Topics in one-dimensional dynamics

Dedicated to the memory of Oleksandr Mykolayovych Sharkovsky

Alexander Blokh[*]

[*]Dept of Math, UAB

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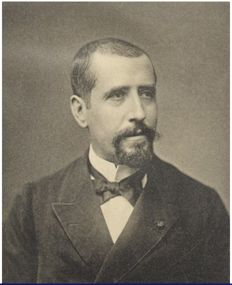
Intermediate Value Teorem

The Intermediate Value Theorem (the IVT) is a part of Calculus 1. In what follows all functions are assumed to be continuous.

Intermediate Value Theorem

Let $\mathbf{f} : [\mathbf{a}, \mathbf{b}] \to \mathbb{R}$ *be a continuous function on the interval* $\mathbf{I}$*. If* $\mathbf{f}(\mathbf{a}) \le \mathbf{T} \le \mathbf{f}(\mathbf{b})$ *or* $\mathbf{f}(\mathbf{a}) \ge \mathbf{T} \ge \mathbf{f}(\mathbf{b})$ *then* $\mathbf{f}(\mathbf{c}) = \mathbf{T}$ *for some* $\mathbf{c} \in [\mathbf{a}, \mathbf{b}]$*. Thus, if* $\mathbf{a}$ *and* $\mathbf{b}$ *map (non-strictly) in opposite directions, then there exists a* **fixed point** $\mathbf{x} \in \mathbf{I}$*, i.e. a point such that* $\mathbf{f}(\mathbf{x}) = \mathbf{x}$*.*



The converse of the IVT fails: some discontinuous functions have the same property. A Darboux function is a function $\mathbf{f}$ satisfying the conclusion of the IVT. Darboux proved that if $\mathbf{f}'$ is defined everywhere, then $\mathbf{f}'$ is a Darboux function.

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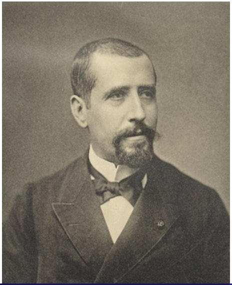

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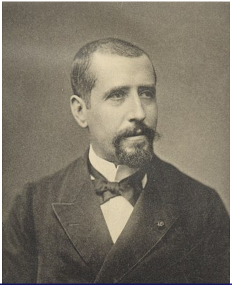

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Intermediate Value Theorem: history

The Intermediate Value Theorem was first “proven” by Bryson of Heraclea Pontica (in modern Turkey, on the Black Sea coast). He was a philosopher (a pupil of Socrates); with Antiphon they were the first to approximate π as follows: inscribe (superscribe) a polygon inside (around) a circle, find the polygon’s area, double the number of sides of the polygon, and repeat the process, resulting in an approximation of the area of a circle.

With these calculations, Bryson was able to approximate π. Aristotle criticized this method, but Archimedes would later use a method similar to that of Bryson and Antiphon to calculate π; however, Archimedes calculated the perimeter of a polygon instead of the area.

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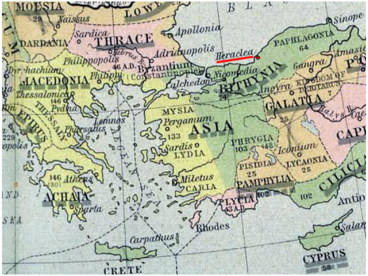
Bryson was concerned with proving that there exists a square of the same area as a given circle. According to the 13th-century English philosopher Robert Kilwardby, Bryson “argued” as follows:

In any genus in which one can find a greater and a lesser than something, one can find what is equal; but in the genus of squares one can find a greater and a lesser than a circle; therefore, one can also find a square equal to a circle.



This is a map of ancient Bulgaria, Greece and Anatolia with Heraclea Pontica shown on it.

