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RENORMALIZATION TOWERS AND THEIR FORCING

ALEXANDER BLOKH AND MICHAŁ MISIUREWICZ

ABSTRACT. A cyclic permutation $\pi : \{1, \dots, N\} \to \{1, \dots, N\}$ has a *block structure* if there is a partition of $\{1, \dots, N\}$ into $k \notin \{1, N\}$ segments of consecutive integers (*blocks*) of the same length, permuted by π; call k the *period* of this block structure. Let $p_1 < \cdots < p_s$ be periods of all possible block structures on π. Call the finite string $(p_1/1,\ p_2/p_1,\ \dots,\ p_s/p_{s-1}, N/p_s)$ the *renormalization tower of* π. The same terminology can be used for *patterns*, i.e., for families of cycles of interval maps inducing the same, up to the flip of the entire orbit, cyclic permutation (thus, there are two permutations, one of whom is a flip of the other one, that define a pattern). A renormalization tower $\mathcal{M}$ *forces* a renormalization tower $\mathcal{N}$ if every continuous interval map with a cycle of pattern with renormalization tower $\mathcal{M}$ must have a cycle of pattern with renormalization tower $\mathcal{N}$. We completely characterize the forcing relation among renormalization towers. Take the following order among natural numbers: $4 \gg 6 \gg 3 \gg 8 \gg 10 \gg 5 \gg \cdots \gg 4n \gg 4n{+}2 \gg 2n{+}1 \gg \cdots \gg 2 \gg 1$ understood in the strict sense (we write consecutive even numbers, starting with 4, then insert m after each number of the form $2(2s + 1)$, and finally append the order with 2 and 1). We show that the forcing relation among renormalization towers is given by the lexicographic extension of this order. Moreover, for any tail T of this order there exists an interval map for which the set of renormalization towers of its cycles equals T.

1. Introduction and statement of the results

From the standpoint of the theory of dynamical systems, the simplest type of limit behavior of a point under iterates of a continuous map is periodic. Thus, *cycles* (called also *periodic orbits*) play an important role in dynamics. The description of possible sets of types of cycles of maps from a certain class is a natural and appealing problem.

Since maps that are topologically conjugate are considered equivalent, it is natural to declare two cycles equivalent if there exists a homeomorphism of the space which sends one of them onto the other one. On the interval this means that if two cycles induce the same cyclic permutations or the cyclic permutations coinciding up to a flip, then these cycles should be viewed as equivalent. Classes of equivalence are then called *cyclic patterns* (since we consider *only* cyclic patterns and permutations, we will call them simply *patterns* and *permutations* from now on). If an interval map f has a cycle that belongs to a specific pattern (equivalently, induces a specific permutation), then one can say that the pattern (the permutation) is

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exhibited by f. Thus, one comes across a problem of characterizing possible sets of patterns (or sets of permutations) exhibited by interval maps. Here the permutations induced by cycles on the interval (and considered up to the flip) should be thought of as types of cycles.

Permutations define corresponding patterns. Vice versa, given a pattern, there are two permutations (one of them is a flip of the other one) that define the pattern. In what follows we will mostly talk about patterns.

How does one describe patterns? This naive question seems to have an obvious answer: patterns should be described by permutations that they are. However an important drawback of that approach is that such description is too detailed and complicated. To have more information may not always be better because then the structure of the set of all patterns exhibited by a map is buried under piles of inessential details. In other words, permutations chosen as types of cycles are not necessarily a good choice because then the set of all permutations induced by cycles of a given interval map may have a very complicated structure not allowing for a transparent description.

A different (opposite in some sense) approach is to describe a pattern by stripping it of all its characteristics but one: the period. Then, of course, a lot of very different patterns will be lumped into one big group of patterns of the same period. This approach may seem to be too coarse and imprecise. However it is this idea that was adopted in one-dimensional dynamics, thanks to a remarkable result obtained by A. N. Sharkovsky in the 1960s (see [Sha64] and [Sha-tr] for its English translation). To state it let us first introduce the *Sharkovsky ordering* for positive integers:

$$3 \succ_s 5 \succ_s 7 \succ_s \cdots \succ_s 2\cdot 3 \succ_s 2\cdot 5 \succ_s 2\cdot 7 \succ_s \cdots \succ_s 4 \succ_s 2 \succ_s 1.$$

Denote by $\mathrm{Sh}(k)$ the set of all integers m with $k \succ_s m$, including k itself, and by $\mathrm{Sh}(2^\infty)$ the set $\{1, 2, 4, 8, \dots\}$. Denote by $\mathrm{P}(g)$ the set of periods of cycles of a map g.

Sharkovsky Theorem. *If $g : [0, 1] \to [0, 1]$ is continuous, $m \succ_s n$, and $m \in \mathrm{P}(g)$, then $n \in \mathrm{P}(g)$, and so there exists $k \in \mathbb{N} \cup 2^\infty$ with $\mathrm{P}(g) = \mathrm{Sh}(k)$. Conversely, if $k \in \mathbb{N} \cup 2^\infty$, then there exists a continuous map $f : [0, 1] \to [0, 1]$ such that $\mathrm{P}(f) = \mathrm{Sh}(k)$.*

The role of the Sharkovsky Theorem in the theory of dynamical systems is, in particular, based upon the fact that its first part can be understood in terms of *forcing relation*. Indeed, it states that if $m \succ_s n$, then the fact that an interval map has a point of period m *forces* the presence of a point of period n among the periodic points of the map. In short, *period m forces period n*. Notice that with this understanding of the concept of forcing, every number forces itself. If we think of the period of a cycle as its type, we can view the Sharkovsky Theorem as a result showing how such types of cycles (i.e., their periods) force each other.

It is natural to try to replace the period by some notion that provides more information but is easy to interpret and handle. To explain our choice we discuss some properties of interval cycles below. Any of the concepts that we introduce will be defined only for, say, permutations, but can be similarly defined for the corresponding cycles and patterns; the same applies to the corresponding notation. We provide an example of how this can be done in the definition below.

Definition 1.1 (Block structure)**.** A cyclic permutation $\pi : \{1, \dots, N\} \to \{1, \dots, N\}$ has a *block structure* if there is a partition of $\{1, \dots, N\}$ into $k \notin \{1, N\}$

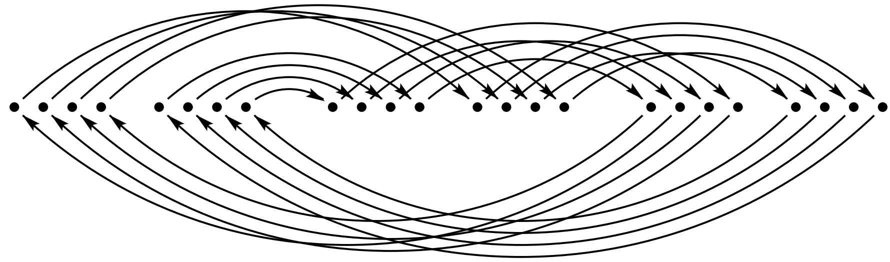

FIGURE 1. A cycle of period 24 with tower $(3, 2, 4)$.

segments of consecutive integers (*blocks*) of the same length, permuted by π. That is, π maps a block to a block, but the order within blocks does not have to be preserved. We will call k the *period* of this block structure. A permutation has a *division* if it has block structure of period 2.

The cycle $P = \{x_1, \dots, x_N\}$ with $x_1 < x_2 < \cdots < x_N\}$ has *block structure* if the corresponding permutation π of $\{1, \dots, N\}$ defined by $x_{\pi(i)} = f(x_i)$ does.

We will show later that if two block structures of the same permutation have periods $p < q$, then q is a multiple of p.

A permutation with a block structure can be studied in two steps: study the factor-permutation obtained if each block is collapsed to a point, while the order among blocks is kept, and then study the restriction of the appropriate power of the permutation to blocks. Evidently, this step-by-step approach is easiest to implement if one uses the most basic and, therefore, the smallest steps. Thus, it is natural to consider all possible block structures and navigate among them moving toward larger and larger periods of blocks while making the smallest possible steps.

Definition 1.2 (Renormalization towers)**.** Let $p_1 < \cdots < p_s$ be periods of all possible block structures on π. Call the finite string $(p_1/1,\ p_2/p_1,\ \dots,\ p_s/p_{s-1}, N/p_s)$ the *renormalization tower of* π and denote it by $\mathrm{RT}(\pi)$. Usually we will call a renormalization tower just a *tower* or, if we want to distinguish it from an infinite tower (see Definition 1.7), a *finite tower*. Call the cardinality $s+1$ of the tower of π its *height* and N its *period*. For consistency, if π has no block structure, we define its tower as (N). While formally the tower of the permutation on $\{1\}$ should be (1) (so the number 1 appears there, unlike in other towers) or $\emptyset$, we will ignore this exception and pretend that the singleton of a fixed point is not a cycle. Figure 1 gives an example of a permutation with non-trivial tower.

