

UNICRITICAL LAMINATIONS

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ABSTRACT. Thurston introduced *invariant (quadratic) laminations* in his 1984 preprint as a vehicle for understanding the connected Julia sets and the parameter space of quadratic polynomials. Important ingredients of his analysis of the angle doubling map σ_2 on the unit circle \mathbb{S}^1 were the Central Strip Lemma, non-existence of wandering polygons, the transitivity of the first return map on vertices of periodic polygons, and the non-crossing of minors of quadratic invariant laminations. We use Thurston's methods to prove similar results for *unicritical* laminations of arbitrary degree d and to show that the set of so-called *minors* of unicritical laminations themselves form a *Unicritical Minor Lamination* UML_d . In the end we verify the *Fatou conjecture* for the unicritical laminations and extend the *Lavaurs algorithm* onto UML_d .

INTRODUCTION

One of the most satisfactory strengths of polynomial dynamics is its successful interplay with symbolic dynamics: even though the Julia sets and the Mandelbrot set have very complicated topological structure, there are very simple combinatorial models that describe the original sets, often up to homeomorphism.

In the “early days of modern holomorphic dynamics”, in the mid-1980’s, Bill Thurston developed his theory of quadratic minors for connected Julia sets of quadratic polynomials, and for their parameter space, the *Mandelbrot set* [Thu85]: he introduced the concept of *invariant quadratic laminations* as subsets of the closed unit disk and described their properties; each of these models a Julia set in the sense that a natural quotient, called *pinched disk* by Douady [Dou93], is homeomorphic to the (filled-in) Julia set if and only if the latter is locally connected; see also [Sch09]. Thurston showed that invariant quadratic lamination is characterized by a unique leaf, called its *minor leaf* (a set of two or possibly one angles), and each angle is part of a

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unique minor leaf. Turning to parameter space, Thurston showed that the union of all minor leaves forms another lamination, the *quadratic minor lamination* QML, that models the Mandelbrot set via its own pinched disk, and is homeomorphic to it exactly when the Mandelbrot set is locally connected, which continues to form a major conjecture in the field. Douady, Hubbard, and Lavaurs showed that QML and hence the Mandelbrot set can be constructed by a very simple algorithm, known as the *Lavaurs algorithm*, that inductively constructs all periodic minor leaves in the order of their period, and so that the closure of all periodic leaves is all of QML.

Thurston already started the question to extend his theory of quadratic minor laminations to higher degrees. For arbitrary degrees $d \geq 3$, this is a substantial and difficult task. The first question that Thurston viewed as fundamental, whether “wandering triangles” exist, was resolved in the affirmative in [B008]; the non-existence of wandering triangles is one of the key lemmas in his quadratic theory (see Theorem II.5.2 from [Thu85]), so this creates a substantial difficulty. A second difficulty lies in the fact that parameter space of degree d polynomials has complex dimension $d - 1$, so the one-dimensional theory of parameter spaces fails fundamentally. Even the combinatorial description of cubic polynomials is most difficult; see for instance [Mil08] and [Kaf06]. Thurston’s work on general laminations of degree d was recently published in [TBGHLLT19]

In this manuscript, we focus on a particular class of degree d laminations called *unicritical*: unicritical polynomials can always be parameterized as $z^d + c$, so they have a one-dimensional parameter space; their connectedness locus is often called the *Multibrot set*, and it is well known to have profound similarities to the Mandelbrot set (see for instance [Sch04]). Fundamental properties of unicritical laminations have been known for a long time, in particular the non-existence of wandering triangles [Lev98, Sch00].

We develop unicritical laminations for degree d in analogy to Thurston’s quadratic laminations. While it has been expected frequently that many of his original methods can be made to work in this setting, we are not aware of a systematic treatment that describes unicritical laminations in detail. Our main results contain the following:

(1) we describe unicritical laminations; in particular, we show that each of its gaps is periodic, and its boundary either consists of finitely many leaves (modeling finitely many rays landing at a periodic point in the Julia set), or it is a Cantor set of leaves that returns back to itself either with degree 1 or degree d (modeling a Siegel disk or a Cremer point, respectively an attracting periodic orbit);

- (2) we show that each unicritical lamination has a unique minor leaf, which may or may not be degenerate, that every angle occurs as the minor leaf of a unicritical lamination, and the union of all such minor leaves of given degree $d \geq 2$ forms another lamination, called the unicritical minor lamination (Theorem 4.15) associated with an equivalence relation on the unit circle;
- (3) in the unicritical minor lamination, the boundary of each gap is either finite (modeling finitely many preperiodic parameter rays landing at a common point), or it is a Cantor set (modeling a hyperbolic component);
- (4) each unicritical minor lamination is the closure of its periodic leaves (all of which are non-degenerate), and these leaves can be constructed recursively by increasing periods by a simple algorithm analogous to Lavaurs' algorithm (Theorem 5.3 and Section 6).

Section 1 contains basic definitions and results. In Section 2 we study *unicritical σ_d -invariant laminations* and show that the minors of *unicritical laminations* form a *lamination* called *Unicritical Minor Lamination of degree d* and denote it by UML_d . In Section 3 we study the basic properties of UML_d . In Section 4 we prove that UML_d is a *q-lamination*. In Section 5, we prove additional facts that help us to devise an algorithm to construct UML_d similar to the *Lavaurs algorithm* [Lav89] for the quadratic case (see Section 6).

1. PRELIMINARIES

1.1. Laminational equivalence relations. Let $\hat{\mathbb{C}}$ be the Riemann sphere. For a compactum $X \subset \mathbb{C}$, let $U^\infty(X)$ be the component of $\hat{\mathbb{C}} \setminus X$ containing ∞ . For X connected, let $\Psi_X : \hat{\mathbb{C}} \setminus \overline{\mathbb{D}} \rightarrow U^\infty(X)$ be a Riemann map with $\Psi_X(\infty) = \infty$, and $\Psi_X(z)$ tending to a positive real limit as $z \rightarrow \infty$. Let P be a monic polynomial of degree $d \geq 2$. Consider the Julia set J_P and the filled Julia set K_P of P . Set θ_d be the map $z^d|_{U^\infty(\overline{\mathbb{D}})}$. If J_P is connected, $\Psi \circ \theta_d = P \circ \Psi$ [DH8485, Mil00]. If J_P is locally connected, Ψ extends to a continuous map $\overline{\Psi} : \hat{\mathbb{C}} \setminus \overline{\mathbb{D}} \rightarrow \overline{\hat{\mathbb{C}} \setminus K_P}$, and $\overline{\Psi} \circ \theta_d = P \circ \overline{\Psi}$ on $\mathbb{C} \setminus \overline{\mathbb{D}}$. A continuous surjection $\overline{\Psi}|_{\mathbb{S}^1} = \psi : \mathbb{S}^1 \rightarrow J_P$ is called the *Caratheodory loop*. We will write σ_d for $\theta_d|_{\mathbb{S}^1}$. Define an equivalence relation \sim_P on \mathbb{S}^1 by $x \sim_P y$ iff $\psi(x) = \psi(y)$ and call \sim_P the *laminational equivalence relation (generated to the polynomial P)*. The map ψ semi-conjugates σ_d and $P|_{J(P)}$, and \sim_P is invariant; \sim_P -classes have pairwise disjoint convex hulls. The quotient space $\mathbb{S}^1 / \sim_P = J_{\sim_P}$ is called the *topological Julia set*. Clearly, J_{\sim_P} is homeomorphic to J_P . The map $f_{\sim_P} : J_{\sim_P} \rightarrow J_{\sim_P}$, induced by σ_d , called the *topological*

polynomial, is topologically conjugate to $P|_{J_P}$. Define an equivalence relation \sim on \mathbb{S}^1 similar to \sim_P but with no references to polynomials.

Definition 1.1. An equivalence relation \sim on the unit circle \mathbb{S}^1 is said to be *laminational* if :

- (1) the graph of \sim is a closed subset of $\mathbb{S}^1 \times \mathbb{S}^1$;
- (2) convex hulls of disjoint equivalence classes are disjoint;
- (3) each equivalence class of \sim is finite.

The equivalence relation \sim is (σ_d) -invariant if:

- (1) \sim is *forward invariant*: for a \sim -class $[g]$, the set $\sigma_d([g])$ is a \sim -class too;
- (2) for any \sim -class $[g]$, the map $\tau = \sigma_d|_{[g]}$ extends to \mathbb{S}^1 as an orientation preserving covering map $\hat{\tau}$ such that $[g]$ is the full preimage of $\tau([g])$ under the covering map $\hat{\tau}$.

Assume that \sim is a σ_d -invariant laminational equivalence relation. Consider the *topological Julia set* $\mathbb{S}^1 / \sim = J_\sim$, and the *topological polynomial* $f_\sim : J_\sim \rightarrow J_\sim$ induced by σ_d . We can use Moore's Theorem to embed J_\sim into \mathbb{C} and to extend the quotient map $\psi_\sim : \mathbb{S}^1 \rightarrow J_\sim$ to a map $\psi_\sim : \mathbb{C} \rightarrow \mathbb{C}$ with the only non-degenerate *fibers* (point preimages) being the convex hulls of non-degenerate \sim -classes. A *Fatou domain* of J_\sim (f_\sim) is a component of $\mathbb{C} \setminus J_\sim$. If U is a periodic *Fatou domain* of f_\sim of period n , $f_\sim^n|_{\text{Bd}(U)}$ is conjugate to an irrational rotation of \mathbb{S}^1 (then U is called a *Siegel domain*) or to σ_k for some $k > 1$ [BL02]. The complement K_\sim of $U(J_\sim)$ is the *filled topological Julia set*, i.e. the union of J_\sim and its bounded *Fatou domains*. If \sim is fixed, we omit it from the notation. We consider f_\sim as a self-mapping of J_\sim .

Definition 1.2. For a \sim -class A , call an edge \overline{ab} of $\text{CH}(A)$ (CH denotes *convex hull*) a (\sim) -*leaf*. All points of \mathbb{S}^1 are called *degenerate \sim -leaves*.

The limit of a converging sequence of \sim -leaves \sim is a \sim -leaf (the family of all \sim -leaves is closed); the union of all \sim -leaves is a continuum. For $X \subset \overline{\mathbb{D}}$ with the property $X = \text{CH}(X \cap \mathbb{S}^1)$, set $\sigma_d(X) = \text{CH}(\sigma_d(X \cap \mathbb{S}^1))$; then for any leaf ℓ of \sim , the set $\sigma_d(\ell)$ is a leaf.

Definition 1.3. The set \mathcal{L}_\sim of all leaves of \sim is called the *lamination generated by \sim* . A lamination generated by some laminational equivalence relation is called a *q-lamination*.

1.2. Laminations as defined by Thurston. [Thu85] By a *prelamination* we mean a collection \mathcal{L} of pairwise unlinked chords in $\overline{\mathbb{D}}$, called *leaves*. For a collection \mathcal{R} of sets, we denote the union of all sets from \mathcal{R} by \mathcal{R}^+ . If all points of the circle are elements of \mathcal{L} (seen as *degenerate*

leaves) and \mathcal{L}^+ is closed in \mathbb{C} , then we call \mathcal{L} a *lamination*. Hence, a lamination can be obtained by closing a *prelamination* and adding all points of \mathbb{S}^1 viewed as degenerate leaves. If $\ell = \overline{ab}$ is a leaf, let $\sigma_d(\ell)$ be the chord with endpoints $\sigma_d(a)$ and $\sigma_d(b)$. If $\sigma_d(a) = \sigma_d(b)$, call ℓ a *critical leaf* and $\sigma_d(a)$ a *critical value*. If $\sigma_d^* : \mathcal{L}^+ \rightarrow \overline{\mathbb{D}}$ is the linear extension of σ_d over all the leaves in \mathcal{L} , then σ_d^* is continuous. Also, σ_d is locally one-to-one on \mathbb{S}^1 , and σ_d^* is one-to-one on any given non-critical leaf. If \mathcal{L} is a lamination, then \mathcal{L}^+ is a continuum.

We call the closure of a component of $\mathbb{D} - \mathcal{L}^+$ a *gap*. For a gap G , call $G \cap \mathbb{S}^1$ the *basis* of G and the points of $G \cap \mathbb{S}^1$ *vertices* of G . A gap G is *infinite* if and only if $G \cap \mathbb{S}^1$ is infinite.

Definition 1.4. Let \mathcal{L} be a lamination. The equivalence relation $\approx_{\mathcal{L}}$ on \mathbb{S}^1 defined by declaring that $x \approx_{\mathcal{L}} y$ if and only if there exists a finite concatenation of leaves of \mathcal{L} joining x and y will be called the *equivalence relation generated by \mathcal{L}* .

Recall that q -laminations were defined above in Definition 1.3. Equivalently, a lamination \mathcal{L} is a q -lamination if and only if the equivalence relation $\approx_{\mathcal{L}}$ is an invariant laminational equivalence relation as defined in Definition 1.1 and \mathcal{L} consists exactly of the edges of the convex hulls of all $\approx_{\mathcal{L}}$ -classes together with all points of \mathbb{S}^1 .

A lamination \mathcal{L} is *Thurston σ_d -invariant* if it is (1) *forward σ_d -invariant*: for any leaf $\ell = \overline{ab} \in \mathcal{L}$, either $\sigma_d(a) = \sigma_d(b)$ or $\sigma_d(\ell) = \overline{\sigma_d(a)\sigma_d(b)} \in \mathcal{L}$, (2) *backward σ_d -invariant*: for any leaf $\ell = \overline{ab} \in \mathcal{L}$ there is a collection of d disjoint leaves in \mathcal{L} each joining a preimage of a to a preimage of b , (3) *gap σ_d -invariant*: for any gap G in \mathcal{L} , the convex hull H of $\sigma_d(G \cap \mathbb{S}^1)$ is a gap of \mathcal{L} , a leaf of \mathcal{L} , or a point of \mathbb{S}^1 . If H is a gap, the map $\sigma_d^*|_{\text{Bd}(G)} : \text{Bd}(G) \rightarrow \text{Bd}(H)$ is the composition of a monotone and a covering map to $\text{Bd}(H)$ with positive orientation, and the image of a point moving in the positive direction around $\text{Bd}(G)$ moves (non-strictly) in the positive direction around $\text{Bd}(H)$.

