2003 UAB MTS: SOLUTIONS

PROBLEM 1 (5 points). Four boys bought a boat for \$60. The first boy paid one half the sum of the amounts paid by the other boys; the second boy paid one third the sum of the amounts paid by the other boys; and the third boy paid one fourth of the sum of the amounts paid by the other boys. How much did the fourth boy pay?

Solution: Suppose that the first boy paid x. Then the equation for x is $x = (60 - x)/2$. Solving it we get that $x = 20$. Similarly, the equation on the amount y paid by the second boy is $y = (60 - y)/3$ which yields that $y = 15$. If the amount paid by the third boy is denoted by z then we see that $z = (60 - z)/4$ and so $z = 12$. We conclude that the fourth boy paid $60 - 20 - 15 - 12 = 13$ dollars.

So the answer is **13**.

PROBLEM 2 (10 points). Ann and Sue bought identical boxes of stationary. Ann used hers to write 1-sheet letters and Sue used hers to write 3-sheet letters. Ann used all her envelopes and had 50 sheets of paper left, while Sue used all her sheets of paper and had 50 envelopes left. What was the number of sheets in the box?

Solution: Denote by e the number of envelopes in a box and by s the number of sheets of paper in a box. Then the way Ann wrote letters yields that $e + 50 = s$ while the same for Sue yields $3(e - 50) = s$. We conclude that $3e - 150 = e + 50$ and hence $e = 100$ and $s = 150$.

So the answer is **150**.

PROBLEM 3 (10 points). Given an equation $|x - 4| + |x - 7| = 3$, what is the number of solutions? Choose from the following options:

A) 0; B) 1; C) 2; D) 3; E) > 3.

Solution: Considering three cases for the location of x which are: 1) $x < 4$; 2) $4 \leq x \leq 7$; 3) $7 < x$ and expanding the absolute value accordingly we see that any x with $4 \leq x \leq 7$ solves the equation. Therefore the equation has infinitely many solutions.

So the answer is **E)** (that is, > 3).

PROBLEM 4 (15 points). 2 positive numbers a, b are inserted between 12 and 36 so that $12, a, b$ are in a geometric progression while

$a, b, 36$ are in arithmetic progression. Find $a + b$.

Solution: Clearly, we can set $a = 12q$ and $b = 12q^2$ with q being the denominator of the geometric progression introduced in the problem. Then the equation which comes from the fact that $12q, 12q^2$ and 36 form an arithmetic progression is as follows: $12q^2 - 12q = 36 - 12q^2$, or, equivalently, $24q^2 - 12q + 36 = 0$. Solving this equation we see that the two roots are $3/2$ and -1. However q must be positive, thus $q = 3/2$ and the sequence in question is $12, 18, 27, 36$ and the sum $a + b$ is $18 + 27 = 45$.

So the answer is **45**.

PROBLEM 5 (15 points). The consecutive angles of a quadrilateral form an arithmetic sequence. If the smallest is 75 degrees, then what is the largest?

Solution: The sum of angles of a quadrilateral is 360 degrees. Since the smallest is 75 degrees and they form an arithmetic progression, the other angles are $75 + d, 75 + 2d$ and $75 + 3d$ degrees. The sum of all angles then can be written as $75 + 75 + d + 75 + 2d + 75 + 3d = 300 + 6d$. Equating this expression and 360 we get the equation $300 + 6d = 360$. Solving it for d we get that $d = 10$ and hence the greatest angle in this quadrilateral is 105 degrees.

So the answer is **105**.

PROBLEM 6 (20 points). Each letter represents uniquely a different digit in base ten: $(MA) \times (HA) = TTT$. Find $M + A + T + H$.

Solution: Observe that TTT is divisible by 111 and hence by 37. Thus, one of the numbers MA and HA is divisible by 37. Suppose that MA is divisible by 37. Then the two cases to be considered are as follows.

1. $MA = 37$. Then $A = 7$ and hence the last digit of $(MA) \times (HA)$ is the last digit of $7 \times 7 = 49$, that is 9. Hence $TTT = 999$ and so $HA = 999/37 = 27$. Clearly, $MA = 37, HA = 27$ and $TTT = 999$ solves the problem and so in this case $M + A + T + H = 21$.

2. $MA = 74$. Then $A = 4$ and hence the last digit of $(MA) \times (HA)$ is the last digit of $4 \times 4 = 16$, that is 6. Hence $TTT = 666$ and so $HA = 666/74 = 9$, a contradiction.

So the the answer is **21**.