Observe that by definition no number in the tower of π equals 1 (except the trivial case of $N = 1$; see Definition 1.2). The concept of the tower extends that of the period of a cyclic permutation and reflects the structure of a cyclic permutation in a more detailed way than the period itself. It combines both the combinatorics and the order among points, but can be viewed as *numeric* in its nature. By this we mean that even though the tower of a permutation can be a string of several numbers, its height does not necessarily increase to infinity with period. In fact, the tower of a permutation with no block structure of *any* period, however big, is 1. This distinguishes towers from permutations themselves.

The concept of renormalization plays a prominent role in smooth dynamics. In a nutshell, it means that under some circumstances one can take a specific region in the space, consider the first return map on this region, normalize the size of the region to the original size of the space, and discover that it is a map of the same type as the original map. In the majority of cases the region in question is itself periodic. This process can be sometimes repeated infinitely many times, leading to regions of smaller and smaller sizes and higher and higher periods. In that case in the end one gets so-called *infinitely renormalizable limit sets* (the corresponding maps are also called *infinitely renormalizable*), and metric properties describing how smaller periodic regions relate to greater periodic regions in terms of size are of utter importance for one's understanding of the smooth dynamical system in question. Our approach is necessarily combinatorial and may be viewed as a discrete version of the above described infinite renormalization process.

In this paper we consider towers as types of cycles and, according to the approach discussed earlier, aim to give a full description of sets of towers of cycles of continuous interval maps. Similar to other cases (for example, to the case of periods of cycles), in order to do so we have to study the problem of the coexistence of towers. More precisely, we want to characterize the forcing relation among them.

Definition 1.3 (Tower forcing)**.** Let $\mathcal{M}$ and $\mathcal{N}$ be two finite strings of integers, each of which is greater than 1. Suppose that every continuous interval map that has a cycle with tower $\mathcal{M}$ has a cycle with tower $\mathcal{N}$. Then we say that the tower $\mathcal{M}$ *forces* the tower $\mathcal{N}$.

Evidently, in the above definition, one can replace "every continuous interval map that has a cycle with tower $\mathcal{M}$ has a cycle with tower $\mathcal{N}$" by "every pattern with tower $\mathcal{M}$ forces a pattern with tower $\mathcal{N}$."

Notice that it is not at all clear if the notion of forcing is meaningful and provides for a transparent order among towers. In this paper we solve this problem, show that the order corresponding to the forcing relation among towers is linear, and provide its full description.

Namely, in a recent paper [BM18] the following order among natural numbers was defined (for terminological reasons the order in [BM18] does not include numbers 2 and 1; it is easy to see that the results of [BM18] hold with the order described below which includes 2 and 1):

$$4 \gg 6 \gg 3 \gg \cdots \gg 4n \gg 4n+2 \gg 2n+1 \gg \cdots \gg 2 \gg 1 \tag{1.1}$$

(we write consecutive even numbers, starting with 4, then insert m after each number of the form $2m$ if m is odd, and finally append the order with 2 and 1). Here we understand the order $\gg$ in the strict sense. In other words, $k \gg k$ is not true for integers k. We will call this order the $\gg$-order (we suggest that $\gg$ be pronounced as "gg" in what follows).

The $\gg$-order covers all positive integers. All positive integers except for 1 and 2 are ordered in a rather transparent fashion. Namely, first we order all even numbers in the increasing way. Then we insert odd numbers $2n+1$ between even numbers $4n+2$ and $4n+4$. At the end we put the so far missing numbers 2 and 1.

Definition 1.4 ($\gg$-tail)**.** A set A of numbers is said to be a *$\gg$-tail* if for any number $m \in A$ and any number n with $m \gg n$ the set A must contain n.

Clearly, the structure of a $\gg$-tail can be described explicitly. Namely, given a $\gg$-tail A, choose the $\gg$-greatest number n of all numbers in A. It is easy to see that the $\gg$-greatest element of A always exists. Hence A is the set $N(n)$ of all numbers x such that $x = n$ or $n \gg x$.

Definition 1.5 (Tower)**.** A finite string of integers larger than 1 is said to be a *tower*.

The order we actually want to introduce for towers is the *lexicographic extension* of $\gg$ onto the family of towers. Let

$$\mathcal{N} = (n_1, n_2, \dots, n_k), \quad \mathcal{M} = (m_1, m_2, \dots, m_l)$$

be two towers (see [Tolkien]). Append each of them by infinite strings of 1s, and denote these *canonical extensions* by $\mathcal{N}'$ and $\mathcal{M}'$. Observe that in any canonical extension constructed above the number 1 does not show in the beginning for some time, but once it shows, the rest of the numbers in the tower must be only 1s. Let s be the first place at which $\mathcal{N}'$ and $\mathcal{M}'$ are different. Then we write $\mathcal{N} \gg \mathcal{M}$ if $n_s \gg m_s$. We keep the same notation $\gg$ for the lexicographic extension of $\gg$, as no confusion arises. Evidently, the family of all towers with $\gg$-order is linearly ordered, as any two towers are comparable in the sense of $\gg$-order.

Before we state our main theorem, we need to introduce additional notions (similar to $\mathrm{Sh}(2^\infty)$ in the Sharkovsky ordering).

Definition 1.6 ($\gg$-tail of towers)**.** A set A of towers is said to be a $\gg$*-tail* if for any tower $\mathcal{M} \in A$ and any tower $\mathcal{N}$ with $\mathcal{M} \gg \mathcal{N}$ the set A must contain $\mathcal{N}$.

It is easy to give a direct and explicit description of all possible $\gg$-tails for towers (based exclusively upon properties of the *gg*-order). Namely, observe that all towers directly generated by cycles have no digits 1 in them (except (1); this is an exception that we will mainly ignore later).

Definition 1.7 (Infinite tower)**.** An *infinite tower* is an infinite string $\mathcal{N}$ of positive integers $(m_0, m_1, \dots)$ such that either (a) $m_i > 1$ for every i or (b) there exists minimal j such that $m_i > 1$ for each $i < j$ and $m_i = 1$ for every $i \geqslant j$. In case (b) we identify the infinite tower $\mathcal{M} = (m_0, \dots)$ with finite tower of all initial numbers $(m_0, m_1, \dots, m_{j-1})$ of $\mathcal{M}$ not equal to 1. The relation $\gg$ extends onto all infinite towers lexicographically.

In particular, when we write $\mathcal{M} \gg \mathcal{N}$ for an infinite tower $\mathcal{M}$ and a finite tower $\mathcal{N}$, we mean that $\mathcal{M} \gg \mathcal{N}'$ in the sense of Definition 1.7 (here the infinite tower $\mathcal{N}'$ is the canonical extension of the tower $\mathcal{N}$).

Definition 1.8 ($\mathrm{Tow}(\mathcal{M})$)**.** Given an infinite tower $\mathcal{M}$, the set $\mathrm{Tow}(\mathcal{M})$ is defined as the set of all finite towers $\mathcal{N}$ such that $\mathcal{M} \gg \mathcal{N}$, together with the finite tower $\widetilde{\mathcal{M}}$ in case $\mathcal{M}$ is the canonical extension of $\widetilde{\mathcal{M}}$.

Observe that infinite towers of type (a) from Definition 1.7, viewed as elements of the ordered space of all towers with the $\gg$-order, are similar to 2^∞ in the Sharkovsky ordering.

Lemma 1.9. *Every $\gg$-tail of towers is of the form* $\mathrm{Tow}(\mathcal{N})$ *for some infinite tower $\mathcal{N}$. Conversely, for every infinite tower $\mathcal{N}$ the set* $\mathrm{Tow}(\mathcal{N})$ *is a $\gg$-tail of towers.*

Proof. The second part of the lemma is obvious, since the order $\gg$ is the same in the set of finite and infinite towers. To prove the first part, assume that $\mathcal{A}$ is a $\gg$-tail of towers. Then we construct the sequence (m_j) by induction. If m_i for $i < j$ are defined, then m_j is the $\gg$-largest number such that there is a tower in $\mathcal{A}$ starting with $(m_0, \dots, m_{j-1}, m_j)$ (and 1 if such number does not exist). Then for the infinite tower $\mathcal{N} = (m_0, m_1, \dots)$ we have $\mathcal{A} = \mathrm{Tow}(N)$. □

Now we can state the main result of the paper.