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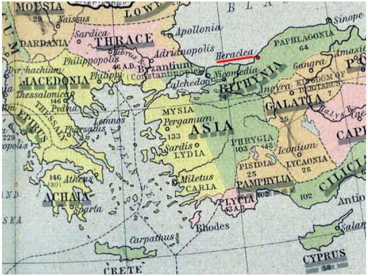

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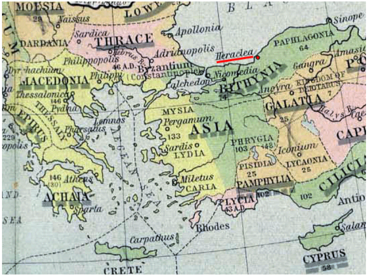

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Dynamical systems: iterations and cycles

Before we go any further, we need to introduce basic concepts of the theory of dynamical systems. The theory studies self-mappings of spaces with certain structure and their iterations, i.e. repeated self-compositions $\mathbf{f}, \mathbf{f}^2, \mathbf{f}^3, \dots$ of a map $\mathbf{f}$ with itself. If we have a map $\mathbf{f} : \mathbf{X} \to \mathbf{X}$ then we can talk about the $(\mathbf{f})$-**orbit** of a point $\mathbf{x} \in \mathbf{X}$ understood as the sequence $(\mathbf{x}, \mathbf{f}(\mathbf{x}), \mathbf{f}^2(\mathbf{x}), \mathbf{f}^3(\mathbf{x}), \dots)$ or as the corresponding set of points.

A point $\mathbf{x} \in \mathbf{X}$ is then said to be **periodic** if for some number $\mathbf{n}$ we have $\mathbf{f}^\mathbf{n}(\mathbf{x}) = \mathbf{x}$. If $\mathbf{n}$ is the minimal number with that property then $\mathbf{n}$ is called the **period** of $\mathbf{x}$, and the set of points $\{\mathbf{x} = \mathbf{f}^\mathbf{n}(\mathbf{x}), \mathbf{f}(\mathbf{x}), \dots, \mathbf{f}^{\mathbf{n}-1}(\mathbf{x})\}$ is said to be **periodic orbit** or a **cycle**. A point of period $\mathbf{1}$ is said to be **fixed**. By the IVT, a continuous self-mapping of an interval must have a fixed point.

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The Sharkovsky order

In 1961 Sharkovsky proved his famous theorem on coexistence of periods of cycles of continuous interval maps (the paper with the proof was published in 1964). It could have been proven any time after Bryson of Heraclea Pontica as it relies only on the IVT and various clever one-dimensional arguments. To state it let us first introduce the **Sharkovsky order** for positive integers:

$$\mathbf{3} \succ_s \mathbf{5} \succ_s \mathbf{7} \succ_s \ldots \succ_s \mathbf{2 \cdot 3} \succ_s \mathbf{2 \cdot 5} \succ_s \mathbf{2 \cdot 7} \succ_s \ldots \succ_s \mathbf{4} \succ_s \mathbf{2} \succ_s \mathbf{1}.$$

This order is extended by transitivity, so that if a number $\mathbf{m}$ is to the left of a number $\mathbf{n}$ in the Sharkovsky order, then we write $\mathbf{m} \succ_s \mathbf{n}$. In this case we say that $\mathbf{m}$ **forces** $\mathbf{n}$.

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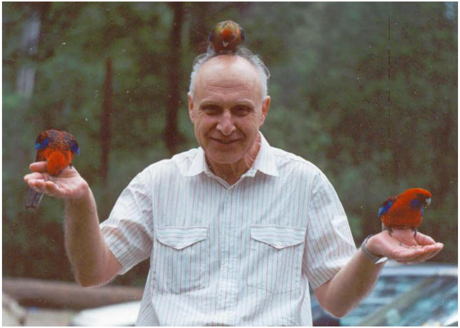
The Sharkovsky theorem

The Sharkovsky theorem can be stated in several ways. Here is the most transparent version.