A gap G of a σ_d -invariant lamination is called *σ_d -critical* if for each $y \in \sigma_d(G)$ the set $\sigma_d^{-1}(y) \cap G$ consists of at least 2 points. A set of critical leaves $\mathcal{C} = \{\overline{c_i}\}_{i=1}^{d-1}$ is *full* if \mathcal{C}^+ contains no polygons. Theorem 1.5 shows how to generate *Thurston invariant laminations*:

Theorem 1.5. [Thu85] *Let \mathcal{L} be a lamination without critical gaps which satisfies the following conditions:*

- (1) \mathcal{L} is forward σ_d invariant and
- (2) there is a full collection of critical leaves in \mathcal{L} .

Then there exists a σ_d -invariant lamination $S(\mathcal{L})$ which contains \mathcal{L} and is obtained by taking pullbacks of the leaves in \mathcal{L} .

Definition 1.6. The lamination $S(\mathcal{L})$ is called the *saturation* of \mathcal{L} .

1.3. Classification of gaps.

Definition 1.7 (Periodic and (pre)periodic gaps). Let G be a gap of a d -invariant lamination \mathcal{L} . G is *(pre)periodic* if $\sigma_d^{m+k}(G) = \sigma_d^m(G)$ for some $m \geq 0$, $k > 0$; if m, k are chosen to be minimal, then G is said to be *preperiodic* if $m > 0$ or *periodic* (of *period* k) if $m = 0$. If the period of G is 1, then G is said to be *invariant*. Define *pre-critical* and *(pre)critical* objects similarly to (pre)periodic objects defined above.

Consider infinite periodic gaps of σ_d -invariant laminations. By [Kiw02], infinite gaps are eventually mapped onto periodic infinite gaps. We state (without a proof) folklore results about (pre)periodic (in particular, infinite) gaps (for q -laminations see [BL02], see also [BOPT20]). Any edge of a (pre)periodic gap is either (pre)periodic or precritical.

Definition 1.8 (Fatou gaps). An infinite gap G is said to be a *Fatou gap* if its basis $G \cap \mathbb{S}^1$ is uncountable.

If G is a Fatou gap, $G \cap \mathbb{S}^1$ can be represented as the union of a non-empty perfect set and a countable set. By [Kiw02] G is *(pre)periodic* under σ_d . If G is a *Fatou gap*, then the map $\psi_G : \text{Bd}(G) \rightarrow \mathbb{S}^1$, collapsing all edges of G to points, monotonically maps $\text{Bd}(G)$ onto \mathbb{S}^1 .

Theorem 1.9 (Siegel gaps and Fatou gaps of degree $k > 1$). *Let G be a periodic Fatou gap of minimal period n . Then ψ_G semiconjugates $\sigma_d^n|_{\text{Bd}(G)}$ to a map $\hat{\sigma}_G = \hat{\sigma} : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ so that either (1) $\sigma_n|_{\text{Bd}(G)}$ is of degree $k \geq 2$ and $\hat{\sigma} = \sigma_k : \mathbb{S}^1 \rightarrow \mathbb{S}^1$, or (2) $\sigma_n|_{\text{Bd}(G)}$ is of degree one and $\hat{\sigma}$ is an irrational rotation.*

If Theorem 1.9(1) holds, G is called a *periodic gap of degree k* , if Theorem 1.9(2) holds, G is called a *Siegel gap*. A (pre)periodic gap eventually mapped to a periodic gap of degree k (Siegel) is also said to be *of degree k (Siegel)*.

1.4. Sibling (σ_d -)invariant laminations. *Sibling invariant laminations* were introduced in [BMOV13]. This notion is slightly more restrictive than that of *Thurston invariant lamination*. However, *sibling invariant geolaminations form a closed set, include all q -laminations, and are Thurston invariant*. For our purposes it suffices to consider sibling invariant laminations only. A major advantage of working with them is that they are defined exclusively through properties of their leaves. A lamination \mathcal{L} is *sibling (σ_d -)invariant* if: (1) for each $\ell \in \mathcal{L}$, we have $\sigma_d(\ell) \in \mathcal{L}$ (2) for each $\ell \in \mathcal{L}$ so that $\sigma_d(\ell)$ is a non-degenerate leaf, there exist d disjoint leaves $\ell_1, \ell_2, \dots, \ell_d$, in \mathcal{L} so that $\ell = \ell_1$ and

$\sigma_d(\ell_i) = \sigma_d(\ell)$ for all $i = 1, 2, \dots, d$. (3) for each $\ell \in \mathcal{L}$, there exists $\ell_1 \in \mathcal{L}$ so that $\sigma_d(\ell_1) = \ell$.

Theorem 1.10 ([BMOV13]). *The space of all sibling σ_d -invariant laminations is compact. All q -laminations are sibling σ_d -invariant.*

In what follows instead of *sibling σ_d -invariant laminations* we talk of *σ_d -invariant laminations*. Also, we talk interchangeably about leaves (gaps) of \sim or of \mathcal{L}_\sim .

2. UNICRITICAL INVARIANT LAMINATIONS: BASIC PROPERTIES

Definition 2.1. A σ_d -invariant lamination is said to be *unicritical* if it has a unique critical set $\mathcal{C}(\mathcal{L})$ (which maps forward d -to-1).

From now on let \mathcal{L} be a σ_d -invariant unicritical lamination. Let us discuss properties of \mathcal{L} . Clearly, $\mathcal{C}(\mathcal{L}) = \mathcal{C}$ is symmetric with respect to rotations by $\frac{1}{d}$. The set \mathcal{C} can be a regular d -gon whose vertices cut the circle into d arcs of length $\frac{1}{d}$ each. Otherwise \mathcal{C} has more than d sides and each circle arc complementary to $\mathcal{C} \cap \mathbb{S}^1$ is shorter than $\frac{1}{d}$. Thus, \mathcal{C} and in general any *unicritical lamination* \mathcal{L} of degree d has a *d -fold rotational symmetry*. In particular, every leaf $\ell \in \mathcal{L}$ has $d - 1$ distinct *siblings*, all of equal length, placed symmetrically with respect to the origin. A *unicritical lamination* \mathcal{L} has at least d leaves of maximal length called the *majors* of \mathcal{L} . Indeed, any unicritical lamination has the d -fold rotational symmetry and so a major has $d - 1$ distinct siblings. However a priori there might exist another group of d siblings of the same length, etc. We will show below that because of the properties of the length function this is not possible. In particular, this will imply that there is only one group of d sibling majors, and, therefore, a unique image of a major of a lamination \mathcal{L} called the *minor* of \mathcal{L} .

Let (a, b) be the arc of \mathbb{S}^1 from the point a to the point b traversed anticlockwise; let $|(a, b)|$ be the length of this arc. Define the *length* $|\overline{ab}|$ of a chord \overline{ab} as $|\overline{ab}| = \min\{|(a, b)|, |(b, a)|\}$. The maximum length of a chord is $\frac{1}{2}$. Call a chord $\ell = \overline{ab}$ *critical* if $\sigma_d(a) = \sigma_d(b)$. The length of a critical chord ℓ is $|\ell| = \frac{j}{d}$ where $j \in \{1, \dots, t \mid t = [\frac{d}{2}]\}$ where $[\frac{d}{2}]$ denotes the greatest integer not exceeding $\frac{d}{2}$. Evidently, $|\sigma_d(\ell)|$ as a function of $|\ell|$ can be described as a “sawtooth” d -to-1 map $\psi : [0, \frac{1}{2}] \rightarrow [0, \frac{1}{2}]$ with d segments of monotonicity (*laps*) and slope d or $-d$ on each of them; increasing and decreasing laps alternate and the first lap $[0, \frac{1}{2d}]$ is increasing. In what follows set $\psi(t)$ to be the *length function* reflecting how $|\sigma_d(\ell)|$ depends on $|\ell|$. The line $y = x$ intersects the graph of $|\sigma_d(\ell)|$ with respect to $|\ell|$ in the interval $[0, \frac{1}{d}]$ at the points 0 and $\frac{1}{d+1}$.

Lemma 2.2. *A leaf ℓ of length $|\ell| < \frac{1}{d+1}$ will keep increasing in length under iteration of σ_d until for some iterate we have $|\sigma_d^j(\ell)| \geq \frac{1}{d+1}$. Thus, $|M(\mathcal{L})| \geq \frac{1}{d+1}$. Hence there are exactly d majors of \mathcal{L} that are all siblings and a unique minor $m(\mathcal{L})$ such that $|m(\mathcal{L})| \leq \frac{1}{d+1}$.*

Proof. We leave to the reader the proof of initial claims of Lemma 2.2 dealing with how the length of a leaf grows. This implies that $|M(\mathcal{L})| \geq \frac{1}{d+1}$. If there exists another group of d majors which have the same image and are of the length $|M(\mathcal{L})|$, this will imply that $1 \geq 2d|M(\mathcal{L})| \geq \frac{2d}{d+1} > 1$, a contradiction (recall that $d \geq 2$). \square

For a non-diameter chord $n = \overline{ab}$, the smaller of the two arcs into which n divides \mathbb{S}^1 , is denoted by $H(n)$. Also, whenever we talk about circle arcs (x, y) of any length we always mean that the movement from x to y inside (x, y) is in the positive direction. In the case of $H(n)$ we always set $n = \overline{ab}$ (thus, the movement inside $H(n)$ from a to b is in the positive direction). In that case we refer to points of $H(n)$ as located “under” n . In this setting a is called the *initial* point of n (and $H(n)$) while b is called the *terminal* point of n (and $H(n)$). Thus, for a non-diameter chord ℓ we can write $\ell = \overline{i(\ell)t(\ell)}$.

Call the *region* of the disk bounded by m and $H(m)$ the *region bounded by m* and denote it by $\mathcal{R}(m)$. Given two non-diameter chords m and n , write $m \succ n$ if $n \subset \mathcal{R}(m)$, and say that n is a *successor* of m . If $m, n \in \text{UML}_d$, $m \succ n$ and there exists no $\ell \in \text{UML}_d$ satisfying $m \succ \ell \succ n$, then n is called an *immediate successor* of m .

Given a minor $m = \overline{ab}$, call the arc $H(m)$ a *minor arc*. Let $\ell = \ell_0 \in \mathcal{L}$; let $\ell_1, \dots, \ell_{d-1}$ be the $d-1$ siblings of the leaf ℓ . There are $d+1$ complementary components to $\bigcup \ell_i$ in $\overline{\mathbb{D}}$. The component $C(\ell)$ bounded by ℓ and its $d-1$ siblings is called the *Central Strip* (defined ℓ). The remaining d components are $\mathcal{R}(\ell_i), 0 \leq i \leq d-1$.

Lemma 2.3 is based on Lemma 2.2; the proof is left to the reader.

Lemma 2.3. *Let \mathcal{L} be a lamination, $\ell \in \mathcal{L}$ be a leaf, $M = M(\mathcal{L})$ and $m = m(\mathcal{L})$. Then the length of images of ℓ increases d -fold on each step, and $H(\sigma_d^i(\ell))$ maps onto $H(\sigma_d^{i+1}(\ell))$ homeomorphically until $\frac{|m|}{d} \leq |\sigma_d^N(\ell)| < |m|$. Then $H(\sigma_d^N(\ell))$ homeomorphically maps onto $H(\sigma_d^{N+1}(\ell))$, and $|m| \leq |\sigma_d^j(\ell)| \leq |M|$ for any $j > N$. Any leaf ℓ' with $|M| \geq \ell' \geq |m|$ is located in $C(m) \setminus C(M)$; in particular, if $i > N$ then $|M| \geq \sigma_d^i(\ell) \geq |m|$, the leaf $\sigma_d^i(\ell)$ is located in $C(m) \setminus C(M)$, and $\sigma_d^i(\ell)$ never separates ℓ and m .*

Define a *wandering triangle* as a triangle which has triangular images under σ_d so that the circular orientation on its vertices is preserved

and, moreover, has images with edges that do not cross. We want to show that a gap G of \mathcal{L} is either (pre)periodic or (pre)critical. By way of contradiction suppose G is a gap which is neither (pre)critical or (pre)periodic. Then any three vertices of G form a convex hull which is, evidently also neither (pre)periodic nor (pre)critical. Then the problem of proving that any gap G of \mathcal{L} is either (pre)periodic or (pre)critical boils down to the next lemma.

Lemma 2.4. *There are no wandering triangles whose images do not cross $\mathcal{C}(\mathcal{L})$. In particular, any gap of \mathcal{L} is (pre)periodic or (pre)critical, and if $\mathcal{C}(\mathcal{L})$ is (pre)periodic then any gap of \mathcal{L} is (pre)periodic.*

Proof. Let $A^0 = \text{CH}(x_0, y_0, z_0)$ be a wandering triangle whose images do not cross $\mathcal{C}(\mathcal{L})$. Set $A^i = \sigma_d(A^0)$, $\sigma_d^i(x) = x_i$, $\sigma_d^i(y) = y_i$, and $\sigma_d^i(z) = z_i$. Let L^i be the length of the longest side of A^i and let s^i be the length of the shortest side of A^i (in defining L^i and s^i we use the above introduced length function so that $L^i \geq \frac{1}{d+1} \geq s^i$ while equalities are only possible if A^i is an equilateral triangle). Evidently, there exists a continuous increasing function $\Delta : [0, \frac{1}{d+1}] \rightarrow \mathbb{R}_{\geq 0}$ such that $\Delta(0) = 0$ and for every $0 \leq t \leq \frac{1}{d+1}$ the area of a triangle inscribed in the unit disk with the shortest side at least t equals $\Delta(t)$. Denote the Lebesgue measure of a set T by $\lambda(T)$.