Main Theorem. *If $\mathcal{N} \gg \mathcal{M}$ and a continuous interval map f has a cycle with tower $\mathcal{N}$, then it has a cycle with tower $\mathcal{M}$, and so there exists an infinite tower $\mathcal{K}$ such that the set of all (finite) towers of cycles of f is* $\mathrm{Tow}(\mathcal{K})$. *Conversely, if $\mathcal{K}$ is an infinite tower, then there exists a continuous interval map g such that the set of all (finite) towers of cycles of g coincides with* $\mathrm{Tow}(\mathcal{K})$.

Since the notion of a tower refines the notion of a period, it should be possible to deduce the Sharkovsky Theorem from our Main Theorem. Since the $\gg$-order on towers is linear and for every period $N > 1$ there are only a finite number of towers of this period, there exists a unique tower of period N that does not force any other tower of the same period. We will call such a tower *primary.*

Proposition 1.10. *If $N > 1$ is odd, then the primary tower of period N is (N). If $N = 2^k \cdot m$, where $k \geqslant 1$ and $m > 1$ is odd, then the primary tower of period N is $(2, \dots, 2, m)$, where 2 is repeated k times. If $N = 2^k$, then the primary tower of period N is $(2, \dots, 2)$, where 2 is repeated k times.*

Proof. If $N > 1$ is odd, then all towers of period N start with some divisor $j > 1$ of N. However, j is odd, so if $j \neq N$, then $j \gg N$. Thus, the primary tower of period N is (N).

If $N = 2^k \cdot m$, where $k \geqslant 1$ and m is odd, then the primary tower of period N has to start with 2, since 2 is at the end of the $\gg$-order (not counting 1). Then we use induction to show that the primary tower of period N has to start with 2 repeated k times. If $m > 1$, we finish the proof by the same argument as for N odd. □

Now the $\gg$-order of the primary towers of all periods gives us immediately the Sharkovsky Theorem.

The paper is organized as follows. In Section 2 we go over preliminary information, including the tools related to the forcing relation among patterns and results of the recent paper [BM18]. In Section 3 we establish a few basic facts concerning towers. In Section 4 we study unimodal cycles and patterns; they play a significant role in the proof of the realization part of the Main Theorem. Section 5 is devoted to the proof of the Main Theorem, as well as illustrating it with some applications.

2. Preliminaries

Here we introduce basic notions and theorems that will be used in the next parts of the paper.

Definition 2.1 (Forcing relation among patterns)**.** If A and B are patterns such that any continuous interval map with a cycle of pattern A has a cycle of pattern B, then one says that A *forces* B. This is a partial order; we might talk of forcing among cycles too.

A nice description of patterns forced by a given pattern can be given if we rely upon the notion of a P*-linear map*. Observe that one can talk of cycles on $\mathbb{R}$ or I even if the map is not defined outside the cycle.

Definition 2.2 (P-linear maps and P-basic intervals)**.** Let $P \subset \mathbb{R}$ be a finite set. Set $I = [\min P, \max P]$. The closure of a component of $I \setminus P$ is said to be a P*-basic* interval. If a map $f : P \to P$ is given, then the P*-linear map* $F : I \to I$ is defined as the continuous extension of f, linear on each P-basic interval. Similarly, a map $g : I \to I$ is said to be P*-monotone* if g is monotone on each P-basic interval.

Mostly, one uses P-linear maps in dealing with cycles.

Theorem 2.3 ([ALM00])**.** *If P is a cycle of pattern A and f is a P-linear map, then the patterns of cycles of f are exactly the patterns forced by A.*

Properties of patterns can be stated in terms of P-linear maps. A map $F : X \to X$ of a topological space X to itself is called *topologically exact* if for any non-empty open set $U \subset X$ there is n such that $F^n(U) = X$. We will often write "basic intervals" meaning "P-basic intervals" if it is clear which cycle (or finite set) P we mean.

Proposition 2.4. *A cycle P of period $n > 2$ has no block structure if and only if the P-linear map f is topologically exact.*

Proof. If P has a block structure, then for any basic interval I in the convex hull of a block X and any integer n there exists a block Y such that $f^n(I)$ is contained in the convex hull of Y. Thus, f is not topologically exact.

Now, assume that P has no block structure. There must be a basic interval $I = [a, b]$ such that $f(a) > a$ and $f(b) < b$. Then $f(I) \supset I$, so consecutive images of I keep growing until some image of I covers P. Let J be any basic interval and consider the union of the intervals $f^k(J)$ over $k = 0, 1, 2, \dots$. If it is not connected, then intersections of its components with P form non-trivial blocks, a contradiction. Hence the images of J eventually cover I and then the entire P.

It follows that if x and $f^m(x)$ (for some $m > 0$) both belong to the interior of a basic interval J, then $|(f^m)'(x)| > 1$ (we will refer to this as an *expanding property*). To see this, take the maximal open interval $K \subset J$ containing x such that $f^i(K) \cap P = \emptyset$ for all $i = 0, 1, \dots, m$. Then $f^m|_K$ is monotone, and for each endpoint y of K there has to be $i \leqslant m$ such that $f^i(y) \in P$, as otherwise K can be slightly enlarged while still fulfilling the property that defines it. Hence $f^m(\overline{K}) = J$, and so if $|(f^m)'(x)| \leqslant 1$, then actually $|(f^m)'(x)| = 1$ and $\overline{K} = J$, a contradiction to the fact that an eventual image of J covers P.

Now let K be an open interval. If there is no k such that $f^k(K)$ contains a point of P, then there is a basic interval J such that the set Z of those integers $i \geqslant 0$ for which $f^i(K) \subset J$ is infinite. Let L be the union of the intervals $f^i(K)$ over $i \in Z$. By the expanding property, the lengths of those intervals $f^i(K)$ are larger than or equal to the length of K. Therefore the set L has finitely many components. However, for the longest component, say H, there is $m > 0$ such that $f^m(H) \subset L$ and the length of $f^m(H)$ is larger than the length of H, a contradiction. This shows that we may assume that K contains a point of P. By shortening K, we may assume that it is a short interval containing a point of P as its endpoint.

By taking $2n$ first images of such K we get $2n$ short intervals containing a point of P as an endpoint, so one of them has to contain some other one. By the

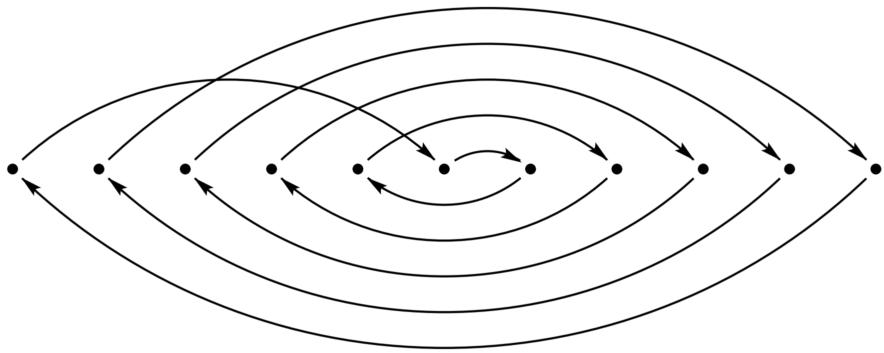

FIGURE 2. A Štefan cycle of period 11.

expanding property, they have different lengths, and the longer one is the image of the shorter one under f^j for some $j > 0$. Iterating f^j further, we get longer and longer intervals, until some image of K contains a basic interval. Now we see that some further image of K is equal to the convex hull of P, as desired. □

Definition 2.5 (Loops of (basic) intervals)**.** Suppose that f is an interval map and there are intervals $I_0, \dots, I_{n-1}$ such that $I_1 \subset f(I_0)$, $I_2 \subset f(I_1)$, $\dots$, $I_0 \subset f(I_{n-1})$. Then a finite string $L = I_0 \to I_1 \to \dots \to I_{n-1} \to I_0$ is said to be a *loop of intervals*. If intervals $I_0, \dots, I_{n-1}$ are P-basic for some finite invariant set (e.g., cycle) P, then L is said to be a *loop of* (P-)*basic intervals.*

The next lemma helps one find various periodic points.

Lemma 2.6 (See, e.g., [Bloc80])**.** *If $L = I_0 \to I_1 \to \dots \to I_{n-1} \to I_0$ is a loop of intervals, then there exists a point x such that $f^j(x) \in I_j$, $0 \leqslant j \leqslant n-1$, while $f^n(x) = x$.*

The orbit of x from Lemma 2.6 is said to *correspond to the loop L.*

Definition 2.7 (Štefan pattern)**.** Consider the cyclic permutation $\sigma : \{1, 2, \dots, 2n+1\} \to \{1, 2, \dots, 2n+1\}$ defined as follows:

- $\sigma(1) = n+1$,
- $\sigma(i) = 2n+3-i$ if $2 \leqslant i \leqslant n+1$,
- $\sigma(i) = 2n+2-i$ if $n+2 \leqslant i \leqslant 2n+2$;

see Figure 2. Then the pattern of this cyclic permutation is called the *Štefan* pattern. The interval $[n+1, n+2]$ is then called the *central* interval of the cycle $\{1, \dots, 2n+1\}$. Moreover, any cycle P of this pattern is said to be a *Štefan* cycle, and the P-basic interval corresponding to the central interval of the pattern is also called the *central* interval of this cycle.