Sharkovsky Theorem

If $\mathbf{g} : [0, 1] \to [0, 1]$ *is continuous,* $\mathbf{m} \succcurlyeq_{\mathbf{s}} \mathbf{n}$ *and there exists a point* $\mathbf{x} \in [0, 1]$ *of period* $\mathbf{m}$ *then* $\mathbf{g}$ *has a periodic point of period* $\mathbf{n}$*.*



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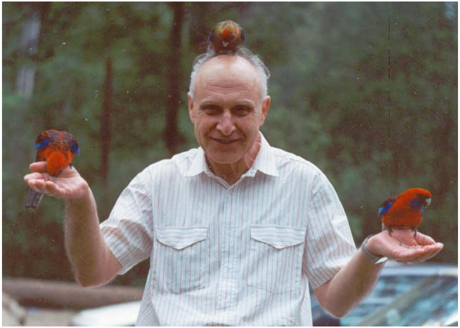

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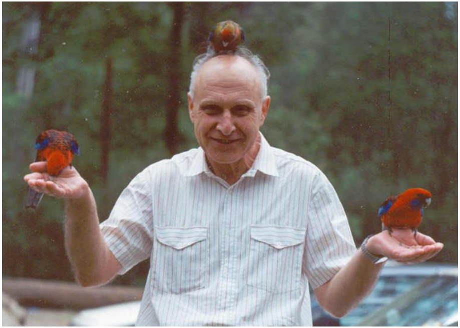

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The Sharkovsky theorem: an anecdote

That a continuous interval map with a point of period **3** must have points of all periods, was rediscovered in 1975 by Li and Yorke in their paper in American Mathematical Monthly titled **"Period Three Implies Chaos"**. Li and Yorke had no idea about the Sharkovsky theorem.

In 1976 Sharkovsky and Yorke visited Max Planck Institute for Mathematics in Bonn. Sharkovsky then could hardly speak English, but a famous Soviet mathematician Anosov was in Bonn, too. When they all took a boat ride on Rhein, Anosov and Sharkovsky explained the matter to Yorke. The issue was resolved, Yorke became a fan of the Sharkovsky theorem and advertised it at various meetings and conferences.

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Sharkovsky theorem: types of cycles

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Let us now consider an interpretation of the Sharkovsky theorem. Think of the period of a cycle as its **"type"**. Then the Sharkovsky Theorem shows how such **types** of cycles (i.e., their **periods**) force each other. What other types can we associate with interval periodic orbits? Periods are rather crude in this respect (many combinatorially different behaviors can be exhibited by cycles of a given period). It turns out that there exists another way of defining the type of a cycle for which a nice order can be established.

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Rotation theory: Poincaré

Rotation theory in the very general sense studies how orbits of points compare to rotations. We say it in a very vague fashion because the concept of rotation can be understood in a variety of ways. However in the case of circle maps the situation is more transparent and intuitive. The first steps here are due to Henri Poincaré who laid foundation for modern theory of dynamical systems.

Poincaré defined **rotation numbers** of circle homeomorphisms preserving orientation, i.e. 1-to-1 continuous self-mappings $\mathbf{f}$ of the circle $\mathbb{S}^1$ such that if $\mathbf{x}$ walks along $\mathbb{S}^1$ clockwise, $\mathbf{f}(\mathbf{x})$ does the same. Then the rotation number $\rho(\mathbf{f})$ describes the circular order among points from an orbit of a point $\mathbf{x} \in \mathbb{S}^1$.