Since $\lambda(\mathbb{D}) < \infty$ and $\lambda(A^i) > 0$, then $\lambda(A^i) \xrightarrow{i \rightarrow \infty} 0$. Since $\lambda(A^i) \geq \Delta(s^i) > 0$, then $s^i \xrightarrow{i \rightarrow \infty} 0$. We can inductively choose a subsequence i_j such that $i_{j+1} > i_j$ is the first moment when $s^{i_{j+1}} < s^{i_j}$. Then, at the preceding moment, the triangle $A^{i_{j+1}-1}$ has a side close to $\frac{1}{d}$ (this side is the one whose image has length s^{i_j}). Hence the critical set $\mathcal{C}(\mathcal{L})$ is a regular polygon with critical edges of length $\frac{1}{d}$. Since $s^i \rightarrow 0$, the triangles $A^{i_{j+1}-1}$ have one short side and two long sides close to an edge of $\mathcal{C}(\mathcal{L})$, of whom the closer to the edge side is of length $L^{i_j} \xrightarrow{i \rightarrow \infty} \frac{1}{d}$.

We may assume that $i_1 = 2$ and A^1 already is close to an edge of $\mathcal{C}(\mathcal{L})$ and has the above described properties. Denote the sides of A^1 by X^1 , Y^1 and Z^1 where $|Z^1| \ll |Y^1| \lesssim |X^1| \lesssim \frac{1}{d}$. Evidently, $s^{i_1} = d(\frac{1}{d} - |X^1|)$. The triangle A^{i_2-1} must lie in the Central Strip $C(X^1)$ of X^1 (because $s^{i_2} < s^2 = s^{i_1}$), thus, A^{i_2-1} is squeezed between $\mathcal{C}(\mathcal{L})$ and the d siblings of X^1 . However this implies that it has at least one side of length less than $\frac{1}{d} - |X^1| < s^1 = d(\frac{1}{d} - |X^1|)$, a contradiction with the choice of sequence i_j (by definition, the first moment the shortest side of the image of A^1 drops below s^1 must be i_2 , not $i_2 - 1$). \square

Now we show that if G is a non-critical periodic gap of \mathcal{L} , then G is finite and the first return map g of G permutes the sides of G transitively as a rational rotation. For a set T let $\text{int}(T)$ be its interior.

Lemma 2.5. *Let G be a periodic gap of \mathcal{L} , and there are no critical gaps in the orbit of G . Then G is finite and if g is the first return map of G , then g permutes the sides of G transitively as a rational rotation.*

Proof. By the assumption and because \mathcal{L} is unicritical, the first return map $g : \text{Bd}(G) \rightarrow \text{Bd}(G)$ is a homeomorphism that acts as a bijection on the sides of G . Suppose that there are $N \geq 2$ cycles $\mathcal{A}_1, \dots, \mathcal{A}_N$ of sides of G . For each i , choose the leaf $\ell_i \in \mathcal{A}_i$ of shortest length s_i ; by the properties of the length function, $s_i < \frac{1}{d+1}$. Let $\ell'_i \in \mathcal{A}_i$ be the leaf with $\sigma_d(\ell'_i) = \ell_i$. Then $L_i = |\ell'_i| > \frac{1}{d+1}$.

Denote the circle arc of length s_i with the endpoints of ℓ_i by H_i . By the above $\sigma_d(\mathcal{C}(\mathcal{L})) \cap \mathbb{S}^1 \subset H_i$ for any i , and we may number these arcs so that $\{H_1 \supset \dots \supset H_N\}$ is a nested collection. Let $N \geq 3$. Then ℓ_N and ℓ_1 must be edges of distinct images of G . However then some image of ℓ_1 is a leaf with endpoints contained in H_2 , a contradiction with the choice of ℓ_1 . Now let $N = 2$. Let ℓ_i be an edge of G_i where G_i is some image of G , $i = 1, 2$. Since G is a gap, each \mathcal{A}_i has at least two leaves in G . If ℓ_1 separates $\mathcal{C}(\mathcal{L})$ from $\text{int}(G_1)$ then this implies that some image of ℓ_1 distinct from ℓ_1 will have endpoints in H_1 , a contradiction with the choice of ℓ_1 . Hence the opposite takes place and it is ℓ_1 that is separated from $\mathcal{C}(\mathcal{L})$ by $\text{int}(G_1)$. Now the fact that $H_1 \supset H_2$ implies that $G_2 \neq G_1$ has all its vertices in H_1 so that, again, some image of ℓ_1 distinct from ℓ_1 , has endpoints in H_1 , a contradiction. \square

A crucial Thurston's result in the *quadratic case* was the *Central Strip Lemma* stating that no forward image of the *minor* of a *quadratic lamination* \mathcal{L} can enter the *Central Strip* defined by the *major*s of the lamination \mathcal{L} . A version of this result holds for *unicritical laminations of arbitrary degree d* . Recall that the Central Strip $C(\ell)$ of a leaf ℓ is closed. Call components of $C(\ell) \cap \mathbb{S}^1$ *holes* of $C(\ell)$, and call their closures *closed holes* of $C(\ell)$. Extending this concept, for a compact set A call a component of $\mathbb{S}^1 \setminus A$ a *hole of A* ; if ℓ is a chord connecting the endpoints of a hole of A , we say that the hole is located *behind* ℓ .

Lemma 2.6 (Central Strip Lemma). *Let $\ell \in \mathcal{L}$, $|\ell| < \frac{1}{d}$. If $\sigma_d^j(\ell)$ is the first time when ℓ re-enters $C(\ell)$ then there are three cases.*

- (1) *If $|\ell| \leq \frac{1}{d(d+1)}$ then $\sigma_d(\ell) \subset C(\ell)$ and the endpoints of $\sigma_d(\ell)$ may belong to one or two closed holes of $C(\ell)$.*
- (2) *If $\frac{1}{d(d+1)} < |\ell| \leq \frac{1}{d+1}$ then $\sigma_d(\ell) \subset C(\ell)$ and the endpoints of $\sigma_d(\ell)$ belong to two consecutive closed holes of $C(\ell)$.*

(3) If $\frac{1}{d+1} < |\ell|$ then $j > 1$ and the endpoints of $\sigma_d^j(\ell)$ belong to two consecutive closed holes of $C(\ell)$.

Proof. Set $|\ell| = t$. If $x < \frac{1}{d+1}$, then $|\sigma_d(\ell)| = \psi(t) > t$ and $\sigma_d(\ell)$ must be contained in $C(\ell)$. The length of each hole of $C(\ell)$ is $\frac{1}{d} - t$. Computations show that if $t \leq \frac{1}{d(d+1)}$ then $\psi(t) \leq \frac{1}{d} - t$. Therefore in this case the endpoints of $\sigma_d(\ell)$ may belong to the same closed hole of $C(\ell)$. On the other hand, it is easy to see that if $\frac{1}{d(d+1)} < t \leq \frac{1}{d+1}$ then $t \leq \psi(t)$ and $\frac{1}{d} - t < \psi(x)$ which implies the desired.

It remains to consider the case when $\frac{1}{d+1} < t < \frac{1}{d}$ ($t \neq \frac{1}{d}$ by the assumptions of the lemma). Clearly, $\psi^i(t) < t$ for $i = 1, \dots, j-1$. We claim that $\psi^i(t) > \frac{1}{d} - t$ for $i = 1, \dots, j-1$. Indeed, if $i < j$ is the least with $\psi^i(t) \leq \frac{1}{d} - t$ then the properties of ψ imply $\psi^{i-1}(t) > t$, a contradiction with $\psi^i(t) < t$ for $i = 1, \dots, j-1$. Thus, $t > \psi^{j-1}(t) > \frac{1}{d} - t$, and, hence, $\psi^j(t) > 1 - dt$. Since each component of $\mathbb{S}^1 \cap C(\ell)$ is of length $\frac{1}{d} - t < 1 - dt$, then $\sigma_d^j(\ell)$ connects distinct closed holes of $C(\ell)$ and $|\sigma_d^j(\ell)| \geq t$. Observe now that if a leaf connects two closed holes of $C(\ell)$ that are not consecutive then its rotation by the angle $\frac{1}{d}$ will cross itself, a contradiction. \square

Lemma 2.6 has important consequences.

Corollary 2.7. *Let \mathcal{L} be a σ_d -invariant unicritical lamination. Let M_1, \dots, M_d be the majors of \mathcal{L} and $m = \sigma_d(M_1)$ be its minor. Let $C(\mathcal{L})$ be the central strip defined by the majors of the lamination \mathcal{L} . Then no forward iterate of m can enter $C(\mathcal{L})$ unless it equals M_s for some s .*

Proof. If $\sigma_d^j(m) = \sigma_d^{j+1}(M_i) \subset C(\mathcal{L})$ then by Lemma 2.6 $|\sigma_d^{j+1}(M_i)| \geq |M_i|$ which by definition of a major implies that $\sigma_d^{j+1}(M_i) = M_s$ for any i and some (unique) s . \square

A d -gon G is said to be *all-critical* if $\sigma_d(G)$ is a point.

Theorem 2.8. *Minors of distinct laminations do not cross.*

Proof. Let $\mathcal{L}_1 \neq \mathcal{L}_2$ be two laminations with minors m_1 and m_2 . We claim that m_1 and m_2 do not cross. Indeed, suppose otherwise, and choose a non-periodic point $x \in \mathbb{S}^1$ located behind both m_1 and m_2 . The σ_d -preimage of the point x is an all-critical d -gon D_x . By Theorem 1.5, there exists an invariant lamination \mathcal{L}' containing D_x . By Corollary 2.7 and since x is not periodic, \mathcal{L}' is compatible with m_1 and m_2 so that m_1 and m_2 are diagonals of a gap G_0 of \mathcal{L}' . By Lemma 2.4, G_0 is a either pre-critical or pre-periodic. It cannot be pre-critical because m_1

and m_2 cannot enter their respective central strips. Thus, G_0 is pre-periodic. Choose $j \in \mathbb{N}$ such that $G = \sigma_d^j(G_0)$ is periodic and $\sigma_d^j(m_1)$ and $\sigma_d^j(m_2)$ are diagonals of G . By Lemma 2.5, the gap G is finite and the first return map permutes the sides of G transitively. Hence, some forward image of m_1 under σ_d will intersect m_1 and some forward image of m_2 under σ_d will intersect m_2 , a contradiction. \square

Theorem 2.9 follows from Theorem 2.8 and Theorem 1.10.

Theorem 2.9. *The space of all σ_d -invariant unicritical laminations is compact. The set of all their minors is a lamination.*

Proof. By Theorem 1.10 if a sequence of σ_d -invariant unicritical laminations converges then the limit is a σ_d -invariant lamination. Since the limit of the sequence of their critical sets then converges to a set that maps forward in the d -to-1 fashion, it follows that the limit lamination is unicritical. This implies the first claim of the theorem. Now, by Theorem 2.8 it remains to prove that minors of σ_d -invariant unicritical laminations form a closed family of chords. Indeed, if a sequence of minors converges to a chord ℓ , then we can choose a subsequence so that the corresponding σ_d -invariant unicritical laminations converge too. Their limit lamination has ℓ as its minor and is, by the first claim of the theorem, unicritical. This proves the theorem. \square

The next definition is similar to Thurston's definition of QML.

Definition 2.10. The set of all chords in the \mathbb{D} which are minors of some σ_d -invariant unicritical lamination is a lamination called the *Unicritical Minor Lamination of degree d* and denoted by UML_d .

3. BASIC PROPERTIES OF UML_d

We begin by studying the criterion under which a non-degenerate chord m drawn in \mathbb{D} or a point on \mathbb{S}^1 can be the minor of some unicritical lamination. By Lemma 2.2 we know that $|m| \leq \frac{1}{d+1}$; this is a necessary condition for m being a minor. However it is not sufficient (e.g., m may have forward images that cross). Thus, to establish the desired criterion we need more conditions on iterations of m . Consider pullbacks of m of length at most $\frac{1}{d}$ and call them '*short*' pullbacks. 'Short' pullbacks of a chord may well be longer than the chord itself. There are $2d$ 'short' pullbacks of m ; they are edges of a well-defined $2d$ -gon. Observe that if $|m| \leq \frac{1}{d+1}$, then there are d 'short' pullbacks of m that are of length $\frac{|m|}{d} < |m|$ and d 'short' pullbacks of m that are of length $\frac{1-|m|}{d} \geq |m|$ (in the latter case equality means that $|m| = \frac{1}{d+1}$). This observation follows from the properties of the length function ψ

described earlier. Given a chord $m = \overline{ab}$ with $|m| \leq \frac{1}{d+1}$, set $D(m)$ to be the convex hull of $\sigma_d^{-1}(\{a, b\})$.

Lemma 3.1. *The following conditions form a criterion for a non-degenerate leaf m to be a minor of unicritical lamination:*

- (1) $|m| \leq \frac{1}{d+1}$;
- (2) m and its forward images do not cross;
- (3) no image of m is shorter than m ;
- (4) If M_1, \dots, M_d are d ‘short’ pullbacks of m of length greater than $\frac{1}{d+1}$, then forward images of m do not cross M_1, \dots, M_d .

Moreover, Thurston’s pullback lamination $\mathcal{L}(m)$ based on m and M_1, \dots, M_d , has m as its minor.

Proof. First we show that if m is the minor of a unicritical lamination \mathcal{L} , then it has the listed properties. Indeed, by Lemma 2.2, $|m| \leq \frac{1}{d+1}$. Moreover, by properties of laminations, m and its forward images do not cross. Consider d ‘short’ pullbacks M_1, M_2, \dots, M_d of m of length greater than $\frac{1}{d+1}$. Then, again by properties of laminations, forward images of m do not cross M_1, M_2, \dots, M_d . Finally, no image of m is shorter than m by Corollary 2.7.

