Department of Mathematics UAB
Calculus II

Review Problems for Test 1

1. (a) Find an equation for the sphere with center $(4, -11, 6)$ and radius 10.
   (b) Write the sphere equation
   $$x^2 + y^2 + z^2 + 2x - 2y - 2z = 33$$
   in standard form, and hence find the center and radius of this sphere.
   (c) Find the equation of the largest sphere with center $(5, 4, 7)$ that is contained in the first octant, i.e. the subset of $\mathbb{R}^3$ given by
   $$\{(x, y, z) : x \geq 0, y \geq 0, z \geq 0\}.$$

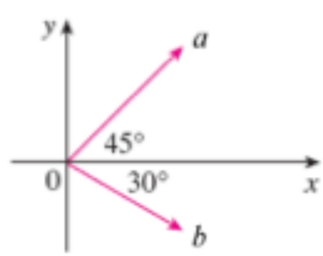
2. Find $\boldsymbol{a} + \boldsymbol{b}$, $2\boldsymbol{a} - 3\boldsymbol{b}$, and $|\boldsymbol{a} - \boldsymbol{b}|$ if
   $$\boldsymbol{a} = \langle 1, 3, -4\rangle, \quad \boldsymbol{b} = \langle -4, -1, 7\rangle.$$

3. Find a unit vector that has the same direction as $2\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$.

4. Find the angle (in radians) between the vector $\boldsymbol{a} = \langle 1, 1\rangle$ and the positive x-axis, and use this angle to write $\boldsymbol{a}$ in the form
   $$\boldsymbol{a} = |\boldsymbol{a}|\langle \cos\theta, \sin\theta\rangle.$$

5. If the vector $\boldsymbol{v} = \langle v_1, v_2\rangle$ lies in the first quadrant and makes an angle of $\pi/3$ with the positive x-axis, and $|\boldsymbol{v}| = 6$ find the components v_1 and v_2.

6. In the figure below let $\boldsymbol{a}$ represent a $\sqrt{2}$-pound force and $\boldsymbol{b}$ a 2-pound force.



   (a) Find the components of the force vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.
   (b) If the forces $\boldsymbol{a}$ and $\boldsymbol{b}$ work together on an object at the point O, find the magnitude and direction of the resultant force $\boldsymbol{a} + \boldsymbol{b}$.

7. A child is pulling a loaded toy-box on a level path with a force of 50 Newtons exerted at an angle of 45° ($\pi/4$ radians) above the horizontal.

   (a) Find the horizontal and vertical components of the force.

   (b) If the initial frictional resistance on the path to be overcome is 25 Newtons, will the child succeed in moving the toy-box?

8. (a) Find $\boldsymbol{a} \cdot \boldsymbol{b}$ if $\boldsymbol{a} = \langle 2, 1, 1\rangle$ and $\boldsymbol{b} = \langle 1, -2, -1\rangle$.

   (b) Find the angle between the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ in (a).

   (c) In general if $\boldsymbol{a} \cdot \boldsymbol{b} < 0$ for two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, what does this tell you about the angle between the vectors?

9. Find the angle between the vector $\boldsymbol{a} = \langle 1, 0, 1\rangle$ and the positive x-axis in $\mathbb{R}^3$.

10. Find the three angles in the space triangle ABC with vertices $A = (1, 0, -1)$, $B = (4, -5, 0)$, and $C = (1, 3, 2)$. [Hint: recall that the angles must add up to π radians.]

11. Use vectors to determine whether the space triangle with vertices

$$P(0, -4, -1), \quad Q(1, -1, -3), \quad R(5, -3, -4)$$

    contains a right-angle.

12. (a) Find the scalar and vector projections of $\boldsymbol{b}$ onto $\boldsymbol{a}$ when

$$\boldsymbol{a} = \langle 1, 1, 1\rangle, \quad \boldsymbol{b} = \langle 1, -1, 1\rangle.$$

   (b) Determine the orthogonal vector $\mathrm{orth}_{\boldsymbol{a}}(\boldsymbol{b})$.

   (c) Sketch the vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\mathrm{proj}_{\boldsymbol{a}}(\boldsymbol{b})$, and $\mathrm{orth}_{\boldsymbol{a}}(\boldsymbol{b})$, indicating precisely how they relate to each other.

13. (a) If $\boldsymbol{a} = \langle 0, 1, 1\rangle$ and $\boldsymbol{b} = \langle 5, 4, 0\rangle$ find $\boldsymbol{a} \times \boldsymbol{b}$ and verify that $\boldsymbol{a} \times \boldsymbol{b}$ is orthogonal (perpendicular) to $\boldsymbol{a}$ and to $\boldsymbol{b}$. [Hint: use dot products to check orthogonality]

   (b) Use vectors to determine the area of the parallelogram with vertices $P(3, 3, 2)$, $Q(3, 4, 3)$, $R(8, 8, 3)$, and $S(8, 7, 2)$. [Hint: first determine the vertices that connect parallel sides of the parallelogram; recall that two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are parallel precisely when $\boldsymbol{u} = c\,\boldsymbol{v}$ for some scalar c]

14. Use the triple scalar product of three vectors to find the volume of the parallelopiped with adjacent edges PQ, PR, and PS when

$$P = (-2, 1, 0), \quad Q = (2, 4, 2), \quad R = (1, 4, -1), \quad S = (3, 6, 3).$$

15. Show that the points

$$A = (1, 2, 2), \quad B = (4, -3, 8), \quad C = (5, 1, 1), \quad D = (2, 6, -5)$$

    all lie in the same plane. [Hint: work out $\overrightarrow{AB} \times \overrightarrow{AC} \cdot \overrightarrow{AD}$]