Let f be the P-linear map defined by the Štefan cyclic permutation from Definition 2.7. Then it has the following properties. Take the central interval $I = [n+1, n+2]$ of P. Then $f(I) = [n, n+2], f^2(I) = [n, n+3]$, etc. In other words, with each application of f the image of the central interval I grows, adding one new P-basic interval to the left or to the right of I, alternately. In particular,

$f^j(I)$ contains $j+2$ points of P. This lasts until on the $2n-1$-st step we have $f^{2n-1} = [1, 2n+1]$. The slow rate of growth of I is "responsible" for the fact that the Štefan pattern is the forcing-weakest among all patterns of the same (odd) period.

Theorem 2.8 ([Šte77]; see also Lemma 2.16 of [ALM00])**.** *A periodic pattern of period $2n+1$ forces the Štefan pattern of period $2n+1$. A periodic pattern of period $2n+1$ which is not Štefan forces the Štefan pattern of period $2n-1$.*

Finally, we would like to make a simple but useful remark concerning the Sharkovsky Theorem. It claims that if a continuous interval map f has a cycle P of period n and $n \succ_{s} m$, then f has a cycle Q of period m. In fact, one can add to this that Q can be chosen *inside* the convex hull of P.

2.1. **Patterns with (no) block structure and the order among their periods.** Let us begin by discussing simple facts concerning patterns A *with* block structure. It turns out that there are several patterns trivially forced by A. In particular, using Lemma 2.6 based upon the techniques of loops of intervals (see Definition 2.5) one can easily prove the following well-known corollary.

Corollary 2.9. *Let P be a cycle of pattern A and let f be the P-linear map. Let $X_1, \dots, X_l$ be blocks of some block structure in P. Then there exists an f-cycle that has exactly one point in the convex hull of each X_i. Moreover, the pattern B of any such f-orbit is well-defined and is forced by A.*

We shall say that the pattern of the cycle whose existence is established in Corollary 2.9 is *generated* by the block structure with blocks $X_1, \dots, X_l$. We will also say that the cycle P (and the pattern A) from Corollary 2.9 has block structure *over* the cycles described in that corollary (or over the pattern B of those cycles).

An important particular case is presented in the next definition.

Definition 2.10 (Doubling)**.** Let P be a cycle of pattern A with block structure such that each block consists of two points. Denote by B the pattern generated by this block structure. Then we say that A is a *doubling* of B. Figure 3 shows a cycle of period 14 which is a doubling, while Figure 4 shows a similar cycle of period 14 which is not a doubling.

Observe that given a pattern B of period larger than 1, there are several (more than one) patterns that are doublings of B. This follows from the fact that the map on each block of A from Definition 2.10 may be either increasing or decreasing.

The next lemma follows immediately from the fact that for a Štefan cycle the image of the leftmost basic interval contains more than half of the points of the cycle.

Lemma 2.11. *Štefan cycles have no block structure.*

Let us now describe the order among periods of the patterns with no block structure induced by the forcing relation. Let $N(p)$ be the set of all integers s with $p \gg s$ and p itself. Given an interval map f, let $NBS(f)$ be the set of periods of all f-cycles with no block structure.

Theorem 2.12 ([BM18])**.** *Let f be a continuous interval map. If $m \gg s$ and f has a cycle with no block structure of period m, then f also has a cycle with no block structure of period s. Moreover, $NBS(f) = N(p)$ for some p, and for every p there exists an interval map f such that $NBS(f) = N(p)$.*

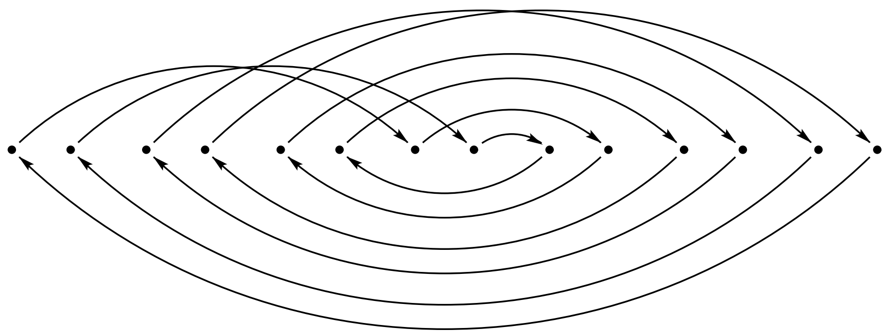

FIGURE 3. A cycle of period 14, which is a doubling.

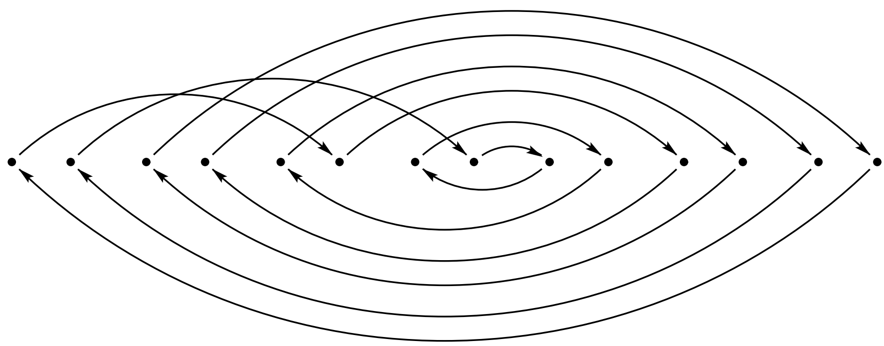

FIGURE 4. A cycle of period 14, which is not a doubling.

Observe that for $p = 2$ and $p = 1$ the existence of a map f with $NBS(f) = N(p)$ is immediate.

For completeness we prove a simple lemma describing some cases of Theorem 2.12.

Lemma 2.13. *The following properties of a continuous interval map f are equivalent:*

(1) $NBS(f) = \{1\}$ *or* $NBS(f) = \{1, 2\}$;
(2) *the map f has no points of odd period greater than* 1.

Proof. Suppose first that $NBS(f) = \{1\}$ or $NBS(f) = \{1, 2\}$. Then f cannot have periodic points of odd periods greater than 1. Indeed, otherwise by Theorem 2.8 f must have a Štefan cycle, and by Lemma 2.11 every Štefan cycle has no block structure, a contradiction to the assumption. On the other hand, suppose that f has no points of odd period greater than 1. Then, by Lemma 2.1.6 of [ALM00], all its f-cycles of periods greater than 1 have division and, hence, a block structure. □

3. Basic facts about renormalization towers

As we mentioned in the introduction, we chose towers as the characteristic of cycles that we want to study as a compromise between periods and permutations. Let us discuss shortly the idea of this choice.

What we want is a notion that on one hand provides some idea of the dynamical structure of a cycle, but on the other hand allows us to study the emerging order without making things too complicated to handle. Ideally, the order should be linear. In particular, this notion should be of a "numerical" type rather than "combinatorial." It turns out that the towers satisfy all our conditions. The order is linear (see Main Theorem). A tower is a finite string of natural numbers. Some important information about the dynamical structure of a cycle can be read from its tower (not all the information, but probably as much as we can count on if we want to keep the other properties of that notion).

In the rest of this section we prove a few statements concerning towers, but not dealing with the $\gg$-order. Let A be a pattern with tower $\mathrm{RT}(A) = (m_1, m_2, \dots, m_k)$. Call the blocks from the block structure of period m_1 *blocks of the first level*. Similarly, the blocks of the next block structure are said to be *blocks of the second level*, and so on. It is easy to see that the block structure of the s-th level is of period $m_1 \cdot m_2 \cdots m_s$; in particular, the period of A is $m_1 \cdots m_k$.

First we relate periods of two block structures of the same cycle.

Lemma 3.1. *Let P be a cycle of period n of an interval map f. Suppose that P has two block structures of periods $p < q$. Then p divides q.*

Proof. Denote by X and Y the leftmost blocks of the block structures of P of periods p and q, respectively. Then, clearly, $Y \subset X$. By definition this implies that $f^s(Y) \cap X = \emptyset$ for every integer s that is not divisible by p. Since $f^q(Y) = Y \subset X$, it follows that q is divisible by p. □

We will also need the next lemma.