On the other hand, we claim that if m has the listed properties then m is the minor of some unicritical lamination. Indeed, consider the $2d$ -gon $D(m)$; it has d edges M_1, M_2, \dots, M_d of length $\lambda = \frac{1-|m|}{d} \geq \frac{1}{d+1}$ and d edges M'_1, M'_2, \dots, M'_d of length $\frac{|m|}{d} < |m|$. The leaf m cuts \mathbb{D} into two pieces: a small piece T_0 and a large piece T_1 . The $2d$ -gon $D(m)$ cuts \mathbb{S}^1 in $2d$ arcs: holes E_1, \dots, E_d of T behind M_1, \dots, M_d , and holes E'_1, \dots, E'_d behind M'_1, \dots, M'_d . Clearly, this entire picture is symmetric with respect to the rotation by $\frac{1}{d}$. Observe that E_1, \dots, E_d map in the homeomorphic fashion to T_1 by σ_d while E'_1, \dots, E'_d map by σ_d to T_0 in the homeomorphic fashion.

We claim that chords $\sigma_d^j(m)$ never enter sets E'_i . Indeed, otherwise the assumption that images of m do not cross chords M_1, \dots, M_d implies that for some j both endpoints of $\sigma_d^j(m)$ belong to sets E'_1, \dots, E'_d and hence $\sigma_d^{j+1}(m) \subset T_0$, a contradiction with the assumption that no image of m is shorter than m . Thus, the forward orbit of m together with T forms a forward invariant lamination \mathcal{L}_0 . Using Thurston’s pullback construction and relying upon the unicriticality, we now construct a desired invariant lamination \mathcal{L} (once we know that \mathcal{L}_0 is forward invariant the pullback construction always goes through). \square

Definition 3.2. If a non-degenerate chord m satisfies the above condition from Lemma 3.1, then we will call the lamination $\mathcal{L}(m)$ constructed

in Lemma 3.1 and having m as the minor as the *lamination generated by m* or the *lamination corresponding to the minor m* . The notation $\mathcal{L}(m)$ will be used from now on for that lamination.

Lemma 3.3. *The following claims hold.*

- (1) *If $n \in \text{UML}_d$, $m \in \mathcal{L}(n)$ is a leaf, $n \succ m$, and there exists no i such that $n \succ \sigma_d^i(m) \succ m$ then $m \in \text{UML}_d$.*
- (2) *If $m, n \in \text{UML}_d$ with $m \succ n$, then m is a leaf of $\mathcal{L}(n)$.*

Proof. (1) To prove that $m \in \text{UML}_d$, we show that m satisfies the three conditions of Lemma 3.1 one by one. Under iteration of σ_d , the length of m will continue to increase until it is at least $\frac{1}{d+1}$ and after that by Lemma 2.3 the image leaves cannot become shorter than n . Since, all leaves on the forward orbit of m are contained in $\mathcal{L}(n)$, the first two of the four hypothesis of Lemma 3.1 are satisfied.

Let M_1, M_2, \dots, M_d be d leaves of length at least $\frac{1}{d+1}$ that are immediate preimages of m (these would be the major leaves of the lamination corresponding to m). They are longer than the major leaves of $\mathcal{L}(n)$ and thus not leaves of $\mathcal{L}(n)$. In order to be able to apply Lemma 3.1 we need to show that no leaf in $\mathcal{L}(n)$ can intersect the interiors of the leaves M_1, M_2, \dots, M_d . Since, no leaf in $\mathcal{L}(n)$ could intersect the major leaves of $\mathcal{L}(n)$, a leaf in $\mathcal{L}(n)$ could intersect M_1, M_2, \dots, M_d only if its length is at most $\frac{1}{d}$ times the length of n .

If a leaf m' in the forward orbit of m is longer than $\frac{1}{d+1}$ (but shorter than the major leaves of $\mathcal{L}(n)$), the image leaf is shorter than m' , but at least as long as n by Lemma 2.3; from then on, no leaf in the orbit of m' can intersect the interior of M_1, M_2, \dots, M_d . Thus, a leaf in the orbit of m can intersect the interior of M_1, M_2, \dots, M_d only during the initial iterations while the length of the leaf increases by a factor of d . If this does not happen, then by Lemma 3.1, m is the minor of an invariant lamination and thus contained in UML_d . Clearly, $n \succ m$.

(2) We will prove the claim for the case n is non-degenerate. The proof for the degenerate case is similar. We first show that m does not intersect any leaf in the lamination $\mathcal{L}(n)$. If $\mathcal{C}(n)$ and $\mathcal{C}(m)$ are the central strips defined by the majors of the laminations $\mathcal{L}(n)$ and $\mathcal{L}(m)$ respectively then $\mathcal{C}(n) \subset \mathcal{C}(m)$. By Lemma 2.6 no image of m crosses a major of $\mathcal{L}(n)$. By construction, no image of m crosses a leaf of $\mathcal{L}(n)$.

We claim that m cannot divide a gap in $\mathcal{L}(n)$. Since, the minor n is non-degenerate, so $\mathcal{L}(n)$ has a critical $2d$ -gon G . Each gap in $\mathcal{L}(n)$ is either an eventually periodic polygon or a gap which eventually maps to G (this follows from the absence of wandering triangles established in Lemma 2.4). However, by Lemma 2.6 no image of m can divide G . On

the other hand, if m divides a pre-periodic polygon, then, by Lemma 2.5, two forward images of m will intersect which is a contradiction. Thus, m cannot divide a gap in $\mathcal{L}(n)$. Therefore, m is a leaf in $\mathcal{L}(n)$. \square

Given a non-diameter chord ℓ , let $A(\ell)$ be the set (possibly empty) of all points that never exit $\overline{H(\ell)}$. Given an arc I we denote by $|I|$ its length and that we normalize the circle length so that it equals 1.

Lemma 3.4. *Suppose that $\ell = \overline{yz}$ is a chord of length at most $\frac{1}{d}$ such that a fixed point a belongs to $[y, z]$. Then the following holds:*

- (1) *The only way $\sigma_d(y)$ and $\sigma_d(z)$ can belong to $[y, z]$ is when one of these points equals a and $|\ell| = \frac{1}{d}$.*
- (2) *The unique minimal invariant set contained in $[y, z]$ is $\{a\}$.*
- (3) *If m is a minor with a fixed endpoint a then $m = \{a\}$.*

Proof. For $0 < [a, z] = \lambda < \frac{1}{d}$ we have $|[a, \sigma_d(z)]| = d\lambda < 1 - (\frac{1}{d} - \lambda) < 1 - |[y, a]|$ which implies that unless one of points y, z equals a and $|\ell| = \frac{1}{d}$, all points of $[y, z]$ except a are eventually mapped outside $[y, z]$. This shows that, except in the case when $|\ell| = \frac{1}{d}$ and either y or z equals a , the set $A(\ell)$ equals $\{a\}$ and claims (1) and (2) are immediate. In the remaining case (1) clearly holds, and $A(\ell)$ consists of a and a countable family of pullbacks of the other endpoint of ℓ towards a . Still, it follows that (2) holds in this case too. To prove (3), let $m = \overline{ab}$ be the non-degenerate minor of a lamination \mathcal{L} . Choose $b' \in H(m)$ with $\sigma_d(b') = b$. Then the major of \mathcal{L} with an endpoint b' must cross m or be located “under” m , a contradiction. \square

If m is the minor of a lamination \mathcal{L} which is not a fixed point, then we denote by M_m the major of \mathcal{L} such that $m \subset \overline{H(M_m)}$ (if m is a fixed point then M_m is not well-defined as there are two majors with the above property).

Lemma 3.5. *Let $M = \overline{yz}$ be a chord with $|M| \leq \frac{1}{d}$ such that $M \succ \sigma_d(M) = m$. Then there are two cases.*

- (1) *M is a chord of length $\frac{1}{d}$ with a fixed endpoint.*
- (2) *Otherwise $y \leq \sigma_d(z) \leq \sigma_d(y) \leq z$, $\sigma_d([y, z]) = [\sigma_d(y), \sigma_d(z)]$, $\sigma_d(\ell) \in [y, z]$ for any critical chord $\ell \succ M$ with $|\ell| = \frac{1}{d}$, and the arc $[z - \frac{1}{d}, y + \frac{1}{d}] = I$ contains no fixed points (so, $\overline{H(M)}$ and $\overline{H(m)}$ contain no fixed points).*

In particular, this holds if m is a minor and $M = M_m$.

Proof. (1) For a fixed point in $\overline{H(M)}$ the claim follows by Lemma 3.4.

(2) Suppose that no fixed point belongs to $\overline{H(M)}$. Then the first two claims follow from the fact that the image of $H(M)$ is an arc of length

at most 1 but greater than $|H(M)|$. Now, since σ_d preserves orientation we see that as we rotate a critical chord ℓ from $y(y + \frac{1}{d})$ to $(z - \frac{1}{d})z$, the point $\sigma_d(\ell)$ moves along the circle in the negative direction from $\sigma_d(y)$ to $\sigma_d(z)$ which proves the third claim. To prove the last claim, observe that if a fixed point a belongs to $[z - \frac{1}{d}, y + \frac{1}{d}]$, then there exists a critical chord ℓ_a of length $\frac{1}{d}$ with both endpoints in I such that $\ell \succ M$. Since by the second claim $\sigma_d(\ell_a) \in [y, z]$, then a coincides with one of the endpoints of M , a contradiction. \square

For a point $x \in \mathbb{S}^1$, let D_x be the convex hull of the full preimage of x under σ_d ; clearly, D_x is an all-critical d -gon.

Lemma 3.6. *Every $x \in \mathbb{S}^1$ is the minor of a unicritical lamination.*

Proof. The set $D_x \cup \mathbb{S}^1$ is forward invariant, has d critical leaves, and, hence, satisfies Thurston's pullback construction conditions stated in Theorem 1.5. Thus, there exists a σ_d -invariant lamination having minor x . Such a lamination is clearly unicritical. \square

Denote the lamination from Lemma 3.6 by \mathcal{L}_x . By our construction both $\mathcal{L}(m)$ (if m is a non-degenerate minor) and \mathcal{L}_x (if $x \in \mathbb{S}^1$ is a point) are sibling invariant laminations. By Lemma 3.6, a point $x \in \mathbb{S}^1$ may be either an endpoint of a non-degenerate minor or there may exist no such non-degenerate minor in which case x is called a *degenerate minor* (i.e., it is a point of the circle disjoint from other minors); by Lemma 3.6 every point of the circle is a minor, but only points disjoint from other minors will be called *degenerate minors*. We will now investigate these phenomena. If x is periodic, one vertex of the d -gon D_x is periodic, the others are not. The simplest case is when x is a fixed point.

Lemma 3.7. *If x is a fixed point, \mathcal{L}_x consists of isolated leaves that accumulate on points of \mathbb{S}^1 . All gaps of \mathcal{L}_x are pullbacks of D_x . If \mathcal{L} is a unicritical lamination such that x belongs to the image of its critical set $\mathcal{C}(\mathcal{L})$, then $\mathcal{L} = \mathcal{L}_x$. Thus, x is a degenerate minor.*

Proof. The gap D_x has d holes. Under the map σ_d they expand to cover the circle exactly once. Moreover, for each such hole I there exists a unique fixed point $y_I \in \bar{I}$. Denote the edge of D_x connecting the endpoints of I by ℓ_I . Then the σ_d pullback of D_x with vertices in I is a $(d + 1)$ -gon attached to D_x at ℓ_I with vertices that include the endpoints of I and otherwise partition I into d equal arcs of length $\frac{1}{d^2}$ each. On the next step we will obtain new pullbacks of D_x with vertices that partition each of those arcs of length $\frac{1}{d^2}$ into yet d equal arcs, etc. Passing to the limit we arrive at the first claim of the lemma.

Now, let \mathcal{L} be a unicritical lamination such that x belongs to the image of its critical set $\mathcal{C}(\mathcal{L}) = \mathcal{C}$. Then $D_x \subset \mathcal{C}$, and all leaves of \mathcal{L} are compatible with \mathcal{L}_x (i.e., do not cross leaves of \mathcal{L}_x). Suppose that \mathcal{L} has a leaf $\ell \notin \mathcal{L}_x$. Then ℓ is contained in one of the pullbacks of D_x which means that an eventual image of ℓ is contained in D_x but is not an edge of D_x . Evidently it is impossible because \mathcal{L} is unicritical. \square

Lemma 3.8 is similar except now x is periodic of period $p > 1$.

Lemma 3.8. *Let x be a non-fixed point. Then:*

- (1) *a unique hole H_1 of D_x contains no σ_d -fixed points while each other hole of D_x contains exactly one σ_d -fixed point;*
- (2) *H_1 contains a non-degenerate minimal set \mathcal{I}_x ;*
- (3) *if x is periodic of period $p > 1$, and \mathcal{L}_c is obtained from \mathcal{L}_x by removing D_x and its backward orbit, then \mathcal{L}_c is non-empty, and there is a Fatou gap U of \mathcal{L}_c of period p that contains D_x .*

Proof. (1) Clearly, $\{x\}$ is the minor of \mathcal{L}_x and is not a vertex of D_x . Choose a hole H_1 of D_x such that $x \in H_1$. Then by Lemma 3.5, no fixed point belongs to H_1 . Since the arcs between two consecutive fixed points are of length $\frac{1}{d-1} > \frac{1}{d}$, H_1 is a *unique* such hole of D_x .

(2) By (1) the set of points that stay forever in $\overline{H_1}$ is an invariant compact set that contains no σ_d -fixed points. Hence, there exists the desired minimal set \mathcal{I}_x .

(3) \mathcal{L}_c is a lamination: all properties of laminations are immediate for \mathcal{L}_c except for the fact that \mathcal{L}_c is closed, and to see that, observe that by construction all edges of D_x are isolated in \mathcal{L}_x ; hence, all pullbacks of edges of D_x are isolated in \mathcal{L}_x and \mathcal{L}_c is closed.

