ID NUMBER:

UAB MATH TALENT SEARCH

This is a two hour and a half contest. Answers are to be written in the spaces provided on the test sheet. Your work **must** be shown on the extra paper which we supply. In case of a tie, the judging panel will use your work to choose the winner. The point value of a problem is posted with it.

PROBLEM 1 (10 points). A sheet of paper is 1/32 inches thick. It is folded in half ten times. What is the resulting thickness?

YOUR ANSWER:

PROBLEM 2 (10 points). A driver averages 60 miles per hour on a trip. He drives 20 minutes on city streets at a constant speed. Then he accelerates and increases his speed by 45 mph for the hour that he is on the interstate highway. For the last half hour he must decrease his speed by 1/3 due to road construction on a four-lane road. How many miles did he drive on the interstate highway?

YOUR ANSWER:

PROBLEM 3 (10 points). A checker board with margins measures $20" \times 20"$ overall. In the center of the board there is a big square which consists of 64 small squares alternating black and red in color. Around it there is a 1" wide border of black. Outside of the black border there is a 1" wide border of red. What percent of the board is colored black?

YOUR ANSWER:

PROBLEM 4 (10 points). Solve the system of two equations:
$$x^3 + y^3 = 35 \quad \text{and} \quad x + y = 5.$$
Find the product of the solutions.

YOUR ANSWER:

PROBLEM 5 (20 points). Three integers form a geometric progression. If the second is increased by 8 they will form an arithmetic progression. However if after that we increase the last number by 64,

the progression becomes geometric again. Find the sum of these integers.

YOUR ANSWER:

PROBLEM 6 (40 points). Inside a right triangle ABC (angle C is 90 degrees) a point O is chosen so that the triangles OAB, OBC, OAC have the same area. Find the length of OC if it is known that $OA^2 + OB^2 = 1445$.

YOUR ANSWER:

PROBLEM 7 (40 points). Two ships A and B sail straight towards the same port P with speeds a and b respectively. In the beginning A, B and P form an equilateral triangle. After some time A and B move to the new positions, B covers 80 miles, and ABP becomes a right triangle. At the moment when A arrives at P, B is 120 miles away from P. What is the distance between A and B in the beginning?

YOUR ANSWER:

PROBLEM 8 (60 points). Angles in a convex polygon form an arithmetic progression $\alpha, \frac{4}{3}\alpha, \frac{5}{3}\alpha, \dots$. What is the maximal number of sides of such a polygon?

YOUR ANSWER:

PROBLEM 9 (80 points). Three lines parallel to the sides of a triangle are drawn through a point inside the triangle. They cut the triangle into 3 small triangles and three small parallelograms. The product of the areas of these triangles equals 13.5. What is the product of the areas of the parallelograms?

YOUR ANSWER:

PROBLEM 10 (130 points). In a convex 12-gon all diagonals are drawn. It is known that no three diagonals intersect at one point.

(1) In how many regions do these diagonals divide the entire 12-gon? Please fully justify your answer. If possible, please give also the formula for the number of regions into which diagonals of a convex n-gon

divide it under the same assumptions (give answer on the next page).

YOUR ANSWER:

(2) Suppose that instead of the assumption that no 3 diagonals intersect at one point we have an assumption that no 3 diagonals intersect at one point **except** that there is one triple of diagonals intersecting each other at one point and one set of 4 diagonals intersecting each other at one point (otherwise no more than 2 diagonals intersect at one point). What is then the answer to the same question?

YOUR ANSWER:

PROBLEM 11 (160 points). On all 8 vertices of a cube there are numbers among which there are some numbers 2, some numbers 7, and maybe some other numbers too. Each number is replaced by the average of the numbers on its 3 adjacent vertices, and this is repeated 20 times after which all the numbers on vertices are the same as in the beginning. What is the sum of all these numbers? Justify your answer.

YOUR ANSWER:

2004 UAB MTS: SOLUTIONS

PROBLEM 1 (10 points). A sheet of paper is 1/32 inches thick. It is folded in half ten times. What is the resulting thickness?

Solution: Each folding doubles the thickness. Therefore ten folds yields $2^{10}\times$ the thickness or $2^{10} \times 2^{-5}$.

So the answer is **32 inches**.