Lemma 3.2. *Let P be a cycle of an interval map f. Let $X_1, \dots, X_k$ be blocks of a block structure of P. Moreover, suppose that there exists i such that X_i is a cycle of f^k with no block structure. Then $X_1, \dots, X_k$ are blocks of the last non-trivial block structure of P.*

Proof. Indeed, otherwise X_i would have a block structure for f^k, which it does not have by the assumption. □

The next lemma immediately follows from definitions and Lemma 3.1.

Lemma 3.3. *Suppose that a pattern A has a block structure over a pattern B. Then we have $\mathrm{RT}(B) = (m_0, m_1, \dots, m_r)$, while*

$$\mathrm{RT}(A) = (m_0, m_1, \dots, m_r, m_{r+1}, \dots, m_k)$$

for appropriate $r < k$. In particular, all patterns B over which A has a block structure have towers $(m_0), (m_0, m_1), \dots, (m_0, m_1, \dots, m_{k-1})$.

The following lemma is actually a simple particular case of the Main Theorem. Given a tower $\mathcal{M} = (m_0, \dots, m_k)$ of height k, we will call any tower $(m_0, \dots, m_i), i \leqslant k$, a *beginning* of $\mathcal{M}$.

Lemma 3.4. *If a tower $\mathcal{N}$ coincides with a beginning of a tower $\mathcal{M}$, then any pattern A with tower $\mathcal{M}$ forces a pattern B with tower $\mathcal{N}$ such that A has a block structure over B. In particular, $\mathcal{M}$ forces $\mathcal{N}$.*

Proof. Let P be a cycle of the P-linear map f, where P has pattern A. Let $X_1, X_2, \dots$ be the blocks of P at the level equal the height of $\mathcal{N}$. By Corollary 2.9 there is a cycle of f generated by the block structure formed by these blocks. By definition and the assumptions of the lemma, this cycle has some pattern B with tower $\mathcal{N}$ and such that A has a block structure over B. By Theorem 2.3 it follows that $\mathcal{M}$ forces $\mathcal{N}$. □

4. Unimodal cycles and patterns

Here we will state several well-known facts about unimodal cycles and patterns. A cycle P of a P-linear map f is *unimodal* if f is unimodal or the period of P is 2 or 1. In particular, all cycles of a unimodal map are unimodal. Patterns of unimodal cycles will also be called *unimodal*.

We will use the *kneading theory* for unimodal maps. It comes in several versions; here we will use the version from [CE], adapted to considering cycles. Suppose that A is a unimodal pattern of period $n > 1$. When we consider its permutation $\sigma : \{1, \dots, n\} \to \{1, \dots, n\}$, we may assume that $\sigma(n) = 1$ (that is, the corresponding unimodal map has a local maximum in the interior of the interval). Then there is $k \in \{1, \dots, n\}$ such that $\sigma(k) = n$. In this situation, σ is increasing on $\{1, \dots, k\}$ and decreasing on $\{k, \dots, n\}$. The *kneading sequence* of A is a sequence $S = (S_1, \dots, S_{n-1})$ of symbols L, R (left, right), where $S_i = L$ if $\sigma^i(k) < k$ and $S_i = R$ if $\sigma^i(k) > k$. Since $\sigma(k) = n$, then $S_1 = R$; since $\sigma(n) = \sigma^2(k) = 1$, then $S_2 = L$. A pattern A is *even* if the number of symbols R in S is even and *odd* if this number is odd. It is easy to see that different patterns have different kneading sequences.

The *star product* $S * T$ of the kneading sequence S of a unimodal pattern A and the kneading sequence $V = (V_1, \dots, V_{m-1})$ of a unimodal pattern B is defined as the concatenation

$$SV_1SV_2, \dots, SV_{m-1}S \text{ if } A \text{ is even, and}$$
$$S\check{V}_1S\check{V}_2, \dots, S\check{V}_{m-1}S \text{ if } A \text{ is odd,}$$

where $\check{L} = R$ and $\check{R} = L$. It turns out that $S * T$ is the kneading sequence of some *unimodal* pattern C. This pattern has a block structure over A. If a cycle P has pattern C and f is the P-linear map, then each block is itself a cycle of f^n (where n is the period of A) and has pattern B. Since the map f is unimodal, it is monotone on each block except perhaps one. Thus, C is a unimodal extension of A with B (see Definition 5.10). We will write $C = A * B$ and will call C the *star product* of A and B.

Theorem 4.1. *There are unimodal patterns with all towers.*

Proof. There are unimodal patterns of all periods with no block structure, for instance, with the kneading sequences RL^{n-2}. Using them and applying the star product we can produce unimodal patterns with any tower. □

Another approach to the unimodal patterns is to look at the cycles of the full tent map with given patterns. The *full tent map* $T : [0, 1] \to [0, 1]$ is given by the formula $T(x) = 2x$ if $x \leqslant 1/2$ and $T(x) = 2 - 2x$ if $x \geqslant 1/2$. If P is a cycle of

T, let us denote by $\alpha(P)$ the largest (rightmost) point of P. For every unimodal pattern A there are one (if A is a doubling) or two (otherwise) cycles of T of that pattern (but if the period of A is 1, we do not treat $\{0\}$ as a cycle). If there is one cycle, say P, we will write $\alpha(A) = \alpha(P)$. If there are two such cycles, say P_1 and P_2, we will write $\alpha(A) = \min(\alpha(P_1), \alpha(P_2))$. It turns out that if A and B are unimodal patterns, then A forces B if and only if $\alpha(A) \geqslant \alpha(B)$. This property is used, in particular, to provide a simple proof of the second ("realization") part of the Sharkovsky Theorem in [ALM00].

There is an important point, 5/6, which is a border between cycles of T with and without a division (observe that $T^2(5/6) = 2/3$, the fixed point of T).

Lemma 4.2. *A unimodal pattern A has a division if and only if $\alpha(A) < 5/6$.*

Proof. We need to show that a cycle P of T has a division if and only if $\alpha(P) < 5/6$. The point 2/3 is the unique fixed point of T in $(0, 1)$, so P has a division if and only if, for $x \in P$, $x < 2/3$ implies that $T(x) > 2/3$ and $x > 2/3$ implies that $T(x) < 2/3$.

If $\alpha(P) < 5/6$, then P is also a cycle of $T_{5/6}$. We have $\bigcap_{n=0}^{\infty} T_{5/6}^n((0, 1]) = [1/3, 5/6]$, so it follows that $P \subset [1/3, 5/6]$. However, $T_{5/6}([1/3, 2/3]) = [2/3, 5/6]$ and $T_{5/6}([2/3, 5/6]) = [1/3, 2/3]$, so P has a division.

If $\alpha(P) > 5/6$, then there exists $x \in P$ with $x > 5/6$. Then both points $T_{5/6}(x)$ and $T_{5/6}^2(x)$ are to the left of 2/3, so P has no division. □

The following theorem follows immediately from this fact.

Theorem 4.3. *Any Štefan pattern forces all unimodal patterns with towers starting from* 2 (*that is, with a division*).

5. Main Theorem

Now we start taking into account the $\gg$-order.

Lemma 5.1. *Suppose that $n = ps$. Then one of the following holds:*

(a) $p = 1$,
(b) $p = 2$,
(c) $n = 4k + 2$ *and* $p = 2k + 1$,
(d) $p \gg n$,
(e) $p = n$.

Proof. If (a), (b), and (e) do not hold, then $2 < p < n$. Under this assumption, we consider various cases. Observe that in the $\gg$-order we have

$$3 \gg 5 \gg 7 \gg 9 \gg \cdots \quad \text{and} \quad 4 \gg 6 \gg 8 \gg 10 \gg \cdots . \tag{5.1}$$

If n is odd, then p is an odd number smaller than n but greater than 2, and by (5.1) (d) holds. Similarly, if both n and p are even, then by (5.1) (d) holds.

If $n = 4k$ and p is odd, we have $p \leqslant 2k$, so $p \leqslant 2k - 1$. Now (d) follows from the fact that $2k - 1 \gg 4k$ and from (5.1).

The remaining case is $n = 4k + 2$ and p odd. If $p = 2k + 1$, then (c) holds. If $p \neq 2k + 1$, then $p \leqslant 2k - 1$, and (d) follows from the fact that $2k - 1 \gg 4k + 2$ and from (5.1). □

Theorem 2.12 deals with patterns with no block structure. Using Lemma 5.1 one can show that it also has consequences related to patterns that admit block structure.

Lemma 5.2. *Suppose that A is a pattern of period n and no division. Then either $n = 4k+2$ and A is a doubling of the Štefan pattern of period $2k+1$ or A forces a pattern of period n with no block structure. In particular, if $n \gg m$ and A has no division, then A forces a pattern of period m and no block structure.*

Proof. If A has no block structure, there is nothing to prove. Assume that it has a block structure. Then the height r of the tower $\mathrm{RT}(A) = (m_1, \dots, m_r)$ of A is at least 2. Let P be a cycle of pattern A of the P-linear map f. By Corollary 2.9, the block structure of the first level of $\mathrm{RT}(A)$ generates a certain cycle Q of pattern B with no block structure. Moreover, B has period m_1 and is forced by A.