Since the period of x is p , the $(p-1)$ -th pullback of D_x is a d -gon Δ_x emanating out of x . Taking σ_d -preimage of Δ_x we get d polygons $\Gamma^1 \subset \overline{H_1}, \dots, \Gamma^d \subset \overline{H_d}$ each of which is a $d+1$ -gon called an *immediate decoration*, hanging out from an edge of D_x . Further pullbacks of D_x hang out of edges of immediate decorations (these are called simply *decorations*), etc.

Consider the convex hull G of \mathcal{I}_x . We claim that pullbacks of D_x can at most “touch” G (i.e., have common vertices with G), but otherwise are disjoint from G . Indeed, edges of G cannot intersect pullbacks of edges of D_x as otherwise a forward image of an edge of G will cross an edge of D_x , a contradiction. Thus, the only way the claim of the lemma fails is if G contains a pullback of an edge of D_x . Since \mathcal{I}_x is invariant, then an edge of D_x contained in the boundary of H_1 , is an edge of G . Since \mathcal{I}_x is minimal, it follows then that \mathcal{I}_x coincides with the periodic orbit of $\sigma_d(x)$. Evidently, this contradicts the assumption

that an edge of G is also an edge of D_x . The claim that the intersection of a pullback of D_x and G is at most a common vertex, follows.

We claim that G is contained in a non-degenerate invariant gap of \mathcal{L}_c . Indeed, by the above G is contained in a non-degenerate gap G' of \mathcal{L}_x ; edges of G' are not pullbacks of edges of D_x as at each edge of a pullback of D_x yet another pullback of D_x is attached. Hence all edges of G' are limits of pullbacks of edges of D_x , and G' remains a non-degenerate gap of \mathcal{L}_c , and \mathcal{L}_c is not an empty lamination. Since \mathcal{L}_c is obtained from \mathcal{L}_x by removing D_x and its backward orbits, one can say that pullbacks of D_x do not fill up the entire unit disk.

The gap U of \mathcal{L}_c that contains D_x is an infinite periodic gap which returns with degree d (it is infinite because by construction it contains immediate decorations as well as other decorations consecutively attached to them). Hence, U is a Fatou gap. Let the period of U be k . If $k < p$, then σ_d^k brings the gap U back to itself but the point \tilde{x} does not return to its initial position and has is of period $\frac{p}{k} > 1$ inside U under the first return map $g = \sigma_d^k$. So, using the same technique as above applied to $\sigma_d^k|_U$, one can show that the gap U contains a non-degenerate gap of \mathcal{L}_c , a contradiction. Thus, $k \not< p$. On the other hand, by construction U contains D_x and all its immediate decorations which implies that $\sigma_d^p(U) = U$. Hence $k = p$ and the claim follows. \square

Lemma 3.8 shows how unicritical laminations with periodic Fatou gaps can be constructed. We will now study such laminations. Throughout this section we use the same notation for the now defined objects. Let $m(\mathcal{L}) = m$ be the minor of \mathcal{L} . As in Lemma 3.1, there are d pullbacks of m denoted by M_1, M_2, \dots, M_d that are of equal length $\lambda = \frac{1-|m|}{d} \geq \frac{1}{d+1}$. Since m is the minor of \mathcal{L} , chords M_1, M_2, \dots, M_d must be leaves of \mathcal{L} (one of them is, hence all of them are since \mathcal{L} is unicritical). Observe that they are $1/d$ -rotations of each other. By Lemma 2.6 forward images of m do not enter the central strip C of \mathcal{L} , i.e. the component of $\mathbb{D} \setminus \bigcup_{i=1}^{d-1} M_i$ that has all M_i 's on its boundary. In particular, no forward image of m coincides with a chord connecting two consecutive endpoints of adjacent majors of \mathcal{L} .

Consider now the case when the minor $m(\mathcal{L}) = m$ of \mathcal{L} is periodic of period, say, n . Since $\sigma_d(\sigma_d^{n-1}(m)) = m$, then $\sigma_d^{n-1}(m)$ is one of the majors M_1, \dots, M_d . Denote $\sigma_d^{n-1}(m)$ by $M(\mathcal{L}) = M$ and call it the *primary major* of \mathcal{L} . The remaining $d - 1$ majors of \mathcal{L} are called *secondary*. Clearly, M and m have the same period.

Lemma 3.9. *Suppose that \mathcal{L} has an n -periodic cycle of Fatou gaps with U critical. Then all d majors of \mathcal{L} are edges of U and exactly one of them is periodic of period n .*

Proof. There are d majors of \mathcal{L} of length at least $\frac{1}{d+1}$. By Lemma 2.6 σ_d^n maps them all to one of them which is periodic of period n . \square

Given a Fatou gap U of period k , we call an edge (or vertex) of U *refixed* if its endpoints are of period k . The periodic major of \mathcal{L} discovered in Lemma 3.9 is, evidently, refixed.

Lemma 3.10. *Suppose that \mathcal{L} has an n -periodic cycle of Fatou gaps and U is a critical gap in that cycle. Then each edge of U is an eventual preimage of $M(\mathcal{L})$. In particular, the first return map $g|_U$ has exactly d fixed points. Two of these points are the endpoints of $M(\mathcal{L})$. Otherwise each component of $\text{Bd}(U)$ from which all majors of \mathcal{L} are removed contains exactly one fixed point of $g|_{\text{Bd}(U)}$.*

Proof. Evidently, U has $d - 1$ edges (possibly, degenerate edges, i.e. vertices) that are refixed. In particular $M(\mathcal{L})$ is a refixed edge of U . By Lemma 2.2, $|M(\mathcal{L})| \geq \frac{1}{d+1}$. Thus, \mathcal{L} has d majors all of which are edges of U of length at least $\frac{1}{d+1}$.

Take an edge N of U . It is well-known that any edge of U is eventually mapped into either critical or periodic edge of U . Since there are no critical edges of U , we may assume that N is itself periodic. Let $\sigma_d^i(N)$ be the longest leaf in the orbit of N . By Lemma 2.2, $|\sigma_d^i(N)| \geq \frac{1}{d+1}$. It follows that there exists a major M' of \mathcal{L} such that the endpoints of $\sigma_d^i(N)$ belong to the hole of U behind M' . However then, as we iterate $\sigma_d^i(N)$, it cannot be mapped to a chord longer than $\sigma_d^i(N)$ (by the choice of $\sigma_d^i(N)$), and neither can it enter its central strip (by Lemma 2.6), a contradiction with N being an eventual image of $\sigma_d^i(N)$. \square

We will now prove the first main result of this section.

Theorem 3.11. *For a p -periodic point $x \in \mathbb{S}^1, p > 1$, there are precisely two possibilities.*

- (1) **Degenerate case:** x is the degenerate minor of some unicritical lamination but there exists no unicritical lamination with a non-degenerate minor one of whose endpoints is x .
- (2) **Non-degenerate case:** x is an endpoint of a unique non-degenerate minor m of some unicritical lamination.

Proof. Suppose that the degenerate case does not hold and there exists a minor m with endpoint x . We claim that such m is unique. Suppose otherwise. We may assume that $m = \overline{xy}$, y is period p , $H(m) = (x, y)$

(the movement from x to y inside (x, y) is clockwise) and for some periodic point $z \in (x, y)$ of period p the chord \overline{xz} is also the minor of some unicritical cubic lamination. Let the corresponding *primary majors* be $M_y = \overline{x'y'}$ and $M_z = \overline{x'z'}$. Also, let U_y be the critical Fatou gap with major M_y and U_z be the critical Fatou gap with major M_z .

Denote by L and R the two siblings of M_y adjacent to M_y . Since by Lemma 2.2 $\frac{1}{d+1} \leq |M_z| < \frac{1}{d}$, then the point z' must belong to an arc I that separates (in the circle) M_y from one of those two siblings L and R , but does not contain the other one. Since this arc contains no p -periodic points that belong to $\text{Bd}(U_y)$, then there is an edge ℓ of U_y with endpoints in I that crosses M_z . Recall that by Lemma 3.10 ℓ is an eventual preimage of M_y . Since σ_d^p acts on U_y as σ_d (after we collapse all edges of U_d to points), then after several iterations of σ_d^p the leaf ℓ will map onto a sibling M_y'' of M_y distinct from M_y ; it is easy to see that, as it happens, the leaf M_z will keep crossing the corresponding images of ℓ . In particular, the appropriate image of M_z will cross M_y'' . However the location of M_z shows that then this image of M_z will cross one of its siblings, a contradiction. \square

4. THE PROOF OF THE FACT THAT UML_d IS A Q-LAMINATION

First let us classify all gaps of UML_d . If m is a minor, then by Lemma 3.5 the arc $H(m)$ contains not fixed points. Thus, if we connect the center $(0, 0)$ of \mathbb{D} with fixed points of σ_d by $d-1$ radii, then each minor is located in one of the thus created $d-1$ circular sectors. Any gap G of UML_d has a side m that “faces” $(0, 0)$ (separates $(0, 0)$ from the rest of G). Other edges n of G are incomparable among themselves because their sets $H(n)$ are pairwise disjoint. Thus, all edges of G are *immediate successors* of m in the partial order \succ . Call m the *leading edge* of G .

Let \mathcal{L} be a lamination with minor $m = \overline{ab}$. \mathcal{L} may have a critical gap $\mathcal{C}(\mathcal{L})$ which is either a pre-periodic polygon with dn sides ($n \geq 3$) or a periodic Fatou gap (this holds by Lemma 2.4). Consider all three cases for $\mathcal{C}(\mathcal{L})$.

Lemma 4.1. *If $\mathcal{C}(\mathcal{L})$ is a polygon with dn sides ($n \geq 3$), then $\sigma_d(\mathcal{C}(\mathcal{L}))$ is a gap of UML_d .*

Proof. The gap $\mathcal{C}(\mathcal{L}) = G$ can be subdivided by adding a collapsing $2d$ -gon that coincides with the convex hull of d sibling edges of G with the same image which is an edge of $\sigma_d(G) = G'$. That collapsing $2d$ -gon, apart from d sibling edges of it shares with G , will have d complementary edges connecting the appropriate endpoints of the sibling edges of

G mentioned above. This can be done for each edge of G' , thus it can be done in n ways. Each edge of G' is the minor of the corresponding lamination obtained by pulling back the appropriate $2d$ -gon inside pullbacks of G . We claim that G' is a gap in UML_d . Indeed, otherwise G' contains a minor which is a diagonal of G' . However, by Lemma 2.4 and Lemma 2.5 every diagonal of G' has crossing forward images and, hence, cannot be a minor. Thus, G' is an actual gap of UML_d . \square

To consider the case when $\mathcal{C}(\mathcal{L})$ is a Fatou gap we need to study the unicritical version of the quadratic Main Cardioid. We first prove the following Lemma:

Lemma 4.2. *Let ℓ be a critical chord of length $\frac{1}{d}$. Then there exists an invariant minimal set $\mathcal{I} \subset \overline{H(\ell)}$, and for each such \mathcal{I} the convex hull G of \mathcal{I} has the longest edge M that separates the rest of \mathcal{I} from the center of \mathbb{D} ; the image $m = \sigma_d(M)$ is a minor, and the unicritical lamination $\mathcal{L}(m)$ includes G as an invariant gap or leaf.*

Proof. By Lemma 3.4 we may consider a critical chord ℓ of length $\frac{1}{d}$ with no fixed point in $\overline{H(\ell)}$. The set $A(\ell) \neq \emptyset$ (since $\sigma_d(\overline{H(\ell)}) = \mathbb{S}^1$) of all points with orbits contained in $\overline{H(\ell)}$ is non-degenerate (by the “no fixed points in $\overline{H(\ell)}$ ” assumption), forward invariant and onto (the latter follows again because $\sigma_d(\overline{H(\ell)}) = \mathbb{S}^1$), with the circular order preserved on $A(\ell)$ except when the endpoints of ℓ belong to $A(\ell)$. Take a minimal set $\mathcal{I} \subset \overline{H(\ell)}$. If $A(\ell)$ consists of two points, then $A(\ell) = \mathcal{I}$ is a two periodic orbit, and we associate with $A(\ell)$ the circle rotation by $\frac{1}{2}$ and the rotation number $\frac{1}{2}$. If $A(\ell)$ consists of more than two points we consider $\text{CH}(A(\ell)) = \overline{U(\ell)}$ with the map $\sigma_d|_{\overline{U(\ell)}}$ (recall that we canonically extend σ_d onto chords of the unit circle). Since the circular orientation on $A(\ell)$ is preserved, the degree of $\sigma_d|_{\text{Bd}(\overline{U(\ell)})}$ is 1.

Thus, we can associate to $\sigma_d|_{\text{Bd}(\overline{U(\ell)})}$ (and, hence, to ℓ) a well-defined rotation number $\rho(\ell) \in (0, 1)$ (there are no fixed points in $A(\ell)$!). Consider the convex hull G of \mathcal{I} . Among edges of G there is a unique edge M that “faces” the center $(0, 0)$ of \mathbb{D} (separates $(0, 0)$ from the rest of G). This is the longest edge of G . Set $m = \sigma_d(M)$. By Lemma 2.2, $|M| \geq \frac{1}{d+1}$, and by Lemma 3.5 $\sigma_d(\ell) \in H(m)$. Consider two cases related to the properties of any minimal invariant set $\mathcal{I} \subset \overline{H(\ell)}$ (later on we will prove that \mathcal{I} is unique).

(1) Suppose that \mathcal{I} is finite, and, hence, is a periodic orbit. This holds if and only if $\rho(\ell)$ is rational. As we map m forward, the images of m are all oriented in the same way inside $H(M)$ because, as long as

the points stay in $H(\ell)$, their circular order is preserved. Their lengths stay under $\frac{1}{d}$, and on each step they are simply multiplied by d .

This allows us to evaluate lengths of all edges of G if we know the period q (not even the rotation number).

Set $\overline{ab} = m = \sigma_d(M)$ and consider the critical $2d$ -gon $D(m)$. It has M and all its siblings as edges and in fact coincides with $\sigma_d^{-1}(\{a, b\})$. Since $D(m)$ and G form a forward invariant lamination, Thurston's pull back construction goes through and gives rise to the canonical pullback lamination $\mathcal{L}(m)$ that has m as its minor. Such minors are said to be *prime rational minors*.

