

SLICES OF THE PARAMETER SPACE OF CUBIC POLYNOMIALS

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ABSTRACT. In this paper, we study slices of the parameter space of cubic polynomials, up to affine conjugacy, given by a fixed value of the multiplier at a non-repelling fixed point. In particular, we study the location of the *main cubioid* in this parameter space. The *main cubioid* is the set of affine conjugacy classes of complex cubic polynomials that have certain dynamical properties generalizing those of polynomials $z^2 + c$ for c in the filled main cardioid.

1. INTRODUCTION

By *classes* of polynomials, we mean affine conjugacy classes. For a polynomial f , let $[f]$ be its class. For any polynomial f , we write $K(f)$ for the filled Julia set of f , and $J(f)$ for the Julia set of f . The *connectedness locus* \mathcal{M}_d of degree d is the set of classes of degree d polynomials whose critical points *do not escape* (i.e., have bounded orbits). Equivalently, \mathcal{M}_d is the set of classes of degree d polynomials f whose Julia set $J(f)$ is connected. The connectedness locus \mathcal{M}_2 of degree 2 is otherwise called the *Mandelbrot set*; the connectedness locus \mathcal{M}_3 of degree 3 is also called the *cubic connectedness locus*. The *principal hyperbolic domain* \mathcal{PHD}_3 of \mathcal{M}_3 can be defined as the set of classes of hyperbolic cubic polynomials with Jordan curve Julia sets. Equivalently, we have $[f] \in \mathcal{PHD}_3$ if both critical points of f are in the immediate attracting basin of the same attracting (or super-attracting) fixed point. Recall that a polynomial is *hyperbolic* if the orbits of all critical points converge to attracting or super-attracting cycles.

We define the *main cubioid* \mathcal{CU} as the set of classes $[f] \in \mathcal{M}_3$ with the following properties: f has a non-repelling fixed point, f has no repelling periodic cutpoints in $J(f)$, and all non-repelling periodic points of f , except at most one fixed point, have multiplier 1. The main cubioid is a cubic analogue of the filled main cardioid (by the *filled main cardioid* we mean the closure of the family of all polynomials $z^2 + c$ that have an attracting fixed point).

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Theorem 1.1 ([BOPT14]). *We have that $\overline{\mathcal{PHD}}_3 \subset \mathcal{CU}$.*

It is conjectured in [BOPT14] that $\overline{\mathcal{PHD}}_3 = \mathcal{CU}$. Some properties of the complement $\mathcal{CU} \setminus \overline{\mathcal{PHD}}_3$ are discussed in [BOPT18].

A *fixed point marked* (or *FP-marked* for short) polynomial is by definition a pair (f, x) consisting of a polynomial f and a fixed point x for f (so that $f(x) = x$). Two FP-marked polynomials (f, x) and (g, y) are said to be *affinely conjugate* if there is a complex affine transformation $A : \mathbb{C} \rightarrow \mathbb{C}$ such that $A(x) = y$ and $g = A \circ f \circ A^{-1}$. Affine conjugacy classes of FP-marked polynomials will be referred to as *FP-classes*. Define the *multiplier* of a FP-marked polynomial (f, x) as $f'(x)$. Clearly, affinely conjugate FP-marked polynomials have the same multiplier.

Let \mathcal{F} be the space of polynomials

$$f_{\lambda,b}(z) = \lambda z + bz^2 + z^3, \quad \lambda \in \mathbb{C}, \quad b \in \mathbb{C}.$$

An affine change of variables reduces any cubic polynomial to the form $f_{\lambda,b}$. Clearly, 0 is a fixed point for every polynomial in \mathcal{F} . From now on we shall regard polynomials from \mathcal{F} as FP-marked polynomials, with marked point 0. The FP-marked class (FP-class for short) of $(f, 0)$ will be denoted as $[f]_{FP}$. Since any polynomial in \mathcal{CU} must have a non-repelling fixed point, it makes sense to mark this point. Thus FP-marked polynomials and FP-classes provide a natural context for studying \mathcal{CU} .

Two polynomials from \mathcal{F} belong to the same FP-class if there exists a map $\psi(z) = \gamma z$ that conjugates them. Define the λ -*slice* \mathcal{F}_λ of \mathcal{F} as the space of all polynomials $g \in \mathcal{F}$ with $g'(0) = \lambda$. Clearly, if two polynomials from \mathcal{F} belong to the same FP-class then they must both belong to the same λ -slice. Any FP-marked polynomial of multiplier λ is affinely conjugate to at most two polynomials from \mathcal{F}_λ . These are $f_{\lambda,b}$ and $f_{\lambda,-b}$ for some $b \in \mathbb{C}$ that are conjugate by means of the map $z \mapsto -z$. Thus, two distinct polynomials $f_{\lambda,b'}$ and $f_{\lambda,b''}$ belong to the same FP-class if and only if $b'' = -b'$. In particular, FP-classes of polynomials $f_{\lambda,b} \in \mathcal{F}_\lambda$ are in one-to-one correspondence with the values of b^2 . The space \mathcal{F} has been studied by Zakeri [Z99], Buff and Henriksen [BuHe01], and other authors.

The main result of this paper is a description of \mathcal{CU} through λ -slices where $|\lambda| \leq 1$. To state our main results, we need a FP-marked version of \mathcal{CU} . Define \mathcal{CU}_{FP} to be the set of FP-classes of all FP-marked cubic polynomials (f, x) with the following properties: x is a non-repelling fixed point, there are no repelling periodic cutpoints in $J(f)$, and all non-repelling periodic points of f , except possibly x , have multiplier 1. Since for $f \in \mathcal{F}$ the marked fixed point is 0, the polynomials from \mathcal{CU}_{FP} that belong to \mathcal{F} can have at most one non-repelling periodic point with multiplier not equal to 1, and this point must be 0.

We use calligraphic (script) font for parameter space objects like \mathcal{F} , \mathcal{M}_3 , etc., to distinguish them from dynamical plane objects. We mostly use German Gothic fonts for various objects in the closed disk related to *laminations*; these objects are used in combinatorial models of polynomials (laminations will be introduced in Section 6). We mostly use Greek letters for *angles*, i.e. elements of \mathbb{R}/\mathbb{Z} .

We need a few combinatorial concepts that play a significant role in complex polynomial dynamics. Their development is based upon several fundamental results that deal with both dynamical and parameter planes and often relate the two. For brevity, in our overview we will focus on polynomials P of degree d with *connected* Julia set. Ultimately, it is the Riemann map for the basin of attraction of infinity that allows one to relate the combinatorics/dynamics of the d -tupling map of the

circle at infinity and the dynamics of P . Douady and Hubbard [DH8485] showed that *all* the external rays to the Julia set $J(P)$ with rational arguments land at points that eventually map to repelling or parabolic periodic points. On the other hand, if $J(P)$ is connected, then *every* repelling or parabolic periodic point or an iterated preimage of such point is a landing point of finitely many rational external rays to $J(P)$.

This allows one to relate P and the d -tupling map σ_d acting on the rational arguments considered as angles of the circle at infinity. For polynomials with locally connected Julia sets this relation extends from rational arguments onto *all* arguments. Thurston's [Thu85] idea was to associate each point z of a locally connected Julia set $J(P)$ to the convex hull of the set of arguments of rays landing at z . Then distinct points give rise to pairwise disjoint convex hulls. Moreover, if one collapses these convex hulls to points, one will create a quotient space of the unit circle \mathbb{S}^1 , and on it a map induced by the d -tupling map; this serves as a precise model of $P|_{J(P)}$. Thurston [Thu85] singled out the most important properties of such collections of chords and called them *laminations*. He studied laminations without necessarily connecting them to polynomials.

Arguably, the most stunning results obtained in [Thu85] deal with the quadratic case when $d = 2$. In particular, Thurston used laminations to construct a model of the Mandelbrot set \mathcal{M}_2 which is given, again, by a lamination that he called the *quadratic minor lamination*. Thus, laminations can be used to model both Julia sets of polynomials (as in the previous paragraph) and parameter spaces of polynomials. We aim at considering cubic polynomial parameter spaces and need tools similar to those in [Thu85] and related to laminations.

Given an angle $\alpha \in \mathbb{R}/\mathbb{Z}$, we write $\bar{\alpha}$ for the corresponding point $e^{2\pi i\alpha}$ of the unit circle $\mathbb{S}^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. The angle tripling map $\sigma_3 : \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}/\mathbb{Z}$, defined by $\sigma_3(\alpha) = 3\alpha$, identifies with the self-map of \mathbb{S}^1 taking z to z^3 . We write $(\bar{\alpha}, \bar{\beta})$ for an open arc of the unit circle with endpoints $\bar{\alpha}$ and $\bar{\beta}$ if the direction from $\bar{\alpha}$ to $\bar{\beta}$ within the arc is positive. A closed chord of the closed unit disk $\bar{\mathbb{D}} = \{z \in \mathbb{C} \mid |z| \leq 1\}$ with endpoints $\bar{\alpha}, \bar{\beta} \in \mathbb{S}^1$ is denoted by $\overline{\alpha\beta}$. Given a closed set $X \subset \bar{\mathbb{D}}$ (e.g., often we will assume that $X \subset \mathbb{S}^1$), define *holes* of X as components of $\mathbb{S}^1 \setminus X$.

For $X \subset \mathbb{C}$, let $\text{CH}(X)$ be its convex hull and $\text{Bd}(X)$ be its boundary. Suppose that $\mathfrak{U} = \text{CH}(\mathfrak{U}')$, where $\mathfrak{U}' = \mathfrak{U} \cap \mathbb{S}^1$. *Edges* of \mathfrak{U} are chords on $\text{Bd}(\mathfrak{U})$. The set \mathfrak{U} is a (*stand alone*) *invariant gap* if $\sigma_3(\mathfrak{U}') = \mathfrak{U}'$, and, for every hole $(\bar{\alpha}, \bar{\beta})$ of \mathfrak{U} , either $\sigma_3(\bar{\alpha}) = \sigma_3(\bar{\beta})$ (then $\overline{\alpha\beta}$ is called *critical*) or the circular arc $(\sigma_3(\bar{\alpha}), \sigma_3(\bar{\beta}))$ is a hole of \mathfrak{U} . Extend $\sigma_3 : \mathfrak{U}' \rightarrow \mathfrak{U}'$ to *edges* of \mathfrak{U} linearly so that a critical edge maps to a point, and a non-critical edge $\overline{\alpha\beta}$ maps to $(3\alpha)(3\beta) = \sigma_3(\alpha)\sigma_3(\beta)$; denote the extension σ_3 too. The *degree* of \mathfrak{U} is the number of its edges mapping *onto* a non-degenerate edge of \mathfrak{U} ; it is well-defined. Degree two gaps are also called *quadratic gaps*.

We measure arc length in \mathbb{S}^1 so that the total length of the entire circle is 1. The *length of a chord* ℓ of $\bar{\mathbb{D}}$ is the length of the shorter circle arc in \mathbb{S}^1 connecting the endpoints of ℓ . A hole of \mathfrak{U} is called a *major hole* if its length is greater than or equal to $1/3$. The edge of \mathfrak{U} connecting the endpoints of a major hole is called a *major edge*, or simply a *major*, of \mathfrak{U} . By [BOPT16, Lemma 3.1], any *quadratic* invariant gap \mathfrak{U} has exactly one major that is either critical or periodic. As easily follows from Lemmas 3.22 and 3.23 of [BOPT16], there exists a Cantor set $Q \subset \mathbb{S}^1$ with the following property. If we collapse every hole of Q to a point, we obtain a

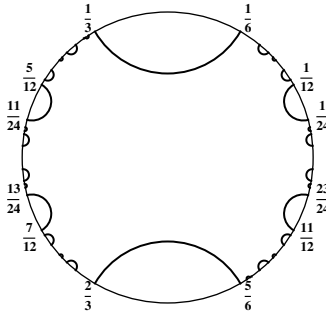


FIGURE 1. The Principal Quadratic Parameter Gap

topological circle whose points are in one-to-one correspondence with *all* quadratic invariant gaps \mathfrak{U} such that $\mathfrak{U} \cap \mathbb{S}^1$ is a Cantor set. Moreover, the following hold:

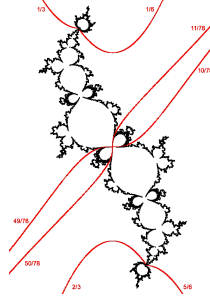
- (1) for each point $\bar{\theta} \in \mathbb{S}^1$ of Q that is not an endpoint of a hole of Q , the critical chord $\overline{(\theta + 1/3)(\theta + 2/3)}$ is the major of a quadratic invariant gap \mathfrak{U} such that $\mathfrak{U} \cap \mathbb{S}^1$ is a Cantor set;
- (2) for each hole $(\bar{\theta}_1, \bar{\theta}_2)$ of Q the chord $\overline{(\theta_1 + 1/3)(\theta_2 + 2/3)}$ is the periodic major of a quadratic invariant gap \mathfrak{U} such that $\mathfrak{U} \cap \mathbb{S}^1$ is a Cantor set.

The convex hull Ω of Q in the plane is called the *Principal Quadratic Parameter Gap* (see Figure 1). The set Q plays a somewhat similar role to that of the following set appearing in quadratic dynamics: the set of arguments of all parameter rays (rays to the Mandelbrot set) landing at points of the main cardioid. Holes of Ω will play an important role. The *period of a hole* (θ_1, θ_2) of Ω is defined as the period of $\theta_1 + 1/3$ under the angle tripling map. This period also equals the period of $\theta_2 + 2/3$ under the angle tripling map. The only period 1 holes of Ω are $(1/6, 1/3)$ and $(2/3, 5/6)$; these will play a special role.

Fix λ with $|\lambda| \leq 1$. The λ -*connectedness locus* \mathcal{C}_λ is defined as the set of all $f \in \mathcal{F}_\lambda$ such that $K(f)$ is connected, equivalently, such that $[f] \in \mathcal{M}_3$. This is a full continuum [BrHu88, Z99] (a compact set $X \subset \mathbb{C}$ is *full* if $\mathbb{C} \setminus X$ is connected). For every polynomial $f \in \mathcal{F}_\lambda$ and every angle $\alpha \in \mathbb{R}/\mathbb{Z}$, we will define the *dynamic ray* $R_f(\alpha)$. Also, for every angle θ , in the parameter plane of \mathcal{F}_λ we define the *parameter ray* $\mathcal{R}_\lambda(\theta)$. We use rays to show that the picture in \mathcal{F}_λ resembles the picture in the parameter plane of quadratic polynomials. Let \mathcal{CU}_λ be the set of all polynomials $f \in \mathcal{F}_\lambda$ with $[f]_{FP} \in \mathcal{CU}_{FP}$.

Main Theorem. *Fix λ with $|\lambda| \leq 1$. The set \mathcal{CU}_λ is a full continuum. The set \mathcal{C}_λ is the union of \mathcal{CU}_λ and a countable family of limbs \mathcal{LI}_H of \mathcal{C}_λ parameterized by holes H of Ω . The union is disjoint (except for holes of period 1). For a hole $H = (\theta_1, \theta_2)$ of Ω , the following hold.*

- (1) *The parameter rays $\mathcal{R}_\lambda(\theta_1)$ and $\mathcal{R}_\lambda(\theta_2)$ land at the same point $f_{\text{root}(H)}$.*
- (2) *Let $\mathcal{W}_\lambda(H)$ be the component of $\mathbb{C} \setminus \overline{\mathcal{R}_\lambda(\theta_1) \cup \mathcal{R}_\lambda(\theta_2)}$ containing the parameter rays with arguments from H . Then, for every $f \in \mathcal{W}_\lambda(H)$, the dynamic rays $R_f(\theta_1 + 1/3)$, $R_f(\theta_2 + 2/3)$ land at the same point, either a periodic and repelling point for all $f \in \mathcal{W}_\lambda(H)$ or the point 0 for all $f \in \mathcal{W}_\lambda(H)$. Moreover, $\mathcal{LI}_H = \mathcal{W}_\lambda(H) \cap \mathcal{C}_\lambda$.*

FIGURE 2. Parameter slice $\mathcal{F}_{e^{2\pi i/3}}$ with some parameter rays

- (3) *The dynamic rays $R_{f_{\text{root}(H)}}(\theta_1 + 1/3)$, $R_{f_{\text{root}(H)}}(\theta_2 + 2/3)$ land at the same parabolic periodic point, and $f_{\text{root}(H)}$ belongs to \mathcal{CU}_λ .*

Figure 2 shows the parameter slice $\mathcal{F}_{e^{2\pi i/3}}$ with some parameter rays and wakes.

A more detailed description of \mathcal{CU}_λ is available for $|\lambda| < 1$. The case $\lambda = 0$ was studied in the PhD thesis [Fau92] of D. Faught. It is claimed there that \mathcal{CU}_0 is a Jordan disk. Unfortunately, there are several gaps in the arguments of [Fau92]; P. Roesch [Roe99, Roe07] corrected these arguments and generalized them to higher degrees. A transition from the case $\lambda = 0$ to the case $|\lambda| < 1$ must be possible through a straightforward quasi-conformal deformation.

Remark. The first version of this paper was written in 2015. We later learned about the book [Z18], which slightly overlaps with our paper, even though it has a different focus. We do not use [Z18] but give references to this book at a few places.

2. DETAILED STATEMENT OF THE MAIN RESULTS

In this section, we break our Main Theorem into steps called Theorem A, Theorem B and Theorem C.

2.1. The structure of \mathcal{F}_λ . A (pre)critical point of a polynomial f is defined as a point mapped to a critical point of f by some iterate f^{or} of the polynomial f (here $r \geq 0$). Let $G_{K(f)}$ be the Green function for $K(f)$. Call unbounded trajectories of the gradient flow for $G_{K(f)}$ *dynamic rays* of f . Dynamic rays of f can be of two types. All but countably many of them accumulate in $K(f)$ and are called *smooth rays* (of f). Otherwise a dynamic ray extends from infinity until it *crashes* at an escaping (pre)critical point of f ; such rays can exist only if $K(f)$ is disconnected and escaping critical points exist. The remaining part of $\mathbb{C} \setminus K(f)$ consists of bounded trajectories of the gradient flow for $G_{K(f)}$ called *ideal rays* (of f). An ideal ray can extend from one escaping (pre)critical point to another escaping (pre)critical point or from an escaping (pre)critical point to the Julia set. Thus, if c is an escaping (pre)critical point, then several dynamic rays crash at c , and some ideal rays emerge from c . We conclude that $\mathbb{C} \setminus K(f)$ is the union of dynamic rays, ideal rays, and escaping (pre)critical points.

Let V_f be the union of all dynamic rays of f . Then V_f is homeomorphic to $\mathbb{C} \setminus \overline{\mathbb{D}}$ and coincides with the basin of attraction of infinity with countably many ideal rays and (pre)critical points removed. The *Böttcher coordinate* is an analytic map $\phi_f : V_f \rightarrow \mathbb{C}$ such that $\phi_f(V_f)$ is the complement in \mathbb{C} of $\overline{\mathbb{D}}$ united with countably

many bounded radial segments that originate at the boundary of $\overline{\mathbb{D}}$ and form a null sequence. Moreover, we can choose ϕ_f so that the derivative $\phi'_f(z)$ tends to a positive real number as $z \rightarrow \infty$ and $\phi_f \circ f = \phi_f^3$. The existence of Böttcher coordinates was established by Douady and Hubbard in [DH8485]. Theorem 2.1 is a consequence of the analytic dependence of the Böttcher coordinate on parameters [DH8485, BrHu88].

Theorem 2.1 ([BuHe01, Proposition 2]). *Fix λ , and let \mathcal{V} be the union of $\{b\} \times V_{f_{\lambda,b}}$ over all $b \in \mathbb{C}$. This set is open in \mathbb{C}^2 . The map $\Psi : \mathcal{V} \rightarrow \mathbb{C}^2$ given by the formula $\Psi(b, z) = (b, \phi_{f_{\lambda,b}}(z))$ is an analytic embedding of \mathcal{V} into \mathbb{C}^2 .*

Dynamic rays of f are preimages under ϕ_f of unbounded radial segments that can be parameterized by *angles*, i.e., elements of \mathbb{R}/\mathbb{Z} : the dynamic ray $R = R_f(\theta)$ of argument $\theta \in \mathbb{R}/\mathbb{Z}$ is such that for every $z \in R$, we have $\phi_f(z) = re^{2\pi i\theta}$, where $r > 1$. A dynamic ray R *lands* at a point $z \in \mathbb{C}$ if $z \in K(f)$, and z is the only accumulation point of R in \mathbb{C} . The complement of V_f in $\mathbb{C} \setminus K(f)$ consists of ideal rays and escaping (pre)critical points. Every accessible point of $K(f)$ is the landing point of some dynamic ray or some ideal ray.

Recall that $|\lambda| \leq 1$. Since 0 is a non-repelling fixed point of $f \in \mathcal{F}_\lambda$, it follows from the Fatou–Shishikura inequality [Fat20, Shi87] that at least one critical point of f is non-escaping. Thus, each f in $\mathcal{F}_\lambda \setminus \mathcal{C}_\lambda$ must have two *distinct* critical points, one of which escapes to infinity while the other is non-escaping. In this case, the non-escaping critical point of f will be denoted by $\omega_1(f)$. Let $\omega_2(f) \neq \omega_1(f)$ denote the other critical point of f . Let $\omega_2^*(f)$ be the corresponding *co-critical point* (i.e., $\omega_2^*(f)$ is the unique point with $\omega_2^*(f) \neq \omega_2(f)$ and $f(\omega_2^*(f)) = f(\omega_2(f))$).

Let \mathcal{P}_λ be the set of polynomials $f \in \mathcal{F}_\lambda$ such that there are polynomials $g \in \mathcal{F}$ arbitrarily close to f with $|g'(0)| < 1$ and $[g]_{FP} \in \mathcal{PHD}_3$. Then clearly $[f]_{FP} \in \mathcal{PHD}_3$. By Theorem A from [BOPT14], if $f \in \mathcal{P}_\lambda$, then $[f] \in \mathcal{CU}$. By [BOPT16a, Theorem 3.2], the points $\omega_1(f)$ and $\omega_2(f)$ can be consistently defined for all $f \in \mathcal{F}_\lambda \setminus \mathcal{P}_\lambda$. Also, for every component \mathcal{W} of $\mathcal{F}_\lambda \setminus \mathcal{P}_\lambda$, the points $\omega_1(f)$ and $\omega_2(f)$ depend holomorphically on f as f moves through \mathcal{W} .

Observe that if $f \in \mathcal{F}_\lambda \setminus \mathcal{C}_\lambda$, then $\omega_2^*(f) \in V_f$. In this case the map $\Phi_\lambda(f) = \phi_f(\omega_2^*(f))$ is a conformal isomorphism between the complement of \mathcal{C}_λ in \mathcal{F}_λ and the complement of the closed unit disk [BuHe01]. This map can be used to define *parameter rays*. Namely, the ray $\mathcal{R}_\lambda(\theta)$ is defined as the preimage of the straight ray (unbounded radial segment) $\{re^{2\pi i\theta} \mid r > 1\}$ under Φ_λ . We emphasize here that for *dynamic rays* we use the notation $R(\cdot)$ (with possible subscripts and superscripts) while for *parameter rays* we use the notation $\mathcal{R}(\cdot)$ (with possible subscripts). This is consistent with our convention to use the calligraphic font for parameter space objects. We have $f \in \mathcal{R}_\lambda(\theta)$ if and only if $\omega_2^*(f)$ belongs to the ray $R_f(\theta)$, i.e., if and only if both $R_f(\theta + 1/3)$ and $R_f(\theta + 2/3)$ crash into the critical point $\omega_2(f)$.

2.2. Immediate renormalization. In [Lyu83, MSS83], the notion of *J-stability* was introduced for any holomorphic family \mathcal{T} of rational functions: a map from \mathcal{T} is *J-stable* with respect to \mathcal{T} if its Julia set admits an equivariant holomorphic motion over some neighborhood of the map in the family. Say that $f \in \mathcal{F}_\lambda$ is *λ -stable* if it is *J-stable* with respect to \mathcal{F}_λ , otherwise we say that f is *λ -unstable*. The set \mathcal{F}_λ^{st} of all stable polynomials $f \in \mathcal{F}_\lambda$ is an open subset of \mathcal{F}_λ . A component of \mathcal{F}_λ^{st} is called a *(λ -)stable component*, or a *domain of (λ -)stability*. It is easy to see that, given λ , the polynomial $f_{\lambda,b}$ has a disconnected Julia set if $|b|$ is sufficiently big.

Hence, if $f = f_{\lambda,b}$ is stable and $J(f)$ is connected, then its domain of stability is bounded. Recall that \mathcal{CU}_λ consists of all polynomials $f \in \mathcal{F}_\lambda$ with $[f]_{FP} \in \mathcal{CU}_{FP}$.

Observe that, given a compact set $A \subset \mathbb{C}$, there is a unique unbounded complementary domain U of A . The set A is said to be *unshielded* if A coincides with the boundary of U . For example, a polynomial Julia set is unshielded whether it is connected or not. For an unshielded compact set A , the *topological hull* $\text{Th}(A)$ of A is by definition the union of A with all bounded complementary components of A . Equivalently, $\text{Th}(A)$ is the complement of U .

Theorem 2.2 ([BOPT14,BOPT16a]). *All bounded components of $\mathcal{F}_\lambda \setminus \mathcal{P}_\lambda$ consist of λ -stable \mathcal{CU} -polynomials. Also, $\mathcal{P}_\lambda \subset \text{Th}(\mathcal{P}_\lambda) \subset \mathcal{CU}_\lambda$.*

Evidently, bounded components of $\mathcal{F}_\lambda \setminus \mathcal{P}_\lambda$ are the same as components of $\text{Th}(\mathcal{P}_\lambda) \setminus \mathcal{P}_\lambda$. In [BOPT14,BOPT16a,BCOT18] we show that polynomials in those components are of so-called *queer* type: for any f in such a component both critical points belong to $J(f)$, are distinct, the Julia set $J(f)$ has positive Lebesgue measure, and carries an invariant line field. It should be noted though that there are no known examples of any bounded components of $\text{Th}(\mathcal{P}_\lambda) \setminus \mathcal{P}_\lambda$.

Definition 2.3 ([DH85]). A *polynomial-like* map is a proper holomorphic map $f : U \rightarrow f(U)$ of degree $k > 1$, where $U, f(U) \subset \mathbb{C}$ are open subsets isomorphic to a disk, and $\bar{U} \subset f(U)$. The *filled Julia set* $K(f)$ of f is the set of points in U that never leave U under iteration. The *Julia set* $J(f)$ of f is defined as the boundary of $K(f)$. Two polynomial-like maps $f : U \rightarrow f(U)$ and $g : V \rightarrow g(V)$ are said to be *hybrid equivalent* if there is a quasi-conformal map φ from a neighborhood of $K(f)$ to a neighborhood of $K(g)$ conjugating f to g in the sense that $g \circ \varphi = \varphi \circ f$ wherever both sides are defined and such that $\bar{\partial}\varphi = 0$ almost everywhere on $K(f)$. If $k = 2$, then the corresponding polynomial-like maps are said to be *quadratic-like*. Note that U can always be chosen as a Jordan domain.