Suppose that A does not force a pattern of period n with no block structure. If $m_1 = 2$, then A has division, a contradiction. If $m_1 > 2$, then by Lemma 5.1 and Theorem 2.12, since B does not force a pattern of period n and no block structure, we have $n = 4k+2$ and $m_1 = 2k+1$. Let us show that B is the Štefan pattern of period $2k+1$. Indeed, suppose that Q is not a Štefan cycle. Then by Theorem 2.8 B forces the Štefan pattern of period $2k-1$. By Theorem 2.12, this pattern forces a pattern of period $n = 4k+2$ with no block structure, a contradiction to the assumption.

Let us now prove the last claim of the lemma. Suppose that $n \gg m$ and A has no division. Consider two cases. First, let $n = 4k+2$ and let A be a doubling of the Štefan pattern of period $2k+1$. Then A forces the Štefan pattern of period $2k+1$ (which has no block structure), and hence the claim is proven in the case when $m = 2k+1$. Otherwise $2k+1 \gg m$ and the claim follows from Theorem 2.12. If (for any n) A is not a doubling of a Štefan pattern, we know already that A forces a pattern of period n and no block structure. Now the claim again follows from Theorem 2.12. □

The first non-trivial step toward a proof of the Main Theorem is to consider two patterns with towers of equal height that differ at the last level. While we could state it using the tower notation, it makes sense to introduce a new notion. Namely, we will say that a pattern A has a *direct block structure* over a pattern D if A has a block structure over D but has no block structure over any pattern B that has block structure over D. In other words, to go from D to A, we take into account only one additional level of the associated tower. Observe that if a cycle P of a map f has a block structure over a pattern D of period n and this structure is not direct, then every block of P over D is a cycle of f^n and this cycle has a block structure.

Lemma 5.3. *Let a pattern A of period nm have a direct block structure over a pattern D of period n. Let $s > 1$ be such that $m \gg s$. Then there exists a pattern B of period ns with a direct block structure over D, forced by A.*

Proof. Consider a cycle Q of pattern A and the Q-linear map g. Let $X_1, X_2, \dots, X_n$ be the blocks in Q of the block structure over D. These blocks are cyclically permuted by g, and each of them consists of m points. For each $i \in \{1, 2, \dots, n\}$, the map $g^n|_{X_i}$ defines some pattern of period m.

Suppose that all cycles of $g^n|_{X_i}$ have a division. Then $X_i = X_i' \cup X_i''$ so that the convex hulls of X_i' and X_i'' are disjoint, $g^n(X_i') = X_i''$, and $g^n(X_i'') = X_i'$. Let us call sets X_i', X_i'' *half-blocks*. Thus, points $x, y \in Q$ belong to the same half-block if and only if $y = g^{2kn}(x)$ for some integer $k \geqslant 0$. Therefore, if $x, y \in Q$ belong to the same half-block, then $g(y) = g^{2kn+1}(x) = g^{2kn}(g(x))$, so $g(x)$ and $g(y)$ belong to the same half-block. This shows that g maps half-blocks to half-blocks, so the block structure of A over D is not direct, a contradiction.

Hence, there exists $i, 1 \leqslant i \leqslant n$, such that $g^n|_{X_i}$ has no division. Then, by Theorem 2.12 and Lemma 5.2, $g^n|_{[\min(X_i),\max(X_i)]}$ must have a cycle S of period s with no block structure. Let P be the g-orbit of any point of S. Then the pattern B of P has period ns and is forced by A. Moreover, by Lemma 3.2, P has a direct block structure over D. □

Now suppose that we apply Lemma 5.3 with $s \geqslant 3$ odd. There may be many patterns of period ns with a direct block structure over D, forced by A. However, at least one of them does not force any other one. Let us investigate this forcing-minimal pattern closer. Observe that studying forcing-minimal patterns from a specific set of patterns gives us an explicit description of the dynamics guaranteed for a map that has cycles with patterns from the set of patterns in question.

Lemma 5.4. *Assume that a pattern B of period ns has a direct block structure over a pattern D of period n. Moreover, suppose that $s \geqslant 3$ is odd and that B does not force any other pattern of the same period that has a direct block structure over D. Let a cycle P of the P-linear map f have pattern B. Denote by X_i the blocks of its block structure over D. Then X_i is a Štefan cycle of f^n for every i.*

Proof. Observe that the map f cyclically permutes the convex hulls of blocks X_i. Our assumptions imply that X_i is a cycle of odd period s of f^n, whose pattern does not force any other pattern of period s with no block structure. Thus, X_i is a Štefan cycle of f^n (see Theorem 2.8). □

On the other hand, observe that in our situation f^n restricted to the convex hull of X_i is a priori not necessarily X_i-linear or even X_i-monotone. Thus, we need some facts concerning arbitrary continuous interval maps that have Štefan cycles. It will be convenient to assume that their periods are strictly larger than 3, as the case of period 3 has to be considered directly. Recall that if P is a Štefan cycle, then by the central P-basic interval we mean the P-basic interval such that the collections of P-basic intervals to the left and to the right of it are of the same cardinalities (see Definition 2.7).

Lemma 5.5. *Suppose that f is a continuous interval map with a Štefan cycle Q of period $s = 2n+1 > 3$. Suppose that the central Q-basic interval I is such that $f^{s-3}(I) \supset Q$. Then f has a point of an odd period less than s but larger than 1 whose orbit is a Štefan cycle. In particular this holds if $f(I)$ contains more than three points of Q.*

Proof. We may assume that $Q = \{q_1, \dots, q_{2n+1}\}$ and that the map $f|_Q$ induces the permutation described in Definition 2.7. By the assumption, $[q_1, q_2] \subset f^{s-3}(I)$. On the other hand, $I \subset f([q_1, q_2])$. By Lemma 2.6 there exists a point $z \in I$ such that $f^{s-3}(z) \in [q_1, q_2]$ while $f^{s-2}(z) = z$. Clearly, z is not f-fixed. Hence the period of z is an (odd) divisor of $s-2$ which is greater than 1. Together with Theorem 2.8 this proves the first claim.

To prove the second claim, notice that $f(I) \supset I$. In addition, observe that $[q_2, q_s] \subset f^{s-3}(I)$. Thus, it suffices to show that $q_1 \in f^{s-3}(I)$. However, by the assumption $f(I)$ contains a point q_r with $r \neq n, n+1, n+2$. Now, it follows from the properties of Štefan cycles that $f^{s-1}(q_n) = q_1$, $f^{s-2}(q_{n+1}) = q_1$, and $f^{s-3}(q_{n+2}) = q_1$. Hence for some $j \leqslant s-4$ we will have that $q_1 = f^j(q_r) \in f^j(f(I))$ so that $q_1 \in f^{j+1}(I) \subset f^{s-3}(I)$ as desired. □

Let us turn our attention back to the cycle P from Lemma 5.4 (we will use the notation from Lemma 5.4 too) and study it using Lemma 5.5. According to our definition, the pattern of P is a forcing-minimal pattern among all patterns of period ns that have direct block structure over a given pattern D of period n (here $s \geqslant 3$ is odd). Recall that $X_i, 1 \leqslant i \leqslant n$, is a Štefan cycle of f^n. To reflect the dynamics of Štefan cycles more closely we can label points in X_i differently than in Lemma 5.5 (the notation from Lemma 5.5 reflects the relative location of points of a cycle rather than the dynamics). Namely, we can denote points of X_i as p_i^j in such a way that $f^n(p_i^j) = p_i^{j+1}$ (adding 1 is modulo s) and either

$$p_i^s < p_i^{s-2} < \cdots < p_i^3 < p_i^1 < p_i^2 < \cdots < p_i^{s-1}$$

or

$$p_i^{s-1} < p_i^{s-3} < \cdots < p_i^2 < p_i^1 < p_i^3 < \cdots < p_i^s.$$

Throughout the following lemmas we shall keep this notation as well as the assumption that s is odd and $s > 3$. We will use notation $\langle x, y\rangle$ for the closed interval with endpoints x, y (here we do not assume that $x < y$). We will rely upon the standard techniques based on Lemmas 2.6 and 5.5 and follow the ideas from [ALM00].