(2) If \mathcal{I} is infinite, then it is known that G is an invariant gap such that when we collapse all its edges to points and transfer the action of σ_d onto the resulting circle we get an irrational rotation. Such gaps are called invariant *Siegel* gaps. Since edges of an invariant gap are (pre)periodic or (pre)critical, edges of G are (pre)critical. By our setup, G can only have one critical edge coinciding with ℓ . Let us show that then no two edges of G can intersect. Indeed, if they do, then after applying σ_d the appropriate number of times we would see that both ℓ and an edge ℓ' of G have a common endpoint, say, b . However ℓ' must eventually map to ℓ thus implying that $\sigma_d(\ell)$ is periodic, a contradiction. Hence no two edges of G have a common endpoint, the entire set $A(\ell)$ is minimal, and coincides with the limit set of, say, $\sigma_d(\ell)$. This case corresponds to an irrational rotation number $\rho(\ell)$. The degenerate minor $\sigma_2(\ell)$ is called a *prime irrational minor*. \square

Definition 4.3. Minors whose existence is proven in Lemma 4.2 are called *prime minors*. If $G(\ell)$ is a finite gap or non-degenerate leaf then the corresponding minor is said to be *rational*. If $G(\ell)$ is a Siegel gap then the corresponding degenerate minor is said to be *irrational*.

Lemma 4.4. *Two distinct prime minors are not \succ -comparable.*

Proof. Indeed, suppose that $m \succ m'$ are two prime minors. Let $M_m = M$ and $M_{m'} = M'$ be the corresponding majors. Then by Lemma 3.5 $M' \succ M$. Choose a critical chord $\ell \succ M'$ of length $\frac{1}{d}$. Since the orbit of m is contained in $\overline{H(M)}$ and the entire orbit of m' is contained in $\overline{H(M')}$, then in fact the orbits of m and m' are contained in $\overline{H(\ell)}$ which implies that both minors are rational and have the same period. However then by the above they must have the same length and, hence, must coincide. \square

Lemma 4.5. *The minimal set \mathcal{I} in Lemma 4.2 is unique*

Proof. Notice that if case (2) of Lemma 4.2 holds, then the claim is immediate. So, assume that $\rho(\ell)$ is rational and show that \mathcal{I} is well-defined. Suppose otherwise. Then there are two prime rational minors m and m' generated by minimal invariant sets $\mathcal{I} \subset \overline{H(\ell)}$ and $\mathcal{I}' \subset \overline{H(\ell)}$. Recall that by Theorem 2.8 m and m' do not cross, but might have a common endpoint. We claim that m and m' are \succ -comparable. Indeed, for geometric reasons $\sigma_d(\ell) \in \overline{H(m)} \cap \overline{H(m')}$. It follows that the only way m and m' are not \succ -comparable is when $\sigma_d(\ell) = y$ is the terminal point of one of them and the initial point of the other one. However then $y \in \mathcal{I} \cap \mathcal{I}'$, and since these sets are minimal, they coincide, a contradiction. Hence, m and m' are \succ -comparable and disjoint. We may assume that $m = \overline{xy}, m' = \overline{x'y'}$ are of period n , and $x < x' < y' < y$. The images of the arc $H(m)$ map onto each other and stay shorter than $\frac{1}{d}$ until m maps to the corresponding major M and m' maps to the corresponding major M' . However then the σ_d^n -image of $[x, x']$ stays shorter than $\frac{1}{d}$, and from that moment on the whole orbit of $[x, x']$ gets repeated. However, $[x, x']$ must expand under σ_d , a contradiction. Thus, $m = m'$. We conclude that \mathcal{I} , and, accordingly, m , as a function of ℓ , is unique. \square

From now on we denote the minor from Lemma 4.5 by $m(\ell)$.

Lemma 4.6. *The family of prime minors forms the boundary of a closed convex set $Q = \text{CH}(Q \cap \mathbb{S}^1)$. Moreover, $Q \cap \mathbb{S}^1$ is a Cantor set in which fixed points in $\text{Bd}(\text{UMC})$ are not isolated from either side.*

Proof. We claim that prime minors are pairwise disjoint. Indeed, the only way they intersect is when two rational prime minors m and m' have a common endpoint z which is the terminal point of, say, m and the initial endpoint of m' . Then z is a periodic point whose orbit B contains the remaining two endpoints of m and m' . Hence m and m' are edges of the same gap $\text{CH}(B)$. However, by the above all edges of $\text{CH}(B)$ are of distinct length, a contradiction.

As we showed above, prime minors are pairwise non-comparable in the sense of \succ , and pairwise disjoint. Let us show that any non-fixed point $y \in \mathbb{S}^1$ is either a prime irrational minor, or has a prime rational minor m such that $y \in H(m)$. Indeed, choose two consecutive fixed point u and v such that the positively oriented arc I from u to v contains y . Then choose an admissible critical chord ℓ with $i(\ell) \in I$ such that $\sigma_d(\ell) = y$. Set $m = m(\ell)$. It follows that $y \in H(m(\ell))$ as desired.

We claim that the union T of prime irrational minors and the endpoints of prime rational numbers is a Cantor set. Indeed, if a sequence

of prime minors converges to a point $y \in \mathbb{S}^1$ then either y is an prime irrational minor, or y is an endpoint of a prime rational number because if y is “under” a prime rational minor m then the minor that converge to y will also be “under” m , a contradiction. Thus, T is closed. On the other hand, an isolated point $z \in T$ may exist only if two prime rational minors meet at z , a contradiction. Since T cannot contain subsegments, T is a Cantor set. Because of the symmetry, if a fixed point a had been isolated from one side, it would have been isolated from either side, a contradiction. It remains to set $Q = \text{CH}(T)$. \square

Definition 4.7. The set Q from Lemma 4.6 is the *unicritical analog of the Quadratic Main Cardioid*. We call it the *Unicritical Main Cardioid* and denote it by \mathcal{UMC} .

Theorem 4.8. *The set \mathcal{UMC} is a gap of UML_d .*

Proof. The edges of \mathcal{UMC} are rational prime minors. It also includes prime irrational minors as its “stand alone” vertices. To show that \mathcal{UMC} is a gap of UML_d it suffices to show that there are no minors inside \mathcal{UMC} . Suppose that $m = m(\mathcal{L})$ is such a minor. By Lemma 3.5 we can choose the major $M_m = \overline{yz}$ of \mathcal{L} such that $M_m \succ m$ and $\sigma_d(\ell) \in \overline{H(m)}$ for any critical chord ℓ with $\ell \succ M_m$ and $|\ell| = \frac{1}{d}$. By Lemma 3.5, $m = \sigma_d(M_m) = \overline{\sigma_d(z)\sigma_d(y)}$. Set $\ell = \overline{y(y + \frac{1}{d})}$ and consider the minor $m(\ell)$ and the associated major $M_{m(\ell)} = M(\ell)$.

By definition, $\ell \succ M(\ell)$. By construction, $\ell \succ M_m$. By Theorem 2.8 m and $m(\ell)$ do not cross. Hence, $M(\ell)$ and M_m do not cross either. If $M(\ell)$ is “under” M_m , then m is “under” $m(\ell)$, a contradiction with m being inside \mathcal{UMC} . Thus, the only possibility left is that $M(\ell)$ has y as the initial endpoint. Then m and $m(\ell)$ share an endpoint $\sigma_d(y)$. Since m is inside \mathcal{UMC} , $m \succ m(\ell)$ and $M(\ell) \succ M_m$. The point $\sigma_d(y)$ belongs to the minimal set $\mathcal{I}(\ell)$ and has the orbit contained in $\overline{H(\ell)}$.

Since m is inside \mathcal{UMC} , the point $\sigma_d(z)$ belongs to a minor $m' \neq m(\ell)$ (otherwise $m = m(\ell)$). Hence for some $N \geq 2$ the point $\sigma_d^N(z)$ does not belong to $\overline{H(\ell)}$. We conclude that $\sigma_d^N(M_m)$ is a chord with one endpoint $\sigma_d^N(y) \in \overline{H(\ell)}$ and the other endpoint $\sigma_d^N(z) \notin \overline{H(\ell)}$. However this contradicts Corollary 2.7 which describes the dynamics of majors of laminations. This completes the proof. \square

One can transfer Theorem 4.8 onto Fatou gaps.

Lemma 4.9. *Let $\mathcal{C}(\mathcal{L})$ is a critical Fatou gap of a unicritical lamination \mathcal{L} of period n . Let $U = \sigma_d(\mathcal{C}(\mathcal{L}))$. Let $\psi : \text{Bd}(U) \rightarrow \mathbb{S}^1$ be a monotone map that collapses edges of U to points. Then UML_d contains a gap \mathcal{UMC}_U whose boundary is $\psi^{-1}(\text{Bd}(\mathcal{UMC}))$. In particular,*

$\mathcal{UMC}_U \cap \mathbb{S}^1$ is a Cantor set, and the minor m of \mathcal{L} has countably infinitely many immediate \succ -successors.

Proof. The existence of the gap \mathcal{UMC}_U is immediate from Theorem 4.8 and properties of the map ψ which semiconjugates $\sigma_d^n|_U$ and σ_d . The only thing that needs to be noticed is that when we lift $\text{Bd}(\mathcal{UMC})$ to \mathcal{UMC}_U using ψ , one fixed point in $\text{Bd}(\mathcal{UMC})$ corresponding to the minor m of \mathcal{L} is lifted to a leaf m which creates isolation on the appropriate sides of the endpoints of m in $\mathcal{UMC}_U \cap \mathbb{S}^1$. However by Lemma 4.6 fixed points in \mathcal{UMC} are non-isolated on either side, hence the endpoints of m are non-isolated in $\mathcal{UMC}_U \cap \mathbb{S}^1$ as desired. \square

Abusing the language, we will call a gap similar to \mathcal{UMC}_U from Lemma 4.9 a *copy* of \mathcal{UMC} .

Lemma 4.10. *If m is a periodic minor then there are two copies of \mathcal{UMC} that share m . Any non-degenerate minor with a periodic endpoint is a periodic minor.*

Proof. Our analysis implies that if m is a non-degenerate prime minor of period n , then in the associated lamination \mathcal{L} there exists a finite invariant gap G such that m is one of its edges. The corresponding major $M = M_m \succ m$ is the edge of G that cuts the rest of G from the center of the circle (equivalently, from all fixed points of σ_d). At M there is a critical Fatou gap $\mathcal{C}(\mathcal{L})$ of \mathcal{L} of period n . Set $U = \sigma_d(\mathcal{C}(\mathcal{L}))$. Then, by Lemma 4.9 there is a counterpart \mathcal{UMC}_U of \mathcal{UMC} in U such that m corresponds to the degenerate minor in \mathcal{UMC} associated with angle 0, and otherwise the correspondence between \mathcal{UMC} and \mathcal{UMC}_U is a homeomorphism. Moreover, \mathcal{UMC}_U is itself a gap of UML_d .

It follows that if we repeatedly apply the same construction, we will discover countable concatenations of copies of \mathcal{UMC} whose existence is established in Lemma 4.9. This completely describes all minors with periodic endpoints. Indeed, let $m = \overline{ab} = m(\mathcal{L})$ be the minor of a unicritical lamination \mathcal{L} and a be a periodic point of period k . By Lemma 3.8 there exists a critical Fatou gap W of period k such that $\sigma_d(W)$ has an edge ℓ of period k and an endpoint a . By the previous paragraph, ℓ is a minor and there are two copies of \mathcal{UMC} adjacent at each other along ℓ . Moreover, the boundaries of these copies of \mathcal{UMC} intersected with \mathbb{S}^1 are Cantor sets. It follows that $m = \ell$ as in UML_d there is no room for any other minor with endpoint a . \square

Definition 4.11. Let $\ell \neq \ell'$ be leaves of a lamination \mathcal{L} . If $\sigma_d(\ell)$ and ℓ' are contained in the same component of $\overline{\mathbb{D}} \setminus \ell$ we say that ℓ maps toward ℓ' , otherwise that it maps away from ℓ' .

We will need the next lemma which is in fact a version of the fixed point theorem from continuum theory.

Lemma 4.12. *Suppose that ℓ' and ℓ are two disjoint leaves of a lamination \mathcal{L} . Moreover, suppose that σ_d^k maps both leaves either toward each other or away from each other. Then there exists a σ_d^k -invariant gap or leaf G that separates ℓ' and ℓ . If ℓ and ℓ' map away from each other then there exists a leaf \bar{y} that separates them and maps to itself.*

Proof. Call leaves of \mathcal{L} that separate ℓ' and ℓ *separating*. Separating leaves are ordered, say, from ℓ' to ℓ family. By our assumptions there must exist a place in this family where the direction of the movement of leaves switches. Thus, either there exists a separating leaf that maps to itself, or there exists a gap G separating ℓ' and ℓ such that $\sigma_d^k(G) = G$. If leaves ℓ' and ℓ map away from each other, then the latter option is impossible unless there is an edge of G that maps to itself because otherwise G would have strictly covered itself under σ_d^k . \square

Let us now study properties of a minor m as a leaf of UML_d .

Lemma 4.13. *Let $m \in \text{UML}_d$ be a non-degenerate non-periodic minor. If there is a sequence of leaves $\ell_i \in \mathcal{L}(m)$ with $\ell_i \succ m$ and $\ell_i \rightarrow m$, then m is the limit of periodic minors n_j with $n_j \succ m$ for any j .*

Proof. By Thurston's pullback construction the assumptions imply that we may assume that all ℓ_i 's are pullbacks of majors. Hence there is a sequence of numbers $k_i \rightarrow \infty$ such that $\sigma_d^{k_j}(\ell_j) = m$. On the other hand $\sigma_d^{k_j}(m) \succ m$, $\sigma_d^{k_j}(m) \neq m$ by properties of minors and because m is not periodic. Thus, $\sigma_d^{k_j}$ maps ℓ_j and m in opposite directions.