Straightening Theorem ([DH85]). *Let $f : U \rightarrow f(U)$ be a polynomial-like map. Then f is hybrid equivalent to a polynomial P of the same degree. Moreover, if $K(f)$ is connected, then P is unique up to (a global) conjugation by an affine map.*

We will need Definition 2.4.

Definition 2.4. Let f be a polynomial, U be an open disk, $f^* = f|_U$ be a polynomial-like map. Let g be a polynomial, V be an open disk whose boundary is a level curve for the Green function of $J(g)$, and suppose that $g|_V$ is hybrid equivalent to f^* . Then the initial segments of dynamic rays of g contained in \bar{V} correspond to half-open curves in $\mathbb{C} \setminus K(f^*)$ that will be called *polynomial-like rays* of f . If the degree of f^* is two, then we will talk about *quadratic-like rays*. We will denote polynomial-like rays $R^*(\beta)$, where β is the argument of the external ray of g corresponding to $R^*(\beta)$.

Note that polynomial-like rays of f are only defined in a bounded neighborhood of $K(f^*)$. Observe also that the polynomial-like (quadratic-like) map will be always specified when we talk about polynomial-like (quadratic-like) rays, which is why we omit it from our notation. Polynomial-like rays (in particular, quadratic-like rays) depend on a choice of U and on a choice of a conjugacy between f^* and g .

We say that a cubic polynomial $f \in \mathcal{F}$ is *immediately renormalizable* if there are Jordan domains U^* and V^* such that $0 \in U^*$, and $f^* = f : U^* \rightarrow V^*$ is a

quadratic-like map (we use the notation f^* at several occasions in the future when talking about immediately renormalizable maps). It is useful to determine if a polynomial P is immediately renormalizable as in that case a lot of machinery that applies to quadratic polynomials will be applicable to P as well. If $f \in \mathcal{F}_\lambda$ with $|\lambda| \leq 1$ is immediately renormalizable, then the quadratic-like Julia set $J(f^*) = J^*$ is connected. Indeed, f^* is hybrid equivalent to a quadratic polynomial g . Since $0 \in K(f^*)$ is a non-repelling f -fixed point, it corresponds to a non-repelling fixed point of g . As a quadratic polynomial with a non-repelling fixed point, g has connected Julia set, and in fact we can choose g to be $g(z) = \lambda z + z^2$. Equivalently, we can choose g to be $g(z) = z^2 + c$, where $c = \lambda/2 - \lambda^2/4$ is in the filled main cardioid. Hence, $J(f^*)$ is connected too. In [BOPT16a], some sufficient conditions on polynomials for being immediately renormalizable are obtained.

Theorem 2.5 ([BOPT16a]). *All polynomials in the unbounded component \mathcal{U}_λ of $\mathcal{F}_\lambda \setminus \mathcal{P}_\lambda$ are immediately renormalizable. In particular, all polynomials in $\mathcal{F}_\lambda \setminus \mathcal{CU}_\lambda$ are immediately renormalizable.*

The proof of Theorem 2.5 unfolds as follows. First, for a polynomial $f \in \mathcal{U}_\lambda$ we show that it has two distinct critical points $\omega_1(f)$ and $\omega_2(f)$ such that a polynomial $g \in \mathcal{F}$ with attracting fixed point 0 sufficiently close to f has a critical point $\omega_1(g)$ close to $\omega_1(f)$ belonging to the immediate basin of attraction $A(g)$ of 0 for g and a critical point $\omega_2(g) \neq \omega_1(g)$ close to $\omega_2(f)$. In fact $\omega_1(g)$ and $\omega_2(g)$ are holomorphic functions of g . Choosing longer and longer parts of the backward orbit of $\omega_1(g)$ contained in $A(g)$ and taking their limit as $g \rightarrow f$ we construct a set $X(f)$ and then prove that, as we vary f in \mathcal{U}_λ , the set $X(f)$ exhibits equivariant holomorphic motion (notice that the proof relies upon the λ -lemma in an essential way). Finally, we show that if $f \in \mathcal{U}_\lambda$, then $X(f)$ is a quadratic-like Julia set. The second claim follows from Theorem 2.2.

The discussion below, following [BCLOS16], aims at relating quadratic-like rays to dynamic and ideal rays (in fact, [BCLOS16] deals with polynomials of arbitrary degree). Define an *external ray* R^e as a homeomorphic copy of \mathbb{R} running from ∞ to $K(f)$ so that either R^e is a smooth dynamic ray or R^e is contained in a one-sided limit of smooth dynamic rays. Abusing the language, we will call such rays either *smooth rays* or *one-sided rays*. Observe that one-sided limits of smooth rays can be defined for any argument θ . If $R^e(\theta)$ is a smooth ray, then the limits of smooth rays with arguments approaching θ from either side are the same and coincide with $R^e(\theta)$. Conversely, if the limits of smooth rays with arguments approaching θ from either side coincide, then there is a smooth ray $R^e(\theta)$ with argument θ . Indeed, the analysis of the structure of $\mathbb{C} \setminus K(f)$ given above shows that every external ray R^e has its initial (unbounded) part coinciding with the dynamic ray of a well-defined argument θ . If R^e is not smooth, then the same analysis shows that R^e is a one-sided limit of smooth rays from exactly one side, in which case we associate to R^e the appropriate *one-sided external argument*, $\theta+$ or $\theta-$, and denote R^e by $R_f^e(\theta-)$ or $R_f^e(\theta+)$, respectively. Initial parts of $R_f^e(\theta-)$ and $R_f^e(\theta+)$ from infinity to a certain precritical point in the plane coincide. However, the remaining parts of $R_f^e(\theta-)$ and $R_f^e(\theta+)$ are disjoint. If R^e is a smooth ray of argument θ , then we set $R^e(\theta+) = R^e(\theta-) = R^e$. Observe that a one-sided ray R depends *only* on the choice of the side from which smooth rays approach R but does *not* depend on the choice of smooth rays in question.

For a non-closed oriented curve l from infinity or a finite point in \mathbb{C} to a bounded region in \mathbb{C} , define its *principal* (or *limit*) set $\text{Pr}(l)$ as the set of all accumulation points of l in the forward direction. We will only consider curves l such that $\text{Pr}(l) = \bar{l} \setminus l$ (e.g., this is the case if l is an external ray). If $Y \subset \mathbb{C}$ is a continuum and $\text{Pr}(l) \subset Y$ say that l is a curve to Y .

Lemma 2.6 (Lemma 6.1, [BCLOS16]). *If two distinct external rays R_0^e, R_1^e have a common point, then R_0^e, R_1^e are both non-smooth. The intersection $L = R_0^e \cap R_1^e$ is connected and can contain a (pre)critical point only as an endpoint. Furthermore, one and only one of the following cases holds:*

- (1) L is a smooth curve joining infinity and a (pre)critical point;
- (2) L is a single (pre)critical point;
- (3) L is a smooth closed arc between two (pre)critical points;
- (4) L is a smooth curve from a (pre)critical point to $J(f)$, in which case the rays R_0^e, R_1^e are not periodic.

Except for the last case, the rays R_0^e, R_1^e have their principal sets in different components of $J(f)$.

In our setting, external rays are closely related to quadratic-like rays. To describe this relation, we need Lemma 2.7, which follows immediately from [BrHu88]. Recall that any $f \in \mathcal{F}_\lambda \setminus \mathcal{C}_\lambda$ with $|\lambda| \leq 1$ is immediately renormalizable.

Lemma 2.7. *If $f \in \mathcal{F}_\lambda \setminus \mathcal{C}_\lambda$ with $|\lambda| \leq 1$, then $J(f^*)$ is an invariant component of $J(f)$, and $\text{Th}(J(f^*)) = K(f^*)$ is a component of $K(f)$.*

Hence, if $J(f)$ is disconnected, then we can choose neighborhoods $V^* \supset U^* \supset K(f^*)$ enclosed by components of equipotentials. We also consider external rays that penetrate U^* ; evidently, any such ray has its entire tail inside U^* so that its principal set is located inside U^* .

Definition 2.8. Let $K \subset \mathbb{C}$ be a compact subset. Consider two smooth curves R_0, R_1 (embedded copies of $[0, \infty)$) in $\mathbb{C} \setminus K$ such that $\overline{R_i} \setminus R_i \subset K$ for each i . This means that each of the curves accumulates in K , and nowhere else. We say that R_0 is isotopic to R_1 relative to K (rel. K for short) if there is a continuous isotopy $h_t : \mathbb{C} \rightarrow \mathbb{C}$ (where $t \in [0, 1]$) such that $h_0 = \text{id}$, the homeomorphism h_1 takes R_0 to R_1 , and $h_t = \text{id}$ on K for all t .

Let $\psi : \mathbb{C} \setminus K \rightarrow \mathbb{C} \setminus \overline{\mathbb{D}}$ be a Riemann map where K is a non-separating continuum. Suppose that $\psi(R_0)$ and $\psi(R_1)$ both land in $\overline{\mathbb{D}}$. By Definition 2.8, if the curves R_0, R_1 are isotopic rel. K then $\psi(R_0), \psi(R_1)$ land at the same point. Observe that no two distinct quadratic-like rays to $K(f^*)$ are isotopic rel. $K(f^*)$. Theorem 2.9 is a special case of a more general result obtained in Theorem 6.9 of [BCLOS16] (we also rely upon Lemma 6.3 from [BCLOS16]).

Theorem 2.9 (cf. [BCLOS16, Theorem 6.9]). *Consider a polynomial $f \in \mathcal{F}_\lambda \setminus \mathcal{C}_\lambda$ with $|\lambda| \leq 1$. Then the set of all arguments of external rays to $K(f^*)$ is closed. An initial segment of every external ray R^e to $K(f^*)$ is isotopic rel. $K(f^*)$ to exactly one quadratic-like ray $R^* = \xi(R^e)$. The mapping $\xi : R^e \mapsto R^*$ from external rays to $K(f^*)$ to quadratic-like rays is onto, and $\xi^{-1}(R^*)$ consists of finitely many rays. Moreover, if $\xi^{-1}(R^*) = \{R_1^e, \dots, R_k^e\}$ with $k > 1$, then one of the following holds:*

- (i) *we have $k = 2$, both rays R_1^e, R_2^e are non-smooth, meet at a precritical point x , and share a common arc from x to $K(f^*)$;*

- (ii) *the rays R_1^e, \dots, R_k^e land at the same preperiodic point, and include at least one pair of disjoint rays.*

2.3. Combinatorics of the angle tripling map: An overview. Here we briefly describe laminational results of [BOPT16]; our discussion in Section 1 shows the importance of laminations for polynomial dynamics. We need these results to describe the mutual location of dynamic rays, parameter rays, and their impressions. A more detailed discussion of [BOPT16] is given later in the paper.

Let $\mathfrak{U} \subset \overline{\mathbb{D}}$ be the convex hull of $\mathfrak{U}' = \mathfrak{U} \cap \mathbb{S}^1$. Set $\sigma_3(\mathfrak{U}') = \mathfrak{V}'$ and $\mathfrak{V} = \text{CH}(\overline{\mathfrak{V}'})$. It is said to map under σ_3 in a *quasi-covering fashion* if for every hole $(\overline{\alpha}, \overline{\beta})$ of \mathfrak{U} , we either have $\sigma_3(\overline{\alpha}) = \sigma_3(\overline{\beta})$ or the circular arc $(\sigma_3(\overline{\alpha}), \sigma_3(\overline{\beta}))$ is also a hole of \mathfrak{V} . The set \mathfrak{U} is a *(stand alone) invariant gap* if $\sigma_3(\mathfrak{U}') = \mathfrak{U}'$, and σ_3 maps \mathfrak{U} in a quasi-covering fashion. The set \mathfrak{U} is a *(stand alone) periodic gap/leaf* if there exists $n > 0$ with $\sigma_3^n(\mathfrak{U}') = \mathfrak{U}'$, the convex hulls of the sets $\sigma_3^{oi}(\mathfrak{U}')$, $i = 0, \dots, n - 1$ intersect **at most** in common edges, and each such convex hull maps under σ_3 in a quasi-covering fashion. Some concepts concerning invariant gaps \mathfrak{U} were introduced a few paragraphs before the Main Theorem.

An invariant gap \mathfrak{g} can have one or two majors [BOPT16]. Any *quadratic* invariant gap \mathfrak{U} has exactly one major $\mathfrak{M}(\mathfrak{U}) = \mathfrak{M}$ which is either critical or periodic. If \mathfrak{M} is critical and has no periodic endpoints, then \mathfrak{U} is said to be of *regular critical type*. If \mathfrak{M} is critical and has a periodic endpoint, then \mathfrak{U} is said to be of *caterpillar type*. If \mathfrak{M} is periodic, then \mathfrak{U} is said to be of *periodic type*. Evidently, any quadratic invariant gap is of one of the three types just introduced.

Quadratic invariant gaps can be generated as follows. Let \mathfrak{c} be any critical chord. Set $L(\mathfrak{c})$ to be the longer closed arc of \mathbb{S}^1 connecting the endpoints of \mathfrak{c} , and $X(\mathfrak{c})$ be the set of all points in $L(\mathfrak{c})$, whose forward orbits stay in $L(\mathfrak{c})$. Let $\mathfrak{U}(\mathfrak{c})$ be the convex hull of $X(\mathfrak{c})$ in the plane. The unique quadratic invariant gap with major $\overline{0(1/2)}$ contained in the upper half of the unit disk (**above** $\overline{0(1/2)}$) is denoted by FG_a while the unique quadratic invariant gap with major $\overline{0(1/2)}$ contained in the lower half of the unit disk (**below** $\overline{0(1/2)}$) is denoted by FG_b .

Theorem 2.10 ([BOPT16]). *If \mathfrak{c} is any critical chord, then $\mathfrak{U}(\mathfrak{c})$ is a quadratic invariant gap. If one of the endpoints of \mathfrak{c} is periodic, then $\mathfrak{U}(\mathfrak{c})$ is of caterpillar type. Otherwise, $\mathfrak{U}(\mathfrak{c})$ is of regular critical type or of periodic type depending on whether or not the forward orbits of the endpoints of \mathfrak{c} are contained in $L(\mathfrak{c})$. There is a unique major hole of $\mathfrak{U}(\mathfrak{c})$; its length is between $1/3$ and $1/2$. The only two invariant quadratic gaps with major hole of length $1/2$ are FG_a and FG_b . Any quadratic invariant gap is of the form $\mathfrak{U}(\mathfrak{c})$ for some \mathfrak{c} .*

For any critical chord \mathfrak{c} , we let $\mathfrak{U}_p(\mathfrak{c})$ denote the convex hull of all non-isolated points in $\mathfrak{U}'(\mathfrak{c})$ (the subscript p stands for “perfect”). Then $\mathfrak{U}_p(\mathfrak{c}) = \mathfrak{U}(\mathfrak{c})$ unless $\mathfrak{U}(\mathfrak{c})$ is of caterpillar type. If the gap $\mathfrak{U}(\mathfrak{c})$ is of caterpillar type, then $\mathfrak{U}_p(\mathfrak{c})$ is a quadratic invariant gap of periodic type while the set $\mathfrak{U}'(\mathfrak{c})$ has isolated points and is obtained from $\mathfrak{U}'_p(\mathfrak{c})$ by adding the non-periodic endpoint of \mathfrak{c} and countably many iterated preimages of it. The boundary of the caterpillar gap $\mathfrak{U}(\mathfrak{c})$ consists of $\mathfrak{U}'_p(\mathfrak{c})$ and countable concatenations of edges inserted into holes of $\mathfrak{U}_p(\mathfrak{c})$.

Lemma 2.11 ([BOPT16]). *Suppose that \mathfrak{U} is a quadratic invariant gap with major \mathfrak{M} and major hole H . If \mathfrak{M} is of regular critical type, then $\sigma_3|_H$ is one-to-one and the only two points of \overline{H} that have the same image are its endpoints. If \mathfrak{M} is of*

periodic type of period $m > 1$ then $\overline{H} = L \cup I \cup R$ where L, I, R are adjacent closed arcs such that $\sigma_3(L) = \sigma_3(R)$ is the closure of a hole T of \mathfrak{U} with $T \cap H = \emptyset$ (both $\sigma_3|_L$ and $\sigma_3|_R$ are one-to-one) while $\sigma_3(I) = \mathbb{S}^1 \setminus T$ (and $\sigma_3|_I$ is one-to-one). If \mathfrak{M} has period 1, then $\sigma_3(L) = \sigma_3(R) = \overline{H}$, and $\sigma_3(I)$ is the closure of $\mathbb{S}^1 \setminus H$. Finally, all majors not equal to $0(1/2)$ cross $0(1/2)$.

A more detailed version of Lemma 2.11 is given closer to the end of the paper.

Lemma 2.12 ([BOPT16]). *Let \mathfrak{U} be a quadratic invariant gap. If $\tilde{\mathfrak{M}} \neq \mathfrak{M}(\mathfrak{U})$ is the major of a quadratic invariant gap of periodic or regular critical type then the endpoints of $\tilde{\mathfrak{M}}$ cannot both belong to the major hole of \mathfrak{U} .*

To describe the parameter space of all quadratic invariant gaps, we first parameterize critical chords by the map taking a point $\bar{\theta} \in \mathbb{S}^1$ to the critical chord $(\theta + 1/3)(\theta + 2/3)$, denoted by \mathfrak{c}_θ . Consider the map π from \mathbb{S}^1 to the set of all quadratic invariant gaps taking $\bar{\theta}$ to $\mathfrak{U}(\mathfrak{c}_\theta)$. The parameter picture of quadratic invariant gaps is somewhat similar to that of the rotation number in an analytic family of circle diffeomorphisms. Below by *fibers* of a map we mean full preimages of points under this map; thus, the fiber of t is simply the full preimage of t .

Theorem 2.13 ([BOPT16]). *The mapping π is surjective but not injective. Non-trivial fibers of π are exactly those of quadratic invariant gaps of periodic type. The fiber of the invariant quadratic gap of periodic type with major $(\theta_1 + 1/3)(\theta_2 + 2/3)$ is the open arc $(\bar{\theta}_1, \bar{\theta}_2)$.*

Non-trivial fibers of π are the holes of a certain compact subset $Q \subset \mathbb{S}^1$. The convex hull \mathfrak{Q} of Q in the plane is called the *Principle Quadratic Parameter Gap*. Holes of \mathfrak{Q} will play an important role in our description of \mathcal{CU}_λ . Sometimes, we will also need the mapping π_p taking $\bar{\theta}$ to the quadratic invariant gap $\mathfrak{U}_p(\mathfrak{c}_\theta)$. Fibers of π_p are the closures of fibers of π .

Let $A \subset \mathbb{S}^1$ be a finite invariant set. If there is an orientation preserving homeomorphism of the circle that conjugates $\sigma_3|_A$ to the restriction of the rotation by the angle p/q onto one or several periodic orbits of this rotation, we associate with A a unique (*combinatorial*) *rotation number* p/q ; we also talk about rotation numbers of periodic angles and finite invariant gaps. Not every periodic point/orbit has a rotation number; to have a rotation number p/q , a periodic point/orbit must be of period q . Any point from such a cycle, and the cycle A itself, are called (p/q -)rotational. Say that the (*combinatorial*) *rotation number* of A (and of every point of A) is p/q . Say that A (and its convex hull $\text{CH}(A)$) is of *type D* (for “disjoint”) if A consists of two disjoint orbits. Notice that then $\text{CH}(A)$ has two majors, whose σ_3 -orbits are disjoint (except for common endpoints).

Every finite invariant gap has a rotation number by definition. Equivalently, we define a rotational periodic point x by saying that σ_d preserves the cyclic order on the orbit X of x . We can also say that a periodic point x is (p/q -)rotational if it is of period q and, for every point y from the orbit X of x , the closed arc from y to $\sigma_3(y)$ in the clockwise direction contains $p + 1$ points of X (notice that these points include y and $\sigma_3(y)$). For example, if x is a σ_d -periodic point of period q such that the order among the points from the orbit of x can be described by inequalities $x < \sigma_3(x) < \dots < \sigma_3^{o(q-1)}(x) < \sigma_3^{oq}(x) = x$, then the point x is $1/q$ -rotational.

A similar notion can be introduced for periodic chords ℓ (here we assume that periodic chords ℓ are such that all their distinct images are pairwise disjoint): a

periodic chord ℓ is *rotational* if (a) all the chords from the orbit L of ℓ are contained in the boundary of one component, say, W , of $\mathbb{D} \setminus L$, and the map does not change the orientation of chords with respect W , and (b) for any i the positively oriented circle arc from $\sigma_3^{\circ i}(\ell)$ to $\sigma_3^{\circ(i+1)}(\ell)$ contains the same number of chords from L . The (*combinatorial*) *rotation number* is then defined similarly to the above. If condition (a) holds, and an endpoint of ℓ is rotational (as a point), then the other endpoint of ℓ and ℓ itself are rotational. As above, we can rephrase assumption (b) by saying that σ_3 preserves the cyclic order on the orbit L of ℓ .

Suppose a hole $(\bar{\theta}_1, \bar{\theta}_2)$ of Ω is such that $\theta_1 + 1/3$ is p/q -rotational; then the other endpoint $\theta_2 + 2/3$ and the major $\mathfrak{M} = \overline{(\theta_1 + 1/3)(\theta_2 + 2/3)}$ are also periodic p/q -rotational. The convex hull of the entire orbit of $\overline{(\theta_1 + 1/3)(\theta_2 + 2/3)}$ is a finite invariant gap \mathfrak{g} of rotation number p/q . Evidently, \mathfrak{g} is of type D (in addition to the orbit of \mathfrak{M} there is one more cycle of edges in $\text{Bd}(\mathfrak{g})$); together with the converse statement this gives Lemma 2.14.

Lemma 2.14 ([BOPT16]). *Suppose that $(\bar{\theta}_1, \bar{\theta}_2)$ is a hole of Ω such that $\theta_1 + 1/3$ and $\theta_2 + 2/3$ have rotation number p/q . Then the chord $\overline{(\theta_1 + 1/3)(\theta_2 + 2/3)}$ is a major of a finite invariant gap of type D. Vice versa, a major of a finite invariant gap of type D is also the major of a quadratic invariant gap of periodic type.*

Definition 2.15 is closely related to Lemma 2.14.

Definition 2.15. A hole $(\bar{\theta}_1, \bar{\theta}_2)$ of Ω is (p/q) -rotational if it satisfies the conditions from Lemma 2.14. The same terminology is used for the major $\overline{(\theta_1 + 1/3)(\theta_2 + 2/3)}$.

Let us give more detail concerning the relation between finite invariant gaps of type D and quadratic invariant gaps; our description is based upon [BOPT16]. Let \mathfrak{g} be a finite invariant gap of type D. A major \mathfrak{M}_1 of \mathfrak{g} defines an invariant quadratic gap \mathfrak{U}_1 of periodic type such that $\mathfrak{U}_1 \cap \mathbb{S}^1$ is the set of all points of the circle whose orbits stay at the same side of \mathfrak{M}_1 as \mathfrak{g} . We have $\mathfrak{g} \subset \mathfrak{U}_1$. Similarly, the other major \mathfrak{M}_2 of \mathfrak{g} determines another invariant quadratic gap \mathfrak{U}_2 .

Lemma 2.16 ([Z18]). *There are q type D finite invariant gaps of rotation number p/q . There are $2q$ rotational holes of Ω corresponding to the rotation number p/q .*

2.4. Main results. The proof of Theorem A is inspired by J. Milnor [Mil00].

Theorem A (Wakes in λ -slices). *If $\lambda \in \mathbb{C}, |\lambda| \leq 1$, then for every hole (θ_1, θ_2) of Ω , the parameter rays $\mathcal{R}_\lambda(\theta_1)$ and $\mathcal{R}_\lambda(\theta_2)$ land at the same point.*

Thus the parameter rays $\mathcal{R}_\lambda(\theta_1)$ and $\mathcal{R}_\lambda(\theta_2)$, together with their common landing point, divide the plane into two (open) parts. Note that results somewhat similar to Theorem A are announced in [Z18] without proof.

Definition 2.17. Let (θ_1, θ_2) be a hole of Ω and $\mathcal{W}_\lambda(\theta_1, \theta_2)$ be the component of $\mathbb{C} \setminus \overline{\mathcal{R}_\lambda(\theta_1) \cup \mathcal{R}_\lambda(\theta_2)}$ containing the parameter rays with arguments from (θ_1, θ_2) . The set $\mathcal{W}_\lambda(\theta_1, \theta_2)$ is called the (parameter) *wake* (of \mathcal{F}_λ). The joint landing point of the rays $\mathcal{R}_\lambda(\theta_1)$ and $\mathcal{R}_\lambda(\theta_2)$ is called the *root point* of the parameter wake $\mathcal{W}_\lambda(\theta_1, \theta_2)$. Let the *period* of the parameter wake $\mathcal{W}_\lambda(\theta_1, \theta_2)$ be the period of $\theta_1 + 1/3$ under the angle tripling map.

Theorem B describes parameter wakes. By definition, for almost all holes (θ_1, θ_2) of Ω the major $\overline{(\theta_1 + 1/3)(\theta_2 + 2/3)}$ is in one-to-one correspondence with (θ_1, θ_2) .

The only exception is the major $\overline{0(1/2)}$, which corresponds to two holes of Ω , namely to holes $(1/6, 1/3)$ and $(2/3, 5/6)$. This in turn is related to the fact that $\overline{0(1/2)}$ as the major of an invariant quadratic gap \mathfrak{U} does not uniquely define \mathfrak{U} . Recall that the unique quadratic invariant gap with major $\overline{0(1/2)}$ contained in the upper half of the unit disk is denoted by FG_a while the unique quadratic invariant gap with major $\overline{0(1/2)}$ contained in the lower half of the unit disk is denoted by FG_b . Both FG_a and FG_b have the same major $\overline{0(1/2)}$. Define the set $\mathcal{W}'_\lambda(\theta_1, \theta_2)$ as the wake $\mathcal{W}_\lambda(\theta_1, \theta_2)$ except for the holes $(1/6, 1/3)$ and $(2/3, 5/6)$ for which we set

$$\mathcal{W}'_\lambda\left(\frac{1}{6}, \frac{1}{3}\right) = \mathcal{W}'_\lambda\left(\frac{2}{3}, \frac{5}{6}\right) = \mathcal{W}_\lambda\left(\frac{1}{6}, \frac{1}{3}\right) \cup \mathcal{W}_\lambda\left(\frac{2}{3}, \frac{5}{6}\right).$$

Theorem B. *Fix a hole $I = (\theta_1, \theta_2)$ of Ω . Then the set $\mathcal{W}'_\lambda(I)$ coincides with the set of polynomials f for which the dynamic rays $R_f(\theta_1 + 1/3)$, $R_f(\theta_2 + 2/3)$ land at the same periodic point that is either repelling for all $f \in \mathcal{W}'_\lambda(I)$ or equals 0 for all $f \in \mathcal{W}'_\lambda(I)$. If f_{root} is the root point of $\mathcal{W}_\lambda(I)$, then the dynamic rays $R_{f_{root}}(\theta_1 + 1/3)$, $R_{f_{root}}(\theta_2 + 2/3)$ land at the same parabolic periodic point, and f_{root} belongs to \mathcal{CU}_λ .*

Define a *limb* of the λ -connectedness locus \mathcal{C}_λ in the λ -slice \mathcal{F}_λ as the intersection of \mathcal{C}_λ with a parameter wake. Theorem C describes some topological properties of \mathcal{C}_λ and \mathcal{CU}_λ . Recall that \mathcal{CU}_λ is the set of all polynomials $f \in \mathcal{F}_\lambda$ with $[f] \in \mathcal{CU}$.