PROBLEM 2 (10 points). A driver averages 60 miles per hour on a trip. He drives 20 minutes on city streets at a constant speed. Then he accelerates and increases his speed by 45 mph for the hour that he is on the interstate highway. For the last half hour he must decrease his speed by 1/3 due to road construction on a four-lane road. How many miles did he drive on the interstate highway?

Solution: The total driving time is $1/3 + 1 + 1/2 = 11/6$ hours. At an average speed of 60 mph, the total distance driven is $60 \times (11/6) = 110$ miles. Using distance being rate $\times$ time, and s as the original speed in mph, the three phases of the trip will total 110:

$$(1/3)s + 1(s + 45) + (1/2)(2/3)(s + 45) = 110;$$

solving for s we get $s = 30$ mph.

So, the answer is $\mathbf{30 + 45 = 75}$ miles.

PROBLEM 3 (10 points). A checker board with margins measures $20" \times 20"$ overall. In the center of the board there is a big square which consists of 64 small squares alternating black and red in color. Around it there is a 1" wide border of black. Outside of the black border there is a 1" wide border of red. What percent of the board is colored black?

Solution: The total area of the board is 400 sq. in. The area *inside* the outer 1" border is $18" \times 18" = 324$ sq. in. So, the red border has an area of $400 - 324 = 76$ sq. in. The checkered area *within* the 1" black border is $16" \times 16" = 256$ sq. in. Hence, the black border has an area of $324 - 256$ or 68 sq. in. The inner checkered area is divided equally between black and red with each color having an area of $256/2 = 128$ sq. in. Thus, the total area of black is $68 + 128 = 196$ sq. in., or 49 percent of the total area of the board. So, the answer is **49** percent.

PROBLEM 4 (10 points). Solve the system of two equations:
$$x^3 + y^3 = 35 \quad \text{and} \quad x + y = 5.$$
Find the product of the solutions.

Solution: Using the fact that $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$ we conclude that the system is equivalent to the system $y = 5 - x$ and $x^2 - xy + y^2 = 7$. Plugging $y = 5 - x$ in the second equation and simplifying we get the quadratic equation $3x^2 - 15x + 18 = 0$. Its two roots are $x = 2$ and $x = 3$. Thus the two solutions are a) x=2, y=3, b) x=3, y=2.

So the answer is **6**.

PROBLEM 5 (20 points). Three integers form a geometric progression. If the second is increased by 8 they will form an arithmetic progression. However if after that we increase the last number by 64, the progression becomes geometric again. Find the sum of these integers.

Solution: Denote the numbers in question u, uq, uq^2 where q is the denominator of the geometric progression. Then we have that
$$u + uq^2 = 2(uq + 8), \; u(uq^2 + 64) = (uq + 8)^2$$
which after simplification reduces to
$$u + uq^2 = 2uq + 16, \; u = \frac{4}{4 - q}.$$
Plugging u from the second equation into the first one and simplifying we get the quadratic equation $q^2 + 2q - 15 = 0$ which has solutions $q + 1 = -5$ and $q_2 = 3$. The corresponding values of u are $u_1 = \frac{4}{9}$ and $u_2 = 4$. Since the three given numbers are integers we see that $q = 3, u = 4$ and integers in question are $4, 12, 36$.

So the answer is **52**.

PROBLEM 6 (40 points). Inside a right triangle ABC (angle C is 90 degrees) a point O is chosen so that the triangles OAB, OBC, OAC have the same area. Find the length of OC if it is known that $OA^2 + OB^2 = 1445$.

Solution: The triangles AOC and ABC have equal bases, and their areas relate to each other as 1 : 3. Hence the height OE of the triangle

AOC is equal to $\dfrac{1}{3}BC$. Similarly, $OD = \dfrac{A}{C}$. This implies that

$$AO^2 = \frac{4}{9}AC^2 + \frac{1}{9}BC^2, OB^2 = \frac{1}{9}AC^2 + \frac{4}{9}BC^2, OC^2 = OD^2 + OE^2 = \frac{1}{9}(AC^2 + BC^2).$$

Hence $\dfrac{5}{9}(AC^2 + BC^2) = 1445$ while on the other hand $OC^2 = \dfrac{1}{5}(1445) = 289$ which implies that $OC = 17$.

So the answer is **17**.

PROBLEM 7 (40 points). Two ships A and B sail straight towards the same port P with speeds a and b respectively. In the beginning A, B and P form an equilateral triangle. After some time A and B move to the new positions, B covers 80 miles, and ABP becomes a right triangle. At the moment when A arrives at P, B is 120 miles away from P. What is the distance between A and B in the beginning?