Lemma 5.6. *If $1 \leqslant t \leqslant n$, then $f^t(\{p_i^j : j = 3, 4, \dots, s-1\}) \cap \langle f^t(p_i^1), f^t(p_i^2)\rangle = \emptyset$. In particular, if $f^t(p_i^1) = p_{i+t}^j$, then either there are no elements of P between p_{i+t}^j and p_{i+t}^{j+1} or there is only one such element, namely $f^t(p_i^s)$.*

Proof. Assume the contrary. Then there exists $j \in \{3, 4, \dots, s-1\}$ such that $f^t(p_i^j) \in \langle f^t(p_i^1), f^t(p_i^2)\rangle$. Hence $f^n(p_i^j) = f^{n-t}(f^t(p_i^j)) \in f^{n-t}(\langle f^t(p_i^1), f^t(p_i^2)\rangle) \subset f^n(\langle p_i^1, p_i^2\rangle)$, which shows that $f^n(\langle p_i^1, p_i^2\rangle)$ contains not only points p_i^1, p_i^2, and p_i^3 but also the point $f^n(p_i^j) = p_i^{j+1}$ distinct from any of them. By Lemma 5.5 this implies that f has a cycle P' of period nk and a block structure over D with $k \geqslant 3$ odd and such that $k < s$. Note that this cycle does not have to have a direct block structure over D. Let J be a convex hull of one of the blocks of P. Then $f^n(J) = J$ and $P' \cap J$ is a cycle of f^n of period k. It forces a Štefan cycle Q of $f^n|_J$ of period s. The set $P'' = \bigcup_{i=0}^{n-1} f^i(Q)$ is a cycle of f of period ns with block structure over D. This block structure is direct, because otherwise Q, as a cycle of f^n, would have a block structure, a contradiction (since Q is a Štefan cycle, Q does not have a block structure). The existence of such P'' contradicts the forcing minimality of P assumed above. The last claim of the lemma follows now from the fact that $i+t$-th block X_{i+t} of P consists of f^t-images of points of X_i. □

Lemma 5.6 allows one to specify the location of points $f^t(p_i^1), 1 \leqslant t \leqslant n$.

Lemma 5.7. *If $1 \leqslant t \leqslant n$, then either $f^t(p_i^1) = p_{i+t}^1$ or $f^t(p_i^1) = p_{i+t}^2$.*

Proof. Let $f^t(p_i^1) = p_{i+t}^j$. Then $f^t(p_i^2) = p_{i+t}^{j+1}$ (recall the dynamical nature of our notation). By Lemma 5.6 either (a) there are no points of P between p_{i+t}^j and p_{i+t}^{j+1} or (b) there is only one such point, namely $f^t(p_i^s)$. Let us also emphasize that

$f^n(p^j_{i+t}) = p^{j+1}_{i+t}$; i.e., we are talking about a point of X_{i+t} and its f^n-image. Since X_{i+t} is Štefan, the properties of Štefan patterns imply that in case (a) we get $j = 1$ as the only possibility. Similarly, in case (b) we get $j = 2$. □

The location of points $f^t(p^1_i), 1 \leqslant t \leqslant n$, is further specified in the next lemma.

Lemma 5.8. *There is exactly one $i \in \{1, 2, \dots, n\}$ such that $f(p^1_i) \neq p^1_{i+1}$.*

Proof. At least one j such that $f(p^1_j) \neq p^1_{j+1}$ must exist because $f^n(p^1_i) = p^2_i$ for all i. Suppose that there are $t, r \in \{1, 2, \dots, n\}$, $t \neq r$, such that $f(p^1_t) \neq p^1_{t+1}$ and $f(p^1_r) \neq p^1_{r+1}$. We may assume that $r < t$ and that t is the smallest of all the integers $i > r$ such that $f(p^1_i) \neq p^1_{i+1}$.

By Lemma 5.7, we have $f(p^1_r) = p^2_{r+1}$ and $f(p^1_t) = p^2_{t+1}$. Since $f^n(p^1_t) = p^2_t$ and $f^n(p^2_{t+1}) = p^3_{t+1}$, it follows that

$$f(p^2_t) = f^{n+1}(p^1_t) = f^n(p^2_{t+1}) = p^3_{t+1}.$$

Moreover, if $r < i < t$, then $f(p^1_i) = p^1_{i+1}$, so similarly, $f(p^2_i) = p^2_{i+1}$. Therefore, $f^{t-r-1}(p^2_{r+1}) = p^2_t$, and thus

$$f^{t-r+1}(p^1_r) = f(f^{t-r-1}(f(p^1_r))) = f(f^{t-r-1}(p^2_{r+1})) = f(p^2_t) = p^3_{t+1},$$

a contradiction to Lemma 5.7. □

Lemma 5.8 implies the following corollary.

Corollary 5.9. *In our situation, i.e., in the case when $s > 3$ is odd, there is exactly one $t \in \{1, 2, \dots, n\}$ such that $f|_{X_t}$ is not monotone.*

The following terminology was introduced in [ALM00].

Definition 5.10 (Extensions)**.** A pattern with a block structure over a pattern A, where the map is monotone on all blocks, except perhaps one, is called an *extension* of A (in [MiNi91] it is called a *simple extension*). If the unique non-monotone restriction of the map on the corresponding block is unimodal, the extension in question is said to be *unimodal*. If an extension of A is such that the first return map to a block belongs to a pattern B, then we say that this is an *extension of A by a pattern B* (observe that if A is an extension of a pattern, then the first return maps to blocks belong to the same pattern). If B is a Štefan pattern, this defines *Štefan extensions*. Figure 5 shows a cycle of period 15, which is a unimodal extension of a Štefan cycle of period 5.

Lemma 5.11 is based on Corollary 5.9, and extra arguments are needed in the case when $s = 3$.

We say that f^k has a *horseshoe* if there are two closed intervals I, J with disjoint interiors which have the property that $f^k(I) \cap f^k(J) \supset I \cup J$.

Lemma 5.11. *Let $s \geqslant 3$ be an odd integer. Consider the family of all patterns of period ns that have a direct block structure over a given pattern A of period n. Let B be a forcing-minimal pattern in this family. Then B is a Štefan extension of A.*

Proof. By Corollary 5.9 and Lemma 5.4 it suffices to consider the case of $s = 3$. Let P be a cycle of pattern B, and let f be a P-linear map. Suppose that B is not an extension of A and show that f^n has a horseshoe on a block of P associated with the block structure of B over A. Let X be a block of P on which f is not monotone, and let r be such that $f|_{f^r(X)}$ is not monotone. Let the convex hull of X

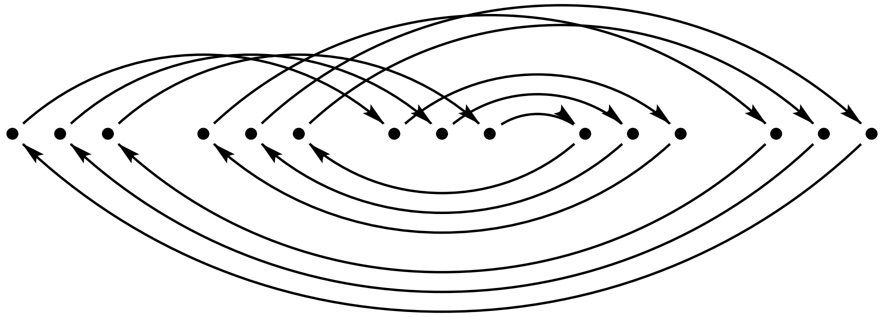

FIGURE 5. A cycle of period 15, which is a unimodal extension of a Štefan cycle of period 5.

be Z, and let the convex hull of $f^r(X)$ be T. Then $Z = I \cup J$ is the union of two P-basic intervals I and J. Similarly, $T = K \cup L$ is the union of two P-basic intervals K and L. Since $f^r|_Z$ is not monotone, we may assume that $f^r(I) = T$. Since $f^{n-r}|_T$ is not monotone, we may assume that $f^{n-r}(K) = Z$, and so $f^n(K) = T$. If $K \subset f^r(J)$, then I and J form a horseshoe of f^n. Otherwise $f^r(J) = L$. Since $f^n(J) \supset I$, then $f^{n-r}(L) \supset I$ and $f^n(L) = T$. Hence K and L form a horseshoe for f^n. It is easy to see that the existence of a horseshoe for f^n on the convex hull of a block of P implies that f has a cycle $Q \neq P$ of period $3n$ whose pattern D has a block structure over A. Thus, B forces D and, therefore, is not forcing-minimal, a contradiction. □

The following result easily follows from Lemmas 5.3, 5.4, and 5.11.

Proposition 5.12. *Let a pattern A of period nm have a direct block structure over a pattern D of period n. Let $s > 1$ be an odd integer such that $m \gg s$. Then A forces a pattern of period ns which is an extension of D by a Štefan pattern.*

In order to prove the Main Theorem we also need two known results. We will state them in a form consistent with our notation.

Theorem 5.13 (Theorem 2.10.8 of [ALM00])**.** *If a pattern C is an extension of B by A and A forces $\widetilde{A}$, then C forces an extension of B by $\widetilde{A}$.*

The second one is a part of Theorem 9.12 from [MiNi91].