Theorem C. *The set \mathcal{CU}_λ is disjoint from all parameter wakes. The λ -connectedness locus \mathcal{C}_λ is the union of \mathcal{CU}_λ and all limbs of \mathcal{C}_λ . The set \mathcal{CU}_λ is a full continuum.*

3. RAYS

3.1. Stability of periodic rays. We first recall Lemma B.1 from [GM93] that goes back to Douady and Hubbard [DH8485].

Lemma 3.1. *Let g be a polynomial, and z be a repelling periodic point of g . If a smooth ray with rational argument θ in the dynamical plane of g lands at z , then, for every polynomial \tilde{g} sufficiently close to g , the ray \tilde{R} with argument θ in the dynamical plane of \tilde{g} is smooth and lands at a repelling periodic point \tilde{z} close to z . Moreover, \tilde{z} depends holomorphically on \tilde{g} .*

Lemma 3.1 easily implies Lemma 3.2 that we will need.

Lemma 3.2. *Let g_t , $t \in [0, 1]$ be a continuous family of polynomials of the same degree, θ be a rational angle, and let R_t be a smooth ray with argument θ in the dynamical plane of g_t . Denote the point at which the ray R_t lands by z_t . Suppose that the points z_t are repelling for $t \in [0, 1)$ but the landing point z_1 is not the limit of landing points z_t as $t \rightarrow 1$. Then the point z_1 is parabolic.*

Lemmas 3.1 and 3.2 deal with continuity of rays landing at repelling periodic points. The situation with parabolic periodic points is studied below. The main objects we consider are repelling petals and rays landing at parabolic periodic points. However, we first state the following basic result concerning rays landing at repelling and parabolic points; the result is attributed to Douady by Milnor and can be found in [Mil06, Theorem 18.11].

Theorem 3.3. *Let f be a polynomial and z be a periodic point of f . Suppose that either $J(f)$ is connected and z is repelling or z is parabolic. Then z is the landing point of at least one and at most finitely many external rays. These rays are periodic, of the same period. If z is parabolic, and the multiplier at z is a primitive q -th root of unity for some $q \geq 1$, then every ray that lands at z is of period q , and all rays landing at z rotate with the same combinatorial rotation number p/q , where p and q are coprime.*

To tackle the issue of stability of periodic rays landing at parabolic periodic points we prove Lemma 3.4.

Lemma 3.4. *Let f be a degree $d \geq 2$ polynomial such that $f : U \rightarrow V$ is a polynomial-like map, where U and V are Jordan domains. Let $R_f = R_f(\alpha)$ be an invariant smooth external ray of f of argument α landing in U . Then, for any degree d polynomial g sufficiently close to f , the invariant external ray $R_g(\alpha)$ of g accumulates in U . In particular, if $f : U \rightarrow V$ is quadratic-like, and $R_f(\alpha)$ is an invariant ray landing at a parabolic fixed point $x_f \in U$, then, for any degree d polynomial g sufficiently close to f such that $R_g(\alpha)$ is smooth, $R_g(\alpha)$ lands at a point $x_g \in U$ close to x_f .*

The argument is somewhat similar to that of [GM93, Lemma B.1] or [DH8485] (Lecture VIII, Section II, Proposition 3).

Proof. A segment of R_f with endpoints a and b will be denoted by $R_f[a, b]$; similarly we define subrays $R_f[a, \infty)$. Define a *fundamental segment* of R_f as a segment of the form $R_f[a, f(a)]$. Since R_f lands in U , there is a fundamental segment I_f of R_f contained in U . It follows that all iterated f -pullbacks of I_f in R_f are also contained in U . Indeed, if an endpoint of an arc belongs to U , and the arc is mapped into U under f , then the arc is contained in U . By continuity of a (local) Böttcher coordinate, there is a fundamental segment I_g of R_g close to I_f . Since U is open, we may assume that $I_g \subset U$. Then all iterated g -pullbacks of I_g in R_g are contained in U , and, hence, R_g accumulates in U .

Now, suppose that R_f lands at a parabolic fixed point x_f while $f : U \rightarrow V$ is quadratic-like. If g is close to f , then all fixed points of $g|_U$ are close to x_f (there are either two repelling fixed points, or one parabolic fixed point, or one repelling and one attracting fixed point, or one neutral and one repelling fixed point). If R_g is smooth, then it lands in U , and the claim follows. \square

To apply Lemma 3.4 we need tools allowing us to find polynomial-like restrictions of polynomials. A useful result in this direction is Theorem B from [BOPT16a].

Theorem 3.5 (Theorem B, [BOPT16a]). *Let $P : \mathbb{C} \rightarrow \mathbb{C}$ be a polynomial, and $Y \subset \mathbb{C}$ be a full P -invariant continuum. The following assertions are equivalent:*

- (1) *the set Y is the filled Julia set of some polynomial-like map $P : U^* \rightarrow V^*$,*
- (2) *the set Y is a component of $P^{-1}(P(Y))$ and, for every attracting or parabolic point y of P in Y , the attracting basin of y or the union of all parabolic domains at y is a subset of Y .*

3.2. Polynomials with parabolic points and their petals. Let g be a polynomial of arbitrary degree such that 0 is a fixed parabolic point of g of multiplier 1 . Suppose that $g(z) = z + az^{q+1} + o(z^{q+1})$, where q is a positive integer and $a \neq 0$. Recall from [Mil06] that an *attracting vector* for g is defined as a vector (=complex

number) v such that av^q is a negative real number, i.e., v and av^{q+1} have opposite directions. Clearly, there are q straight rays consisting of attracting vectors that divide the plane of complex numbers into q repelling sectors.

Consider a repelling sector S . Note that the set $S^{-q} = \{z \in \mathbb{C} \mid z^{-q} \in S\}$ is the complement of the ray $\{-ta \mid t > 0\}$ in \mathbb{C} . Let U be a sufficiently small disk around 0. We will write F for the composition of the function $w \mapsto w^{-1/q}$ mapping $(S \cap U)^{-q}$ onto $S \cap U$, the function g mapping $S \cap U$ onto $g(S \cap U)$, and the function $z \mapsto z^{-q}$ mapping $g(S \cap U)$ into \mathbb{C} . We have $F(w) = w - qa + \alpha(w)$, where $\alpha(w)$ denotes a power series in $w^{-1/q}$ that converges in a neighborhood of infinity, and whose free term is zero (note that this function is single valued and holomorphic on S^{-q}). It follows that there exists a positive real number r with the property $|\alpha(w)| < |a|/2$ whenever $|w| > r|a|$. Consider the half-plane Π given by the inequality $\Re(w/a) > r$. Since this inequality implies that $|w| > r|a|$, we have $F(\Pi) \supset \Pi$, and also that the shortest distance from a point on the boundary of Π to a point on the boundary of $F(\Pi)$ is at least $(q - 1/2)|a|$. The preimage of the half-plane Π under the map $S \rightarrow S^{-q}$, $z \mapsto z^{-q}$ is called a *repelling petal* of g .

Every repelling sector includes a repelling petal; thus, our polynomial $g(z) = z + az^{q+1} + o(z^{q+1})$ has q repelling petals. Hence there are at least q external rays landing at 0. A repelling petal P of g has the property $g(P) \supset P$.

3.3. Stability of rays and their perturbations. In this subsection, we fix $\lambda = \exp(2\pi ip/q)$ for some relatively prime p and q (i.e., λ is a root of unity). The ratio p/q is called the *rotation number* (of the fixed point 0). We discuss conditions that imply that a dynamic ray $R_f(\theta)$ in the dynamic plane of a polynomial $f \in \mathcal{F}_\lambda$ landing at 0 is stable (i.e., for $\tilde{f} \in \mathcal{F}_\lambda$ close to f , the ray $R_{\tilde{f}}(\theta)$ also lands at 0).

Proposition 3.6 (cf. Proposition 3.3, [BOPT14]). *We have $f_{\lambda,b}^{\circ q}(z) = z + T_{p/q}(b)z^{q+1} + o(z^{q+1})$, where $T_{p/q}(b)$ is a non-zero polynomial in b . The degree of $T_{p/q}$ is at most q and if, for some b , we have $T_{p/q}(b) = 0$, then $f_{\lambda,b}(z)$ has $2q$ parabolic Fatou domains at 0 forming two cycles under f as well as $2q$ external rays landing at 0.*

The representation for $f_{\lambda,b}^{\circ q}(z)$ and the fact that $T_{p/q}$ is a non-zero polynomial are proved in [BOPT14, Proposition 3.3]. The claim about the degree follows from Lemma 3.7. The last claim in Proposition 3.6 follows from [Bea00] (see the Petal Theorem 6.5.4 and Theorem 6.5.8) and the fact that our maps are cubic.

Lemma 3.7. *Let \mathbb{V} be the vector space of all polynomials in b and z given by $p(b, z) = \sum_{n=1}^N a_n(b)z^n$, where a_n is a polynomial of degree $\leq n - 1$. Then $f_{\lambda,b}^{\circ r}(p) \in \mathbb{V}$ for each $p \in \mathbb{V}$ and each integer $r > 0$.*

Proof. Let $p(b, z) = a_1(b)z + \cdots + a_N(b)z^N$ be a polynomial such that the degree of $a_j(b)$ is at most $j - 1$, for every j . It follows that $p(b, z)^k = c_k(b)z^k + \cdots + c_{Nk}(b)z^{Nk}$, where the degree of each $c_i(b)$ is at most $i - k$ for each $i \geq k$. Thus, $f_{\lambda,b}^{\circ r}(p(b, z)) = \lambda^r p(b, z) + b p(b, z)^2 + p(b, z)^3$ can be written as $d_1(b)z + \cdots + d_{3N}(b)z^{3N}$, where the degree of each $d_i(b)$ is at most $i - 1$. The result follows. \square

Proposition 3.8 deals with rays landing at parabolic points.

Proposition 3.8 ([BOPT14, Proposition 3.4]). *Suppose that an external ray $R_{f_{\lambda,b_*}}(\theta)$ with periodic argument $\theta \in \mathbb{R}/\mathbb{Z}$ lands at 0, and $T_{p/q}(b_*) \neq 0$. Then, for all b sufficiently close to b_* , the ray $R_{f_{\lambda,b}}(\theta)$ lands at 0.*

4. POLYNOMIALS WITH DISCONNECTED JULIA SETS

Let us now study cubic polynomials with disconnected Julia sets.

4.1. External and quadratic-like rays of polynomials in $\mathcal{F}_\lambda \setminus \mathcal{C}_\lambda$. In this subsection, λ is a complex number of modulus at most 1. We study the parameter plane \mathcal{F}_λ . The set $\mathcal{F}_\lambda \setminus \mathcal{C}_\lambda$ is foliated by parameter rays. Let us describe the dynamics of a polynomial $f \in \mathcal{F}_\lambda \setminus \mathcal{C}_\lambda$ choosing f from the parameter ray $\mathcal{R}_\lambda(\varkappa)$. We will establish a correspondence between the dynamical properties of f and the properties of the invariant quadratic gap $\mathfrak{U}_p(\varkappa)$. Let us emphasize that in what follows we deal with both *external* rays (defined after Theorem 2.5), *quadratic-like* rays (defined in Subsection 2.2), and *dynamic* rays (defined in Subsection 2.1), so it is important to distinguish between these types of rays.

A polynomial $f \in \mathcal{F}_\lambda$ belongs to a *parameter ray* $\mathcal{R}_\lambda(\varkappa)$ if and only if the dynamic rays $R_f(\varkappa+1/3)$ and $R_f(\varkappa+2/3)$ crash into $\omega_2(f) = \omega_2$; by Theorem 2.1 then no other dynamic ray crashes into ω_2 . Set $R_f(\varkappa+1/3) \cup R_f(\varkappa+2/3) \cup \{\omega_2\} = \Gamma_{\omega_2}$; clearly, Γ_{ω_2} is a curve dividing \mathbb{C} , the dynamic plane of f , in two parts. Let $\Sigma_f = W_f(\varkappa+1/3, \varkappa+2/3)$ be the part of the plane bounded by Γ_{ω_2} and containing the rays $R_f(\theta)$ for all $\theta \in (\varkappa+1/3, \varkappa+2/3)$; we call such sets Σ_f (enclosed by two dynamic rays of f landing or crashing at the same point) *dynamic wedges*. In general, a *dynamic wedge* $W_f(\alpha, \beta)$ is a part of the plane bounded by $R_f(\alpha) \cup R_f(\beta)$ and containing the rays $R_f(\theta)$ for all $\theta \in (\alpha, \beta)$, where the dynamic rays $R_f(\alpha)$ and $R_f(\beta)$ crash or land at the same point.

Then Σ_f maps one-to-one onto the complement of $f(\Gamma_{\omega_2})$; moreover, Σ_f contains the dynamic rays with arguments in $\mathbb{S}^1 \setminus L(\varkappa)$, where $L(\varkappa) = L(\mathfrak{c}_\varkappa)$ is the longer closed arc with endpoints $\varkappa+1/3$ and $\varkappa+2/3$ (recall that by definition $\mathfrak{c}_\varkappa = (\varkappa+1/3)(\varkappa+2/3)$). By Theorem 2.5, the polynomial f is immediately renormalizable. The quadratic-like filled Julia set $K(f^*)$ is disjoint from Γ_{ω_2} , since every point of Γ_{ω_2} is escaping. Clearly, $K(f^*) \not\subset \Sigma_f$ because $f|_{\Sigma_f}$ is one-to-one while $f|_{K(f^*)}$ is two-to-one except for the critical point of $f|_{K(f^*)}$. Hence, $K(f^*) \cap \overline{\Sigma}_f = \emptyset$.

Let us make a convenient choice of the Jordan domains U^* and V^* , for which the map $f|_{U^*} = f^* : U^* \rightarrow V^*$ is quadratic-like, cf. [BrHu88]. Recall that G_K is the Green function for a compact set K . The set E_f of points $z \in \mathbb{C}$ with $G_{K(f)}(z) = G_{K(f)}(\omega_2(f))$ is a figure eight; observe that $K(f)$ is a compact subset of the set V_f of all points $z \in \mathbb{C}$ such that $G_{K(f)}(z) < G_{K(f)}(\omega_2(f))$ (clearly, $E_f = \text{Bd}(V_f)$). Define V^* as the connected component of V_f containing $K(f^*)$ (clearly, V^* is disjoint from Γ_{ω_2}). Define the domain U^* as the f -pullback of V^* in V^* . If $z \in K(f) \cap V^*$ then either $f(z) \in V^*$ and $z \in U^*$ or $f(z) \in \Sigma_f$ and $z \notin U^*$.

Lemma 4.1. *Suppose that $f \in \mathcal{R}_\lambda(\varkappa)$. A dynamic ray $R_f(\alpha)$ does not accumulate in $K(f^*)$ if and only if there is an integer $k \geq 0$ such that $R_f(3^k\alpha) \subset \overline{\Sigma}_f$. In particular, a smooth ray $R_f(\alpha)$ does not accumulate in $K(f^*)$ if and only if there is an integer $k \geq 0$ such that $R_f(3^k\alpha) \subset \Sigma_f$.*

Proof. Since all non-smooth dynamic rays eventually map into Γ_{ω_2} , none of them can accumulate in $K(f^*)$. Fix a smooth ray $R_f(\alpha)$. Evidently, no image of $R_f(\alpha)$ can intersect Γ_{ω_2} . If $R_f(\alpha)$ accumulates in $K(f^*)$, then, since $K(f^*) \subset \mathbb{C} \setminus \overline{\Sigma}_f$ is invariant, all images of $R_f(\alpha)$ are contained in $\mathbb{C} \setminus \overline{\Sigma}_f$. Now, suppose that all images of $R_f(\alpha)$ are contained in $\mathbb{C} \setminus \overline{\Sigma}_f$. The ray $R_f(\alpha)$ accumulates on its principal set $\text{Pr}(R_f(\alpha))$; we claim that $\text{Pr}(R_f(\alpha)) \subset K(f^*)$. Indeed, otherwise there must exist

the minimal i such that $f^{\circ i}(z) \notin U^*$ for some $z \in \text{Pr}(R_f(\alpha))$. By the remark before the lemma, $f^{\circ(i+1)}(z) \in \Sigma_f$. Since $f^{\circ(i+1)}(z)$ belongs to the principal set of $f^{\circ(i+1)}(R_f(\alpha))$, then $f^{\circ(i+1)}(R_f(\alpha)) \subset \Sigma_f$, a contradiction. \square

We describe *external* rays accumulating in $K(f^*)$. Recall that external rays R^e have one-sided arguments. If R^e is a smooth ray with argument θ , then it is associated with both one-sided external arguments $\theta+$ and $\theta-$. If R^e is not smooth, then R^e is the one-sided limit of smooth rays from exactly one side, and we associate to R^e the appropriate *one-sided* external argument, $\theta+$ or $\theta-$. In what follows, we write θ^τ for a one-sided argument $\theta+$ or $\theta-$ according to whether $\tau = +$ or $\tau = -$. For any set $A \subset \mathbb{S}^1$ and angle $\alpha \in \mathbb{R}/\mathbb{Z}$, say that $\alpha+$ (respectively, $\alpha-$) is the *one-sided argument (of α) in A* or that $\alpha+$ (respectively, $\alpha-$) *belongs to A* if $\alpha \in A$, and α is not isolated in A from the positive side (resp., from the negative side).

Proposition 4.2. *Consider a polynomial $f \in \mathcal{R}_\lambda(\mathcal{X})$ and its immediate renormalization $f^* : U^* \rightarrow V^*$. Then the set of arguments of external rays to $K(f^*)$ coincides with the union of the set of arguments $\theta \pm$ where $\theta \in \mathcal{U}'_p(\mathbf{c}_\mathcal{X})$ never maps to an endpoint of $\mathbf{c}_\mathcal{X}$ (these external rays are smooth) and the set of one-sided arguments γ^τ , where $\gamma \in \mathcal{U}'_p(\mathbf{c}_\mathcal{X})$ eventually maps to an endpoint of $\mathbf{c}_\mathcal{X}$, and γ^τ belongs to $\mathcal{U}'_p(\mathbf{c}_\mathcal{X})$ (these external rays are not smooth).*

In particular, the (one-sided) argument of an external ray accumulating in $K(f^*)$ always belongs to $\mathcal{U}'_p(\mathbf{c}_\mathcal{X})$.

Proof. Smooth rays are exactly the rays with arguments that never map to the endpoints of $\mathbf{c}_\mathcal{X}$. Moreover, smooth rays are both external and dynamic. Thus, by Lemma 4.1, a smooth ray $R_f^e(\theta)$ accumulates in $K(f^*)$ if and only if, for any $k \geq 0$, the ray $R_f(3^k\theta)$ is disjoint from $\overline{\Sigma}_f$. This is equivalent to the fact that $3^k\theta$ belongs to the interior of $L(\mathbf{c}_\mathcal{X})$ for every integer $k \geq 0$, which, in turn, is equivalent to the fact that $\theta \in \mathcal{U}'_p(\mathbf{c}_\mathcal{X})$ and never maps to the endpoints of $\mathbf{c}_\mathcal{X}$.

Now we describe the arguments of non-smooth external rays accumulating in $K(f^*)$. Let R^e be a non-smooth external ray with argument γ^τ accumulating in $K(f^*)$. There is a minimal $k \geq 0$ such that $3^k\gamma$ is an endpoint of $\mathbf{c}_\mathcal{X}$. Now, if some image of R^e is contained in Σ_f then its closure is contained in $\overline{\Sigma}_f$ which is disjoint from $K(f^*)$, a contradiction with the assumption. So, all points $\overline{3^m\gamma}$ are in the interior of $L(\mathcal{X})$ for $m < k$, and the orbit of γ is contained in $\overline{L(\mathcal{X})}$. Hence γ and its entire orbit under the angle-tripling map are contained in $\mathcal{U}'(\mathbf{c}_\mathcal{X})$. Moreover, $\overline{3^k\gamma}$ is an endpoint of $\mathbf{c}_\mathcal{X}$. The latter belongs to $\mathcal{U}'(\mathbf{c}_\mathcal{X})$ too, which implies that $\mathcal{U}'(\mathbf{c}_\mathcal{X})$ is of caterpillar type or of regular critical type. Clearly, $R^e((3^k\gamma)^\tau) = f^{\circ k}(R^e)$ accumulates in $K(f^*)$ and the orbit of the principal set of R^e is disjoint from $\overline{\Sigma}_f$.

We claim that $\gamma \in \mathcal{U}'_p(\mathbf{c}_\mathcal{X})$. Indeed, γ can be isolated in $\mathcal{U}'(\mathbf{c}_\mathcal{X})$ only if $\mathcal{U}'(\mathbf{c}_\mathcal{X})$ is a caterpillar gap, and γ eventually maps to the preperiodic endpoint β of $\mathbf{c}_\mathcal{X}$. Let $\mathbf{c}_\mathcal{X} = \overline{\alpha\beta}$ where α is periodic of period n and $\beta = \alpha + 1/3$. Then the one-sided external ray \tilde{R} with argument $\alpha- \in \mathcal{U}'_p(\mathbf{c}_\mathcal{X})$ lands at a periodic point $z \in J(f^*)$ of period n , and there are countably many dynamic rays that crash into iterated $f^{\circ n}$ -preimages of $\omega_2(f) \in \tilde{R}$. These dynamic rays and appropriate segments of \tilde{R} separate the dynamic ray with argument β from $K(f^*)$. See Figure 3. Hence the rays $R^e(\beta^\pm)$ do not accumulate in $K(f^*)$. Indeed, the rays $R^e(\beta^\pm)$ follow the dynamic ray with argument β all the way to the critical point $\omega_2(f)$. At the point $\omega_2(f)$ the ray $R^e(\beta-)$ turns inside Σ_f and cannot accumulate in $K(f^*)$. The ray

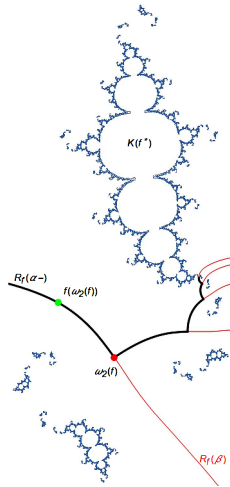


FIGURE 3. This figure illustrates the proof of Proposition 4.2. The polynomial f whose dynamic plane is displayed belongs to $\mathcal{R}_{-1}(1/6)$. The limit ray $R_f(\alpha-)$ (here $\alpha = 1/2$) is shown as thick. Thin lines represent the dynamic ray $R_f(\beta)$ up to the point $\omega_2(f)$ where it crashes and the dynamic rays crashing at iterated preimages of $\omega_2(f)$. In our case, $\beta = \alpha + 1/3 = 5/6$ and $n = 1$.

$R^e(\beta+)$ turns in $\mathbb{C} \setminus \bar{\Sigma}_f$. However there is a dynamic ray \widehat{R} that crashes into the n -th preimage of $\omega_2(f)$ that belongs to \widehat{R} ; the ray \widehat{R} shields $K(f^*)$ from $R^e(\beta+)$ so that $R^e(\beta+)$ does not accumulate in $K(f^*)$ either. Hence $\gamma \in \mathcal{U}'_p(\mathbf{c}_\varkappa)$.

It remains to consider rays with one-sided arguments $\varkappa + 1/3 \pm, \varkappa + 2/3 \pm$ under the assumption that $\mathcal{U}'(\mathbf{c}_\varkappa)$ is of caterpillar type or of regular critical type. A simple analysis shows that $R^e(\theta^\tau)$ (where $\bar{\theta}$ is an endpoint of \mathbf{c}_\varkappa) accumulates in $K(f^*)$ if and only if θ^τ is a one-sided argument in $\mathcal{U}'_p(\mathbf{c}_\varkappa)$. Indeed, if θ^τ is not a one-sided argument in $\mathcal{U}'_p(\mathbf{c}_\varkappa)$ then, similar to the above, we see that $R^e(\theta^\tau)$ enters Σ_f and cannot accumulate in $K(f^*)$. Hence if θ^τ is a one-sided argument in $\mathcal{U}'_p(\mathbf{c}_\varkappa)$ then by Lemma 4.1 we can approximate θ from the τ -side by smooth rays accumulating in $K(f^*)$, and, thus, $R^e(\theta^\tau)$ accumulates in $K(f^*)$. Thus, in the regular critical case both endpoints of \mathbf{c}_\varkappa have one-sided argument in $\mathcal{U}'_p(\mathbf{c}_\varkappa)$, in the periodic case neither endpoint of \mathbf{c}_\varkappa belongs to $\mathcal{U}'_p(\mathbf{c}_\varkappa)$, and in the caterpillar case only the periodic endpoint of \mathbf{c}_\varkappa has one-sided argument in $\mathcal{U}'_p(\mathbf{c}_\varkappa)$. \square

We defined the *isotopy* rel. $K(f^*)$ of external and quadratic-like rays in Definition 2.8.

Proposition 4.3. *For any hole $(\bar{\alpha}, \bar{\beta})$ of $\mathcal{U}_p(\mathbf{c}_\varkappa)$, the external rays $R_f^e(\alpha-)$ and $R_f^e(\beta+)$ are isotopic rel. $K(f^*)$ to the same quadratic-like ray. Moreover, one of the following holds:*

- (1) *the gap $\mathcal{U}_p(\mathbf{c}_\varkappa)$ is of regular critical type, the external rays $R_f^e(\alpha-)$ and $R_f^e(\beta+)$ meet at an eventual preimage x of $\omega_2(f)$, and then continue along a joint curve from x to $K(f^*)$;*

- (2) the gap $\mathfrak{U}_p(\mathfrak{c}_\varkappa)$ is of periodic type with major hole $(\overline{\alpha_*}, \overline{\beta_*})$, the external rays $R_f^e(\alpha_-)$ and $R_f^e(\beta_+)$ both land at an eventual preimage of the common periodic landing point for the rays $R_f^e(\alpha_*^-)$ and $R_f^e(\beta_*^+)$.

Proof. Relying upon Theorem 2.9, suppose that the external rays $R_f^e(\alpha_-)$ and $R_f^e(\beta_+)$ are isotopic to distinct quadratic-like rays $T_\alpha \neq T_\beta$ rel. $K(f^*)$. Then, clearly, both components U_1^*, U_2^* of $U^* \setminus (T_\alpha \cup T_\beta \cup K(f^*))$ contain segments of quadratic-like rays to $K(f^*)$. Therefore, by Theorem 2.9, both U_1^* and U_2^* contain segments of external rays to $K(f^*)$. However, by Proposition 4.2, there are no external rays to $K(f^*)$ with arguments in (α, β) , a contradiction. Hence, $R_f^e(\alpha_-)$ and $R_f^e(\beta_+)$ are isotopic to the same quadratic-like ray T rel. $K(f^*)$.