PROBLEM 7 (25 points). Which of these triples could not be the lengths of the three altitudes of a triangle:

A) $1/5, 1/12, 1/13$; B) $1/3, 1/6, 1/10$; C) $1/5, 1/8, 1/9$;
D) $1/3, 1/4, 1/5$; E) $1/2, 1/3, 1/4$; F) $1/10, 1/13, 1/21$;
G) $1/4, 1/7, 1/9$; H) $1/4, 1/5, 1/8$.

Solution: In any triangle we have that $ah_a = 2A$ where a is the length of a side of the triangle, h_a is the height of the triangle perpendicular to the side of the length a, and A is the area of the triangle. Therefore for the lengths of the sides of the triangle we get the formulas $a = \frac{2A}{h_a}, b = \frac{2A}{h_b}, c = \frac{2A}{h_c}$. This together with the triangle inequality implies that the lengths of the heights of a triangle should satisfy the following inequalities: $\frac{1}{h_a} + \frac{1}{h_b} \geq \frac{1}{h_c}$; $\frac{1}{h_b} + \frac{1}{h_c} \geq \frac{1}{h_a}$; $\frac{1}{h_c} + \frac{1}{h_a} \geq \frac{1}{h_b}$. A direct verification shows that the only triple of given heights not satisfying this condition is B) $1/3, 1/6, 1/10$.

So the the answer is **B)** (that is, $1/3, 1/6, 1/10$).

PROBLEM 8 (40 points). A square ABCD is given. Then the circle of radius AB centered at A is drawn. Inside the square the circle intersects the perpendicular bisector of BC at the point O. Find the value of angle AOC in degrees.

Solution: The triangle AOD is equilateral. Hence all is angles are 60 degrees. Thus the angle ODC is $90 - 60 = 30$ degrees. Since in the triangle ODC we have $OD = CD$ then the angles COD and OCD are the same. Relying upon the fact that the angle ODC is 30 degrees and that the sum of the angles in a triangle is 180 we see that the angle COD is 75 degrees. Thus, the angle AOC which is the sum of the angles AOD and DOC is equal to $60 + 75 = 135$ degrees.

So the the answer is **135**.

PROBLEM 9 (65 points). Let D be a subset of $\{1, 2, \ldots, 700\}$ such that no two numbers from D differ by 2 or 5. What is the largest number of elements that the set D can have?

Solution: Consider the intersection $D \cap \{1, 2, 3, 4, 5, 6, 7\} = R$. Of the numbers $1, 3, 5, 7$ only two may belong to R, and of the numbers $2, 4, 6$ either only one belongs to R, or numbers of 2 and 6 belong to R. Suppose indeed that 2 and 6 belong to R. Then neither 1 nor 7 can belong to R, and of the remaining odd numbers $3, 5$ only one can belong to R which implies that there are no more than 3 numbers in R in this case. Now, if only one even number belongs to R then since no more than 2 odd numbers belong to R we again conclude that there are no more than 3 numbers in R. So we prove that in any case there are no more than 3 numbers in R. Clearly, the same arguments apply to any consecutive seven integer numbers which shows that there no more than three elements in $D \cap \{8, 9, 10, 11, 12, 13, 14\}$, no more than three elements in $D \cap \{15, 16, \ldots, 21\}$ etc. Since the collection of numbers $\{1, \ldots, 700\}$ can be divided into 100 groups of seven numbers as above, we see that overall there are no more than 300 elements in D.

Now, the example of a possible set D with all the required properties and 300 elements could be the set of all numbers among $1, \ldots, 700$ which have the remainders $2, 3$ or 6 when divided by 7.

So the the answer is **300**.

PROBLEM 10 (80 points). Find the largest possible number n such that $n!$ can be expressed as the product of $n-5$ consecutive integers.

Solution: Suppose that n has the desired property. Then we have that $n! = (k+1)(k+2)\ldots(k+n-5)$ for some k. Multiplying both sides by $k!$ we have that $n!k! = (k+n-5)!$. Observe that then $k \geq 6$ since otherwise the left hand side of this equation is less than the right hand side. Therefore $n+k-5 \geq n+1$ and we can cancel out $n!$ from either side of the equation. This gives $k! = (n+1)(n+2)\ldots(n+k-5)$. Observe that by the same token as before $n \geq 6$. Hence $k+n-5 \geq k, k+n-6 \geq k-1, \ldots, n+2 \geq 7$. We conclude that the remaining terms in the products on either side of the equation are to satisfy the opposite inequality, namely that $6! = 720 \geq n+1$ and so $n \leq 719$. It remains to observe that $k = 6$ and $n = 719$ solve our equation: $719! = 720 \cdot 719 \cdot \cdots \cdot 7$ as desired.

So the the answer is **719**.