By Lemma 4.12 there exists a $\sigma_d^{k_j}$ -invariant gap or leaf of $\mathcal{L}(m)$ that separates m and ℓ_j . Now, consider the gap (or leaf) G . It has a leaf \bar{g} that "faces" m (so that \bar{g} separates m and the rest of G). We may also assume that the two arcs between the endpoints of \bar{g} and the endpoints of m are shorter than $|m|$. Consider the forward orbit of \bar{g} . By Lemma 2.3 the length of any image of \bar{g} is at least $|m|$. Hence $\bar{g} \succ \sigma_d^i(\bar{g})$ for some i then $\sigma_d^i(\bar{g})$ separates g and m (any other location of $\sigma_d^i(\bar{g})$ would imply that the length of $\sigma_d^i(\bar{g})$ is smaller than $|m|$). Since \bar{g} is periodic, we may assume that some image \bar{h} of \bar{g} separates m and G and has no images $\sigma_d^j(\bar{h})$ such that $\bar{h} \succ \sigma_d^i(\bar{h})$.

It is easy to see that the d pullbacks H_1, \dots, H_d of \bar{h} that are leaves of $\mathcal{L}(m)$ are located outside of the Central Strip $C(m)$. Forward images of \bar{h} cannot cross those pullbacks. By Lemma 3.1 the leaf \bar{h} is a minor. Since it is periodic, it is a part of a copy of UMC_U of UMC for the appropriate Fatou gap U . \square

Consider now non-degenerate and non-periodic minors.

Lemma 4.14. *Let \mathcal{L} be a unicritical lamination with non-degenerate and non-periodic minor. Then there exists a lamination $\mathcal{L}' \subset \mathcal{L}$ with finite critical set $\mathcal{C}(\mathcal{L}')$ such that the set $\sigma_d(\mathcal{C}(\mathcal{L}')) = H'$ is approached by leaves of \mathcal{L}' from all sides and one of the following two cases holds:*

- (1) $\mathcal{L}' = \mathcal{L}$;
- (2) *the set $\mathcal{C}(\mathcal{L}')$ is a gap with $dn, n \geq 3$ edges and \mathcal{L} is obtained from \mathcal{L}' by inserting d sibling edges in $\mathcal{C}(\mathcal{L}')$ so that \mathcal{L} has a critical $2d$ -gon G inside $\mathcal{C}(\mathcal{L}')$ and then pulling G back along the backward orbit of $\mathcal{C}(\mathcal{L}')$ in \mathcal{L}' .*

Proof. We claim that the critical gap $\mathcal{C}(\mathcal{L})$ of \mathcal{L} is finite. Indeed, if $\mathcal{C}(\mathcal{L})$ is infinite then it is periodic of period, say, N ; thus, $\mathcal{C}(\mathcal{L})$ is an infinite gap that maps onto itself d -to-1 under σ_d^N . Yet then by Lemma 3.9 the associated minor must be periodic, a contradiction. Hence $\mathcal{C}(\mathcal{L})$ is a finite gap with dn edges, and $n \geq 2$ ($n = 1$ would imply that the minor m is degenerate, a contradiction).

Consider several cases. Suppose that $n \geq 3$ so that $\sigma_d(\mathcal{C}(\mathcal{L})) = H$ is a gap (not a leaf). We claim that then each edge of H is a limit edge of leaves of \mathcal{L} . Indeed, if ℓ is an edge of H isolated in \mathcal{L} , then there is a gap G that shares ℓ with H . Since $\mathcal{C}(\mathcal{L})$ is a finite gap, then G cannot be infinite because any infinite gap must have an image on which σ_d is not 1-to-1 while the only gap of \mathcal{L} on which σ_d is not 1-to-1 is a finite gap $\mathcal{C}(\mathcal{L})$. Thus, G is finite.

Suppose that G at some point maps onto $\mathcal{C}(\mathcal{L})$. Then on the next step G maps to H ; we may assume that $\sigma_d^N(G) = H$ for some N . It follows that G is a dn -gon while H is an n -gon. Therefore σ_d^N cannot act on G and H “swapping” them (in other words, σ_d^N cannot be “flipping” ℓ). Properties of laminations imply that σ_d^N cannot map ℓ to itself as identity either. So, σ_d^N maps ℓ to an edge $\ell' \neq \ell$ of H . Properties of laminations imply now that H maps under σ_d^N to a gap adjacent to H along ℓ' , etc. Thus, iterating σ_d^N and using properties of invariant laminations we will find an infinite concatenation of images of H which is clearly impossible (e.g., because H is (pre)periodic by Lemma 2.4).

So, G is not precritical; hence, by Lemma 2.4 G is preperiodic and never maps to H . The same can be said about H , but for an even more trivial reason: H has n edges and $\mathcal{C}(\mathcal{L})$ had $dn > n$ edges, hence H is not precritical and by Lemma 2.4 H is (pre)periodic. As we apply σ_d over and over to the union $G \cup H$, both gaps G and H stay away from the critical gap $\mathcal{C}(\mathcal{L})$ and eventually map onto two distinct adjacent finite periodic gaps \widehat{G} and \widehat{H} that share a leaf $\widehat{\ell}$. Let us show that this is impossible. Indeed, by Lemma 2.5 at every edge of G' there is an

image of H' attached to G' . Then we can say the same about each such image of H' , and so on. This yields infinite family of images of \widehat{G} or \widehat{H} , a contradiction. So, if H is a gap, then all edges of H are limit edges in \mathcal{L} . Then we can set $\mathcal{L}' = \mathcal{L}$ and case (1) holds.

Otherwise $\mathcal{C}(\mathcal{L})$ is a $2d$ -gon and $\sigma_d(\mathcal{C}(\mathcal{L}))$ is a leaf m . If m is a limit leaf from both sides, then, again, we can set $\mathcal{L}' = \mathcal{L}$ and case (1) holds. Suppose that $H = m$ is an edge of a gap G of \mathcal{L} . As before, G cannot be infinite because an infinite gap must have an image which maps forward no 1-to-1, and the only way it is possible under the circumstances is when an image of G equals $\mathcal{C}(\mathcal{L})$, a contradiction since $\mathcal{C}(\mathcal{L})$ is a $2d$ -gon. This, G is a finite gap.

We claim that G never maps onto $\mathcal{C}(\mathcal{L})$. Indeed, if it happens then m must be periodic, a contradiction; the same reason implies that G never maps onto itself. By Lemma 2.4 G is preperiodic. There are d immediate preimages of G denoted G_1, \dots, G_d ; these are gaps attached to the appropriate edges of $\mathcal{C}(\mathcal{L})$ denoted $\bar{g}_1, \dots, \bar{g}_d$, resp. Then $\bar{g}_1, \dots, \bar{g}_d$ are isolated sibling leaves of \mathcal{L} with the same image m . Evidently, if we remove their backward orbits from \mathcal{L} we will obtain a new *closed* (because \bar{g}_i 's are isolated) invariant lamination $\mathcal{L}' \subsetneq \mathcal{L}$ in which G remains a gap (no edge of G belongs to the union of the backward orbits of \bar{g}_i 's because G is not periodic). By construction, the critical gap $\mathcal{C}(\mathcal{L}')$ of the new lamination \mathcal{L}' coincides with the convex hull of $\bigcup G_i$ or, equivalently, is the union of $\mathcal{C}(\mathcal{L})$ and $\bigcup G_i$ from which all leaves \bar{g}_i are removed.

Now, the just described critical set $\mathcal{C}(\mathcal{L}')$ of the new lamination \mathcal{L}' is a finite gap with dk edges and $k \geq 3$. Hence by the above proven, all edges of $G = \sigma_d(\mathcal{C}(\mathcal{L}'))$ are limit edges of \mathcal{L}' as claimed. \square

Finally, we can prove that UML_d is a q -lamination.

Theorem 4.15. *The lamination UML_d is a q -lamination.*

Proof. Consider a minor $m = m(\mathcal{L}) \in \text{UML}_d$. If it is periodic then by Lemma 4.10 it is isolated in UML_d and is disjoint from all other minors. Assume that endpoints of m are non-periodic. By Lemma 4.14 there are two possibilities. First, m is the limit leaf of leaves of $\mathcal{L}(m)$ from both sides. Then by Lemma 4.13 and Lemma 3.3(1) m is the limit of minors from both sides. Finally, assume that $m = m(\mathcal{L})$ is not the limit of leaves of \mathcal{L} from one side. Then by Lemma 4.14 there exists a lamination \mathcal{L}' with a critical gap $\mathcal{C}(\mathcal{L}')$ with $dn, n \geq 3$ edges such that m is an edge of the gap $\sigma_d(\mathcal{C}(\mathcal{L}'))$ which is approached by leaves of \mathcal{L}' from the outside of $\sigma_d(\mathcal{C}(\mathcal{L}'))$. By Lemma 3.3(1) m is a minor, approached by other minors from the outside of $\sigma_d(\mathcal{C}(\mathcal{L}'))$. Thus, any

non-periodic minor is either itself approximated from all sides by other minors, or is an edge of a finite gap whose all edges are approached by minors from the outside of $\sigma_d(\mathcal{C}(\mathcal{L}'))$. Evidently, all this implies that UML_d is a q -lamination. \square

Lemma 4.16 is a UML_d -version of the density of hyperbolicity.

Lemma 4.16 (Fatou conjecture). *Copies of UMC are dense in UML_d .*

Proof. It suffices to prove that periodic minors are dense in UML_d . Let m be a non-periodic non-degenerate minor. By Lemma 4.13 if m is the limit of leaves $\ell_i \succ m$ then m is the limit of periodic minors ℓ'_i such that $\ell'_i \succ m$. Now, if m is not the limit of leaves $\ell_i \succ m$ with $\ell_i \in \mathcal{L}(m)$ then m is an edge of a finite gap G in UML_d and no edge of this gap is “under” m . Then m is the limit of minors all of which are “under” m . Moreover, these minors can be chosen to be limits of leaves of their respective laminations located so that the minors are “under” those leaves; hence, again by Lemma 4.13, we can find periodic minors in a small neighborhood of m located “under” m . \square

5. CONSTRUCTING UML_d

Let \mathcal{L} be a unicritical σ_d -invariant lamination with a unique critical set $\mathcal{C}(\mathcal{L}) = \mathcal{C}$. If the σ_d -orbit of \mathcal{C} is finite, then \mathcal{C} is either periodic or strictly pre-periodic. In the latter case we call the lamination \mathcal{L} a *Misiurewicz lamination* and \mathcal{C} a *Misiurewicz critical set*.

Lemma 5.1. *Let \mathcal{L} be a Misiurewicz lamination with critical set $\mathcal{C}(\mathcal{L})$. Then there exists a q -lamination $\mathcal{L}' \subset \mathcal{L}$ with critical set $\mathcal{C}(\mathcal{L}')$ such that all gaps of \mathcal{L}' are finite, $\mathcal{C}(\mathcal{L}) \subset \mathcal{C}(\mathcal{L}')$, and one of the following holds:*

- (1) $\mathcal{C}(\mathcal{L})$ is an all-critical d -gon and $\mathcal{C}(\mathcal{L}) \subset \mathcal{C}(\mathcal{L}')$;
- (2) $\mathcal{C}(\mathcal{L})$ is a critical $2d$ -gon that map onto an edge of $\sigma_d(\mathcal{C}(\mathcal{L}'))$ and $\mathcal{C}(\mathcal{L}) \subset \mathcal{C}(\mathcal{L}')$;
- (3) $\mathcal{C}(\mathcal{L}) = \mathcal{C}(\mathcal{L}')$.

In particular, all leaves of \mathcal{L}' are non-isolated from at least one side.

Proof. Clearly, \mathcal{C} must be finite (if \mathcal{C} is infinite then \mathcal{C} is periodic but we have assumed that \mathcal{C} is strictly pre-periodic). Also, all gaps of \mathcal{L} are finite. Indeed, if G is an infinite gap of \mathcal{L} , then there exists a critical forward iterate U of G contradicting the fact that \mathcal{C} is finite.

If G and H are gaps that never map to \mathcal{C} , then G and H cannot share a common leaf. Indeed, if G and H share a common leaf ℓ , we can σ_d over and over to the union $G \cup H$ and see that G and H stay away from \mathcal{C} and eventually map onto two distinct adjacent finite periodic

gaps G' and H' that share a leaf ℓ' . By Lemma 2.5 at every edge of G' there is an image of H' attached to G' , at every edge of these images of H' there is an image of G' , and so on. This yields infinite family of concatenated images of G' and H' , a contradiction.

Finally, if G is an image of \mathcal{C} , then by Lemma 4.14 only the three cases from Lemma 5.1 are possible. \square

Define an equivalence relation $\approx_{\mathcal{L}}$ on \mathbb{S}^1 by declaring that $x \sim y, x, y \in \mathbb{S}^1$ if a finite concatenation of leaves of \mathcal{L} joins x and y (see Definition 1.4). It is well-known (and easy to see) that $\approx_{\mathcal{L}}$ is a laminational equivalence relation (see the remark after Definition 1.4). Let $\psi : \mathbb{S}^1 \rightarrow \mathbb{S}^1 / \approx_{\mathcal{L}} = J_{\approx_{\mathcal{L}}} = J$ be the quotient map that semiconjugates $\sigma_3 : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ and the induced topological polynomial $f_{\mathcal{L}} = f : J \rightarrow J$.

If \mathcal{L} is a Misiurewicz lamination with critical set \mathcal{C} then we set $\mathcal{C}' = \psi(\mathcal{C})$ and observe that the f -orbit $O(\mathcal{C}')$ of \mathcal{C}' is finite and contained in J . Moreover, by Lemma 5.1 all gaps of \mathcal{L} are finite and, hence, J is a *dendrite* (i.e. it is locally connected and contain no subsets homeomorphic to the circle since such a subset would correspond to an infinite gap of \mathcal{L}). This allows us to use a standard interval notation in J and denote by $[a, b]$ the unique interval with endpoints $a, b \in J$. Let $\mathcal{H}(\mathcal{L})$ be the smallest connected set containing $O(\mathcal{C}')$ in J . Then $\mathcal{H}(\mathcal{L})$ is a tree (because $O(\mathcal{C}')$ is finite) $\mathcal{H}(\mathcal{L})$ called the *Hubbard tree (of \mathcal{L})*.