Suppose that $\mathfrak{U}_p(\mathfrak{c}_\varkappa)$ has major $\mathfrak{M} = \overline{\alpha_*\beta_*}$. If it is of regular critical type, then by Lemma 2.6, the external rays $R_f^e(\alpha_*^-)$, $R_f^e(\beta_*^+)$ meet at $\omega_2(f)$ and then extend together towards $K(f^*)$. If $\overline{\alpha\beta}$ is an edge of $\mathfrak{U}_p(\mathfrak{c}_\varkappa)$, then $\overline{\alpha\beta}$ is an iterated σ_3 -pullback of \mathfrak{M} , so that α and β are appropriate preimages of α_* and β_* , and the external rays $R_f^e(\alpha_-)$, $R_f^e(\beta_+)$ meet at a preimage x of $\omega_2(f)$ and then extend together towards $K(f^*)$. In fact, their union is a pullback of the union of rays $R_f^e(\alpha_*^-)$, $R_f^e(\beta_*^+)$. This covers case (1) of the proposition and corresponds to case (i) of Theorem 2.9. Now, if $\mathfrak{U}_p(\mathfrak{c}_\varkappa)$ is of periodic type, then, by Lemma 2.6, the rays $R_f^e(\alpha_*^-)$, $R_f^e(\beta_*^+)$ do not intersect. Since, by the above, they are isotopic to the same quadratic-like ray rel. $K(f^*)$, they land at the same periodic point of $K(f^*)$. Since all edges of $\mathfrak{U}_p(\mathfrak{c}_\varkappa)$ are preimages of $\overline{\alpha_*\beta_*}$, claim (2) follows.

There are two distinct cases within claim (2). If $\mathfrak{U}(\varkappa) = \mathfrak{U}_p(\varkappa)$ is of periodic type, the rays $R_f^e(\alpha_*)$ and $R_f^e(\beta_*)$ are smooth, disjoint, and land at the same periodic point. The situation is a little more complicated if $\mathfrak{U}(\varkappa)$ is of caterpillar type. Without loss of generality, assume that $\alpha_* = \varkappa + 1/3$ is periodic. Then the dynamic ray $R_f(\alpha_*)$ crashes (together with the dynamic ray $R_f(\alpha_* + 1/3)$) at the escaping critical point $\omega_2(f)$, the external non-smooth ray $R_f^e(\alpha_*^-)$ still lands at the same periodic point as the smooth external ray $R_f^e(\beta_*^+)$, and both rays correspond to the same quadratic-like ray to $K(f^*)$. \square

Evidently, if, for some argument α , $R_f^e(\alpha_+) \neq R_f^e(\alpha_-)$, then their principal sets are contained in distinct components of $K(f)$. Hence, if for some α, β and some choice of $\rho, \tau \in \{+, -\}$ the set $\text{Pr}(R_f^e(\alpha^\rho)) \cap \text{Pr}(R_f^e(\beta^\tau)) \cap K(f^*) \neq \emptyset$, then there is a unique choice of one-sided external rays with these arguments for which this holds. In this case the chord $\overline{\alpha\beta}$ determines a unique cut $\overline{R_f^e(\alpha^\rho)} \cup \overline{R_f^e(\beta^\tau)} = \Gamma(\alpha\beta)$ of \mathbb{C} called the *cut (generated by $\overline{\alpha\beta}$)*. If the (one-sided) rays defining a cut $\Gamma(\overline{\alpha\beta})$ land at the same point z , then z is called the *vertex (of $\Gamma(\overline{\alpha\beta})$)* and the cut itself is denoted by $\Gamma_z(\overline{\alpha\beta})$ (or sometimes simply by Γ_z).

Suppose now that f lies in the parameter ray $\mathcal{R}_\lambda(\varkappa)$. By Proposition 4.3, there are cuts generated by edges of $\mathfrak{U}_p(\mathfrak{c}_\varkappa)$. In the regular critical case, $\Gamma_{\omega_2}(\overline{\alpha\beta})$ is formed by dynamic rays $R_f(\alpha)$ and $R_f(\beta)$ crashing into the same critical point $\omega_2 = \omega_2(f)$; this cut is disjoint from $K(f^*)$. This corresponds to case (1) of Proposition 4.3. In the periodic and caterpillar cases the external rays $R_f^e(\alpha_-)$, $R_f^e(\beta_+)$ are disjoint but land at the same (pre)periodic point. This corresponds to case (2) of Proposition 4.3. In the caterpillar case, exactly one of the rays forming a cut is non-smooth. Clearly, $\Gamma_{\omega_2}(\overline{\alpha\beta})$ separates all dynamic rays of f , whose arguments belong to (α, β) , from all dynamic rays of f , whose arguments belong to (β, α) .

By Proposition 4.2, the (one-sided) argument of an external ray accumulating in $K(f^*)$ belongs to $\mathfrak{U}'_p(\mathfrak{c}_\varkappa)$. No chord connecting two points of $\mathfrak{U}'_p(\mathfrak{c}_\varkappa)$ crosses an edge of $\mathfrak{U}_p(\mathfrak{c}_\varkappa)$ (two distinct chords of $\overline{\mathbb{D}}$ cross if they intersect in \mathbb{D}). This yields Lemma 4.4.

Lemma 4.4. *Suppose that a polynomial f lies in $\mathcal{R}_\lambda(\varkappa)$ with $|\lambda| \leq 1$. Each edge of $\mathfrak{U}_p(\mathfrak{c}_\varkappa)$ generates a cut in the dynamical plane of f consisting of two rays that accumulate either on a (pre)critical point (in the regular critical case) or on a (pre)periodic point (in the periodic and caterpillar cases). In either case, the associated external rays correspond to the same quadratic-like ray. If a chord of \mathbb{D} generates a cut and crosses an edge of $\mathfrak{U}_p(\mathfrak{c}_\varkappa)$ in \mathbb{D} , then it coincides with this edge.*

4.2. Landing properties. In this subsection, we fix a hole $(\bar{\theta}_1, \bar{\theta}_2)$ of Ω , consider a polynomial $f \in \mathcal{R}_\lambda(\varkappa) \subset \mathcal{F}_\lambda \setminus \mathcal{C}_\lambda$ with $|\lambda| \leq 1$, and study conditions on the mutual location of the point \varkappa and the hole $(\bar{\theta}_1, \bar{\theta}_2)$, under which the dynamic rays $R_f(\theta_1 + 1/3)$ and $R_f(\theta_2 + 2/3)$ can have a common landing point in $K(f^*)$. Note that, by Theorem 2.13, the hole $(\bar{\theta}_1, \bar{\theta}_2)$ defines an invariant quadratic gap \mathfrak{U} with periodic major $\mathfrak{M} = (\theta_1 + 1/3)(\theta_2 + 2/3)$. We refer the reader to Subsection 2.3 for the notation and the main concepts we deal with in the present subsection.

Lemma 4.5. *Suppose that $f \in \mathcal{R}_\lambda(\varkappa)$, that $(\bar{\theta}_1, \bar{\theta}_2)$ is a hole of Ω , and that the dynamic rays $R_f(\theta_1 + 1/3)$ and $R_f(\theta_2 + 2/3)$ are contained in external rays R_1^e, R_2^e , respectively, with the same landing point z . Then $z \in K(f^*)$. Moreover, if neither $(\theta_1, \theta_2) = (2/3, 5/6)$ nor $(\theta_1, \theta_2) = (1/6, 1/3)$, then the following are equivalent:*

- (1) $\varkappa \notin [\theta_1, \theta_2]$;
- (2) the plane cut $R_1^e \cup \{z\} \cup R_2^e$ separates $K(f^*)$.

Proof. Since $f \in \mathcal{R}_\lambda(\varkappa) \subset \mathcal{F}_\lambda \setminus \mathcal{C}_\lambda$, the dynamic rays $R_f(\varkappa + 1/3)$ and $R_f(\varkappa + 2/3)$ crash at the critical point $\omega_2(f) = \omega_2$ and form a cut Γ_{ω_2} of the plane that separates \mathbb{C} in two wedges. One of them (W_1) contains dynamic rays with arguments from $(\varkappa + 1/3, \varkappa + 2/3)$ and the other one (W_2) contains dynamic rays with arguments from $(\varkappa + 2/3, \varkappa + 1/3)$. It follows that $K(f^*) \subset W_2$. Suppose that the sets $\{\theta_1 + 1/3, \theta_2 + 2/3\}$ and $\{\varkappa + 1/3, \varkappa + 2/3\}$ are disjoint. Then Γ_{ω_2} and the external rays R_1^e, R_2^e are disjoint too. Since arcs $(\theta_1 + 1/3, \theta_2 + 2/3)$ and $(\theta_2 + 2/3, \theta_1 + 1/3)$ are longer than $1/3$, then angles $\theta_1 + 1/3, \theta_2 + 2/3$ are in the arc $(\varkappa + 2/3, \varkappa + 1/3)$, the rays R_1^e and R_2^e are contained in W_2 , and the point z belongs to W_2 . Similar arguments show that in general (i.e. if $\{\theta_1 + 1/3, \theta_2 + 2/3\}$ and $\{\varkappa + 1/3, \varkappa + 2/3\}$ are allowed to intersect) angles $\theta_1 + 1/3, \theta_2 + 2/3$ are in the arc $[\varkappa + 2/3, \varkappa + 1/3]$, the rays R_1^e and R_2^e are contained in \overline{W}_2 , one of these rays lies in W_2 , and $z \in W_2$.

Observe that the endpoints $\varkappa + 1/3, \varkappa + 2/3$ of \mathfrak{c}_\varkappa can belong to either $[\theta_1 + 1/3, \theta_2 + 2/3]$ or to $[\theta_2 + 2/3, \theta_1 + 1/3]$; in the latter case there are significant restrictions upon the exact location of \mathfrak{c}_\varkappa (see Lemma 4.6 for an explicit description).

Let $W_{(1)}, W_{(2)}$ be the two wedges defined by R_1^e, R_2^e : say, $W_{(1)}$ is the wedge containing rays with arguments from $(\theta_1 + 1/3, \theta_2 + 2/3)$ and $W_{(2)}$ is the wedge containing rays with arguments from $(\theta_2 + 2/3, \theta_1 + 1/3)$. If $K(f^*)$ meets both these wedges, then $z \in K(f^*)$ as desired. Moreover, then $\varkappa \notin [\theta_1, \theta_2]$ because otherwise $\mathfrak{U}'_p(\mathfrak{c}_\varkappa) \subset [\theta_2 + 2/3, \theta_1 + 1/3]$, a contradiction with the fact that $K(f^*)$ meets both wedges defined by R_1^e, R_2^e .

Assume that $K(f^*)$ is contained in one of the wedges $W_{(1)}, W_{(2)}$, union $\{z\}$. We claim that then, with one exception, $\varkappa \in [\theta_1, \theta_2]$. Indeed, otherwise the arc

$[\varkappa + 1/3, \varkappa + 2/3]$ is contained in the arc $(\theta_2 + 2/3, \theta_1 + 1/3)$. If $K(f^*)$ is contained in W_2 and $W_{(2)} \cup \{z\}$ then σ_3 is one-to-one on arguments of external rays landing in $K(f^*)$ except possibly for $\varkappa + 1/3, \varkappa + 2/3$, a contradiction. Suppose that $K(f^*)$ is contained in W_2 and $W_{(1)} \cup \{z\}$. Since by our assumptions $[\theta_1 + 1/3, \theta_2 + 2/3] \subset (\varkappa + 2/3, \varkappa + 1/3)$, then $W_{(1)} \cup \{z\} \subset W_2$, and the above is equivalent to $K(f^*) \subset W_{(1)} \cup \{z\}$. By Theorem 2.10, $|(\theta_1 + 1/3, \theta_2 + 2/3)| \leq 1/2$. However, $K(f^*) \subset W_{(1)} \cup \{z\}$ implies that $|(\theta_1 + 1/3, \theta_2 + 2/3)| \geq 1/2$. Hence $|(\theta_1 + 1/3, \theta_2 + 2/3)| = 1/2$. By Theorem 2.10, either $\theta_1 + 1/3 = 0, \theta_2 + 2/3 = 1/2$, or $\theta_1 + 1/3 = 1/2, \theta_2 + 2/3 = 0$. In both cases by Proposition 4.3 the external rays R_1^e, R_2^e are smooth, periodic, and land at the same (periodic) point z of $J(f^*)$ as desired.

Thus, we may assume that the endpoints of c_\varkappa belong to $[\theta_1 + 1/3, \theta_2 + 2/3]$, and at least one of the endpoints of c_\varkappa belongs to $(\theta_1 + 1/3, \theta_2 + 2/3)$. Assume first that both endpoints of c_\varkappa belong to $(\theta_1 + 1/3, \theta_2 + 2/3)$. Then by Proposition 4.3 the external rays R_1^e, R_2^e are smooth, periodic, and land at the same (periodic) point z of $J(f^*)$. Now, suppose that $\varkappa = \theta_1$ (the case when $\varkappa = \theta_2$ is similar). Then the dynamic and external rays with argument $\theta_2 + 2/3$ coincide, and form a smooth ray. This rays lands at a periodic point $w \in K(f^*)$. Moreover, the ray $R^e(\theta_1 + 1/3-)$ lands at the same point. On the other hand the ray $R^e(\theta_1 + 1/3+)$ accumulates inside Σ_f and, therefore, cannot land at w . Hence the rays R_1^e, R_2^e are in fact $R^e(\theta_2 + 2/3)$ and $R^e(\theta_1 + 1/3-)$, and they land at the point $z = w \in K(f^*)$. \square

Observe that if $(\theta_1, \theta_2) = (2/3, 5/6)$ or $(\theta_1, \theta_2) = (1/6, 1/3)$, then the plane cut $R_1^e \cup \{z\} \cup R_2^e$ defined in Lemma 4.5 cannot separate $K(f^*)$.

Lemma 4.6. *The following claims are equivalent.*

- (1) *The dynamic rays $R_f(\theta_1 + 1/3)$ and $R_f(\theta_2 + 2/3)$ are contained in a pair of external rays with the same periodic landing point.*
- (2) *One of the following holds.*
 - (a) *The angle \varkappa is in $[\theta_1, \theta_2]$.*
 - (b) *The chord $\overline{(\theta_1 + 1/3)(\theta_2 + 2/3)}$ is a major edge of a type D invariant gap/leaf \mathfrak{g} while $[\varkappa + 1/3, \varkappa + 2/3]$ is contained in the closure of the hole behind the other major edge of \mathfrak{g} . Moreover, for every $\alpha\beta$ in the σ_3 -orbit of $(\theta_1 + 1/3)(\theta_2 + 2/3)$, the rays $R_f^e(\alpha+)$ and $R_f^e(\beta-)$ land at a fixed point z ; we have $z = 0$ unless $(\theta_1, \theta_2) = (2/3, 5/6)$ or $(1/6, 1/3)$.*

Except for the case when $\mathfrak{g} = \overline{0(1/2)}$, the set \mathfrak{g} is a gap.

Proof. We first prove that (2) implies (1). Assume that $\varkappa \in [\theta_1, \theta_2]$. Then, by Proposition 4.3, the rays $R_f^e(\theta_1 + 1/3-)$ and $R_f^e(\theta_2 + 2/3+)$ land at a common point, as desired. If (b) holds, the claim follows immediately.

Assume now that (1) holds. Denote the external rays containing $R_f(\theta_1 + 1/3)$ and $R_f(\theta_2 + 2/3)$ and landing at a common point z by R_1^e, R_2^e ; then, by Lemma 4.5, we have $z \in K(f^*)$. Suppose that $\varkappa \notin [\theta_1, \theta_2]$, and prove that (b) holds.

To begin with, consider the case when $\overline{(\theta_1, \theta_2)} = (2/3, 5/6)$ so that $(\theta_1 + 1/3, \theta_2 + 2/3) = (0, 1/2)$. In this case $\mathfrak{g} = \overline{0(1/2)}$. This is a leaf but it can be informally viewed as a gap with two distinct sides $\overline{0(1/2)}$ and $\overline{(1/2)0}$. Our assumption that $\varkappa \notin [\theta_1, \theta_2]$ means that $\varkappa \notin [2/3, 5/6]$. Since the critical cut formed by the rays $R_f(\varkappa + 1/3)$ and $R_f(\varkappa + 2/3)$ cannot cross the cut formed by the rays R_1^e and R_2^e , then, as in the proof of Lemma 4.5, it follows that $[\varkappa + 1/3, \varkappa + 2/3] \subset [1/2, 0]$. In other words, $\overline{[\varkappa + 1/3, \varkappa + 2/3]}$ is contained in the closure of the hole behind

the other major side $\overline{(1/2)0}$ of \mathfrak{g} as desired. Moreover, by Proposition 4.3, then $R_f^e(1/2-)$ and $R_f^e(0+)$ land at the same fixed point as desired. The same can be proven in the case when $(\theta_1, \theta_2) = (1/6, 1/3)$.

Thus, from now on we may assume that $(\theta_1, \theta_2) \neq (2/3, 5/6)$ and $(\theta_1, \theta_2) \neq (1/6, 1/3)$. Recall that we suppose that $\varkappa \notin [\theta_1, \theta_2]$, and want to prove that then (b) holds. By Theorem 2.9, the external rays R_1^e and R_2^e correspond to quadratic-like rays. If they correspond to the same quadratic-like ray, then the cut $R_1^e \cup \{z\} \cup R_2^e$ does not separate $K(f^*)$. By Lemma 4.5, then $\varkappa \in [\theta_1, \theta_2]$, a contradiction. Hence the external rays R_1^e and R_2^e correspond to different quadratic-like rays to $K(f^*)$.

By Lemma 4.4, the chord $\mathfrak{M} = \overline{(\theta_1 + 1/3)(\theta_2 + 2/3)}$ lies inside $\mathfrak{U}_p(c_\varkappa)$, except for its endpoints. Since the arguments of both rays are periodic, z is periodic. By the assumption, the external rays R_1^e and R_2^e are isotopic to different quadratic-like rays rel. $K(f^*)$, and z is a cutpoint of $K(f^*)$. Since $|\lambda| \leq 1$, there are no periodic cutpoints of $K(f^*)$ that are not fixed, and the fixed cutpoint 0 must be parabolic. Therefore, $|\lambda| = 1$ and $z = 0$ is parabolic. Since \mathfrak{M} is a major of some quadratic invariant gap \mathfrak{U} , the orbit of \mathfrak{M} consists of pairwise disjoint chords; since z is parabolic, σ_3 restricted to them preserves their circular order. Completing the orbit of \mathfrak{M} to its convex hull \mathfrak{g} , we see that \mathfrak{g} is an invariant finite gap of type D. Hence $\overline{[\varkappa + 1/3, \varkappa + 2/3]}$ is contained in the closure of the hole behind a major edge of \mathfrak{g} ; by definition, \mathfrak{M} must be the other major edge of \mathfrak{g} . The remaining claim that for every $\overline{\alpha\beta}$ in the orbit of $\overline{(\theta_1 + 1/3)(\theta_2 + 2/3)}$ the rays $R_f^e(\alpha+)$ and $R_f^e(\beta-)$ land at the same point follows from Proposition 4.3. \square

5. FIXED AND PERIODIC POINTS

Let us recall some topological results of [BFMOT13]. First define *special pieces* of a polynomial f with connected Julia set; loosely, these are subcontinua X of $K(f)$ carved in it by *exit continua* E_1, \dots, E_n located on the boundary of X so that $f(X)$ grows out of X only “through” the exit continua of X . Definition 5.1 is a bit more special and less general than that in [BFMOT13].

Definition 5.1 (Central component and exit continua). Let E_1, \dots, E_n be a finite (perhaps empty) collection of (possibly degenerate) pairwise disjoint continua in $J(f)$, each consisting of principal sets of two or more external rays. Denote the union of E_i with these external rays by \tilde{E}_i . If there is a component T of $\mathbb{C} \setminus \cup \tilde{E}_i$, whose boundary intersects all E_1, \dots, E_n , then E_i ’s are called *exit continua* (of T) while T is called a *central component* (of the exit continua E_1, \dots, E_n).

If there are no exit continua, then a central component T coincides with \mathbb{C} .

Definition 5.2 (Special pieces). Let T be the central component of the exit continua E_1, \dots, E_n . For every i , let the wedge W_i be the component of $\mathbb{C} \setminus \tilde{E}_i$ containing T . Any non-separating (in \mathbb{C}) continuum X is said to be a *special piece* (of f) if the following hold:

- (1) $X \subset [T \cap K(f)] \cup (\cup E_i)$ is a continuum and X contains $\cup E_i$;
- (2) each E_i is either a fixed point or maps forward in such a way that $f(E_i) \subset W_i$ (loosely, sets E_i are mapped “towards T ”);
- (3) the set $f(X) \setminus X$ is disjoint from T (loosely, X can only grow “through exit continua”).

Evidently, the set $[T \cap K(f)] \cup (\cup E_i)$ above is a continuum.

Theorem 5.3 ([BFMOT13]). *Let f be a polynomial with connected Julia set, and X be a special piece of f . Then X contains a fixed Cremer or Siegel point, or an invariant attracting or parabolic Fatou domain, or a fixed repelling or parabolic point at which at least two rays land so that f non-trivially rotates these rays.*

In particular, Theorem 5.3 applies to invariant continua in $K(f)$ that are non-separating in \mathbb{C} (with an empty collection of exit continua).

Definition 5.4. Given a (possibly disconnected) Julia set J , choose an angle α and consider the set L^- of all limit points of sequences $x_i \in R_f(\alpha_i)$ where $R_f(\alpha_i)$ are smooth rays with arguments α_i , and α_i converge to α from the negative side in the usual sense (i.e., $\alpha_i < \alpha$). Then $\text{Imp}(\alpha-) = L^- \setminus R_f^c(\alpha-)$ is the *one-sided impression of α from the negative side*. Similarly we define the *one-sided impression $\text{Imp}(\alpha+)$ of α from the positive side*. In case J is connected, the *impression $\text{Imp}(\alpha)$* is defined as $\text{Imp}(\alpha-) \cup \text{Imp}(\alpha+)$.

Let us apply these tools to Julia sets and their periodic points.

Corollary 5.5. *Let f be a polynomial with connected Julia set and let T be the central component of exit continua E_1, \dots, E_n , each of which is a repelling or parabolic periodic point. Set $Y = [T \cap K(f)] \cup (\cup E_i)$. Suppose that Y contains no periodic Cremer points, no periodic Siegel points, and no periodic attracting or parabolic Fatou domains. Then there are infinitely many periodic points in Y , at each of which finitely many external rays land so that the minimal iterate of f that fixes the periodic point non-trivially rotates these external rays.*

Proof. Choose the iterate $f^{\circ r}$ of f that fixes all rays landing at E_1, \dots, E_n . It is easy to see that $Y = Y_0$ is a special piece for $f^{\circ r}$. By Theorem 5.3, there exists an $f^{\circ r}$ -fixed repelling point $y_0 \in Y_0$ with several external rays of $f^{\circ r}$ -period $m_0 > 1$ landing at y_0 . Observe that, by the choice of r , the point y_0 is not equal to any of the points E_0, \dots, E_n . Consider one of the wedges formed by the external rays of f landing at y_0 , say, W_1 , and set $Y_1 = W_1 \cap Y_0$. It follows that Y_1 is a special piece for a suitable iterate of f , whose exit continua are periodic points of f . We can now apply the same argument to Y_1 , etc. \square

The following is a consequence of Corollary 5.5.

Corollary 5.6. *Suppose that Z is a non-separating continuum in \mathbb{C} that is a finite union of one-sided impressions of periodic external angles and that contains no periodic Cremer points and no periodic Siegel points. Then Z is a singleton.*

Proof. Observe that one-sided impressions cannot contain parabolic or attracting periodic Fatou domains. Passing to a suitable iterate of the map, say, $f^{\circ n}$, we may assume that all the arguments of the external rays involved are invariant. Then, by Corollary 5.5, we conclude that $f^{\circ n}|_Z$ has a fixed point z at which several external rays land, and these external rays rotate under $f^{\circ n}$. Since the arguments of the external rays whose impressions form Z are invariant, we get a contradiction in the case when Z is non-degenerate. Hence, Z is a singleton as desired. \square

6. LAMINATIONS AND LAMINATIONAL MODELS

A *geodesic lamination*, which we will call simply a *lamination*, is a closed collection \mathcal{L} of closed chords in $\overline{\mathbb{D}}$ that do not cross in \mathbb{D} . We will always assume that

all points of \mathbb{S}^1 (i.e., all degenerate chords) are included into \mathcal{L} . The collection \mathcal{L} being closed means that the union \mathcal{L}^+ of all chords in \mathcal{L} is a closed subset of $\overline{\mathbb{D}}$. Elements of \mathcal{L} are called *leaves*.

As in the beginning of Subsection 2.3, we identify \mathbb{S}^1 with \mathbb{R}/\mathbb{Z} . Then the d -tupling map $t \mapsto d \cdot t \pmod{1}$ identifies with the map $\sigma_d : z \mapsto z^d$ on the unit circle \mathbb{S}^1 in the complex plane \mathbb{C} . If a chord ℓ has endpoints $\overline{\alpha}$ and $\overline{\beta}$, then we denote this chord by $\overline{\alpha\beta}$. We will always extend σ_d over all leaves of a given lamination \mathcal{L} by linearly mapping any leaf $\ell = \overline{\alpha\beta}$ onto the chord $\sigma_d(\overline{\alpha})\sigma_d(\overline{\beta}) = \overline{(d\alpha)(d\beta)}$.

Definition 6.1 (Invariant laminations, cf. [BMOV13]). A lamination \mathcal{L} is said to be (sibling) σ_d -invariant if:

- (1) for each $\ell \in \mathcal{L}$, we have $\sigma_d(\ell) \in \mathcal{L}$,
- (2) for each $\ell \in \mathcal{L}$ there exists $\ell^* \in \mathcal{L}$ such that $\sigma_d(\ell^*) = \ell$,
- (3) for each $\ell \in \mathcal{L}$ such that $\sigma_d(\ell)$ is a non-degenerate leaf, there exist d **pairwise disjoint** leaves ℓ_1, \dots, ℓ_d in \mathcal{L} such that $\ell_1 = \ell$ and $\sigma_d(\ell_i) = \sigma_d(\ell)$ for all $i = 2, \dots, d$.

The leaves ℓ_1, \dots, ℓ_d are called *siblings* of ℓ .

Gaps of \mathcal{L} are the closures of components of $\overline{\mathbb{D}} \setminus \mathcal{L}^+$; they are *finite* or *infinite* according to whether they have finitely many or infinitely many points in \mathbb{S}^1 . We say that a σ_3 -invariant lamination \mathcal{L} *co-exists* with a stand-alone invariant quadratic gap \mathfrak{U} if there are no leaves of \mathcal{L} crossing edges of \mathfrak{U} in \mathbb{D} . The lamination \mathcal{L} *tunes* the gap \mathfrak{U} if all edges of \mathfrak{U} are leaves of \mathcal{L} .

Definition 6.2. Suppose that a closed equivalence relation \sim on \mathbb{S}^1 is such that the convex hulls of \sim -classes are pairwise disjoint; then we call \sim a *laminal equivalence relation*. The collection of all edges of these convex hulls together with all singletons from \mathbb{S}^1 form a lamination denoted by \mathcal{L}_\sim ; this lamination is said to be *generated* by \sim . If \mathcal{L}_\sim is invariant, then we call \sim *invariant* too.

If a laminal equivalence relation is invariant, the map σ_d induces a continuous map on the corresponding quotient space.

Definition 6.3 ([BCO11]). If \sim is a σ_d -invariant laminal equivalence relation, we denote \mathbb{S}^1/\sim by J_\sim and call it the *topological Julia set (generated by \sim)*. The induced map $f_\sim : J_\sim \rightarrow J_\sim$ is called the *topological polynomial (generated by \sim)*.