Solution: Since A comes to P before B then $a > b$. Hence if we denote by A', B' the positions of the ships at the moment when they form a right triangle with P then we can say that the angle $B'A'P$ equals 90 degrees. Observe that the angle APB is 60 degrees. Denote by x the unknown distance. Then we have that $PB' = x - 80$ and hence $PA' = \dfrac{x}{2} - 40$. It follows that $\dfrac{.5x + 40}{a} = \dfrac{80}{b}$. Similarly, at the moment when A arrives at P we have $\dfrac{x}{a} = \dfrac{x - 120}{b}$. Excluding $\dfrac{a}{b}$ from these equations we get $x^2 - 200x + 9600 = 0$ and hence $x = 240$ (the only sensible root of this equation is the positive one).

So the answer is **240**.

PROBLEM 8 (60 points). Angles in a convex polygon form an arithmetic progression $\alpha, \dfrac{4}{3}\alpha, \dfrac{5}{3}\alpha, \dots$. What is the maximal number of sides of such a polygon?

Solution: Suppose that we deal with an n-gon. Then the angles are $\alpha, \dfrac{4}{3}\alpha, \dots, \dfrac{n+2}{3}\alpha$. Since the n-gon is convex all its angles are less than π, in particular $\dfrac{n+2}{3}\alpha < \pi$ and so $\alpha < \dfrac{3\pi}{n+2}$. Now, if we sum all the angles up we will get $S = \dfrac{2n}{3}\alpha + \dfrac{\alpha}{3}\dfrac{n(n+1)}{2} < \dfrac{\pi}{n+2}(2n + \dfrac{n(n+1)}{2})$.

On the other hand the sum of all angles of the n-gon must be $(n-2)\pi$. We conclude that $\pi(n-2) < \dfrac{\pi}{n+2}(2n + \dfrac{n(n+1)}{2})$. Solving this inequality we get after simplification that $n^2 - 5n - 8 < 0$ which implies that maximal n for which it holds is 13.

So, the answer is **13**.

PROBLEM 9 (80 points). Three lines parallel to the sides of a triangle are drawn through a point inside the triangle. They cut the triangle into 3 small triangles and three small parallelograms. The product of the areas of these triangles equals 13.5. What is the product of the areas of the parallelograms?

Solution: Let us assume that the triangle in question is ABC (counterclockwise) and the side AB of the triangle is horizontal and equals 1 (we can always scale our triangle so that this is so). Also, we assume that the product of the areas of the triangles is a. Denote the point chosen inside ABC by O. Let the line parallel to AB and passing through O intersect AC at a point D and CB at a point E. Similarly, the line parallel to AC and passing through O intersects AB at F and CB at G, and the line parallel BC and passing through O intersects AC at H and AB at K. Assume that $AF : FK : KB = x : y : z$. We will use these parameters to solve the problem.

Denote the area of a triangle with vertices P, Q, R as A(PQR) and the area of a quadrilateral PQRS as A(PQRS). Then it follows from the formulas for the area of a triangle and a parallelogram that $A(ADOF)/A(OFK) = 2AF/FK = 2\dfrac{x}{y}$.

Draw a line parallel to HK through D so that it intersects AB at J. It follows that $JK = DO = AF, AJ = FK$. On the other hand it is clear that $AD : DH : HC = AJ : JK : KB = y : x : z$. Hence $A(CGOH)/A(DOH) = 2\dfrac{z}{x}$. Similarly we see that $A(OKEB)/A(OGE) = 2\dfrac{y}{z}$. Multiplying these ratios we see that the product of the parallelograms divided by the product of the triangles equals 8. Hence in general the product of the parallelograms is $8a$.

So, the answer is **108**.

PROBLEM 10 (130 points). In a convex 12-gon all diagonals are drawn. It is known that no three diagonals intersect at one point. In how many regions do these diagonals divide the entire 12-gon? Please fully justify your answer. If possible, please give also the formula for the number of regions into which diagonals of a convex n-gon divide it under the same assumptions.