Theorem 5.14. *Suppose that a pattern A forces a pattern B but does not have a block structure over B and B is not a doubling. Then A forces extensions of B by all unimodal patterns.*

Now we are at last ready to prove the first claim of the Main Theorem. We state it as a separate theorem below. Recall that by Lemma 1.9, $\gg$-tails of towers coincide with the sets $\mathrm{Tow}(\cdot)$ for infinite towers.

Theorem 5.15 (Strong version of the first claim of the Main Theorem)**.** *Let $\mathcal{N}'$ and $\mathcal{M}'$ be two towers such that*

$$\mathcal{N}' = (m_1, \dots, m_t, m_{t+1}, \dots, m_r), \quad \mathcal{M}' = (m_1, \dots, m_t, s),$$

with $m_{t+1} \gg s$. Then any pattern A of tower $\mathcal{N}'$ forces a pattern C of tower $\mathcal{M}'$ and all unimodal extensions of C. Thus, if $\mathcal{N} \gg \mathcal{M}$ and if a continuous interval *map f has a cycle with tower $\mathcal{N}$, then it has a cycle with tower $\mathcal{M}$. Hence, there exists an infinite tower $\mathcal{K}$ such that the set of all finite towers of cycles of f is* $\mathrm{Tow}(\mathcal{K})$.

Proof. If a pattern A has tower $\mathcal{N}'$, then by Lemma 3.4 it forces a pattern B with tower $(m_1, \dots, m_t, m_{t+1})$ over which A has a block structure (A might also coincide with B). Then, by Lemma 5.3, B forces a pattern C with tower $(m_1, \dots, m_t, s)$.

To prove the second claim, consider two cases. If $s > 2$, then C is not a doubling. Moreover, evidently the pattern B does not have block structure over the pattern C. Therefore, by Theorem 5.14, B forces all unimodal extensions of C.

Let $s = 2$ (and, hence, $m_{t+1} > 2$). Denote by D the pattern with tower $(m_1, \dots, m_t)$ over which B has a block structure. Properties of the $\gg$-order imply that there exists an odd number $l > 1$ such that $m_{t+1} \gg l \gg s_1 = 2$. By Proposition 5.12, B forces an extension of the pattern D by the Štefan pattern of period l. By Theorem 5.13, B forces extensions of D by all patterns forced by the Štefan pattern of period l. Therefore, by Theorem 4.3, B forces all extensions of D by a unimodal pattern with division as desired.

In view of Theorem 4.1 and Lemma 1.9, the last two claims of the theorem immediately follow from the first one. □

Remark 5.16. One can make a statement equivalent to Theorem 5.15, dealing with patterns rather than cycles. Namely, if $\mathcal{N} \gg \mathcal{M}$, then every pattern with tower $\mathcal{N}$ forces a pattern with tower $\mathcal{M}$.

To prove the second part of the Main Theorem we will look at the family of truncated tent maps T_a, $0 < a \leqslant 1$, given by $T_a(x) = \min(a, T(x))$, where T is the full tent map (see Section 4). This is the same strategy that was used in order to prove the second part of the Sharkovsky Theorem in [ALM00]. Recall that given a unimodal pattern D we define the number $\alpha(D)$ right after Theorem 4.1.

Theorem 5.17 (Second claim of the Main Theorem)**.** *If $\mathcal{K}$ is an infinite tower, then there exists a continuous interval map g such that the set of all (finite) towers of cycles of g coincides with* $\mathrm{Tow}(\mathcal{K})$.

Proof. For every n there is a unimodal pattern B_n which is forcing-minimal among unimodal patterns of period n with no block structure; they are identified in [Mis94]. In particular, it is easy to see that B_4 is the unimodal pattern with kneading sequence RL_2 corresponding to the permutation ξ of the set $\{1, 2, 3, 4\}$ such that $\xi(1) = 2$, $\xi(2) = 3$, $\xi(3) = 4$, and $\xi(4) = 1$. By Theorem 5.13, for a finite tower $\mathcal{M} = (m_1, \dots, m_s)$, the unimodal pattern $A_{\mathcal{M}} = B_{m_1} * \dots * B_{m_s}$ (see Section 4; the star product of unimodal patterns is by the definition a unimodal pattern) is forcing-minimal among unimodal patterns with tower $\mathcal{M}$. In other words, $\alpha(A_{\mathcal{M}})$ is smaller than $\alpha(C)$ for every other unimodal pattern C with tower $\mathcal{M}$. Consider the map $T_{\mathcal{M}} = T_{\mathcal{M}} = T_{\alpha(A_{\mathcal{M}})}$. By the first claim of the Main Theorem, the map $T_{\mathcal{M}}$ has cycles of all towers from $\mathrm{Tow}(\mathcal{M})$. On the other hand, suppose that it has a cycle of some other tower. Then again by the first claim of the Main Theorem it would imply that A forces another pattern with tower $\mathcal{M}$, a contradiction to the fact that A is forcing-minimal among unimodal patterns with tower $\mathcal{M}$. Thus, $T_{\mathcal{M}}$ has cycles of all towers from $\mathrm{Tow}(\mathcal{M})$ and no cycles with other towers. We conclude that the function $\psi : \mathcal{M} \mapsto \alpha(A_{\mathcal{M}})$ is increasing as a map from the space of all towers with order $\gg$ to the interval $[0, 1]$; in other words, $\mathcal{M} \gg \mathcal{N}$ implies that $\psi(\mathcal{M}) > \psi(\mathcal{N})$.

Let $\mathcal{K} = (k_1, k_2, \dots)$ be an infinite tower with $k_n > 1$ for every n. For each n we define a finite tower $\mathcal{K}_n$ by $\mathcal{K}_n = (k_1, k_2, \dots, k_n)$. Then the sequence $\alpha(A_{\mathcal{K}_n})$ is increasing, so it has the limit, which we will denote $\beta(\mathcal{K})$. Consider the map $T_\mathcal{K} = T_{\beta(\mathcal{K})}$. We claim that it has cycles of all towers from $\mathrm{Tow}(\mathcal{K})$ and no cycles of other towers.

The first part of the claim is immediate. Indeed, $\beta(\mathcal{K}) > \alpha(A_{\mathcal{K}_n})$ for every n, so $T_\mathcal{K}$ has cycles of all towers $\mathcal{M}$ for which there is n with $\mathcal{K}_n \gg \mathcal{M}$. However, these are exactly all towers from $\mathrm{Tow}(\mathcal{K})$.

To prove the second part of the claim, consider a tower $\mathcal{M} = (m_1, \dots, m_s) \notin \mathrm{Tow}(\mathcal{K})$. Then $\mathcal{M} \gg \mathcal{K}_n$ for every n. Since ψ is monotone, $\psi(\mathcal{M}) = \alpha(A_\mathcal{M}) > \psi(\mathcal{K}_n)$, which implies that $\psi(\mathcal{M}) \geqslant \beta(\mathcal{K})$. We claim that $\psi(\mathcal{M}) > \beta(\mathcal{K})$. We may assume that $m_1 = k_1, \dots, m_{j-1} = k_{j-1}$, but $m_j \gg k_j$. Let D be the unimodal pattern with the kneading sequence RL^3; in other words, it corresponds to the permutation φ of $\{1, 2, 3, 4, 5\}$ such that $\varphi(1) = 2$, $\varphi(2) = 3$, $\varphi(3) = 4$, $\varphi(4) = 5$, and $\varphi(5) = 1$.

It is easy to check that the unimodal pattern D forces the unimodal pattern B_4 (equivalently, $\alpha(D) > \alpha(B_4)$). Since by definition $4 \gg n$ for any integer n, we see that by Theorem 2.12 D forces all patterns B_n. Therefore, by Theorem 5.15, $A_\mathcal{M}$ forces $B_{k_1} * \cdots * B_{k_j} * D$, which in turn forces all patterns A_{K_n}. Thus, $\alpha(B_{k_1} * \cdots * B_{k_j} * D)$ is an upper bound of the set $\{\alpha(A_{\mathcal{K}_n}) : n = 1, 2, \dots\}$, so $\alpha(C) > \alpha(B_{k_1} * \cdots * B_{k_j} * D)$ cannot be its supremum. This is a contradiction, and hence $T_\mathcal{K}$ has no cycles with towers not in $\mathrm{Tow}(\mathcal{K})$. □

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Department of Mathematics, University of Alabama at Birmingham, Birmingham, Alabama 35294

Email address: ablokh@math.uab.edu

Department of Mathematical Sciences, Indiana University- Purdue University Indianapolis, 402 N. Blackford Street, Indianapolis, Indiana 46202

Email address: mmisiure@math.iupui.edu