7. THE INVARIANT QUADRATIC GAP $\mathfrak{U}(f)$ AND THE JULIA SET

In this section, we consider an immediately renormalizable polynomial $f \in \mathcal{F}_\lambda$ with $|\lambda| \leq 1$ (in particular, by Theorem 2.5 any polynomial f in the unbounded component of $\mathcal{F}_\lambda \setminus \mathcal{P}_\lambda$ is immediately renormalizable), and define an invariant quadratic gap $\mathfrak{U}(f)$ associated with 0; by $f^* : U^* \rightarrow V^*$, we denote the corresponding polynomial-like map. When $J(f)$ is disconnected, the gap that is the convex hull of all arguments of rays to $K(f^*)$ was introduced in Subsection 4.1 (though using different approach based upon the fact that $J(f)$ is disconnected in an essential way). Our approach here is necessarily different as we deal with immediately renormalizable polynomials and their quadratic-like Julia sets in both connected and disconnected cases. Once we introduce $\mathfrak{U}(f)$, it will be easy to see that the gap introduced in the disconnected case in Subsection 4.1 does coincide with $\mathfrak{U}(f)$.

Theorem 7.1 (Theorem 5.11, [McM94]). *Let $f_i : U_i \rightarrow V_i$ be polynomial-like maps of degree d_i , for $i = 1, 2$. Assume $f_1 = f_2 = f$ on $U = U_1 \cap U_2$. Let U' be a component of U with $U' \subset f(U') = V'$. Then $f : U' \rightarrow V'$ is polynomial-like of degree $d = \min(d_1, d_2)$, and $K(f) = K(f_1) \cap K(f_2) \cap U'$. If $d = d_i$, then $K(f) = K(f_i)$.*

Lemma 7.2. *If f is a complex cubic polynomial with a non-repelling fixed point a , and there exists a quadratic-like filled Julia set K^* with $a \in K^*$, then K^* is unique.*

Proof. Suppose that U_1 and U_2 are Jordan disks such that $f_1^* = f|_{U_1}$ and $f_2^* = f|_{U_2}$ are quadratic-like maps with filled Julia sets K_1^* , K_2^* , both containing the non-repelling fixed point a . Observe that, by definition, the map $f : U_1 \cap U_2 \rightarrow f(U_1 \cap U_2)$ is proper. Let U be the component of $U_1 \cap U_2$ containing a . Then $f : U \rightarrow f(U)$ is also proper. Moreover, by definition $f(\text{Bd}(U)) \cap \bar{U} = \emptyset$. Since $f(a) = a$, it follows from the Maximum Modulus Principle that $\bar{U} \subset f(U)$. By Theorem 7.1 it follows that $f|_U$ is quadratic-like and that $K_1^* = K_2^*$ as desired. \square

To define the quadratic invariant gap $\mathfrak{U}(f)$ associated with $K(f^*)$, we use (pre)periodic points of f . Observe that, by definition, $K(f^*)$ is a component of $f^{-1}(K(f^*))$.

Definition 7.3. Let $\widehat{X}(f) = \widehat{X}$ be the set of all σ_3 -(pre)periodic points $\bar{\alpha} \in \mathbb{S}^1$ such that $R_f(\alpha)$ lands in $K(f^*)$. Let $X(f) = X$ be the closure of \widehat{X} . Let $\mathfrak{U}(f)$ be the convex hull of X . Let $\widetilde{K}(f^*) \neq K(f^*)$ be a component of $f^{-1}(K(f^*))$ ($\widetilde{K}(f^*)$ exists because $f|_{K(f^*)}$ is two-to-one). Let $Y(f) = Y$ be the closure of the set of all preperiodic points $\bar{\alpha} \in \mathbb{S}^1$ with $R_f(\alpha)$ landing in $\widetilde{K}(f^*)$.

By Lemma 7.2, the sets introduced in Definition 7.3 (such as $\widehat{X}(f)$, $\mathfrak{U}(f)$ etc.) are *well-defined* for a given f as long as f is immediately renormalizable. The gap $\mathfrak{U}(f)$ is the combinatorial counterpart of the set $K(f^*)$. Clearly, $X \neq \emptyset$ and the map $\sigma_3 : X \rightarrow X$ is such that any point of X is the image of two or three different points of X . Denote the length of any circle arc T by $|T|$. The lengths of circle arcs are measured in radians/ 2π so that the total length of the unit circle is equal to 1.

Lemma 7.4. *The map $\sigma_3|_{\widehat{X}}$ is exactly two-to-one. Moreover, Y lies in the closure of a hole $I = (\bar{\alpha}, \bar{\beta})$ of X .*

Proof. The set Y lies in the closure $[\bar{\alpha}, \bar{\beta}]$ of a hole $I = (\bar{\alpha}, \bar{\beta})$ of X as otherwise $K(f^*)$ and $\widetilde{K}(f^*)$ are non-disjoint. Let $\bar{\theta} \in \widehat{X}$; let $z \in K(f^*)$ be the landing point of $R_f(\theta)$. If z is critical, we may assume that $R_f(\theta + 1/3)$ lands at z , and so $\overline{\theta + 1/3} \in \widehat{X}$; if z is not critical, we can find a point $z' \in K(f^*)$ with the same image as z and may assume that $R_f(\theta + 1/3)$ lands at z' , and hence $\overline{\theta + 1/3} \in \widehat{X}$. Thus the map $\sigma_3|_{\widehat{X}}$ is at least two-to-one. Now, suppose that $\theta, \theta + 1/3, \theta + 2/3 \in \widehat{X}$. Let $z \in K(f^*)$ be the landing point of $R_f(\theta)$. Choose a point $\tilde{z} \in \widetilde{K}(f^*)$ with the same image as z ; then there exists $\bar{\theta} \in Y$ such that $R_f(\bar{\theta})$ lands at \tilde{z} . Thus, $\bar{\theta} \notin \widehat{X}$ whereas $\bar{\theta}$ must coincide with either $\overline{\theta + 1/3}$ or $\overline{\theta + 2/3}$, a contradiction. \square

From now on, in the setting when f is immediately renormalizable, $I = (\bar{\alpha}, \bar{\beta})$ is the hole of X whose closure $[\bar{\alpha}, \bar{\beta}]$ contains the set Y .

Lemma 7.5. *The set $\widetilde{K}(f^*)$ is disjoint from U^* . The gap $\mathfrak{U}(f)$ is an invariant quadratic gap of regular critical or periodic type, and I is its major hole.*

Proof. The set $\widetilde{K}(f^*)$ is disjoint from U^* because otherwise points of $\widetilde{K}(f^*) \cap U^*$ belong to $K(f^*)$, a contradiction. Suppose that $|I| < 1/3$. If $\overline{\alpha} \in \widehat{X}$, then points $\overline{\alpha + 1/3}, \overline{\alpha + 2/3}$ belong to $(\overline{\beta}, \overline{\alpha})$ and cannot belong to $Y \subset [\overline{\alpha}, \overline{\beta}]$. Hence in this case three points in \widehat{X} have the same σ_3 -image, a contradiction with Lemma 7.4. If $\overline{\alpha} \notin \widehat{X}$, then we can find a point $\overline{\theta} \in \widehat{X}$ so close to α that points $\overline{\theta}, \overline{\theta + 1/3}, \overline{\theta + 2/3}$ belong to $(\overline{\beta}, \overline{\alpha})$, which similarly leads to a contradiction. Hence $|I| \geq 1/3$.

We claim that if $|I| = 1/3$, then neither $\overline{\alpha}$ nor $\overline{\beta}$ is periodic. Otherwise let $\overline{\alpha}$ be periodic and set $\overline{\alpha\beta} = \mathbf{c}$. Since X is invariant, $X \subset \mathcal{U}'(\mathbf{c})$. By [BOPT16] (see Subsection 2.3), the gap $\mathcal{U}(\mathbf{c})$ is of caterpillar type, $\overline{\beta}$ is isolated in $\mathcal{U}'(\mathbf{c})$ and in X . Thus, $\overline{\beta} \in \widehat{X}$, and so $\overline{\alpha} \in \widehat{X}$ by definition and because an eventual σ_3 -image of $\overline{\beta}$ equals $\overline{\alpha}$. Since $\overline{\alpha + 2/3} \in (\overline{\beta}, \overline{\alpha})$, then $\overline{\alpha + 2/3} \notin Y$; hence the landing point of $R_f(\alpha + 2/3)$ belongs to $K(f^*)$. Therefore, $\overline{\alpha + 2/3} \in \widehat{X}$, a contradiction with Lemma 7.4. We conclude that if $|I| = 1/3$, then $\mathcal{U}(f)$ is of regular critical type.

Assume that $|I| > 1/3$. Choose a critical chord \mathbf{c} with endpoints in $[\overline{\alpha}, \overline{\beta}]$; clearly, we can always choose \mathbf{c} to be a critical chord with no periodic endpoint. Then $X \subset \mathcal{U}'(\mathbf{c})$, where $\mathcal{U}(\mathbf{c})$ is the invariant quadratic gap defined in Subsection 2.3; by [BOPT16], the set $\mathcal{U}'(\mathbf{c})$ has no isolated points, and, for any point $\overline{\gamma} \in \mathcal{U}'(\mathbf{c})$, the backward orbit of $\overline{\gamma}$ in $\mathcal{U}'(\mathbf{c})$ is dense in $\mathcal{U}'(\mathbf{c})$ (see Subsection 2.3 for details). Hence $X = \mathcal{U}'(\mathbf{c})$ is a quadratic invariant gap of periodic type as desired. \square

If $f \in \mathcal{F}_\lambda, |\lambda| \leq 1$ has a disconnected Julia set then by Theorem 2.5, the map f is immediately renormalizable, and $\mathcal{U}(f)$ is defined. Also, results of Subsection 4.1 apply to f . We now relate $\mathcal{U}(f)$ and the gaps introduced in Subsection 4.1.

Proposition 7.6. *If $f \in \mathcal{F}_\lambda, |\lambda| \leq 1$ has a disconnected Julia set and f belongs to a parameter ray $\mathcal{R}_\lambda(\mathbf{x})$ then $\mathcal{U}(f) = \mathcal{U}_p(\mathbf{c}_\mathbf{x})$. Thus, if $\overline{\alpha\beta}$ is an edge of $\mathcal{U}(f)$, then the principal sets of $R^e(\alpha-)$ and $R^e(\beta+)$ intersect. Moreover, if α and β are periodic, then the rays $R^e(\alpha-)$ and $R^e(\beta+)$ land at the same point of $K(f^*)$.*

Proof. By Proposition 4.2, we have $\mathcal{U}(f) \subset \mathcal{U}_p(\mathbf{c}_\mathbf{x})$. Since both gaps in question are invariant quadratic gaps not of caterpillar type, it follows by Lemma 2.12 that $\mathcal{U}(f) = \mathcal{U}_p(\mathbf{c}_\mathbf{x})$. Then the remaining claims follow from Proposition 4.3. \square

In the rest of this section, we suppose that $J(f)$ is connected. For an edge $\overline{\alpha\beta}$ of $\mathcal{U}(f)$, we let $\Delta(\alpha, \beta)$ stand for a closed set $\text{Imp}(\alpha-) \cup \text{Imp}(\beta+) \cup R_f^e(\alpha-) \cup R_f^e(\beta+)$.

Theorem 7.7 [BOPT17] is similar to Theorem 2.9 but applies in the case when the Julia set is connected. Given a set $T \subset \mathbb{S}^1$ and a map $h : T \rightarrow \mathbb{S}^1$ we say that h is *monotone extendable* if it has a monotone extension $m : \mathbb{S}^1 \rightarrow \mathbb{S}^1$.

Theorem 7.7 (Lemma 4.9, [BOPT17]). *Let a polynomial f of degree d have a connected Julia set and $K(f^*) \subset K(f)$ be a polynomial-like connected filled Julia set of degree k . Let $\mathcal{B}_f(K(f^*)) = \mathcal{B}$ be the set of all angles $\theta \in \mathbb{S}^1$ such that the external ray $R_f^e(\theta)$ lands in $J(f^*)$. Then there is a monotone extendable continuous map $\psi : \mathcal{B} \rightarrow \mathbb{S}^1$ such that:*

- (1) *the set of all (pre)periodic angles from $\psi(\mathcal{B})$ is dense in \mathbb{S}^1 ;*
- (2) *for every $\theta \in \mathcal{B}$, the polynomial-like ray $R^*(\psi(\theta))$ lands at the same point y_θ as $R_f^e(\theta)$ and is isotopic to $R_f^e(\theta)$ rel. $K(f^*)$;*
- (3) *we have $\psi \circ \sigma_d(\theta) = \sigma_k \circ \psi(\theta)$ for every $\theta \in \mathcal{B}$.*

By Theorem 7.7, the ψ -images of σ_d -(pre)periodic points of \mathcal{B} are σ_k -(pre)periodic. Since for any connected polynomial Julia set J a dense set in \mathbb{S}^1 of (pre)periodic

angles gives rise to a dense set in J of their landing points, it follows that the set of landing points of all (pre)periodic angles from \mathcal{B} is dense in $J(f^*)$.

Unshielded compacta A and their topological hulls $\text{Th}(A)$ were defined in Subsection 2.2. The topological hull of a compact unshielded subset of the plane is the union of this subset with all bounded complementary domains. Thus, the topological hull of a polynomial Julia set $J(P)$ is the filled Julia set $K(P)$.

Lemma 7.8. *If $A, B \subset \mathbb{C}$ are continua such that A is disjoint from bounded complementary domains of B and B is disjoint from bounded complementary domains of A , then $\text{Th}(A \cap B) = \text{Th}(A) \cap \text{Th}(B)$.*

Proof. (1) Since $A \cap B \subset A$, then $\text{Th}(A \cap B) \subset \text{Th}(A)$; similarly, $\text{Th}(A \cap B) \subset \text{Th}(B)$. Thus, $\text{Th}(A \cap B) \subset \text{Th}(A) \cap \text{Th}(B)$.

(2) BWOC, choose $x \in \text{Th}(A) \cap \text{Th}(B) \setminus \text{Th}(A \cap B)$. Connect x and infinity with a curve $S = \psi([0, \infty)) \subset \mathbb{C} \setminus (A \cap B)$ (here $\psi(0) = x$ and ψ is a continuous injective map). Set τ_A to be the least number with $\psi(t) \in \text{Th}(A) \forall t \in [0, \tau_A]$ while there are $t > \tau_A$ arbitrarily close to τ_A with $\psi(t) \notin \text{Th}(A)$; then $\psi(\tau_A) \in A$. Define τ_B similarly. If $\tau_A < \tau_B$, then $\psi(\tau_A) \in \text{Th}(B)$ while $\psi(\tau_A) \notin B$ (for $S \cap A \cap B = \emptyset$). Hence $\psi(\tau_A)$ belongs to a bounded complementary domain of B , a contradiction. Thus, $\tau_A < \tau_B$ fails; analogously, $\tau_B < \tau_A$ fails. Hence $\tau_A = \tau_B = \tau$, and, hence, $\psi(\tau)$ is a point of S which belongs to $A \cap B$, a contradiction. \square

Proposition 7.9 is similar to Proposition 7.6, but deals with connected Julia sets.

Proposition 7.9. *If $\overline{\alpha\beta}$ is an edge of $\mathfrak{U}(f)$, then $\text{Imp}(\alpha-) \cap \text{Imp}(\beta+) \cap J(f^*)$ is non-empty. Moreover, either $J(f^*) \subset \text{Imp}(\alpha-) \cup \text{Imp}(\beta+)$ or, otherwise, $J(f^*) \setminus [\text{Imp}(\alpha-) \cup \text{Imp}(\beta+)]$ is contained in the unbounded component of $\mathbb{C} \setminus \Delta(\alpha, \beta)$ containing external rays with arguments from (β, α) .*

Recall that one-sided impressions are continua.

Proof. Clearly, any bounded complementary component of $J(f^*)$ (or any bounded complementary component of $\text{Imp}(\alpha-)$) is a Fatou component of f . Therefore, bounded complementary components of either $J(f^*)$ or $\text{Imp}(\alpha-)$ are disjoint from $J(f^*) \cup \text{Imp}(\alpha-)$. Moreover, if a bounded complementary domain of $\text{Imp}(\alpha-)$ and a bounded complementary domain of $J(f^*)$ are non-disjoint, then they coincide.

Set $T = \text{Imp}(\alpha-) \cap J(f^*)$. By Lemma 7.8, the topological hull $\text{Th}(T)$ of T coincides with the intersection of the topological hull $\text{Th}(\text{Imp}(\alpha-))$ of $\text{Imp}(\alpha-)$, and $K(f^*)$. We claim that T is a continuum. We will use Theorem 7.7, including the notation from that theorem. If T is disconnected, then $\text{Th}(T)$ is disconnected too. Divide the components of $\text{Th}(T)$ in two groups whose unions A and B are disjoint non-separating compact sets. Choose open Jordan disks $U \supset A$ and $V \supset B$ such that \overline{U} and \overline{V} are disjoint closed Jordan disks. Then, by Moore's theorem [Dav86], a map Ψ that collapses \overline{U} and \overline{V} maps \mathbb{C} to a space homeomorphic to the plane, in which the continuum $\Psi(\text{Th}(\text{Imp}(\alpha-)))$ and the continuum $\Psi(K(f^*))$ intersect over a two-point set. By Theorem 61.4 from [Mun00], this implies that $\Psi(\text{Th}(\text{Imp}(\alpha-))) \cup \Psi(K(f^*))$ is a separating continuum, which implies that $\text{Th}(\text{Imp}(\alpha-)) \cup K(f^*)$ is a separating continuum.

Since both $\text{Th}(\text{Imp}(\alpha-))$ and $K(f^*)$ are non-separating, there exists a bounded complementary component W of $\text{Th}(\text{Imp}(\alpha-)) \cup K(f^*)$ and an open (in relative topology of $\text{Bd}(W)$) set $E \subset \text{Bd}(W) \cap K(f^*)$. Then, by Theorem 7.7, we can find a (pre)periodic angle $\gamma \in \mathcal{B}$ such that the (pre)periodic polynomial-like ray $R^*(\psi(\gamma))$

has a terminal interval that is contained in W , lands in E , and is isotopic to the external ray $R_f^e(\gamma)$ rel. $K(f^*)$, a contradiction (recall that W is a Fatou domain of f). Thus, T is connected. Similarly, $\text{Imp}(\beta+) \cap J(f^*)$ is a continuum.

By way of contradiction, assume that the continua $\text{Imp}(\alpha-) \cap J(f^*)$ and $\text{Imp}(\beta+) \cap J(f^*)$ are disjoint. By choosing U^* sufficiently tight around $K(f^*)$ we may assume that there are Jordan disks V_α, V_β with

$$\text{Imp}(\alpha-) \cap J(f^*) \subset V_\alpha \subset \overline{V}_\alpha \subset U^*, \quad \text{Imp}(\beta+) \cap J(f^*) \subset V_\beta \subset \overline{V}_\beta \subset U^*$$

such that $\overline{V}_\alpha \cap \overline{V}_\beta = \emptyset$. By Theorem 7.7, we can find a pair of (pre)periodic angles $\psi(\gamma), \psi(\eta)$ in $\psi(\mathcal{B})$ such that $R^*(\psi(\gamma)) \cup R^*(\psi(\eta)) \cup J(f^*)$ separates $\overline{V}_\alpha \setminus K(f^*)$ from $\overline{V}_\beta \setminus K(f^*)$. Then $R_f(\gamma) \cup R_f(\eta) \cup J(f^*)$ also separates $\overline{V}_\alpha \setminus K(f^*)$ from $\overline{V}_\beta \setminus K(f^*)$. It follows that $\{\gamma, \eta\}$ separates $\{\alpha, \beta\}$ in \mathbb{R}/\mathbb{Z} . Thus γ or η belongs to (α, β) ; assume that it is γ . The ray $R_f(\gamma)$ is (pre)periodic and lands in $J(f^*)$, a contradiction with the definition of $\mathfrak{U}(f)$. So, $\text{Imp}(\alpha-) \cap \text{Imp}(\beta+) \cap J(f^*) \neq \emptyset$.

Let us prove the remaining claims. If $J(f^*)$ has any points outside of $\text{Imp}(\alpha-) \cup \text{Imp}(\beta+)$, then, by definition of $\mathfrak{U}(f)$, these points cannot belong to the unbounded component of $\mathbb{C} \setminus \Delta(\alpha, \beta)$ that contains external rays with arguments from (α, β) . On the other hand, no point of $J(f^*)$ can belong to a bounded component of $\mathbb{C} \setminus \Delta(\alpha, \beta)$ because otherwise some points of $J(f^*)$ do not belong to the closure of $\mathbb{C} \setminus K(f)$. Hence $J(f^*) \setminus (\text{Imp}(\alpha-) \cup \text{Imp}(\beta+))$ is contained in the unbounded component of $\mathbb{C} \setminus \Delta(\alpha, \beta)$ containing external rays with arguments from (β, α) . \square

8. CUTS AND THE QUADRATIC INVARIANT GAP $\mathfrak{U}(f)$

By a *strict [(pre)periodic] cut* $\Gamma_x(\overline{\alpha\beta})$ we mean two external rays with arguments α and β landing at the same [(pre)periodic] point x union the singleton $\{x\}$; then x is said to be the *vertex* of $\Gamma_x(\overline{\alpha\beta})$.

Lemma 8.1 ([BOPT16]). *Let \mathfrak{U} be a quadratic invariant gap of regular critical type. Then, for every chord $\overline{\alpha\beta}$ of $\overline{\mathbb{D}}$ such that $\sigma_3^{on}(\overline{\alpha\beta})$ neither crosses nor coincides with edges of \mathfrak{U} for all n , the chord $\overline{\alpha\beta}$ is eventually mapped to a chord inside \mathfrak{U} .*

Definition 8.2 (cf. [BOPT16]). Let \mathfrak{U} be a quadratic invariant gap of periodic type. Let $\overline{(\theta_1 + 1/3)(\theta_2 + 2/3)} = \mathfrak{M}$ be its major; assume that it is of period k with sibling $\overline{(\theta_2 + 1/3)(\theta_1 + 2/3)} = \mathfrak{M}^*$. Let \mathfrak{V}' be the set of all points $\overline{\alpha}$ of the circle such that, for every n , we have $3^n\alpha \in [3^n(\theta_1 + 1/3), 3^n(\theta_2 + 2/3)]$, and let \mathfrak{V} be the convex hull of \mathfrak{V}' . In this setting, \mathfrak{U} is called the *senior gap* while \mathfrak{V} is called the *vassal gap* (of \mathfrak{U}). If $\mathfrak{U} = \mathfrak{U}(f)$ is defined by an immediately renormalizable polynomial $f \in \mathcal{C}_\lambda$, we also consider the continuum $I(f) = I = \text{Imp}(\theta_1 + 1/3-) \cup \text{Imp}(\theta_2 + 2/3+)$ (the fact that I is a continuum follows from Proposition 7.9).

The relation between senior and vassal gaps is described in Theorem 8.3. Recall that, given a lamination \mathcal{L} , the map σ_3 extends to all leaves of \mathcal{L} so that leaves are mapped to leaves, possibly degenerate. As in [Thu85], we use the *barycentric construction* and extend σ_3 over all gaps of \mathcal{L} so that gaps are mapped to gaps or (possibly degenerate) leaves and the extension is a continuous self-map of $\overline{\mathbb{D}}$.

Theorem 8.3 ([BOPT16]). *There is a unique invariant laminational equivalence relation $\sim_{\mathfrak{U}}$ with a given senior gap \mathfrak{U} of periodic type and such that the vassal gap $\text{CH}(\mathfrak{V}') = \mathfrak{V}$ is a gap of $\sim_{\mathfrak{U}}$. If the major \mathfrak{M} of \mathfrak{U} is of period k , then the gap \mathfrak{V} is of period k too. Moreover, under σ_3^k the gap \mathfrak{V} maps two-to-one onto itself, all other*

gaps of $\sim_{\mathfrak{U}}$ are one-to-one pullbacks of \mathfrak{U} or \mathfrak{V} and all leaves of the corresponding lamination $\mathcal{L}_{\mathfrak{U}}$ are edges of these gaps. Also, there are no points of period k located between the major \mathfrak{M} and its sibling \mathfrak{M}^* except for the endpoints of \mathfrak{M} .

We will need the following consequences of Theorem 8.3.

Corollary 8.4. *If $f \in \mathcal{F}_\lambda$ with $|\lambda| \leq 1$ is immediately renormalizable, then:*

- (1) *there is no strict (pre)-periodic cut $\Gamma_x(\overline{\alpha\beta})$ with vertex x such that $\overline{\alpha\beta}$ crosses an edge of $\mathfrak{U}(f)$;*
- (2) *if there exists a strict periodic cut with vertex $y \neq 0$, then $\mathfrak{U}(f)$ is of periodic type;*
- (3) *no periodic ray with argument $\alpha \in (\theta_1 + 1/3, \theta_2 + 2/3)$ lands at a landing point of $R_f(\theta_1 + 1/3)$ or $R_f(\theta_2 + 2/3)$;*
- (4) *for any strict periodic cut $\Gamma_x(\overline{\alpha\beta})$ the chord $\overline{\alpha\beta}$ does not cross the major \mathfrak{M} of $\mathfrak{U}(f)$ or its siblings.*

Proof. (1) Suppose that there exists a cut $\Gamma_x(\overline{\alpha\beta})$ with (pre)periodic vertex x such that $\overline{\alpha\beta}$ crosses an edge of $\mathfrak{U}(f)$. Then, by definition, there are points of $J(f^*)$ in each component of $\mathbb{C} \setminus \Gamma_x(\overline{\alpha\beta})$, which implies that $x \in J(f^*)$, and hence $\overline{\alpha}, \overline{\beta} \in \widehat{X} \subset \mathfrak{U}(f)$, a contradiction (indeed, by the assumption at least one of $\overline{\alpha}, \overline{\beta}$ must not belong to $\mathfrak{U}(f)$).

(2) Suppose now that $y \neq 0$ is the vertex of a periodic cut and that $\mathfrak{U}(f)$ is of regular critical type. Let \mathfrak{h} be the convex hull of the finite set $\mathfrak{h}' \subset \mathbb{S}^1$ of points $\overline{\gamma}$, where γ are arguments of rays landing at y . By the previous paragraph, images of \mathfrak{h} do not cross edges of $\mathfrak{U}(f)$. Then, by Lemma 8.1, the set \mathfrak{h} is eventually mapped inside $\mathfrak{U}(f)$; since \mathfrak{h} is periodic we may assume that $\mathfrak{h} \subset \mathfrak{U}(f)$. Since $\mathfrak{U}(f)$ is regular critical, then there are angles from $\mathfrak{U}(f)$ on either side of \mathfrak{h} which implies that $y \neq 0$ is a periodic cutpoint of $J(f^*)$. However, this is impossible since $|\lambda| \leq 1$ and, hence, there are no periodic cutpoints of $J(f^*)$ different from 0. This proves (2).