By Lemmas 3.8 and 5.1, there exists an invariant finite gap or non-degenerate leaf G of \mathcal{L} . It corresponds to a fixed point $\psi(G) \in J$. The map $\sigma_d|_G$ is a transitive rotation of period $n > 1$ (by Lemma 2.5 this can fail only if G is a leaf with fixed endpoints, but each hole of \mathcal{C} is of length at most $\frac{1}{d}$ and cannot contain two σ_d -fixed points, a contradiction). This implies that if I is a short subinterval of the interval $[\mathcal{C}', \psi(G)]$ with endpoint $\psi(G)$ then $I, f(I), \dots, f^{n-1}(I)$ are pairwise disjoint (except for the common endpoint $\psi(G)$ of them all) and $f^n(I) \supsetneq I$ (the latter follows from the fact that locally f expands because σ_d is locally expanding). Hence $\psi(G) \in \mathcal{H}(\mathcal{L})$.

Clearly, $\mathcal{H}(\mathcal{L})$ is the union all intervals $T = [f^i(\mathcal{C}'), f^j(\mathcal{C}')]$, and there are finitely many of them. If $\mathcal{C}' \notin T$ then $f(T) = [f^{i+1}(\mathcal{C}'), f^{j+1}(\mathcal{C}')] \subset \mathcal{H}(\mathcal{L})$. If $\mathcal{C}' \in T$ then $f(T) = [f^{i+1}(\mathcal{C}'), f(\mathcal{C}')] \cup [f(\mathcal{C}'), f^{j+1}(\mathcal{C}')] \subset \mathcal{H}(\mathcal{L})$. Hence $f(T) \subset T$. To show that $f(T) = T$, note that $\mathcal{H}(\mathcal{L})$ is the union of all intervals $[f^i(\mathcal{C}'), \psi(G)]$ (this is because by the previous paragraph $\psi(G) \in \mathcal{H}(\mathcal{L})$). Then each $[f^i(\mathcal{C}'), \psi(G)]$ is covered by the image of $[f^{i-1}(\mathcal{C}'), \psi(G)]$, and it remains to show that some interval $[f^k(\mathcal{C}'), \psi(G)]$ contains \mathcal{C}' . Indeed, otherwise $O(\mathcal{C})$ is contained in the hole of \mathcal{C} that contains G . However G is the unique minimal subset of that hole. Hence the periodic portion of $O(\mathcal{C})$ must in fact coincide

with G . However all preimages of G are separated from G by the set \mathcal{C} which implies the desired. We have proven the following lemma.

Lemma 5.2. *We have that $f(\mathcal{H}(\mathcal{L})) = \mathcal{H}(\mathcal{L})$ and $\psi(G) \in \mathcal{H}(\mathcal{L})$.*

The quadratic version of the next lemma is due to Lavaurs [Lav89].

Theorem 5.3. *If two leaves in UML_d have the same period and are \succ -comparable, then these two leaves are separated in \mathbb{D} by a leaf in UML_d of lower period.*

Proof. Let $m \succ n$ be minors of period p ; then their endpoints are of period p . Choose a preperiodic point $z \in H(n)$ such that the period of a periodic eventual image of z is larger than p . By Theorem 4.15 and Lemma 5.1 there exists a Misiurewicz q -lamination \mathcal{L} and a (possibly degenerate) preperiodic leaf or gap δ of UML_d such that $z \in \sigma_d(\mathcal{C}(\mathcal{L})) = \delta$ and $\delta \cap \mathbb{S}^1 \subset H(n)$. By Lemma 3.3(2) m and n are leaves of \mathcal{L} . Set $\mathcal{C}(\mathcal{L}) = \mathcal{C}$, $\psi(\mathcal{C}) = \mathcal{C}'$, $\psi(m) = m'$, $\psi(n) = n'$ and $\psi(\delta) = \delta'$; these are points of $\mathcal{H}(\mathcal{L})$. Then $\mathcal{H}(\mathcal{L}) \setminus \mathcal{C}'$ has ld components for some $l \geq 1$.

We claim that $f^k(m')$ and $f^k(n')$ lie in different components of $\mathcal{H}(\mathcal{L}) \setminus \mathcal{C}'$ for some $k > 0$. Indeed if, for all $k > 0$, $f^k(m')$ and $f^k(n')$ lie in the same component of $\mathcal{H}(\mathcal{L}) \setminus \mathcal{C}'$, then m and n are edges of the same periodic gap of the lamination \mathcal{L} . Then, by Lemma 2.5, there exists i such that $\sigma_d^i(m) = n$ contradicting the fact that a minor cannot map under itself. Hence, $f^k(m')$ and $f^k(n')$ lie in different components of $\mathcal{H}(\mathcal{L}) \setminus \mathcal{C}'$ for the minimal $k > 0$. Then, by continuity of f , there exists a gap \mathcal{C}_{-1} that separates m and n and is such that $\sigma_d^k(\mathcal{C}_{-k}) = \mathcal{C}$. Moreover, k is the least number for which such gap exists. Since m and n are p -periodic, then $k \leq p - 1$. Consider two cases.

(1) Let $k < p - 1$. Then $f^{k+1}(\mathcal{C}'_{-k}) = \delta'$ and so \mathcal{C}_{-k} maps away from m ; also, by the properties of minors m maps away from \mathcal{C}_{-k} . Hence by Lemma 4.12 there is a leaf t of \mathcal{L} that separates m and \mathcal{C}_{-k} and maps to itself under σ_d^{k+1} . Let us now choose the lest number j such that there exists a leaf t' of period j that separates m and n . It follows that $j \leq k + 1 < p$. Moreover, we may assume that t' is chosen among all such leaves as the closest to m . We claim that t' is a minor. Indeed, the only reason why t' may not be a minor is if $\sigma_d^i(t')$ is “under” t' for some $i < j$. However then by Lemma 4.12 there exists a leaf t'' that separates m and t' and is such that $\sigma_d^i(t'') = t''$, a contradiction with the choice of t' . So, t' is a minor as desired.

(2) Let $k = p - 1$. Then f^{k+1} maps n' to itself while mapping \mathcal{C}'_{-k} “under” itself. Since f^{k+1} is monotone on $[n', \mathcal{C}'_{-k}]$, a small subinterval $I = [y, m']$ of $[\mathcal{C}'_{-k}, m']$ “flips” under the action of f^p which means that

the leaf m flips under the action σ_d^p , a contradiction with the fact that the endpoints of n are f^p -fixed. Thus, case (2) is impossible. \square

Recall that by Lemma 3.10 a critical n -periodic Fatou gap U of a unicritical lamination \mathcal{L} has one n -periodic major and otherwise $d - 2$ periodic points on its boundary that are not endpoints of any edge of U . Therefore, the minor $m(\mathcal{L})$ of \mathcal{L} is a unique n -periodic non-degenerate edge of $\sigma_d(U)$ while all other n -periodic points from $\text{Bd}(\sigma_d(U))$ are disjoint from all edges of $\sigma_d(U)$. In the forthcoming arguments we will use the semiconjugacy $\psi : \text{Bd}(\sigma_d(U)) \rightarrow \mathbb{S}^1$ that collapses all edges of $\sigma_d(U)$ to points and seminconjugates $\sigma_d^n|_{\sigma_d(U)}$ and σ_d .

Lemma 5.4. *Let \mathcal{L} be a unicritical lamination with critical Fatou gap U of period n . Then all non-degenerate edges of $\sigma_d(U)$ are non-degenerate minors. In particular, periodic minors are either disjoint from $\sigma_d(U)$ or contained in it and then their periods are multiples of n . Moreover, all points of $\text{Bd}(\sigma_d(U))$ that are of period n and are not the endpoints of the minor $m(\mathcal{L})$ are degenerate minors.*

Proof. Let ℓ be a non-degenerate edge of $\sigma_d(U) = V$. Consider the convex hull C of $\sigma_d^{-1}(\ell)$. Clearly, C and its forward orbit (which begins with ℓ) is a forward invariant lamination. Then Thurston's pullback construction yields a lamination with minor ℓ as desired. It follows that if a minor m' is non-disjoint from the interior of V then in fact $m' \subset V$. One may say that the gap V with all the minors contained in it forms a part of UML_d . Since all edges of V are pullbacks of $m(\mathcal{L})$, the claim about periodic minors from the lemma easily follows.

Let $x \in \text{Bd}(V)$ be a point of period n but not an endpoint of $m(\mathcal{L}) = m$ (m is a non-degenerate edge of V such that $\sigma_d^n(m) = m$). Assume, by way of contradiction, that there exists a non-degenerate minor $\bar{y} = m(\mathcal{L}')$ with an endpoint x (here \mathcal{L}' is some unicritical lamination). By the previous paragraph $\bar{y} \subset V$. Construct the lamination $\mathcal{L}(\bar{y})$ as usual: first construct the critical set which is the convex hull $\mathcal{C}(\bar{y})$ of $\sigma_d^{-1}(\bar{y})$ and then pull it back in the compatible with itself way.

Evidently, all pullbacks of $\mathcal{C}(\bar{y})$ as well as images of \bar{y} are contained in the corresponding images and pullbacks of V that are gaps of \mathcal{L} . The same applies to the leaves from the closure of these pullbacks of $\mathcal{C}(\bar{y})$ and images of \bar{y} , i.e. to all leaves of $\mathcal{L}(\bar{y})$. Thus, the restriction $\mathcal{L}(\bar{y})|_V$ of $\mathcal{L}(\bar{y})$ onto V satisfies all properties of invariant laminations except that instead of σ_d we need to consider σ_d^n . Applying ψ to it, we will obtain a σ_d -invariant unicritical lamination with $\psi(x)$ belonging to the image of its critical set. However then by Lemma 3.7 no leaf of

this lamination may have an endpoint $\psi(x)$, a contradiction with the existence of the leaf $\psi(\bar{y})$. \square

Now, suppose that for some k all periodic minors of periods at most k , non-degenerate and degenerate, are already constructed. Denote their union by A_k . Take a component B of $\overline{\mathbb{D}} \setminus A_k$. Let z be a periodic point of period $k + 1$ that belongs to B . By Lemma 3.8 there is a lamination \mathcal{L} with a critical Fatou gap U of period $k + 1$ such that $z \in \sigma_d(U) = V$. By Lemma 5.4 the entire V is contained in B . Either z is an endpoint of the minor $m(\mathcal{L})$, or z is a degenerate minor located “under” $m(\mathcal{L})$. All of this can happen more than once inside B and will exhaust all periodic points of period $k + 1$ that belong to B .

We need to make another observation based upon our study of the Unicritical Main Cardioid UMC . By construction, leaves from UMC of periods at most k are included in A_k , and no minor intersects the interior of UMC . Hence each minor of period $k + 1$ contained in B is located between two consecutive σ_d -fixed points, and the same applies to any gap similar to the gap V considered above. Together with the previous paragraph, this yields an algorithm for constructing UML_d . The quadratic version of this algorithm was proved by Lavaurs in [Lav89].

6. THE LAVAURS ALGORITHM IN THE UNICRITICAL CASE

Let us construct leaves of UML_d that have periodic endpoints; the closure of their collection is UML_d . The map σ_d has $d - 1$ fixed points on the circle: $0, \frac{1}{d-1}, \frac{2}{d-1}, \dots, \frac{d-2}{d-1}$. By Lemma 3.7 they are *degenerate minors*. We place a *dot* on the circle at each of these points.

Now, take points of *period 2* under σ_d . The map σ_d has $d^2 - 1$ points of period 2 on the circle given by $\{\frac{k}{d^2-1} | k = 0, 1, 2, \dots, d^2 - 1\}$. Some of these will correspond to *degenerate minors* and some of them will be the endpoints of *non-degenerate minors*. It follows that for any *non-degenerate* minor γ of period 2, there exist $d - 2$ *degenerate* minors δ_i such that $\gamma \succ \delta_i$ for every i . Moreover, any non-degenerate minor of period 2 is located between two consecutive fixed points. Take the smallest point of period 2: $\frac{1}{d^2-1}$, connect it to the point $\frac{d}{d^2-1}$, and place a *dot* at the intermediate points $\frac{2}{d^2-1}, \frac{3}{d^2-1}, \dots, \frac{d-1}{d^2-1}$ (by Lemma 5.4 they are *degenerate minors*). Continue this way around the circle.

In general, suppose the periodic points on the circle of period less than k have been *connected* or *dotted*. Connect those of period k starting with the smallest point p of period k . Let q_1, q_2, \dots, q_{d-1} be the next $d - 1$ points *ordered anticlockwise* and belonging to the same component of $\overline{\mathbb{D}}$ with minors of periods less than k removed as p . Connect p to q_{d-1} and place a *dot* at the intermediate points: q_1, \dots, q_{d-2} . Choose

the next available (not yet *connected* or *dotted*) point of period k and proceed in the same way; continue similarly around the circle. We do this for periodic points on the circle in the order of increasing periods.

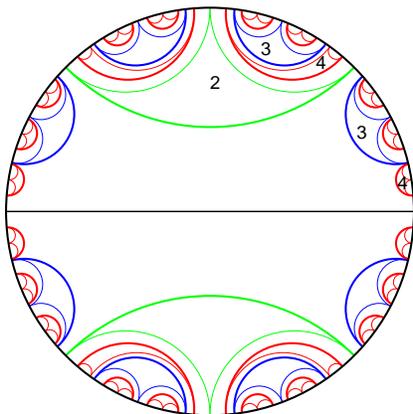


FIGURE 1. Construction of the periodic minors of the Unicritical Minor Lamination UML_3 up to period 4; the gaps for different periods are coded by colors, and some periods are labeled.

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