\lambda$ of queer type. Then, for any polynomial $f \in \mathcal{W}$, the Julia set $J(f)$ has positive Lebesgue measure and carries an invariant line field.*

The most difficult case of Theorem 4.11 is covered by the following theorem of S. Zakeri:

Theorem 4.12 ([Zak99], Theorem 3.4)**.** *Let $0 < \theta < 0$ be a Brjuno number, and set $\lambda = e^{2\pi i\theta}$. Consider a polynomial f from a queer component of $\mathcal{C}_\lambda \setminus \mathrm{Bd}(\mathcal{C}_\lambda)$. Then $J(f)$ pas positive Lebesgue measure and carries an invariant line field.*

Brjuno numbers are irrational numbers satisfying a certain number theoretic condition. If θ is Brjuno, then any holomorphic germ at 0 of the form $z \mapsto \lambda z + \dots$, where dots denote terms of degree 2 and higher, is linearizable [Yoc]. The converse is known only for quadratic polynomials. The proof of [Zak99, Theorem 3.4] does not use the Brjuno condition. It works verbatim for any stable component $\mathcal{W}$ of $\mathcal{F}_\lambda$

such that polynomials $f \in \mathcal{W}$ have a Siegel disk around 0 (obviously, this condition does not depend on the choice of f within $\mathcal{W}$).

Proof of Theorem 4.11. The argument follows the same lines as [McS, Zak99]. There is an equivariant holomorphic motion μ of the set $J(f)$ for $f \in \mathcal{W}$. If we fix some $f_0 \in \mathcal{W}$, then μ can be regarded as a map from $J(f_0) \times \mathcal{W}$ to $\mathbb{C}$ such that $\mu(z, f_0) = z$ for all $z \in J(f_0)$ and, for any fixed z, the map $f \mapsto \mu(z, f)$ is holomorphic. Suppose that μ can be extended to a holomorphic motion of the entire complex plane in such a way that $z \mapsto \mu(z, f)$ is holomorphic on the Fatou set of f_0, for any f sufficiently close to f_0. Then $J(f_0)$ must have positive Lebesgue measure (otherwise, by the Ahlfors Lemma, the map $z \mapsto \mu(z, f)$ would be a global holomorphic conjugacy between f_0 and f). An invariant line field on $J(f_0)$ is given by the pullback of the standard conformal structure under the quasi-conformal map $z \mapsto \mu(z, f)$: say, one can take major axes of the ellipses representing the described quasi-conformal structure.

The desired extension of μ is easy if there are no bounded Fatou components of f_0. Indeed, in this case, it suffices to define $\mu(z, f)$ for z outside of $K(f_0)$. We set $\mu(z, f)$ to be a point in the dynamical plane of f, whose (suitably normalized) Böttcher coordinate with respect to f is the same as the Böttcher coordinate of z with respect to f_0. If the Fatou set of f_0 has bounded components, then we can also extend μ to each of the Fatou components by equating the (suitably normalized) linearizing coordinates. The only problem is with Siegel disks, for which the linearizing coordinates have no obvious normalization, especially if the boundary of the Siegel disk is not locally connected and there are no critical points on the boundary. This (most difficult) case is covered by [Zak99, Theorem 3.4] stated above as Theorem 4.12. □

Obviously, Lemma 4.10 and Theorem 4.11 imply our Main Theorem.

Acknowledgements

The authors are grateful to M. Lyubich for useful discussions. We are also indebted to the referee for thoughtful remarks.

The first and the third named authors were partially supported by NSF grant DMS–1201450. The second named author was partially supported by NSF grant DMS-0906316. The fourth named author was partially supported by the RFBR grant 16-01-00748a. The article was prepared within the framework of a subsidy granted to the HSE by the Government of the Russian Federation for the implementation of the Global Competitiveness Program.

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