(3) Now, suppose that $\Gamma_x(\overline{\alpha\beta})$ is a strict periodic cut with $\beta \in \{\theta_1 + 1/3, \theta_2 + 2/3\}$ and $\alpha \in (\theta_1 + 1/3, \theta_2 + 2/3)$. Then β must be of the same period m as α . By Theorem 8.3, the point $\overline{\alpha}$ cannot be located between \mathfrak{M} , the major of $\mathfrak{U}(f)$, and \mathfrak{M}^* . Hence $\ell = \overline{\alpha\beta}$ must cross \mathfrak{M}^* . This implies that either $\sigma_3(\ell) \subset \mathfrak{U}(f)$, and all leaves from the orbit ℓ are contained in $\mathfrak{U}(f)$, a contradiction with ℓ eventually mapped onto itself (ℓ is periodic!), or $\sigma_3(\ell)$ crosses an edge of $\mathfrak{U}(f)$, a contradiction with (1).

(4) Let $\Gamma_x(\overline{\alpha\beta})$ be a strict periodic cut with vertex x ; set $\ell = \overline{\alpha\beta}$. Then, by (2), the gap $\mathfrak{U}(f)$ is of periodic type. By (1), the chord ℓ cannot cross \mathfrak{M} or its sibling which is an edge of $\mathfrak{U}(f)$. Suppose that ℓ crosses \mathfrak{M}^* . If ℓ is disjoint from $\mathfrak{U}(f)$, then, by Lemma 2.11, the image $\sigma_3(\ell)$ crosses an edge of $\mathfrak{U}(f)$, a contradiction. Otherwise an endpoint of ℓ must coincide with an endpoint of \mathfrak{M} and, by (3), the chord ℓ does not cross \mathfrak{M}^* either. □

Theorem 8.3 implies Corollary 8.5 which uses the notation from Theorem 8.3.

Corollary 8.5. *Let \mathfrak{U} be a senior gap of periodic type with vassal gap \mathfrak{V} . Let ℓ be a chord. Then one of the following holds.*

- (1) *An eventual σ_3 -image of ℓ crosses an edge of \mathfrak{U} .*
- (2) *An eventual σ_3 -image of ℓ is contained in \mathfrak{U} .*
- (3) *An eventual σ_3 -image of ℓ separates \mathfrak{M} from \mathfrak{M}^* in \mathbb{D} .*

Proof. Set $\ell = \overline{\alpha\beta}$. Assume that neither (1) nor (2) holds. If an eventual image of ℓ intersects \mathfrak{U} , then this image is a chord having a common endpoint with an edge of \mathfrak{U} but otherwise located outside \mathfrak{U} . Suppose that $\mathfrak{M} = \overline{(\theta_1 + 1/3)(\theta_2 + 2/3)}$ is of period k . By Theorem 8.3, we may assume that $\alpha = \theta_1 + 1/3$. Let us discuss possible locations of the point β . Since we assume that neither (1) nor (2) holds then $\beta \in [\theta_2 + 2/3, \theta_1 + 1/3]$ is impossible. If $\beta \in [\theta_1 + 2/3, \theta_2 + 2/3]$, then ℓ separates \mathfrak{M} from \mathfrak{M}^* in \mathbb{D} and (3) holds. If $\beta \in (\theta_1 + 1/3, \theta_2 + 1/3]$, then we multiply β by 3^k . Observe that multiplication by 3^k maps $[\theta_1 + 1/3, \theta_2 + 1/3]$ monotonically (with constant expansion factor) onto $[\theta_1 + 1/3, \theta_2 + 2/3]$. Hence $3^{ks}\beta \in [\theta_2 + 1/3, \theta_2 + 2/3]$ for some s . If $3^{ks}\beta \in [\theta_1 + 2/3, \theta_2 + 2/3]$, we are done by the above. Hence we may assume that $3^{ks}\beta \in (\theta_2 + 1/3, \theta_1 + 2/3)$. Set $\sigma_3^{\circ ks}(\ell) = \tilde{\ell}$. It follows that $\tilde{\ell}$ will have an endpoint $\theta_1 + \frac{1}{3}$ and the other endpoint in $(\theta_2 + 1/3, \theta_1 + 2/3)$. Since $\sigma_3^{\circ k}[\theta_2 + 1/3, \theta_1 + 2/3] = [\theta_2 + 2/3, \theta_1 + 1/3]$, this implies that either (1) or (2) must hold, a contradiction. Thus we may assume that eventual images of ℓ are disjoint from \mathfrak{U} .

We claim that then (3) holds. Indeed, suppose first that ℓ is eventually mapped inside \mathfrak{V} . Then the claim follows from the fact that $\sigma_3^k|_{\mathfrak{V}}$ is semiconjugate to σ_2 and the well-known (and easy) fact that an eventual σ_2 -image of any non-degenerate chord separates 0 and $1/2$ or contains 0. Suppose that ℓ is never mapped inside \mathfrak{V} . Combining this and the fact that all iterated σ_3 -images of ℓ are disjoint from \mathfrak{U} we see that ℓ crosses a leaf ℓ_1 of $\mathcal{L}_{\mathfrak{U}}$. Suppose that no image of ℓ separates \mathfrak{M} and \mathfrak{M}^* . Then for every n we have that $\sigma_3^{\circ n}(\ell)$ and $\sigma_3^{\circ n}(\ell_1)$ are two chords located in the part of the unit disk cut off the disk by an edge \mathfrak{N} of \mathfrak{U} . If \mathfrak{N} is not a major, then the length of the hole of \mathfrak{U} behind \mathfrak{N} is less than $1/3$, and hence the images of $\sigma_3^{\circ n}(\ell)$ and $\sigma_3^{\circ n}(\ell_1)$ are still crossing. Now, suppose that $\mathfrak{N} = \mathfrak{M}$. Moreover, suppose that $\sigma_3^{\circ n}(\ell)$ does not separate \mathfrak{M} from \mathfrak{M}^* in \mathbb{D} . Then either leaves $\sigma_3^{\circ n}(\ell)$ and $\sigma_3^{\circ n}(\ell_1)$ have endpoints in $[\theta_1 + 1/3, \theta_1 + 2/3]$ or leaves $\sigma_3^{\circ n}(\ell)$ and $\sigma_3^{\circ n}(\ell_1)$ have endpoints in $[\theta_2 + 1/3, \theta_2 + 2/3]$. In both cases, again, images of $\sigma_3^{\circ n}(\ell)$ and $\sigma_3^{\circ n}(\ell_1)$ are still crossing. This implies that all the images of ℓ continue crossing the corresponding images of ℓ_1 . However, ℓ_1 is eventually mapped to \mathfrak{M} , a contradiction. \square

9. PROPERTIES OF POLYNOMIALS FROM $\mathcal{C}_\lambda \setminus \mathcal{CU}_\lambda$

We need the *Separation Lemma* of Jan Kiwi [Kiw00] inspired by [GM93] (we give a convenient version while a stronger statement is proved in [Kiw00]).

Lemma 9.1 (Separation Lemma, [Kiw00]). *Suppose that f is a polynomial with connected Julia set. Then there is a finite collection of periodic dynamic rays for f (landing at periodic repelling or parabolic points) such that the closure of the union of these rays divides the plane into parts with the following property: every part contains at most one non-repelling periodic point of f or one parabolic basin of f .*

A cut formed by periodic dynamic rays from Lemma 9.1 is called a *K-cut*. Let Υ be the closure of the union of the periodic rays from Lemma 9.1; let $x \neq y$ be periodic non-repelling points of f ; let Γ^K be a K-cut. Say that Γ^K *separates domains at x from y* if Γ^K separates x and y and, if x is attracting or parabolic, Γ^K separates *all* attracting or parabolic domains at x from y simultaneously. A *cut* was defined right after Proposition 4.3. Recall that by a *strict cut (with vertex y)* we mean the union of two external rays landing at y , and the singleton $\{y\}$.

Lemma 9.2. *Suppose that $f \in \mathcal{F}_\lambda$ is immediately renormalizable, the gap $\mathfrak{U}(f) = \mathfrak{U}$ is of periodic type with major $\mathfrak{M} = \overline{(\theta_1 + 1/3)(\theta_2 + 2/3)}$ of period k , and, for some strict periodic cut $\Gamma_x(\overline{\alpha_1\beta_1})$ with vertex $x \neq 0$, $\overline{\alpha_1\beta_1}$ does not belong to the grand orbit of \mathfrak{M} . Then there is a cut $\Gamma_y(\overline{\alpha_2\beta_2})$, an eventual pullback of $\Gamma_x(\overline{\alpha_1\beta_1})$, such that $\overline{\alpha_2\beta_2}$ is separated from \mathfrak{U} by \mathfrak{M} and $\sigma_3^{ok}(\overline{\alpha_2\beta_2})$ is separated from \mathfrak{U} by $\overline{\alpha_2\beta_2}$.*

Proof. Put $\overline{\alpha_i\beta_i} = \ell_i$. Suppose that no image of ℓ_1 separates \mathfrak{M} and \mathfrak{M}^* . Let us show that all images of ℓ_1 are disjoint from \mathfrak{U} . Assume otherwise. Then by Corollary 8.4, the iterated images of ℓ_1 do not cross edges of \mathfrak{U} . Hence an eventual image ℓ'_1 of ℓ_1 has either $\theta_1 + 1/3$ or $\theta_2 + 2/3$ as its endpoint. On the other hand, by Corollary 8.4, the chord ℓ'_1 cannot cross \mathfrak{M}^* , the sibling of \mathfrak{M} disjoint from \mathfrak{U} . By Theorem 8.3, except for $\theta_1 + 1/3$ or $\theta_2 + 2/3$, there are no other periodic points of period k between \mathfrak{M} and \mathfrak{M}^* . Hence $\ell'_1 = \mathfrak{M}$, a contradiction.

Since ℓ_1 and all its images are disjoint from \mathfrak{U} , then, by Corollary 8.5, there is an image of ℓ_1 separating \mathfrak{M} and \mathfrak{M}^* . We may assume that ℓ_1 separates \mathfrak{M} and \mathfrak{M}^* . Recall that k is the period of \mathfrak{M} . Let $\overline{\alpha_1}$ and $\overline{\beta_1}$ be the endpoints of ℓ_1 chosen so that $\theta_1 + \frac{1}{3} < \alpha_1 < \beta_1 < \theta_2 + \frac{2}{3}$. Since both arcs of \mathbb{S}^1 between \mathfrak{M} and \mathfrak{M}^* are mapped onto the major hole of \mathfrak{M} homeomorphically under σ_3^{ok} , there is a unique angle α_2 in $(\theta_1 + 1/3, \alpha_1)$ with $3^k\alpha_2 = \alpha_1$, and there is a unique angle β_2 in $(\beta_1, \theta_2 + 2/3)$ with $3^k\beta_2 = \beta_1$. Consider the chord $\ell_2 = \overline{\alpha_2\beta_2}$; then we have $\sigma_3^{ok}(\ell_2) = \ell_1$. By Corollary 8.4, no pullback of ℓ_1 can cross an edge of $\mathfrak{U}(f)$. Hence the cut that maps onto $\Gamma_x(\overline{\alpha_1\beta_1})$ under f^{ok} and contains $R_f(\alpha_2)$ coincides with $\Gamma_y(\overline{\alpha_2\beta_2})$, and ℓ_2 is the desired pullback of ℓ_1 . □

Lemma 9.3. *Let $f \in \mathcal{F}_\lambda \setminus \mathcal{CU}_\lambda$ where $|\lambda| \leq 1$. Then f is immediately renormalizable. Suppose that there is one cycle of parabolic domains at 0 and two cycles of external rays landing at 0. Let \mathfrak{g} be the gap whose vertices have the same arguments as all external rays landing at 0. Then \mathfrak{g} is of type D. There exists a unique major $\mathfrak{M} = \overline{(\theta_1 + 1/3)(\theta_2 + 2/3)}$ of \mathfrak{g} such that the quadratic invariant gap defined by \mathfrak{M} coincides with $\mathfrak{U}(f)$.*

Proof. The map f is immediately renormalizable by Theorem 2.5; thus, $\mathfrak{U}(f)$ is well-defined. The fact that \mathfrak{g} is of type D is immediate. Moreover, by the assumptions 0 is parabolic. It follows that $\mathfrak{U}(f)$ contains arguments of all external (pre)periodic rays landing at points from the cycle of parabolic domains at 0. Denote by $\mathfrak{M} = \overline{(\theta_1 + 1/3)(\theta_2 + 2/3)}$ the major of \mathfrak{g} coming from the cycle of edges of \mathfrak{g} corresponding to the cycle of planar wedges at 0 that do not contain parabolic domains at 0. Then the wedge $W_f(\theta_1 + 1/3, \theta_2 + 2/3)$ contains no points of $J(f^*)$. By Lemma 2.12 the quadratic invariant gap defined by \mathfrak{M} is $\mathfrak{U}(f)$. □

In the situation of Lemma 9.3 denote the major \mathfrak{M} by $\mathfrak{D}(f)$.

Lemma 9.4. *Let $f \in \mathcal{F}_\lambda \setminus \mathcal{CU}_\lambda$, where $|\lambda| \leq 1$. Then f is immediately renormalizable and the following hold.*

- (1) *If f admits a strict periodic cut with vertex $x \neq 0$, then $\mathfrak{U}(f)$ is of periodic type with major $\mathfrak{M} = \overline{(\theta_1 + 1/3)(\theta_2 + 2/3)}$. The rays $R_f(\theta_1 + 1/3)$, $R_f(\theta_2 + 2/3)$ land at the same point $z_f \in K(f^*)$. Moreover, either $z_f \neq 0$ is repelling, or $z_f = 0$ is parabolic with exactly one cycle of parabolic Fatou domains at 0 and two cycles of external rays landing on 0.*

- (2) If f does not admit a strict periodic cut with vertex $y \neq 0$, then either $K(f)$ is disconnected and $\mathfrak{U}(f)$ is of regular critical type or 0 is parabolic with exactly one cycle of parabolic Fatou domains at 0 and two cycles of external rays landing on 0, and $\mathfrak{U}(f)$ is of periodic type with major $\mathfrak{D}(f)$.

Thus, if $J(f)$ is connected, then $\mathfrak{U}(f)$ is of periodic type with major \mathfrak{M} , and there exists a strict periodic cut $\Gamma_z(\mathfrak{M})$.

Proof. Since $f \in \mathcal{F}_\lambda \setminus \mathcal{CU}_\lambda$, then, by Theorem 2.5, the polynomial f is immediately renormalizable. Moreover, by definition $f \notin \mathcal{CU}_\lambda$ means that $J(f)$ is disconnected, or $J(f)$ is connected and there exists a strict periodic cut with repelling vertex, or there exists a periodic neutral point not equal to 0 at which the multiplier is not equal to 1. Taking this into account, consider cases (1) and (2) separately.

(1) Suppose that there exists a periodic cut $\Gamma_x(\overline{\alpha_1\beta_1})$ with vertex $x \neq 0$. Then, by Corollary 8.4, the gap $\mathfrak{U}(f)$ is of periodic type. Assume that $\mathfrak{M} = \overline{(\theta_1 + 1/3)(\theta_2 + 2/3)}$ is the major of $\mathfrak{U}(f)$. We claim that the rays $R_f(\theta_1 + 1/3)$, $R_f(\theta_2 + 2/3)$ land at the same point. By way of contradiction suppose otherwise. Then the cut $\Gamma_x(\overline{\alpha_1\beta_1})$ satisfies conditions of Lemma 9.2. This implies that there exists a cut $\Gamma_y(\overline{\alpha_2\beta_2})$ with properties from that lemma. Consider the first $k - 1$ iterated σ_3 -pullbacks of the chord ℓ_2 whose endpoints are located in holes of $\mathfrak{U}(f)$ behind the periodic edges. The corresponding cuts together with a suitably chosen equipotential bound a Jordan disk such that the restriction of f to this Jordan disk is a quadratic-like map. Let T be the corresponding quadratic-like Julia set. By Lemma 7.2 we have $T = J(f^*)$; by definition the landing point z_1 of $R_f(\theta_1 + 1/3)$ and the landing point z_2 of $R_f(\theta_2 + 2/3)$ belong to $J(f^*)$ (here f^* is the quadratic-like restriction of f).

Suppose that $z_1 \neq z_2$. Consider the component W of $\mathbb{C} \setminus [R_f(\theta_1 + 1/3) \cup R_f(\theta_2 + 2/3) \cup J(f^*)]$ containing external rays with arguments from $(\theta_1 + 1/3, \theta_2 + 2/3)$. Since $z_1 \neq z_2$, there are periodic points of $J(f^*)$ that belong to $\text{Bd}(W)$ and such that parts of quadratic-like rays near $K(f^*)$ are contained in W . Then, by Theorem 7.7 (or by Theorem 2.9 in the disconnected case), there are repelling periodic points $y \in J(f^*) \cap \text{Bd}(W)$ at which the corresponding external rays with arguments from $(\theta_1 + 1/3, \theta_2 + 2/3)$ land; these external rays are isotopic rel. $K(f^*)$ to certain quadratic-like rays. However, by definition of $\mathfrak{U}(f)$ (see Definition 7.3) all arguments of external rays landing at such periodic points belong to $(\theta_2 + 2/3, \theta_1 + 1/3)$, a contradiction. Thus $z_1 = z_2 = z_f \in K(f^*)$, as desired.

Let us prove the claims of the lemma concerning the point z_f . First assume that $z_f \neq 0$. Clearly, z_f is repelling (a quadratic-like Julia set cannot contain two cycles of non-repelling points) as stated in the lemma. On the other hand, if $z_f = 0$ then z_f is parabolic. We claim that there is exactly one cycle of parabolic domains at 0. Indeed, if there are two cycles of parabolic Fatou domains at 0, then there are no quadratic-like Julia sets containing 0, which implies that f is not immediately renormalizable, a contradiction. Thus, there is exactly one cycle, say, \mathcal{P} , of parabolic domains at 0. Since the orbits of $\theta_1 + 1/3$ and $\theta_2 + 2/3$ are distinct, then there are two cycles of external rays landing at 0. Hence there is exactly one cycle of parabolic Fatou domains at 0 and there are exactly two cycles of external rays landing on 0 as claimed.

(2) Assume now that there are no periodic cuts with non-zero vertex. Consider first the case when $J(f)$ is disconnected. We may assume that $\mathcal{U}(f)$ is of periodic type. Then, by Theorem 2.9, the rays $R_f(\theta_1 + 1/3)$ and $R_f(\theta_2 + 2/3)$ land at the same point, say, x . If $x \neq 0$, we get a contradiction with our assumptions. Hence $x = 0$ is parabolic. It follows that there are two cycles of rays landing at 0, that of $\theta_1 + 1/3$ and that of $\theta_2 + 2/3$. Thus, there are two cycles of wedges at 0, and only one of them contains a cycle of parabolic domains at 0 (if both do, then f cannot be immediately renormalizable, similarly to the above).

Now, suppose that $J(f)$ is connected. Since $f \notin \mathcal{CU}_\lambda$, there exists a non-repelling periodic point $y \neq 0$ of multiplier not equal to 1. If y were parabolic then a strict periodic cut with non-zero vertex would exist, a contradiction. Hence y is not parabolic. By Lemma 9.1 (Kiwi’s Separation Lemma) there are strict periodic cuts separating 0 (or possibly existing parabolic domains at 0) from points of the orbit of y . Since the only strict periodic cuts of f are cuts with vertex 0, then 0 is parabolic, and cuts at 0 separate parabolic domains at 0 from the points of the orbit of y . This is only possible if there are two cycles of external rays landing at 0 and forming two cycles of wedges at 0: one of them contains a cycle of parabolic domains at 0, and the other one contains the entire orbit of y . \square

10. NON-SPECIAL PARAMETER WAKES AND THEIR ROOTS

Let us introduce notation which is used throughout this section. Assume that (θ_1, θ_2) is a hole of \mathcal{Q} and fix it, together with $\lambda \in \mathbb{C}$, $|\lambda| \leq 1$; assume that the period of $\theta_1 + 1/3$ is m (this notation will be used throughout the section). We consider polynomials from \mathcal{F}_λ . Define the set $\widehat{\mathcal{W}} = \widehat{\mathcal{W}}_\lambda(\theta_1, \theta_2)$ of all polynomials such that the smooth *dynamic* rays $R_f(\theta_1 + 1/3)$ and $R_f(\theta_2 + 2/3)$ land at the same *repelling* periodic point z_f of f . Together with the point z_f these rays form a strict periodic cut $\Gamma_{z_f} = \Gamma_{z_f}(\overline{(\theta_1 + 1/3)(\theta_2 + 2/3)})$ of the plane. Define an open dynamic wedge $W_f(\theta_1 + 1/3, \theta_2 + 2/3) = W_f$ as the component of $\mathbb{C} \setminus \Gamma_{z_f}$ that contains external rays with arguments from the arc $H = (\theta_1 + 1/3, \theta_2 + 2/3)$. By Lemma 3.1, the set $\widehat{\mathcal{W}}$ is open. Denote $\overline{(\theta_1 + 1/3)(\theta_2 + 2/3)}$ by \mathfrak{M} ; the chord \mathfrak{M} is the major of a quadratic invariant gap $\mathfrak{U} = \mathfrak{U}_p$ of periodic type. These are mostly combinatorial notions; in this section we, in particular, will relate them to the previously introduced dynamical and analytic concepts defined for maps from $\widehat{\mathcal{W}}$. In this section we assume that $\widehat{\mathcal{W}} \neq \emptyset$.

Finally, we want to make an observation concerning the difference between the cases when $m = 1$ and $m > 1$. In the latter case \mathfrak{M} determines the quadratic invariant gap \mathfrak{U} with major \mathfrak{M} and other related objects. Thus, when proving certain statements, in the case when $m > 1$ we only need to consider one case. On the other hand, if $m = 1$ then $\mathfrak{M} = \overline{0(1/2)}$ may be associated with two distinct (but very similar) quadratic invariant gaps, FG_a and FG_b (see Subsection 2.3). However, otherwise in a lot of cases the arguments are not different. The proof of Corollary 10.1 gives a good example of what we mean.

Corollary 10.1. *Suppose that $g \in \widehat{\mathcal{W}} = \widehat{\mathcal{W}}_\lambda(\theta_1, \theta_2)$, where $|\lambda| \leq 1$, and the gap (θ_1, θ_2) of \mathcal{Q} corresponds to an invariant quadratic gap \mathfrak{U} with major $\mathfrak{M} = \overline{(\theta_1 + 1/3)(\theta_2 + 2/3)}$. Then g is immediately renormalizable. Moreover, if g has a strict periodic cut $\overline{\Gamma_x(\alpha\beta)}$ with vertex x , where $\alpha\beta$ is the major of a quadratic invariant gap, then $\alpha\beta = \mathfrak{M}$. In particular, the repelling periodic point x belongs*

to the quadratic-like filled Julia set $K(g^*)$. Moreover, if $m > 1$, then the gap $\mathfrak{U}(g)$ coincides with \mathfrak{U} , and if $m = 1$, then $\mathfrak{U} \in \{FG_a, FG_b\}$ and $\mathfrak{U}(g) \in \{FG_a, FG_b\}$.

Proof. Since $g \in \widehat{\mathcal{W}} = \widehat{\mathcal{W}}_\lambda(\theta_1, \theta_2)$, then g has a strict periodic cut $\Gamma_{z_g}(\mathfrak{M})$ with repelling vertex z_g . By Theorem 2.5, g is immediately renormalizable. Now let us prove the second claim of the corollary. Let us first assume that $m > 1$. Since $K(g^*)$ has no periodic cutpoints other than 0, then, it is easy to see by Lemma 2.12 that $K(g^*) \subset \mathbb{C} \setminus W_g$ (recall that $z_g \neq 0$ as z_g is repelling). It follows that $\mathfrak{U}(g) \subset \mathfrak{U}$. However, this implies $\mathfrak{U}(g) = \mathfrak{U}$ since the backward orbit of a point in $\mathfrak{U}(g)$ under the map $\sigma_3 : \mathfrak{U} \rightarrow \mathfrak{U}$ is dense in \mathfrak{U} .

Now consider two cases: $x \neq 0$ and $x = 0$.

Case $x \neq 0$. The chord ℓ cannot cross the interior of \mathfrak{U} since otherwise $x \neq 0$ would be a periodic cutpoint of $K(g^*)$, and $K(g^*)$ has no periodic cutpoints other than 0. Therefore, $\mathfrak{U} \subset \mathfrak{U}(\ell)$, which again implies that $\mathfrak{U} = \mathfrak{U}(\ell)$ and, hence, $\ell = \mathfrak{M}$.

Case $x = 0$. The invariant gap \mathfrak{g} whose vertices have the same arguments as rays landing at 0 is contained in \mathfrak{U} (indeed, all rays landing at 0 stay on the same side of $\Gamma_g(\mathfrak{M})$, and since this is the same side on which parabolic domains at 0 are located, it follows that $\mathfrak{g} \subset \mathfrak{U}$). Since there are *two* orbits of vertices of \mathfrak{g} and in the quadratic case every periodic gap has only *one* orbit of vertices, then this is only possible if vertices of \mathfrak{g} are formed by the orbits of $\theta_1 + 1/3$ and $\theta_2 + 2/3$. However then 0 must coincide with the repelling point z_g , a contradiction.

The rest follows from Lemma 9.4.

Consider now the case $m = 1$ and, hence, $\mathfrak{M} = \overline{0(1/2)}$. Then, depending on θ_1 and θ_2 , the gap \mathfrak{U} may be either FG_a or FG_b . The same options are possible for the gap $\mathfrak{U}(g)$, but, unlike in the case $m > 1$, we cannot conclude that $\mathfrak{U} = \mathfrak{U}(g)$. Still, $\ell = \mathfrak{M}$ because ℓ is the major of a quadratic invariant gap, and, by Lemma 2.11, all other majors of quadratic invariant gaps cross $\overline{0(1/2)}$, a contradiction.

As before, the rest follows from Lemma 9.4. □

10.1. Non-special parameter wakes: $m > 1$. Let us study dynamics of polynomials $f \in \widehat{\mathcal{W}} = \widehat{\mathcal{W}}_\lambda(\theta_1, \theta_2)$ assuming that $|\lambda| \leq 1$ and $m > 1$, where m is the period of $\theta_1 + 1/3$ under the angle tripling map. Observe that, by Theorem 2.5, the polynomial $f \in \widehat{\mathcal{W}}$ is immediately renormalizable, and one can talk about the gap $\mathfrak{U}(f)$; also, by Lemma 7.2, the filled quadratic-like Julia set $K(f^*)$ is unique.

In the following series of lemmas, we establish properties of the set $\widehat{\mathcal{W}}$. Fix an integer $r > 1$. Define a finite subset $\mathcal{Z}_{\lambda,r} \subset \mathcal{F}_\lambda$ as follows. Let $\mathcal{Z}'_{\lambda,r}$ be the set of $f \in \mathcal{F}_\lambda$ such that f^{or} has a parabolic fixed point $y \neq 0$ of multiplier 1. This is an algebraic set in \mathcal{F}_λ different from \mathcal{F}_λ ; hence it is finite. If $\lambda^r \neq 1$, then we set $\mathcal{Z}''_{\lambda,r} = \emptyset$. If $\lambda^r = 1$ and $\lambda = e^{2\pi ip/q}$, where q divides r and p and q are coprime, then we set $\mathcal{Z}''_{\lambda,r} = \{f_{\lambda,b} \mid T_{p/q}(b) = 0\}$. In any case $\mathcal{Z}''_{\lambda,r}$ is a finite set. Observe that $\mathcal{Z}'_{\lambda,r} \cap \mathcal{Z}''_{\lambda,r} = \emptyset$. Indeed, by Proposition 3.8, if $f \in \mathcal{Z}''_{\lambda,r}$, then there are two cycles of parabolic domains at 0; since f is cubic, this implies that both critical points of f belong to these cycles and no other parabolic point of f exists.