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Rotation theory: modeling complicated maps

To state one related and nice result by Poincaré recall that a continuous map $\psi : \mathbb{S}^{\mathbf{1}} \to \mathbb{S}^{\mathbf{1}}$ is said to be **monotone** if for any point $\mathbf{x} \in \mathbb{S}^{\mathbf{1}}$ its preimage $\psi^{-\mathbf{1}}(\mathbf{x})$ is a closed arc or a point. Also, suppose that we have maps $\mathbf{f} : \mathbb{S}^{\mathbf{1}} \to \mathbb{S}^{\mathbf{1}}$ and $\mathbf{g} : \mathbb{S}^{\mathbf{1}} \to \mathbb{S}^{\mathbf{1}}$. These maps are said to be **monotonically semiconjugate** is there exists a monotone map $\psi : \mathbb{S}^{\mathbf{1}} \to \mathbb{S}^{\mathbf{1}}$ such that $\psi \circ \mathbf{f} = \mathbf{g} \circ \psi$ (if the map ψ is in fact a homeomorphism, we say that $\mathbf{f}$ and $\mathbf{g}$ are **conjugate**). Here is the corresponding commutative diagram:

$$\begin{tikzcd} \mathbb{S}^1 \arrow[r, "\mathbf{f}"] \arrow[d, "\psi"] & \mathbb{S}^1 \arrow[d, "\psi"] \\ \mathbb{S}^1 \arrow[r, "\mathbf{g}"] & \mathbb{S}^1 \end{tikzcd} \tag{1}$$

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Rotation theory: modeling complicated maps

Suppose now that $\rho(\mathbf{f})$ is irrational. Then Poincaré proved that $\mathbf{f}$ is monotonically semiconjugate to a true rigid irrational rotation $\tau_{\rho(\mathbf{f})}$ of $\mathbb{S}^{\mathbf{1}}$ by the angle $\rho(\mathbf{f})$. What it means is that one can collapse some (perhaps, infinitely many) closed arcs of $\mathbb{S}^{\mathbf{1}}$ to points and by doing so transform a given homeomorphism $\mathbf{f}$ to the irrational rotation $\tau_{\rho(\mathbf{f})}$ by the angle $\rho(\mathbf{f})$. In fact, Denjoy later proved that if $\mathbf{f}$ is twice differentiable, then $\mathbf{f}$ and $\tau_{\rho(\mathbf{f})}$ are in fact conjugate.

Poincaré's approach is key in dynamical systems. The idea is to model a given, seemingly very complicated map, using a more transparent map to which a given map is conjugate (or at least semiconjugate by a nice map). Finding good models for various classes of maps is an important part of the dynamical systems theory.

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Rotation theory on the interval

Take a periodic point $\mathbf{x}$ of period $\mathbf{q}$ on the interval. Take a vector from $\mathbf{x}$ to $\mathbf{f(x)}$ and consider its evolution as we apply $\mathbf{f}$ to its endpoints. Each time vector changes its direction we think of the rotation of the vector in the positive direction by $\frac{1}{2}$. In the end the vector comes back, so the overall angle by which it rotated is an integer $\mathbf{p}$. Then the **rotation pair** of $\mathbf{x}$ is $(\mathbf{p}, \mathbf{q})$ and the **rotation number** of $\mathbf{x}$ is $\frac{\mathbf{p}}{\mathbf{q}}$. Then, clearly, $\frac{\mathbf{p}}{\mathbf{q}} \leq \frac{1}{2}$.

Forcing of Rotation Pairs (B. and Misiurewicz, 1997)

If an interval map $\mathbf{f}$ has a periodic point of rotation pair $(\mathbf{m}, \mathbf{n})$ and $\frac{\mathbf{m}}{\mathbf{n}} < \frac{\mathbf{p}}{\mathbf{q}}$, then $\mathbf{f}$ has a periodic point of rotation pair $(\mathbf{p}, \mathbf{q})$.

One can say that if $(\mathbf{p}, \mathbf{q})$ and $(\mathbf{m}, \mathbf{n})$ are rotation pairs, and $\frac{\mathbf{m}}{\mathbf{n}} < \frac{\mathbf{p}}{\mathbf{q}}$, then $(\mathbf{m}, \mathbf{n})$ **forces** $(\mathbf{p}, \mathbf{q})$.