Solution: (1) We suggest an argument which is applicable in general, i.e. to an arbitrary convex n-gon. The idea of the proof is based upon the idea to draw the diagonals in a step-by-step fashion and investigate how the number of regions grows. To do so, think of a few diagonals being already drown and new one being added to them. By the assumption, there are already several regions formed by the existing diagonals. If a new diagonal intersects k of them then we associate each point of intersection X of the new diagonal AB with an old one with one old region divided into 2 by the subsegment in AB which ends at X (thus, the number of regions grows by 1). Then when the diagonal comes to the ending vertex B, this also adds 1 to the number of regions; we associate this adding of 1 with the new diagonal AB itself.

This shows that while drawing diagonals we add to the initial number of regions (that is, 1) the number which is equal to the number of all intersections of diagonals plus the number of all diagonals themselves.

Now, each two vertices of an n-gon may be connected with the segment, so overall there are $n(n-1)/2$ such segments. However n segments are sides of the polygon, so there are $n(n-1)/2 - n = n(n-3)/2$ diagonals.

On the other hand, every point of intersection of two diagonals is determined by the diagonals themselves, which are in turn determined by the choice of four vertices: two vertices-endpoints for every diagonal. The choice of 4 vertices can be done by choosing first one, then second one, then third one, the fourth one, and that can be done in $n(n-1)(n-2)(n-3)$ ways. On the other hand, each such choice will be repeated in different order $4! = 24$ times. So, the formula for the number of points of intersections of diagonals is $n(n-1)(n-2)(n-3)/24$. Then the sum of the two just found numbers and 1 (as the initial number if regions when no diagonal is there) is (after simplification)

$$N = (n-1)(n-2)(n^2 - 3n + 12)/24.$$

So, in the case when $n = 12$ the answer is **550**.

(2) Take the 12-gon and perturb (change) it a little bit so that there are no more than 2 diagonals intersecting at one point. Then the 3 diagonals which used to intersect at one point form a triangle, and the 4 diagonals which used to intersect at one point form 3 triangles. In the resulting 12-gon there are 550 regions into which the diagonals divide it, so we need to subtract 1+3=4 from 550 to see what is the answer.

So, the answer is **546**.

PROBLEM 11 (160 points). On all 8 vertices of a cube there are numbers among which there are some numbers 2, some numbers 7, and maybe some other numbers too. Each number is replaced by the average of the numbers on its 3 adjacent vertices, and this is repeated 20 times after which all the numbers on vertices are the same as in the beginning. What is the sum of all these numbers? Justify your answer.

Solution: According to the algorithm, on each step we get the next cube with the numbers on its vertices. Thus, we can denote these cubes with numbers C_0 (this is the initial cube), $C_1, \ldots, C_{20}$. Let M_i be the maximum of all the numbers obtained on the step i. Then the conditions imply that $M_0 \geq M_1 \geq \ldots, M_{20}$ while on the other hand $M_0 = M_{20}$. Hence in fact $M_0 = M_1 = \cdots = M_{20} = M$. The same applies to the minima, so $m_0 = m_1 = \cdots = m_{20} = m$.

Clearly, to begin with the maximum M_0 and the minimum m_0 of the numbers on the vertices of the cube are distinct. Hence $m < M$. Now, to get $M_1 = M_0$ we must have three vertices "surrounding" the vertex on which M_1 is assumed on the cube C_1. The same can be said about m_1. We see that at least 3 vertices of C_0 have values M and at least 3 vertices of C_0 have values m. The same arguments show that there are at least 3 vertices of value M and 3 vertices of value m in every cube through C_{19}, and since $C_1 = C_{20}$ the same applies to C_{20} too.

Let us show that no vertex of C_i can be of value not equal to m or M. Indeed, if so then three vertices next to this one on the next step will have the values distinct from m or M, a contradiction. Hence the possible values are M and m only. Now, if M-vertex and m-vertex of some C_i are adjacent to the same vertex x then the value of x on the next step will not be equal to m or M, a contradiction.

Now, denote the vertices of the cube C_0 as follows: A, B, D, F on the bottom face and A', B', D', F' above them respectively. Assume

that the value $v(A) = M$. Then it follows that $v(D) = M, v(B') = M, v(D') = M$. One of the remaining vertices must have value m, and it easily follows that then all remaining vertices must have value m. Now it is easy to see, that on each step M-vertices and m-vertices simply swap their places. Hence in all the cubes there are 4 vertices of value M and 4 vertices of value m. Since there are vertices of C_0 of values 2 and 7 we see that $M = 7, m = 2$.

The answer then is **36**.