Our aim is to describe the boundary of the set $\widehat{\mathcal{W}}$. Suppose that $f \in \text{Bd}(\widehat{\mathcal{W}})$ and $g \in \widehat{\mathcal{W}}$ is a polynomial sufficiently close to f . By definition, the rays $R_g(\theta_1 + 1/3)$ and $R_g(\theta_2 + 2/3)$ are smooth and land at the same repelling periodic point z_g of g . Since the set $\widehat{\mathcal{W}}$ is open, then $f \notin \widehat{\mathcal{W}}$; in particular, it is not true that

$R_f(\theta_1 + 1/3)$ and $R_f(\theta_2 + 2/3)$ are smooth and land at the same repelling periodic point. Evidently, the two possible reasons for that are:

- (1) one of the rays, say, $R_f(\theta_1 + 1/3)$, fails to be smooth;
- (2) both rays $R_f(\theta_1 + 1/3)$ and $R_f(\theta_2 + 2/3)$ are smooth, but they fail to land at the same repelling periodic point.

Consider these cases separately; first consider case (1).

Lemma 10.2. *Let $\widehat{W} = \widehat{W}_\lambda(\theta_1, \theta_2)$ be as above. Suppose that $f \in \text{Bd}(\widehat{W})$ and one of the rays $R_f(\theta_1 + 1/3)$, $R_f(\theta_2 + 2/3)$ fails to be smooth. Then $f \in \mathcal{R}_\lambda(\theta_1) \cup \mathcal{R}_\lambda(\theta_2)$.*

Proof. We will write g for a polynomial in \widehat{W} sufficiently close to f . Assume, e.g., that $R_f(\theta_1 + 1/3)$ crashes into a (pre)critical point. In fact, it must be the critical point $\omega_2(f)$ since $\omega_1(f) \in J(f)$. Either $R_f(\theta_1)$ and $R_f(\theta_1 + 1/3)$ crash into $\omega_2(f)$ (and then $f \in \mathcal{R}_\lambda(\theta_1 - 1/3)$) or $R_f(\theta_1 + 1/3)$ and $R_f(\theta_1 + 2/3)$ crash into $\omega_2(f)$ (and then $f \in \mathcal{R}_\lambda(\theta_1)$). We claim that the latter option takes place.

Suppose otherwise: $f \in \mathcal{R}_\lambda(\theta_1 - 1/3)$, and the rays $R_f(\theta_1)$ and $R_f(\theta_1 + 1/3)$ crash into $\omega_2(f)$. We may assume that $g \in \widehat{W}$, that g satisfies one of the conditions (2)(a) or (2)(b) from Lemma 4.6. However, since $m > 1$, then $(\theta_1, \theta_2) \neq (2/3, 5/6)$ and $(\theta_1, \theta_2) \neq (1/6, 2/3)$, and we see that (2)(b) is impossible as it would imply that the rays $R_g(\theta_1 + 1/3)$, $R_g(\theta_2 + 2/3)$ land at 0, which is parabolic, whereas they land at the repelling periodic point z_g . Thus, g belongs to an external parameter ray $\mathcal{R}_\lambda(\varkappa)$ with $\varkappa \in (\theta_1, \theta_2)$, a contradiction with $f \in \mathcal{R}_\lambda(\theta_1 - 1/3)$. We established that $f \in \mathcal{R}_\lambda(\theta_1)$. Similarly, if $R_f(\theta_2 + 2/3)$ crashes at $\omega_2(f)$, then we conclude that $f \in \mathcal{R}_\lambda(\theta_2)$. Thus, $f \in \mathcal{R}_\lambda(\theta_1) \cup \mathcal{R}_\lambda(\theta_2)$ as desired. \square

Let us now consider case (2) above.

Lemma 10.3. *Suppose that $f \in \text{Bd}(\widehat{W})$, both rays $R_f(\theta_1 + 1/3)$ and $R_f(\theta_2 + 2/3)$ are smooth, but they fail to land at the same repelling periodic point. Then $R_f(\theta_1 + 1/3)$, or $R_f(\theta_2 + 2/3)$, or both, land at a parabolic periodic point y such that $f^{\circ m}(y) = y$, an $f^{\circ m}$ -invariant ray lands at y , and $f^{\circ m}$ has multiplier 1 at y . Moreover, one of the following holds:*

- (a) y can be chosen to be not equal to 0 and $f \in \mathcal{Z}'_{\lambda, m}$,
- (b) the point $y = 0$ is the only parabolic periodic point at which at least one of the rays $R_f(\theta_1 + 1/3)$, $R_f(\theta_2 + 2/3)$ lands, and $f \in \mathcal{Z}''_{\lambda, m}$.

Proof. It follows from Lemma 3.2 that $R_f(\theta_1 + 1/3)$, or $R_f(\theta_2 + 2/3)$, or both, land at a parabolic periodic point y such that $f^{\circ m}(y) = y$, an $f^{\circ m}$ -invariant ray lands at y , and $f^{\circ m}$ has multiplier 1 at y . To prove the rest it suffices to consider case (b). Then $\lambda^m = 1$, so that $\lambda = e^{2\pi i p/q}$, where q divides m . Suppose that $T_{p/q}(b) \neq 0$, and let $R_f(\theta_1 + 1/3)$ land at 0. Then, by Proposition 3.8, the ray $R_g(\theta_1 + 1/3)$ lands at 0 too provided that g is close enough to f , a contradiction as 0 is not repelling. This shows that $T_{p/q}(b) = 0$ and therefore $f \in \mathcal{Z}''_{\lambda, m}$. \square

The following is an immediate corollary of Lemmas 10.2 and 10.3.

Corollary 10.4. *We have $\text{Bd}(\widehat{W}) \subset \mathcal{R}_\lambda(\theta_1) \cup \mathcal{R}_\lambda(\theta_2) \cup \mathcal{Z}'_{\lambda, m} \cup \mathcal{Z}''_{\lambda, m}$. In particular, parameter rays $\mathcal{R}_\lambda(\theta_1)$ and $\mathcal{R}_\lambda(\theta_2)$ land at the same point.*

We write $f_{\text{root}}(\lambda, \theta_1, \theta_2) = f_{\text{root}}$ for the common landing point of the parameter rays $\mathcal{R}_\lambda(\theta_1)$ and $\mathcal{R}_\lambda(\theta_2)$. Define the wake $\mathcal{W}_\lambda(\theta_1, \theta_2) = \mathcal{W}$ as a complementary

component of the set $\overline{\mathcal{R}_\lambda(\theta_1) \cup \mathcal{R}_\lambda(\theta_2)}$ containing all dynamic rays with arguments in (θ_1, θ_2) . The polynomial f_{root} is called the *root point* (or just the *root*) of the wake $\mathcal{W}_\lambda(\theta_1, \theta_2)$. Now we study a few hypothetic dynamical patterns that potentially might be exhibited by polynomials $f \in \text{Bd}(\widehat{\mathcal{W}})$ and polynomials $g \in \widehat{\mathcal{W}}$ close to f .

The map $\sigma_3|_{\text{Bd}(\mathcal{U})}$ is semiconjugate to $\sigma_2 : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ by means of a map collapsing all edges of \mathcal{U} . The orbit of the edge $\mathfrak{M} = \overline{(\theta_1 + 1/3)(\theta_2 + 2/3)}$ becomes a σ_2 -periodic orbit of period m under σ_2 (recall that \mathfrak{M} is of period m under σ_3). Denote this periodic orbit by \mathfrak{P} . The major \mathfrak{M} may be s/m -rotational for some s , in which case \mathfrak{P} can also be called s/m -rotational (under the action of σ_2). Recall that $\widehat{\mathcal{W}} \neq \emptyset$ by our assumption.

Lemma 10.5. *Suppose that \mathfrak{M} is s/m -rotational. Then $\lambda \neq e^{2\pi is/m}$.*

Proof. By way of contradiction, suppose that $\lambda = e^{2\pi is/m}$. Choose $g \in \widehat{\mathcal{W}}$. Since 0 is a parabolic fixed point of g with multiplier λ , there is a cycle of external rays R_g^1, \dots, R_g^m of g landing at 0, “rotating” around 0 with combinatorial rotation number s/m , and never entering W_g . Hence their arguments belong to $\mathcal{U}(g) = \mathcal{U}$ (the latter equality is proven in Corollary 10.1). The map $\sigma_3|_{\text{Bd}(\mathcal{U})}$ is semiconjugate to $\sigma_2 : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ by the map collapsing to points all edges of \mathcal{U} . Since for any $\rho \in \mathbb{R}/\mathbb{Z}$ there is a unique periodic orbit of rotation number ρ under σ_2 , and since \mathfrak{M} “rotates” around the circle with combinatorial rotation number s/m , the only possibilities for the rays R_g^1, \dots, R_g^m are to be either the rays with arguments from the orbit of $\theta_1 + 1/3$ or the rays with the arguments from the orbit of $\theta_2 + 2/3$. This is a contradiction as the rays with such arguments land at repelling periodic points. □

Lemma 10.6 relies upon Lemma 10.5.

Lemma 10.6. *There are no polynomials $f \in \text{Bd}(\widehat{\mathcal{W}}) \cap \mathcal{Z}''_{\lambda,m}$.*

Proof. By way of contradiction, suppose that $f \in \text{Bd}(\widehat{\mathcal{W}}) \cap \mathcal{Z}''_{\lambda,m}$; then case (b) of Lemma 10.3 holds, and one of the rays $R_f(\theta_1 + 1/3), R_f(\theta_2 + 2/3)$ lands at 0. Hence \mathfrak{M} is rotational and its rotation number equals the argument of λ , contradicting Lemma 10.5. □

It remains to consider case (a) of Lemma 10.3.

Lemma 10.7. *If $f \in \text{Bd}(\widehat{\mathcal{W}}) \cap \mathcal{Z}'_{\lambda,m}$ then $f = f_{root} \in \mathcal{CU}_\lambda$ is the common landing point of the parameter rays $\mathcal{R}_\lambda(\theta_1)$ and $\mathcal{R}_\lambda(\theta_2)$. The rays $R_{f_{root}}(\theta_1 + 1/3)$ and $R_{f_{root}}(\theta_2 + 2/3)$ are smooth and land at a common point $z_{root} \neq 0$. The point z_{root} is a parabolic periodic point of f_{root} . Also, as $f \in \mathcal{W}$ tends to f_{root} , the common landing point z_f of $R_f(\theta_1 + 1/3)$ and $R_f(\theta_2 + 2/3)$ tends to z_{root} . Thus, $\widehat{\mathcal{W}} = \mathcal{W}$.*

Proof. By Lemma 10.3 we may assume that $R_f(\theta_1 + 1/3)$ lands at a parabolic periodic point $y \neq 0$ such that $f^{om}(y) = y$; since $R_f(\theta_1 + 1/3)$ is of period m then f^{om} has multiplier 1 at y . It is easy to see that $J(f)$ is connected. Indeed, the orbit of y attracts a critical point, and one more critical point is associated with non-repelling fixed point 0 (by [Fat20] if 0 is attracting or parabolic, or by [Man93] if 0 is Siegel or Cremer). Thus, $f \in \mathcal{C}_\lambda$. We claim that $f \in \mathcal{CU}_\lambda$. By way of contradiction let $f \notin \mathcal{CU}_\lambda$. Then by Lemma 9.4, the map f is immediately renormalizable with filled quadratic-like Julia set $K(f^*)$, the quadratic invariant

gap $\mathfrak{U}(f)$ has a periodic major ℓ with endpoints, say, $\bar{\alpha}$ and $\bar{\beta}$, and $\Gamma_{z_f}(\overline{\alpha\beta})$ is a strict periodic cut with periodic vertex z_f . By Lemma 9.4, there are two cases.

(1) The point $z_f \neq 0$ is repelling. If now $\ell = \mathfrak{M}$, then $f \in \widehat{\mathcal{W}}$ while we assume that $f \in \text{Bd}(\widehat{\mathcal{W}})$, a contradiction. On the other hand, if $\ell \neq \mathfrak{M}$, then, by Lemma 3.1, polynomials $g \in \widehat{\mathcal{W}}$ close to f must have two strict periodic cuts with repelling vertices x_g and y_g , namely cuts $\Gamma_{x_g}(\ell)$ and $\Gamma_{y_g}(\mathfrak{M})$ respectively, corresponding to two distinct majors of quadratic invariant gaps, ℓ and \mathfrak{M} , which is impossible by Corollary 10.1. Hence z_f cannot be repelling.

(2) The point $z_f = 0$ is parabolic with exactly one cycle of parabolic Fatou domains at 0 and two cycles of external rays landing at 0. By Proposition 3.8, if $g \in \widehat{\mathcal{W}}$ is close to f , the rays of g with the same arguments land at 0 too. By Corollary 10.1, we have $\ell = \mathfrak{M}$, a contradiction as the common landing point z_g of rays $R_g(\theta_1 + 1/3)$ and $R_g(\theta_2 + 2/3)$ is repelling and hence cannot be equal to 0.

Thus, $f \in \mathcal{CU}_\lambda$, and the point $y \neq 0$ is of period m . Indeed, suppose that y is of period $s < m$; then $f^{\circ s}$ rotates rays landing at y with a non-zero (combinatorial) rotation number which by definition implies that $f \notin \mathcal{CU}_\lambda$, a contradiction.

The only rays of $g \in \widehat{\mathcal{W}}$ that land at z_g are $R_g(\theta_1 + 1/3)$ and $R_g(\theta_2 + 2/3)$. Indeed, otherwise a ray $R_g(\alpha)$ lands at z_g , and $\alpha \notin \{\theta_1 + 1/3, \theta_2 + 2/3\}$ is of period m . By Corollary 8.4, $\theta_2 + 2/3 < \alpha < \theta_1 + 1/3$. Yet, this would imply that z_g is a periodic repelling cutpoint of $K(g^*)$ which is impossible as $f|_{K(g^*)}$ is conjugate to a quadratic polynomial from the Main Cardioid. This proves our claim.

Consider how the dynamics of maps $g \in \widehat{\mathcal{W}}$ changes as $g \rightarrow f$. We may assume that as $g \rightarrow f$, the repelling $g^{\circ m}$ -fixed points z_g converge to an $f^{\circ m}$ -fixed point w . Suppose that $w \neq y$. Consider the parabolic domain U of f at y . Since y is of period m and multiplier of $f^{\circ m}$ at y is 1, then, by Theorem 3.5, the continuum \bar{U} is a quadratic-like filled Julia set. By Lemma 3.4, polynomials g close to f are such that their points z_g are close to y . On the other hand, points z_g must be close to $w \neq y$, a contradiction. Thus, $y = w$. Recall that we are considering case (a) of Lemma 10.3 and so $y \neq 0$. The point y is the point z_{root} from the statement of the lemma, and this is how we will denote this point in the last paragraph of the proof.

Our detailed description of the dynamics of f implies that $f = f_{\text{root}}$. Indeed, otherwise f is a puncture of $\widehat{\mathcal{W}}$. The point z_{root} is an $f^{\circ m}$ -fixed neutral point which means that the modulus of the multiplier of $g^{\circ m}$ at z_g has a local minimum at $g = f$. This is a contradiction with the maximum modulus principle applied to the function $g \mapsto 1/(g^{\circ m'}(y_g))$. Hence $f = f_{\text{root}}$ as claimed. \square

10.2. Non-special parameter wakes: $m = 1$. It remains to consider the case where the hole (θ_1, θ_2) of Ω has period $m = 1$. There are two holes like that: $(1/6, 1/3)$ and $(2/3, 5/6)$. Let us consider how the main concepts change in this case. By definition, the set $\widehat{\mathcal{W}}$ is the set of polynomials $f \in \mathcal{F}_\lambda$ such that the rays $R_f(\theta_1 + 1/3)$ and $R_f(\theta_2 + 2/3)$ are smooth and land at the same repelling periodic point of f . In the case at hand this means that the rays $R_f(1/2)$ and $R_f(0)$ (both are σ_3 -invariant) land at the same fixed repelling point of f . As always, this creates a strict periodic (in this case invariant) cut $\Gamma_{z_f}(\overline{0(1/2)}) = \Gamma_{z_f}$. Otherwise we keep the same notation and conventions as before. As before, we assume that $|\lambda| \leq 1$.

The non-repelling point 0 and the entire filled quadratic-like Julia set $K(f^*)$ are on the same side of Γ_{z_f} . It can be either side. Observe that this ambiguity concerning the location of 0 with respect to Γ_{z_f} was not present before. Indeed, if

$\mathfrak{M} = \overline{(\theta_1 + 1/3)(\theta_2 + 2/3)}$ is not $\overline{0(1/2)}$, then the location of \mathfrak{U} with respect to \mathfrak{M} is unique: \mathfrak{U} is contained in the closure of the component of $\overline{\mathbb{D}} \setminus \mathfrak{M}$ containing the circle arc $\overline{[\theta_2 + 2/3, \theta_1 + 1/3]}$ (the longer of the two arcs into which \mathfrak{M} partitions the circle). The point 0 and the set $K(f^*)$ then were contained in the corresponding component of $\mathbb{C} \setminus \Gamma_{z_f}$. However if $\mathfrak{M} = \overline{0(1/2)}$, the location of \mathfrak{U} with respect to \mathfrak{M} (and the location of 0 and $K(f^*)$ with respect to Γ_{z_f}) is not unique.

Note that by definition $\widehat{\mathcal{W}}_\lambda(1/6, 1/3) = \widehat{\mathcal{W}}_\lambda(2/3, 5/6)$.

Lemma 10.8. *Suppose that (θ_1, θ_2) is either $(1/6, 1/3)$ or $(2/3, 5/6)$. Then the boundary of $\widehat{\mathcal{W}}_\lambda(\theta_1, \theta_2)$ lies in the union*

$$\mathcal{R}_\lambda\left(\frac{1}{6}\right) \cup \mathcal{R}_\lambda\left(\frac{1}{3}\right) \cup \mathcal{R}_\lambda\left(\frac{2}{3}\right) \cup \mathcal{R}_\lambda\left(\frac{5}{6}\right) \cup \mathcal{Z}'_{\lambda,1} \cup \mathcal{Z}''_{\lambda,1}.$$

Proof. The lemma is similar to Corollary 10.4 summarizing Lemma 10.2 and Lemma 10.3. The analog of Lemma 10.3 can be proven in the same fashion. To prove the analog of Lemma 10.2 we argue in almost the same fashion except that we do not rule out appearance of $\mathcal{R}_\lambda(2/3)$ and $\mathcal{R}_\lambda(5/6)$ in the boundary of $\widehat{\mathcal{W}}(1/6, 1/3)$. \square

Lemma 10.9. *Suppose that $\widehat{\mathcal{W}} = \widehat{\mathcal{W}}_\lambda(1/6, 2/3) \neq \emptyset$. The parameter rays $\mathcal{R}_\lambda(1/6)$ and $\mathcal{R}_\lambda(1/3)$ land at the same point f_{root} . In the dynamical plane of f_{root} , the rays $R_{f_{root}}(0)$ and $R_{f_{root}}(1/2)$ land at the same point z_{root} . It is the limit of the common landing point z_f of the rays $R_f(0)$, $R_f(1/2)$ as $f \in \widehat{\mathcal{W}}$ tends to f_{root} . Similar statements hold for parameter rays $\mathcal{R}_\lambda(2/3)$ and $\mathcal{R}_\lambda(5/6)$.*

Proof. By Lemma 10.8, at least two of the four parameter rays with arguments $1/6, 1/3, 2/3, 5/6$ have a common landing point. The parameter b -plane is symmetric with respect to the transformation $b \mapsto -b$; hence, if the parameter rays with arguments α, β have the same landing point, then so do the parameter rays with arguments $\alpha + 1/2$ and $\beta + 1/2$. Also, by Lemma 4.6, a parameter ray with argument \varkappa can belong to $\widehat{\mathcal{W}}$ only if $\varkappa \in (1/6, 1/3) \cup (2/3, 5/6)$. It leaves us with two cases:

- (1) The set $\widehat{\mathcal{W}}$ consists of two components; one of which is bounded by $\overline{\mathcal{R}_\lambda(1/6) \cup \mathcal{R}_\lambda(1/3)}$, and the other by $\overline{\mathcal{R}_\lambda(2/3) \cup \mathcal{R}_\lambda(5/6)}$.
- (2) The set $\widehat{\mathcal{W}}$ consists of one component bounded on one side by $\overline{\mathcal{R}_\lambda(1/3) \cup \mathcal{R}_\lambda(2/3)}$ and on the other side by $\overline{\mathcal{R}_\lambda(1/6) \cup \mathcal{R}_\lambda(5/6)}$.

Similarly to the case $m > 1$, in both cases there are no punctures. In case (1), the parameter rays with arguments $1/6, 1/3$ have the same landing point, as do the parameter rays with arguments $2/3, 5/6$. In case (2), the parameter rays with arguments $1/3, 2/3$ have the same landing point, as do the parameter rays with arguments $1/6, 5/6$.

Case (1) implies the statement of the lemma in the same way as when $m > 1$. Thus it remains to rule out case (2). In case (2), we must have $f_{\lambda,0} \in \widehat{\mathcal{W}}$, since $\widehat{\mathcal{W}}$ is simply connected and symmetric with respect to this point. The dynamical plane of $f_0 = f_{\lambda,0}$ is symmetric with respect to the transformation $z \mapsto -z$ (this transformation maps orbits to orbits, the Julia set to the Julia set, rays to rays, etc.). It follows that the common landing point of $R_{f_0}(0)$ and $R_{f_0}(1/2)$ must be 0. However, 0 is not repelling, a contradiction. \square

Proposition 10.10 gives the summary of results obtained in this section.

Proposition 10.10. *If $|\lambda| \leq 1$ and $\widehat{\mathcal{W}}_\lambda(\theta_1, \theta_2) \neq \emptyset$, then the following hold.*

- (1) *The parameter rays $\mathcal{R}_\lambda(\theta_1)$ and $\mathcal{R}_\lambda(\theta_2)$ land at f_{root} , where f_{root} is a polynomial with a parabolic periodic point $z_{root} \neq 0$ at which the smooth dynamic rays $R_{f_{root}}(\theta_1 + 1/3)$ and $R_{f_{root}}(\theta_2 + 2/3)$ land. Moreover, as $g \in \widehat{\mathcal{W}}_\lambda(\theta_1, \theta_2)$ converges to f_{root} , the common landing point z_g of the smooth dynamic rays $R_g(\theta_1 + 1/3)$ and $R_g(\theta_2 + 2/3)$ converges to z_{root} .*
- (2) *The set $\widehat{\mathcal{W}}_\lambda(\theta_1, \theta_2)$ coincides with the component of the set $\mathbb{C} \setminus \overline{\mathcal{R}_\lambda(\theta_1) \cup \mathcal{R}_\lambda(\theta_2)}$ containing the parameter rays with arguments from (θ_1, θ_2) except when $(\theta_1, \theta_2) = (1/6, 1/3)$ or $(\theta_1, \theta_2) = (2/3, 5/6)$. In both latter cases $\widehat{\mathcal{W}}_\lambda(1/6, 1/3) = \widehat{\mathcal{W}}_\lambda(2/3, 5/6)$ is the union of the two components of the set*

$$\mathbb{C} \setminus \overline{\mathcal{R}_\lambda\left(\frac{1}{6}\right) \cup \mathcal{R}_\lambda\left(\frac{1}{3}\right) \cup \mathcal{R}_\lambda\left(\frac{2}{3}\right) \cup \mathcal{R}_\lambda\left(\frac{5}{6}\right)}$$

containing the parameter rays with arguments from $(1/6, 1/3)$ for one component and from $(2/3, 5/6)$ for the other one.

11. SPECIAL PARAMETER WAKES AND THEIR ROOTS

Assume that (θ_1, θ_2) is a hole of Ω and fix it, together with $\lambda \in \mathbb{C}, |\lambda| \leq 1$; assume that the period of $\theta_1 + 1/3$ is q . Set $\mathfrak{M} = \overline{(\theta_1 + 1/3)(\theta_2 + 2/3)}$; then \mathfrak{M} is the major of a quadratic invariant gap $\mathfrak{U} = \mathfrak{U}_p$ of periodic type. We consider polynomials from \mathcal{F}_λ . Unlike before, let us now assume that the dynamic rays $R_f(\theta_1 + 1/3)$ and $R_f(\theta_2 + 2/3)$ never land at the same repelling periodic point, no matter what polynomial $f \in \mathcal{F}_\lambda$ we choose; this assumption remains valid throughout this entire section. Still, the rays $R_f(\theta_1 + 1/3)$ and $R_f(\theta_2 + 2/3)$ may land at the same point z_f which then is necessarily parabolic. Together with this point these rays form a strict periodic cut $\Gamma_{z_f}(\mathfrak{M}) = \Gamma_{z_f}$ of the plane. Define an open dynamic wedge $W_f(\theta_1 + 1/3, \theta_2 + 2/3) = W_f$ as the component of $\mathbb{C} \setminus \Gamma_{z_f}$ that contains external rays with arguments from the arc $H_{\mathfrak{M}} = (\theta_1 + 1/3, \theta_2 + 2/3)$.

11.1. Special parameter wakes. Recall that in Subsection 2.3 we introduced p/q -rotational periodic points, p/q -rotational periodic chords (e.g., majors of quadratic invariant gaps and finite invariant gaps of type D) and p/q -rotational holes of Ω .

Lemma 11.1. *Under our assumptions $\lambda = e^{2\pi ip/q}$ is a root of unity while the major \mathfrak{M} and the hole (θ_1, θ_2) are p/q -rotational. If $f \in \mathcal{R}_\lambda(\varkappa)$ with $\varkappa \in (\theta_1, \theta_2)$, then the dynamic rays $R_f(\theta_1 + 1/3)$ and $R_f(\theta_2 + 2/3)$ land at 0.*

Proof. Take an angle $\varkappa \in (\theta_1, \theta_2)$. By Theorem 2.13, the chord \mathfrak{M} is the major of the quadratic invariant gap $\mathfrak{U}(\varkappa)$ of periodic type. It follows from Proposition 4.3 that, for every $f \in \mathcal{R}_\lambda(\varkappa)$, the dynamic rays $R_f(\theta_1 + 1/3)$ and $R_f(\theta_2 + 2/3)$ land at the same periodic point x . This point is either repelling or parabolic. By our assumption, x is not repelling, hence x is parabolic. Since $f \notin \mathcal{C}_\lambda$ and by the Fatou–Shishikura inequality, there is at most one non-repelling cycle of f . Thus we must have $x = 0$, which means that 0 is a parabolic point, i.e., the multiplier λ is a root of unity. Moreover, it follows that $\lambda = e^{2\pi ip/q}$ where p/q is the combinatorial rotation number of f at 0 coinciding with the rotation number of, say, $\theta_1 + 1/3$, which implies that \mathfrak{M} is p/q -rotational. □

For $f \in \mathcal{F}_\lambda$, let \mathfrak{g}_f be the convex hull of all points $\bar{\theta}$ such that $R_f(\bar{\theta})$ lands at 0.