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Take a periodic point $\mathbf{x}$ of period $\mathbf{q}$ on the interval. Take a vector from $\mathbf{x}$ to $\mathbf{f(x)}$ and consider its evolution as we apply $\mathbf{f}$ to its endpoints. Each time vector changes its direction we think of the rotation of the vector in the positive direction by $\frac{1}{2}$. In the end the vector comes back, so the overall angle by which it rotated is an integer $\mathbf{p}$. Then the **rotation pair** of $\mathbf{x}$ is $(\mathbf{p}, \mathbf{q})$ and the **rotation number** of $\mathbf{x}$ is $\frac{\mathbf{p}}{\mathbf{q}}$. Then, clearly, $\frac{\mathbf{p}}{\mathbf{q}} \leq \frac{1}{2}$.

Forcing of Rotation Pairs (B. and Misiurewicz, 1997)

If an interval map $\mathbf{f}$ has a periodic point of rotation pair $(\mathbf{m}, \mathbf{n})$ and $\frac{\mathbf{m}}{\mathbf{n}} < \frac{\mathbf{p}}{\mathbf{q}}$, then $\mathbf{f}$ has a periodic point of rotation pair $(\mathbf{p}, \mathbf{q})$.

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Rotation theory on the interval

Notice that $\frac{1}{3} < \frac{2}{5} < \frac{3}{7} < \cdots < \frac{n}{2n+1} < \frac{n+1}{2n+3} < \cdots < \frac{1}{2}$. The closest to $\frac{1}{2}$ rotation number of a cycle is $\frac{n}{2n+1}$. Hence the first countable segment in the Sharkovsky Theorem follows; also, any odd period > 1 forces any even period. Considering f^2, we get the second countable segment (dealing with numbers of the form $2(2n+1)$), etc. The last countable segment with descending powers of two needs a separate proof which is rather easy.

Indeed, first one proves that a cycle of period 4 forces a cycle of period 2.

Now, suppose that f has a cycle of period 2^n and consider a map $g = f^{2^{n-2}}$. It follows that g has a cycle of period 4. By the above, g must have a cycle of period 2. Since $g = f^{2^{n-2}}$, then this implies that f has a cycle of period 2^{n-1}. In other words, 2^n forces 2^{n-1}.

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Indeed, first one proves that a cycle of period $\mathbf{4}$ forces a cycle of period $\mathbf{2}$.

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A couple of lemmas

To give you a flavor of one-dimensional combinatorial dynamics, here are some lemmas from the original diploma of Sharkovsky (1961).

L-scheme

Suppose that there are points $\mathbf{a}, \mathbf{x}$ such that $\mathbf{f}^2(\mathbf{x}) \leq \mathbf{a} = \mathbf{f}(\mathbf{a}) < \mathbf{x} < \mathbf{f}(\mathbf{x})$. Then $\mathbf{f}$ has cycles of all periods. If a map has such points one says that it has **L-scheme**.

Forcing L-scheme

Suppose that in a cycle there are points $\mathbf{y}$ and $\mathbf{x}$ with $\mathbf{f}(\mathbf{y}) < \mathbf{y} < \mathbf{x} < \mathbf{f}(\mathbf{x})$. Then $\mathbf{f}$ has L-scheme, and therefore, cycles of all periods.

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Another look at rotation and one more lemma

By the above the only cycles of interest are the ones that do not force $\mathbf{f}$ to have L-scheme. These are the cycles in which there are no points $\mathbf{x}, \mathbf{y}$ with $\mathbf{f(y) < y < x < f(x)}$. It means that in a cycle all points mapped to the right are located to the left, and all points mapping to the left are located to the right. Hence there is a fixed point, say, $\mathbf{a}$ between these two collections of points. We can connect $\mathbf{a}$ and a point from the cycle with a rubber string, and count every "jump" of a point of the cycle over $\mathbf{a}$ as rotation by $\frac{1}{2}$. This is another way to figure the rotation pair of a cycle.

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