Lemma 11.2. *Suppose that there exists $f \in \mathcal{F}_\lambda \setminus \mathcal{C}_\lambda$ such that the dynamic rays $R_f(\theta_1 + 1/3)$ and $R_f(\theta_2 + 2/3)$ both land at 0 and the parabolic domains of f at 0 are disjoint from $W_f(\theta_1 + 1/3, \theta_2 + 2/3)$. Then \mathfrak{g}_f is a finite invariant gap of type D. Moreover, if $f \in \mathcal{R}_\lambda(\varkappa)$, then $\varkappa \in (\theta_1, \theta_2)$.*

Proof. Let $f \in \mathcal{R}_\lambda(\varkappa)$. Then \varkappa belongs to (θ_1, θ_2) . Indeed, the rays $R_f(\varkappa + 1/3)$ and $R_f(\varkappa + 2/3)$ crash at the escaping critical point $\omega_2(f)$. Thus, both $\varkappa + 1/3$ and $\varkappa + 2/3$ belong to the same major hole of \mathfrak{g}_f . Since $(\theta_1 + 1/3)(\theta_2 + 2/3)$ is the major of an invariant quadratic gap, the σ_3 -orbits of $\theta_1 + 1/3$ and $\theta_2 + 2/3$ do not enter the arc $(\theta_1 + 1/3, \theta_2 + 2/3)$. Hence, $(\theta_1 + 1/3, \theta_2 + 2/3)$ is a major hole of \mathfrak{g}_f . There are two major holes of \mathfrak{g}_f , each giving rise to the corresponding dynamic wedge containing a critical point of f . Since, by the assumption, the parabolic domains of f at 0 are disjoint from $W_f(\theta_1 + 1/3, \theta_2 + 2/3)$ and one of these dynamic wedges contains a non-escaping critical point of f , then the escaping critical point $\omega_2(f)$ is in $W_f(\theta_1 + 1/3, \theta_2 + 2/3)$, i.e., we have $\varkappa \in (\theta_1, \theta_2)$. \square

Consider the set $\widetilde{\mathcal{W}}_\lambda(\theta_1, \theta_2)$ of all polynomials $f = f_{\lambda,b}$ such that the rays $R_f(\theta_1 + 1/3)$ and $R_f(\theta_2 + 2/3)$ land at 0, and the parabolic domains of f at 0 are disjoint from $W_f(\theta_1 + 1/3, \theta_2 + 2/3)$. Since there are at most two cycles of dynamic rays landing at 0, then f can have only one cycle of parabolic domains at 0. Then, by Proposition 3.6, $T_{p/q}(b) \neq 0$.

Proposition 11.3. *If $\widetilde{\mathcal{W}}_\lambda(\theta_1, \theta_2) \neq \emptyset$, then the parameter rays $\mathcal{R}_\lambda(\theta_1)$, $\mathcal{R}_\lambda(\theta_2)$ land at the same point.*

Proof. The proof is similar to that of Proposition 10.10. If $\widetilde{\mathcal{W}}_\lambda(\theta_1, \theta_2) \neq \emptyset$, then, by Lemma 11.2, the boundary of $\widetilde{\mathcal{W}}_\lambda(\theta_1, \theta_2)$ lies in the union of the rays $\mathcal{R}_\lambda(\theta_1)$, $\mathcal{R}_\lambda(\theta_2)$ and the polynomials $f_{\lambda,b}$ corresponding to roots b of the polynomial $T_{p/q}$. Thus, the rays $\mathcal{R}_\lambda(\theta_1)$ and $\mathcal{R}_\lambda(\theta_2)$ land at the same point. \square

Let us now define *special (parameter) wakes*.

Definition 11.4. Let $(\bar{\theta}_1, \bar{\theta}_2)$ be a hole of Ω . Suppose that there exists $f \in \mathcal{F}_\lambda \setminus \mathcal{C}_\lambda$ such that the dynamic rays $R_f(\theta_1 + 1/3)$ and $R_f(\theta_2 + 2/3)$ both land at 0 and the parabolic domains of f at 0 are disjoint from $W_f(\theta_1 + \frac{1}{3}, \theta_2 + \frac{2}{3})$. Then the set $\mathcal{W}_\lambda(\theta_1, \theta_2)$ bounded by the parameter rays $\mathcal{R}_\lambda(\theta_1)$, $\mathcal{R}_\lambda(\theta_2)$ and their common landing point is called a *special (parameter) wake*. The landing point $f_{root} = f_{root}(\theta_1, \theta_2)$ of the rays $\mathcal{R}_\lambda(\theta_1)$ and $\mathcal{R}_\lambda(\theta_2)$ is called the *root point* of the parameter wake $\mathcal{W}_\lambda(\theta_1, \theta_2)$.

The next claim complements Proposition 11.3; the proof is similar to that of Proposition 10.10 (the last claim holds by Proposition 3.6).

Proposition 11.5. *If $\widetilde{\mathcal{W}}_\lambda(\theta_1, \theta_2) \neq \emptyset$, and $f_{root} = f_{\lambda,b_0}$ is the common landing point of the parameter rays $\mathcal{R}_\lambda(\theta_1)$, $\mathcal{R}_\lambda(\theta_2)$, then $T_{p/q}(b_0) = 0$. Moreover, $\widetilde{\mathcal{W}}_\lambda(\theta_1, \theta_2)$ coincides with $\mathcal{W}_\lambda(\theta_1, \theta_2)$, possibly with several punctures f_{λ,b_i} , where $1 \leq i \leq N$ and $T_{p/q}(b_i) = 0$. In particular, the map f_{λ,b_i} , for every $0 \leq i \leq N$, has two cycles of parabolic domains at its unique parabolic point 0.*

Proposition 11.6. *Let $\mathcal{W}_\lambda(\theta_1, \theta_2)$ be a special parameter wake, and f_{root} be its root point. Then, for every $f \in \mathcal{W}_\lambda(\theta_1, \theta_2) \cup \{f_{root}\}$, the rays $R_f(\theta_1 + 1/3)$ and $R_f(\theta_2 + 2/3)$ land at 0.*

Proof. Let $f = f_{\lambda,b} \in \mathcal{W}_\lambda(\theta_1, \theta_2)$ or $f = f_{root}$. If $T_{p/q}(b) \neq 0$, then, by Propositions 11.3 and 11.5, the rays $R_f(\theta_1 + 1/3)$ and $R_f(\theta_2 + 2/3)$ land at 0. Suppose that $T_{p/q}(b) = 0$ but at least one of the rays $R_f(\theta_1 + 1/3)$, $R_f(\theta_2 + 2/3)$ (say, the former) fails to land at 0. By Proposition 3.6 the map $f_{\lambda,b}$ has $2q$ parabolic Fatou domains at 0 forming two cycles under f . The ray $R_f(\theta_1 + 1/3)$ lands, say, at a point $z \neq 0$. The point z is repelling or parabolic. It cannot be repelling because then, by Lemma 3.1, for polynomials g sufficiently close to f the ray $R_g(\theta_1 + 1/3)$ will have to land at a point close to z while, by Proposition 11.5, there are polynomials g arbitrarily close to f such that $R_g(\theta_1 + 1/3)$ lands at 0, a contradiction. On the other hand, by the Fatou–Shishikura inequality, the point $z \neq 0$ cannot be parabolic either (both critical points of f are in parabolic domains at 0). \square

In Subsection 2.3 we introduce p/q -rotational holes of \mathfrak{Q} ; by Lemma 2.16, there are $2q$ rotational holes of \mathfrak{Q} corresponding to the rotation number p/q .

Theorem 11.7 (Special parameter wakes vs. rotational holes). *The parameter wake $\mathcal{W}_\lambda(\theta_1, \theta_2)$ is a special parameter wake corresponding to p/q if and only if the hole (θ_1, θ_2) of \mathfrak{Q} is p/q -rotational. There are $2q$ special parameter wakes in \mathcal{F}_λ .*

Proof. Let $\mathcal{W}_\lambda(\theta_1, \theta_2)$ be a special parameter wake. Then, by Lemma 11.1, the hole (θ_1, θ_2) of \mathfrak{Q} is p/q -rotational. Now, assume that the hole $(\bar{\theta}_1, \bar{\theta}_2)$ is p/q -rotational. Then, by Lemma 10.5, we have $\widehat{\mathcal{W}}_\lambda(\theta_1, \theta_2) = \emptyset$. Hence the assumptions of this section hold, and the parameter wake $\mathcal{W}_\lambda(\theta_1, \theta_2)$ is a special parameter wake. The last claim of the theorem follows from Lemma 2.16. \square

Special parameter wakes appear in Section 5.4 of [Z18]; their existence and properties are announced there without proof.

11.2. Roots of special parameter wakes. In this subsection, we still assume that $\lambda = e^{2\pi ip/q}$ is a fixed root of unity. We will prove that all zeros of $T_{p/q}$ correspond to root points of special parameter wakes. In particular, it will follow that the set $\widehat{\mathcal{W}}_\lambda(\theta_1, \theta_2)$ coincides with the special parameter wake $\mathcal{W}_\lambda(\theta_1, \theta_2)$, where $(\bar{\theta}_1, \bar{\theta}_2)$ is any p/q -special hole of \mathfrak{Q} .

Lemma 11.8. *If two special parameter wakes $\mathcal{W}_\lambda(\theta_1, \theta_2)$ and $\mathcal{W}_\lambda(\theta'_1, \theta'_2)$ have the same root point, then $(\bar{\theta}_1 + 1/3, \bar{\theta}_2 + 2/3)$ and $(\bar{\theta}'_1 + 1/3, \bar{\theta}'_2 + 2/3)$ are the two major holes of the same type D finite invariant gap.*

Proof. Let $f = f_{root}$ be the common root point of the parameter wakes $\mathcal{W}_\lambda(\theta_1, \theta_2)$ and $\mathcal{W}_\lambda(\theta'_1, \theta'_2)$. By Proposition 11.6, the four dynamic rays $R_f(\theta_1 + \frac{1}{3})$, $R_f(\theta_2 + \frac{2}{3})$, $R_f(\theta'_1 + \frac{1}{3})$, $R_f(\theta'_2 + \frac{2}{3})$ land at 0. Therefore, the four arguments of these rays correspond to the vertices of the same type D finite invariant gap \mathfrak{g}_f . Clearly, these vertices are on the boundaries of major holes of \mathfrak{g}_f . \square

Two majors \mathfrak{M} and \mathfrak{M}' of the same type D finite invariant gap \mathfrak{g} , as well as the corresponding major holes of \mathfrak{g} , will be called *conjugate*.

Proposition 11.9. *Suppose that $\lambda = e^{2\pi ip/q}$. Every zero b of the polynomial $T_{p/q}$ corresponds to a common root point $f_{\lambda,b}$ of two special parameter wakes $\mathcal{W}_\lambda(\theta_1, \theta_2)$ and $\mathcal{W}_\lambda(\theta'_1, \theta'_2)$, where arcs $(\bar{\theta}_1 + 1/3, \bar{\theta}_2 + 2/3)$ and $(\bar{\theta}'_1 + 1/3, \bar{\theta}'_2 + 2/3)$ are conjugate major holes depending on b . The degree of the polynomial $T_{p/q}$ is q .*

Proof. By Theorem 11.7 and Lemma 11.8, the $2q$ special parameter wakes in \mathcal{F}_λ have at least q different root points. By Proposition 11.5, each special parameter wake has a zero of the polynomial $T_{p/q}$ as its root point. Since, by Proposition 3.6, the degree of $T_{p/q}$ is at most q , the degree of $T_{p/q}$ equals q , and each of the zeros of $T_{p/q}$ corresponds to a common root point of two special parameter wakes. \square

An alternative way of proving that the degree of the $T_{p/q}$ equals q may follow the methods of a paper by Buff, Écalle and Epstein [BEE13], in which the authors prove a similar statement for parameter slices of quadratic rational functions.

Theorem 11.10 (Dynamics of special parameter wakes). *Assume that the wake $\mathcal{W}_\lambda(\theta_1, \theta_2)$ is a special parameter wake. A polynomial $f = f_{\lambda,b}$ belongs to $\mathcal{W}_\lambda(\theta_1, \theta_2)$ if and only if the dynamic rays $R_f(\theta_1 + 1/3)$, $R_f(\theta_2 + 2/3)$ land at 0, the parabolic domains at 0 are disjoint from the wedge $W_f(\theta_1 + 1/3, \theta_2 + 2/3)$, and $T_{p/q}(b) \neq 0$. A polynomial f is the root point of the parameter wake $\mathcal{W}_\lambda(\theta_1, \theta_2)$ if and only if $T_{p/q}(b) = 0$, and the rays $R_f(\theta_1 + 1/3)$, $R_f(\theta_2 + 2/3)$ land at 0; the polynomial f has a parabolic point 0 with two cycles of parabolic domains at 0.*

Proof. Let $f = f_{\lambda,b} \in \mathcal{W}_\lambda(\theta_1, \theta_2)$. Then, by Proposition 11.6, the dynamic rays $R_f(\theta_1 + 1/3)$, $R_f(\theta_2 + 2/3)$ land at 0. By Proposition 11.9, we have $T_{p/q}(b) \neq 0$ for all $f = f_{\lambda,b}$ in the parameter wake, and, by Proposition 11.5, the parabolic domains at 0 are disjoint from the wedge $W_f(\theta_1 + 1/3, \theta_2 + 2/3)$. On the other hand, if the dynamic rays $R_f(\theta_1 + 1/3)$, $R_f(\theta_2 + 2/3)$ land at 0, the parabolic domains at 0 are disjoint from the wedge $W_f(\theta_1 + 1/3, \theta_2 + 2/3)$, and $T_{p/q}(b) \neq 0$, then, by Proposition 11.5 and by Proposition 11.9, we have $f \in \mathcal{W}_\lambda(\theta_1, \theta_2)$. The characterization of the root point of $\mathcal{W}_\lambda(\theta_1, \theta_2)$ follows from Proposition 11.9. \square

12. MAIN THEOREMS

Proof of Theorem A. Take a hole $(\bar{\theta}_1, \bar{\theta}_2)$ of \mathfrak{Q} . Choose $z \in (\theta_1, \theta_2)$ and a polynomial $f \in \mathcal{R}_\lambda(z)$. By Lemma 4.6, both dynamic rays $R_f(\theta_1 + 1/3)$ and $R_f(\theta_2 + 2/3)$ land at a periodic point z . If z is repelling, then, by Proposition 10.10, the parameter rays $\mathcal{R}_\lambda(\theta_1)$ and $\mathcal{R}_\lambda(\theta_2)$ land at the same point. If z is parabolic, then, by Proposition 11.3, the parameter rays $\mathcal{R}_\lambda(\theta_1)$ and $\mathcal{R}_\lambda(\theta_2)$ land at the same point. \square

Now we finalize preparations to the proofs of Theorems B and C. Suppose that λ is a fixed complex number such that $|\lambda| \leq 1$. Recall that the *main cubioid* \mathcal{CU} is the set of classes $[f] \in \mathcal{M}_3$ with the following properties: f has a non-repelling fixed point, f has no repelling periodic cutpoints in $J(f)$, and all non-repelling periodic points of f , except at most one fixed point, have multiplier 1.

Our main result is a description of \mathcal{CU} through λ -slices where $|\lambda| \leq 1$. To state it, we need a FP-marked version of \mathcal{CU} . Recall that we define \mathcal{CU}_{FP} to be the set of FP-classes of all FP-marked cubic polynomials (f, x) with the following properties: x is a non-repelling fixed point, there are no repelling periodic cutpoints in $J(f)$, and all non-repelling periodic points of f , except possibly x , have multiplier 1. Since for $f \in \mathcal{F}$ we have an agreement that the marked fixed point is 0, the polynomials from \mathcal{CU}_{FP} that belong to \mathcal{F} can have at most one non-repelling periodic point with multiplier not equal to 1, and this point must be 0. Recall that by \mathcal{CU}_λ we denote the set of all polynomials $f \in \mathcal{F}_\lambda$ with $[f]_{FP} \in \mathcal{CU}_{FP}$. We will need Theorem 5.3 when proving that certain polynomials $f \in \mathcal{F}_\lambda$ do not belong to \mathcal{CU}_λ .

We will prove that all parameter wakes in the λ -slice \mathcal{F}_λ are disjoint from \mathcal{CU}_λ , for $\lambda \neq 1$. Recall that the λ -connectedness locus \mathcal{C}_λ is the set of all $f \in \mathcal{F}_\lambda$ with $K(f)$ connected. A *limb* of \mathcal{C}_λ is the intersection of \mathcal{C}_λ with a parameter wake.

Theorem 12.1 (A part of Theorem 4.3, [BCLOS16]). *Let a be a Cremer periodic point or a Siegel periodic point of a polynomial Q . Then there exists a recurrent critical point c and an iterated image b of a such that no strict (pre)periodic cut separates c from b .*

Let us investigate the dynamics of polynomials in a limb of \mathcal{C}_λ .

Lemma 12.2. *Suppose that f lies in a limb of \mathcal{C}_λ . Then either f has a repelling periodic cutpoint in $K(f)$ or f belongs to a special wake $\mathcal{W}_\lambda(\theta_1, \theta_2)$ of some period k and the dynamic wedge $W_f(\theta_1 + 1/3, \theta_2 + 2/3)$ contains a non-repelling f -periodic point $x \neq 0$ of period k of multiplier different from 1 such that for every i , $0 < i < k$, the point $f^{\circ i}(x)$ belongs to the dynamic wedge $W_f(3^i\theta_1, 3^i\theta_2)$.*

Recall that the wedge $W_f(\alpha, \beta)$ contains external rays with arguments from (α, β) and is bounded by the rays $R_f(\alpha)$, $R_f(\beta)$ and the point z where they land or crash.

Proof. By our assumptions, the filled Julia set $K(f)$ is connected, and f lies in some parameter wake $\mathcal{W}_\lambda(\theta_1, \theta_2)$ of some period k . The rays $R_f(\theta_1 + 1/3)$ and $R_f(\theta_2 + 2/3)$ land at the point x_1 , which is either a repelling periodic point (if the parameter wake is non-special) or the parabolic point 0 (if the parameter wake is special). If the parameter wake is non-special, we are done. Hence we may assume that the parameter wake is special. Set $X_1 = \overline{W_f(\theta_1 + \frac{1}{3}, \theta_2 + \frac{2}{3})} \cap K(f)$. Then X_1 is a *special piece* for $f^{\circ k}$ with *exit continuum* $\{0\}$ (see Definitions 5.1 and 5.2). By Theorem 5.3 applied to X_1 and $f^{\circ k}$, the set X_1 contains either (1) a non-repelling $f^{\circ k}$ -fixed point in X_1 , whose multiplier is different from 1, or (2) an $f^{\circ k}$ -invariant parabolic domain with the $f^{\circ k}$ -fixed point on its boundary and multiplier 1, or (3) a repelling $f^{\circ k}$ -fixed cutpoint of X_1 . We may assume that f does not have repelling periodic cutpoints, and so we may assume that case (3) fails.

Let us rule out case (2). Suppose that there is an $f^{\circ k}$ -invariant parabolic domain Ω in X_1 . Let x be the corresponding parabolic $f^{\circ k}$ -fixed point of multiplier 1. Since the parameter wake $\mathcal{W}_\lambda(\theta_1, \theta_2)$ is special, then, by Theorem 11.10, we have $x \neq 0$. Since the multiplier at x is 1, there exists an external $f^{\circ k}$ -fixed ray $R_f(\gamma)$ landing at x with $\gamma \in (\theta_1 + 1/3, \theta_2 + 2/3)$. We claim that this is impossible for combinatorial reasons. Indeed, the map f is two-to-one on Ω as it cannot be two-to-one on any other iterated image of Ω (all $f^{\circ i}$ -images of Ω for $0 < i < k$ are contained in wedges that map forward one-to-one). Hence, by Lemma 2.11, the angle γ must be located in the strip between the major $(\theta_1 + 1/3)(\theta_2 + 2/3)$ and its sibling $(\theta_2 + 1/3)(\theta_1 + 2/3)$; however, by Theorem 8.3, this is impossible.

Finally, consider case (1). Since x is non-repelling, there exists a critical point c and a non-negative integer i such that no strict (pre)periodic cut separates c from $f^{\circ i}(x)$. This follows from Theorem 12.1 (in the Cremer and Siegel cases) or is obvious (in the parabolic or attracting cases). Indeed, the critical point of f in a parabolic domain at 0 is separated from any $f^{\circ i}(x)$ with $i \geq 0$ by a strict (pre)periodic cut. It follows that x belongs to the strip bounded by the two cuts: the cut formed by the rays $R_f(\theta_1 + 1/3)$, $R_f(\theta_2 + 2/3)$ (these rays land at the point 0) and the cut formed by the rays $R_f(\theta_2 + 1/3)$, $R_f(\theta_1 + 2/3)$ (these rays land at

the preimage of 0 not equal to 0). It follows that for every i with $0 < i < k$, the point $f^{\circ i}(x)$ belongs to the dynamic wedge $W_f(3^i\theta_1, 3^i\theta_2)$, as claimed. \square

Theorem 12.3. *Parameter wakes in \mathcal{F}_λ are disjoint from \mathcal{CU}_λ .*

Proof. Let $f \in \mathcal{W}_\lambda(\theta_1, \theta_2)$. If $K(f)$ is disconnected, then, by definition, $f \notin \mathcal{CU}_\lambda$. If $K(f)$ is connected, then, by Lemma 12.2, the map f has a non-repelling periodic point $x \neq 0$ with multiplier not equal to 1, or a periodic repelling cutpoint in $K(f)$. In the latter case, we have (by definition of \mathcal{CU}_λ) that $f \notin \mathcal{CU}_\lambda$. In the former case, since $\lambda \neq 1$, we again see that, by definition of \mathcal{CU}_λ , we have $f \notin \mathcal{CU}_\lambda$. \square

Proof of Theorem B. Theorem B follows from Theorem 11.10 and Proposition 10.10, except for its last claim, according to which, for every root polynomial f_{root} , we have $f_{root} \in \mathcal{CU}_\lambda$. Indeed, suppose otherwise. Then f_{root} would have properties listed in Lemma 9.4, and it clearly does not because of the following. Suppose first that f_{root} is the vertex of a non-special wake $\mathcal{W}_\lambda(\theta_1, \theta_2)$. Then, by Proposition 10.10, the point $z \neq 0$, the common landing point of the dynamic rays $R_f(\theta_1 + 1/3)$ and $R_f(\theta_2 + 2/3)$, is parabolic. However, by Lemma 9.4, in this case z is repelling, a contradiction. Otherwise f_{root} is the vertex of a special wake, and, by Theorem 11.10, has two cycles of parabolic domains at 0 while, by Lemma 9.4, it would have exactly one if $f_{root} \notin \mathcal{CU}_\lambda$. Thus, f_{root} belongs to \mathcal{CU}_λ . \square

Proof of Theorem C. The first claim of Theorem C is proven in Theorem 12.3. To complete the proof of Theorem C, it remains to prove that \mathcal{CU}_λ , where $|\lambda| \leq 1$, is a full continuum. The set \mathcal{C}_λ is a full continuum [BuHe01]; this is very similar to the fact that the standard Mandelbrot set is a full continuum [DH8485]. By Theorem 12.3 and Lemma 9.4, the set \mathcal{CU}_λ is obtained from the full continuum \mathcal{C}_λ by removing all limbs. Note that, if we remove finitely many limbs from \mathcal{C}_λ , then we are left with a full continuum; indeed, a limb does not separate \mathcal{C}_λ . Being the intersection of a nested sequence of full continua, the set \mathcal{CU}_λ is also a full continuum. This concludes the proof of Theorem C. \square

We now complete the characterization of non-special wakes by providing conditions on polynomials equivalent to being root points of non-special wakes.

Theorem 12.4. *Suppose that $|\lambda| \leq 1$ and $\mathcal{W}_\lambda(\theta_1, \theta_2)$ is a non-special wake. If $f \in \mathcal{F}_\lambda$, then the following hold.*

- (1) *Suppose that $(\theta_1, \theta_2) \neq (2/3, 5/6)$ and $(\theta_1, \theta_2) \neq (1/6, 1/3)$. Then the dynamic rays $R_f(\theta_1 + 1/3)$, $R_f(\theta_2 + 2/3)$ land at the same periodic parabolic point $z \neq 0$ of multiplier 1 if and only if f is the root point of $\mathcal{W}_\lambda(\theta_1, \theta_2)$.*
- (2) *Suppose that either $(\theta_1, \theta_2) = (2/3, 5/6)$ or $(\theta_1, \theta_2) = (1/6, 1/3)$. Then the dynamic rays $R_f(0)$ and $R_f(1/2)$ land at the same periodic parabolic point $z \neq 0$ of multiplier 1 if and only if f is the root point of either $\mathcal{W}_\lambda(2/3, 5/6)$ or $\mathcal{W}_\lambda(1/6, 1/3)$.*

In case (1) the arc (θ_1, θ_2) is in one-to-one correspondence with the major $\mathfrak{M} = (\theta_1 + 1/3)(\theta_2 + 2/3) \neq 0(1/2)$ of a quadratic invariant gap \mathfrak{U} not equal to FG_a or FG_b ; in case (2) both arcs $(\theta_1, \theta_2) = (2/3, 5/6)$ and $(\theta_1, \theta_2) = (1/6, 1/3)$ are associated with the same major $0(1/2)$ that can serve as the major of either FG_a or FG_b .

Proof. The “if” part in both cases follows from Proposition 10.10. It remains to prove that if the dynamic rays $R_f(\theta_1+1/3)$, $R_f(\theta_2+2/3)$ land at the same periodic parabolic point $z \neq 0$ of multiplier 1, then f is the root point of $\mathcal{W}_\lambda(\theta_1, \theta_2)$ (in case (1)) or the root point of either $\mathcal{W}_\lambda(2/3, 5/6)$ or $\mathcal{W}_\lambda(1/6, 1/3)$ (in case (2)). In either case we need to show that arbitrarily close to f there are maps g such that $R_g(\theta_1+1/3)$, $R_g(\theta_2+2/3)$ land at the same periodic repelling point z .

If the dynamic rays $R_f(\theta_1+1/3)$, $R_f(\theta_2+2/3)$ land at the same periodic parabolic point $z \neq 0$ of multiplier 1 and q is the minimal period of z , then, by Theorem 3.5, the parabolic domain U at z is a quadratic-like Julia set.

On the other hand, since the multipliers of the periodic points of $g \in \mathcal{F}_\lambda$ are branches of certain multivalued analytic functions of g , there exists a stable component $\mathcal{U} \subset \mathcal{F}_\lambda$ bounded by a real analytic curve such that,

- we have $f \in \text{Bd}(\mathcal{U})$;
- for all $g \in \mathcal{U}$, there is an attracting periodic point z_g of period q depending analytically on g ;
- we have $z_g \rightarrow z$ as $g \in \mathcal{U}$ approaches f .

Choose $g \in \mathcal{U}$ close to f . By Lemma 3.4, the external rays $R_g(\theta_1+1/3)$, $R_g(\theta_2+2/3)$ land at point(s) of period q very close to z . However, our choice of g implies that there is only one such point, say, w , close to z , namely, the unique point of period q on the boundary of the basin of immediate attraction of the point z_g . It follows that w is repelling with rays $R_g(\theta_1+1/3)$, $R_g(\theta_2+2/3)$ landing at w as desired. \square

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